J. Math. Soc. Japan
Vol. 20, No. 3, 1968

Differential geometry of complex hypersurfaces II*

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(Received Jan. 8, 1968)

In this paper we continue the study of complex hypersurfaces of complex space forms (i. e. Kählerian manifolds of constant holomorphic sectional curvature) begun in [8]. The main results are: the determination of the holonomy groups of such hypersurfaces, a generalization of the main theorem of [8] on Einstein hypersurfaces, the non-existence of a certain type of hypersurface in the complex projective space, and some results concerning the curvature of complex curves.

Let $\tilde{M}$ be a complex space form (which in general will not be complete) of complex dimension $n+1$ and let M be an immersed complex hypersurface in $\tilde{M}$. In § 1 we show that the rank of the second fundamental form of M is intrinsic and that M is rigid in $\tilde{M}$, if the latter is simply connected and complete. The local version of rigidity is contained as a special case in the work of Calabi [1], but our method is more direct and more in the line of classical differential geometry.

The holonomy group of M (with respect to the induced Kähler metric) is studied in § 2. If the holomorphic sectional curvature $\tilde{c}$ of $\tilde{M}$ is negative, the holonomy group is always $U(n)$. In the case where $\tilde{c}>0$ (e. g. $\tilde{M}=P^{n+1}(C)$), the holonomy group of M is either $U(n)$ or $SO(n)\times S^1$ (S^1 denotes the circle group), the latter case arising only when M is locally holomorphically isometric to the complex quadric Q^n in $P^{n+1}(C)$. When $\tilde{c}=0$ (i. e. when $\tilde{M}$ is flat), the holonomy group of M depends on the rank of the second fundamental form and we obtain a result of Kerbrat [3] more directly.

In § 3 we first obtain the following generalized local version of the classification theorem of [8]. If the Ricci tensor S of M is parallel (i. e. $\nabla S=0$), then M is totally geodesic in $\tilde{M}$ or else $\tilde{c}>0$ and M is locally a complex quadric. To prove this we modify Theorem 2 [8] to show that M is locally symmetric when its Ricci tensor is parallel, and obtain the local classification without using the list of irreducible Hermitian symmetric spaces. This local version was proved by Chern [2] with the original assumption that M is Einstein, and Takahashi [9] has shown that M is Einstein if its Ricci tensor

* This work was partially supported by grants from the National Science Foundation.

is parallel. It is worth noting that when $\tilde{c} \neq 0$ this latter result follows immediately from Theorem 2 of § 2. We conclude this section with a better global version of the classification theorem of [8]—here the proof is made considerably more elementary than the original one and simple-connectedness of the hypersurfaces is no longer assumed in the case $\tilde{c} \leqq 0$.

We show, in § 4, that the rank of the second fundamental form cannot be identically equal to 2 on a compact complex hypersurface in $P^{n+1}(C)$, $n \geqq 3$. In § 5 we discuss the Gaussian mapping of a complex hypersurface M in C^{n+1} into $P^n(C)$; we find that its Jacobian is essentially the second fundamental form and we show how the Gaussian mapping relates the Kählerian connections of M and $P^n(C)$.

The study of complex curves in a 2-dimensional complex space form is taken up in § 6. First we take care of the case $n=1$ in Theorems 4 and 5. We then obtain some characterizations of P^1 and Q^1 among closed nonsingular complex curves in $P^2(C)$ by curvature conditions.

We shall use the same notation as in [8].

§ 1. Rigidity.

Let M be a Kähler manifold of complex dimension n and let f be a Kählerian immersion (i. e. a complex isometric immersion) of M as a complex hypersurface in a space $\tilde{M}$ of constant holomorphic curvature $\tilde{c}$. For each point $x_0 \in M$ there is a neighborhood $U(x_0)$ of x_0 in M on which Gauss' equation for the immersion f may be written as

$$R(X, Y) = \tilde{R}(X, Y) + D(X, Y)$$

with

$$\tilde{R}(X, Y) = \frac{\tilde{c}}{4}\{X \wedge Y + JX \wedge JY + 2g(X, JY)J\}$$

and

$$D(X, Y) = AX \wedge AY + JAX \wedge JAY\,,$$

where $X \wedge Y$ denotes the skew-symmetric endomorphism which maps Z upon $g(Y, Z)X - g(X, Z)Y$, and X, Y, Z are tangent vectors to M (see Proposition 3 [8]). Whereas A depends on the immersion f and on a local choice of unit vector field normal to M, the following lemma shows that its kernel does not.

LEMMA 1. *At each point* $x \in U(x_0)$ *we have*

$$\begin{aligned}\operatorname{Ker} A &= \{X \in T_x(M) \,|\, D(X, Y) = 0 \quad \text{for all } Y \in T_x(M)\} \\ &= \{X \in T_x(M) \,|\, (R - \tilde{R})(X, Y) = 0 \quad \text{for all } Y \in T_x(M)\}\,.\end{aligned}$$

PROOF. Clearly $\operatorname{Ker} A$ is contained in the subspace defined by D. On the other hand, if $X \notin \operatorname{Ker} A$ then $D(X, JX) = -2AX \wedge JAX \neq 0$, and the first equality follows. The second equality follows immediately from Gauss' equation.

REMARK. The (even) integer rank A_x will be called the *rank* of M at x since Lemma 1 shows that it is intrinsic (i. e. depends only on M). It is in fact twice the type number in the sense of Allendoerfer.

Let $f, \bar{f}: M \to \tilde{M}$ be two Kählerian immersions. For each $x_0 \in M$ there is a neighborhood $U(x_0)$ of x_0 in M on which we can choose of a unit normal vector field ξ (resp. $\bar{\xi}$) for the immersion f (resp. $\bar{f}$), thereby giving rise to tensor fields A and s (resp. $\bar{A}$ and $\bar{s}$) on $U(x_0)$, as indicated in [8].

LEMMA 2. *At each point* $x \in U(x_0)$

i) $A=0$ *if and only if* $\bar{A}=0$,

ii) *if* $A=\bar{A}\neq 0$ *and* $\nabla A=\nabla\bar{A}$, *then* $s=\bar{s}$.

PROOF. i) This follows from Lemma 1, since each of these conditions is equivalent to $R=\tilde{R}$ on $T_x(M)$.

ii) The equations of Codazzi for the two immersions yield

$$(\bar{s}(X)-s(X))JAY=(\bar{s}(Y)-s(Y))JAX \qquad \text{for} \quad X, Y\in T_x(M)\,.$$

If $X\notin \operatorname{Ker} A$, then we get, by setting $Y=JX$, $(\bar{s}(X)-s(X))AX=(\bar{s}(JX)-s(JX))JAX$. Since $AX(\neq 0)$ and JAX are linearly independent, we conclude that $\bar{s}(X)=s(X)$. If $X\in \operatorname{Ker} A$, then we choose $Y\notin \operatorname{Ker} A$ and get $(\bar{s}(X)-s(X))JAY=0$, that is, $\bar{s}(X)=s(X)$.

LEMMA 3. *Assuming* $R\neq\tilde{R}$ *(that is,* $A\neq 0$*) at some point of* M, *let* x_0 *be a point where the rank of* M *is maximal. There exists a neighborhood* $U(x_0)$ *of* x_0 *on which we may choose unit normal vector fields* ξ *and* $\bar{\xi}$, *with respect to the immersions* f *and* $\bar{f}$, *respectively, such that* $A=\bar{A}$ *and* $s=\bar{s}$ *on* $U(x_0)$.

PROOF. On a neighborhood of x_0 on which the rank of M is constant and equal to k, say, we choose unit normal vector fields ξ and $\bar{\xi}$, with respect to the immersions f and $\bar{f}$, respectively. At each point x of this neighborhood we choose an orthonormal basis $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ of $T_x(M)$ for which the matrix of A is of the form

$$\begin{pmatrix} \lambda_1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \lambda_k & & & & & & & \\ & & & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & & & & \\ & & & & & & -\lambda_1 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & -\lambda_k & & \\ & & & & & & & & & 0 & \\ & & & & & & & & & & \ddots & \\ & & & & & & & & & & & 0 \end{pmatrix}.$$

Now

$$(R-\tilde{R})(e_i, Je_i) = -2Ae_i \wedge JAe_i = -2\bar{A}e_i \wedge J\bar{A}e_i$$

and the middle form of this identity being nonzero when $i \leqq k$, we see that $\bar{A}e_i$ is a linear combination of Ae_i and JAe_i, say

$$\bar{A}e_i = \alpha_i Ae_i + \beta_i JAe_i .$$

It is then clear that $\alpha_i^2 + \beta_i^2 = 1$. From

$$R(e_i, e_j) - \tilde{R}(e_i, e_j) = Ae_i \wedge Ae_j + JAe_i \wedge JAe_j$$
$$= \bar{A}e_i \wedge \bar{A}e_j + J\bar{A}e_i \wedge J\bar{A}e_j$$

we can easily deduce that $\alpha_i = \alpha_j = \alpha$, say, and $\beta_i = \beta_j = \beta$, say, for $1 \leqq i, j \leqq k$. However $\operatorname{Ker} A = \operatorname{Ker} \bar{A}$, by virtue of Lemma 1, and therefore $\bar{A} = \alpha A + \beta JA$ with $\alpha^2 + \beta^2 = 1$ at each point of a neighborhood of x_0. By virtue of the assumption on the rank of M at x_0 we can find a differentiable vector field X on a neighborhood of x_0 such that $AX \neq 0$; and, since $\alpha = \dfrac{g(\bar{A}X, AX)}{g(AX, AX)}$, it follows that α (and similarly β) is a differentiable function on a neighborhood of x_0. We may then define a differentiable function θ on a neighborhood $U(x_0)$ of x_0 such that $\alpha = \cos\theta$ and $\beta = \sin\theta$. Then $\xi' = \cos\theta\xi + \sin\theta J\xi$ is a unit normal vector field on $U(x_0)$ with respect to the immersion f and clearly $A' = \bar{A}$. By Lemma 2, it follows that $s' = \bar{s}$ also.

THEOREM 1. *A connected Kählerian hypersurface M of complex dimension $n \geqq 1$ of a simply connected complete complex space form $\tilde{M}$ is rigid in $\tilde{M}$.*

PROOF. If $R = \tilde{R}$ at every point of M, then M has constant holomorphic sectional curvature $\tilde{c}$. Therefore, by Corollary 2 of [8, § 3], M is totally geodesic in $\tilde{M}$ and thus is rigid. If $R \neq \tilde{R}$ at some point of M, let x_0 be a point where the rank of M is maximal. Let $f, \bar{f}: M \to \tilde{M}$ be two Kählerian immersions. By virtue of Lemma 3, there exists a neighborhood $U(x_0)$ of x_0 and suitably chosen unit normal vector fields ξ and $\bar{\xi}$ on $U(x_0)$ with respect to the immersions f and $\bar{f}$ respectively such that $A = \bar{A}$ and $s = \bar{s}$ on $U(x_0)$. We now resort to local coordinates to show that f and $\bar{f}$ differ by a holomorphic motion ϕ of $\tilde{M}$ on $U(x_0)$, that is, $\bar{f} = \phi \circ f$ on $U(x_0)$; and, by analyticity, this will then hold on all of M. In fact, since the group of holomorphic isometries of $\tilde{M}$ is transitive on the set of unitary frames, we may assume without loss of generality that

$$f(x_0) = \bar{f}(x_0), \qquad f_*(x_0) = \bar{f}_*(x_0), \qquad \xi(x_0) = \bar{\xi}(x_0),$$

where f_* and $\bar{f}_*$ denote the differentials of f and $\bar{f}$, respectively, and prove that $f = \bar{f}$ in a neighborhood of x_0. Let $(x^1, \cdots, x^{2n})$ be a system of local coordinates on $U(x_0)$ and let $(u^1, \cdots, u^{2n+2})$ be a system of local coordinates on a neighborhood of $f(x_0)$ in $\tilde{M}$ derived from a system of complex coordinates.

We agree on the following ranges for the indices:

$$1 \leqq i, j, k, l \leqq 2n, \qquad 1 \leqq p, q, r, s \leqq 2n+2.$$

Our notation (in the summation convention) will be

$$f^p(x) = u^p(f(x)), \quad f^p_i = \frac{\partial f^p}{\partial x^i}, \quad f^p_{ij} = \frac{\partial^2 f^p}{\partial x^i \partial x^j}, \quad \text{etc.}, \quad f_*\left(\frac{\partial}{\partial x^i}\right) = f^p_i\left(\frac{\partial}{\partial u^p}\right),$$

$$\xi = \xi^r \frac{\partial}{\partial u^r}, \quad J\xi = (J\xi)^r \frac{\partial}{\partial u^r}, \quad \xi^r_i = \frac{\partial \xi^r}{\partial x^i}, \quad \xi^r_{ij} = \frac{\partial^2 \xi^r}{\partial x^i \partial x^j}, \quad \text{etc.}.$$

The corresponding notation for $\bar{f}$ is then self-explanatory. We also use

$$h_{ij} = h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \qquad k_{ij} = k\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

$$A\frac{\partial}{\partial x^i} = a^j_i \frac{\partial}{\partial x^j}, \qquad s\left(\frac{\partial}{\partial x^i}\right) = s_i.$$

(Note that we have $A = \bar{A}$ and $s = \bar{s}$ so that we do not need the corresponding notation for $\bar{f}$ here). The Christoffel symbols are denoted by Γ^i_{jk} for $(x^1, \cdots, x^{2n})$ and by Γ^p_{qr} for $(u^1, \cdots, u^{2n+2})$. We note that $(J\xi)^r = -\xi^{r+n+1}$ and $(J\xi)^{r+n+1} = \xi^r$ (indices are understood here modulo $2n+2$) because of the nature of the coordinate system $(u^1, \cdots, u^{2n+2})$. The equations

$$\tilde{\nabla}_{f_*\left(\frac{\partial}{\partial x^i}\right)} f_*\left(\frac{\partial}{\partial x^j}\right) = f_*\left[\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right] + h\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]\xi + k\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]J\xi,$$

$$\tilde{\nabla}_{f_*\left(\frac{\partial}{\partial x^i}\right)} \xi = -f_*\left[A\frac{\partial}{\partial x^i}\right] + s\left[\frac{\partial}{\partial x^i}\right]J\xi$$

for the immersion f then yield

(I) $$f^r_{ij} = -f^p_i f^q_j \Gamma^r_{pq} + f^r_k \Gamma^k_{ij} + h_{ij}\xi^r + k_{ij}(J\xi)^r,$$

(II) $$\xi^r_i = -f^p_i \xi^q \Gamma^r_{pq} - a^j_i f^r_j + s_i (J\xi)^r.$$

We denote the corresponding equations for the immersion $\bar{f}$ by $(\bar{\text{I}})$ and $(\overline{\text{II}})$. At x_0 we have

(1) $$f^p(x_0) = \bar{f}^p(x_0), \quad f^p_i(x_0) = \bar{f}^p_i(x_0), \quad \xi^r(x_0) = \bar{\xi}^r(x_0), \quad (J\xi)^r(x_0) = (J\bar{\xi})^r(x_0).$$

We wish to show that $f = \bar{f}$ in a neighborhood of x_0; since f^p and $\bar{f}^p$ are real analytic it suffices to prove

(2) $$f^p_{ij}(x_0) = \bar{f}^p_{ij}(x_0),$$

(4) $$f^p_{ijk}(x_0) = \bar{f}^p_{ijk}(x_0),$$

and so on for all higher-order derivatives at x_0. (2) follows from (I), $(\bar{\text{I}})$, (1) and the equation $A = \bar{A}$ on $U(x_0)$, while

(3) $$\xi^r_i(x_0) = \bar{\xi}^r_i(x_0)$$

follows from (II), $(\overline{\text{II}})$, (1) and the equations $A = \bar{A}$ and $s = \bar{s}$ on $U(x_0)$. Now

f_{ijk}^r and $\bar{f}_{ijk}^r$ are obtained by differentiating (I) and ($\bar{\text{I}}$) and we deduce (4) from the equations (1), (2), (3) and the equation $A=\bar{A}$ on $U(x_0)$. In the same manner ξ_{ij}^r and $\bar{\xi}_{ij}^r$ are obtained by differentiating (II) and ($\overline{\text{II}}$). Using the previous equations together with the equations $A=\bar{A}$ and $s=\bar{s}$ on $U(x_0)$, we infer

$$\xi_{ij}^r(x_0)=\bar{\xi}_{ij}^r(x_0)\,. \tag{5}$$

We can then easily obtain

$$f_{ijkl}^p(x_0)=\bar{f}_{ijkl}^p(x_0)\,. \tag{6}$$

The equalities for higher-order derivatives are obtained in the same fashion. Thus $f=\bar{f}$ in a neighborhood of x_0 and this completes the proof.

§ 2. Holonomy.

In this section we study the restricted holonomy group H of a complex hypersurface M in a space $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$. When the complex dimension n of M equals 1 it is clear that either $H=U(1)$ or M is flat. In this latter case the results of § 6 will show that $\tilde{c}=0$. It will then be clear that Theorems 2 and 3 in this section are valid for $n=1$. We therefore assume $n\geqq 2$ in the following.

On a neighborhood $U(x_0)$ of any point $x_0\in M$, the Riemannian and Ricci curvature tensors of M may be written as

$$R(X,Y)=\frac{\tilde{c}}{4}\{X\wedge Y+JX\wedge JY+2g(X,JY)J\}+AX\wedge AY+JAX\wedge JAY\,, \tag{7}$$

$$S(X,Y)=\frac{(n+1)\tilde{c}}{2}g(X,Y)-2g(A^2X,Y)\,, \tag{8}$$

where $X, Y\in T_x(M)$ and $x\in U(x_0)$ [8]. We pick an orthonormal basis $\{e_1,\cdots,e_n, Je_1,\cdots,Je_n\}$ of $T_{x0}(M)$ with respect to which the matrix of A is of the form

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -\lambda_n \end{pmatrix},$$

where $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n\geqq 0$. With respect to this basis the Lie algebra of the group of unitary transformations of the tangent space $T_{x0}(M)$ may be identified with the Lie algebra of all block matrices of the form $\begin{bmatrix} C & -D \\ D & C\end{bmatrix}$, where C and D are respectively skew-symmetric and symmetric $n\times n$ real matrices. The holonomy algebra $\mathfrak{h}$ is thereby identified with a Lie subalgebra (also denoted

$\mathfrak{h}$) of this matrix algebra. For the sake of brevity we frequently use the same symbol to denote an endomorphism of $T_{x_0}(M)$ and its matrix with respect to the above basis. We shall say that M is *nondegenerate* when $J\in\mathfrak{h}$ and this definition is independent of the point x_0 (see [4], where the notion of non-degeneracy was defined to mean $J\in H$).

In this section all indices range from 1 to n and *we agree that* $i\neq j$. Let E^i_j denote the $n\times n$ matrix whose (i, j) entry (i-th row, j-th column) is 1 and whose (j, i) entry is -1, all other entries being zero. For $p\neq q$ as well as $p=q$, let F^p_q denote the $n\times n$ matrix whose (p, q) and (q, p) entries equal 1, all other entries being zero. Setting $K^i_j=\begin{bmatrix} E^i_j & 0 \\ 0 & E^i_j \end{bmatrix}$ and $S^p_q=\begin{bmatrix} 0 & -F^p_q \\ F^p_q & 0 \end{bmatrix}$ (including $p=q$), the following identities are readily verified (assuming $i\neq j$ as agreed):

$$(9)\qquad \begin{cases} [K^i_j, S^i_k]=-S^j_k & (k\neq j)\,, \\ [K^i_j, S^i_j]=2(S^i_i-S^j_j)\,, \\ [S^i_j, S^i_i]=K^i_j\,, \end{cases}$$

where [,] denotes the usual bracket operation.

The holonomy algebra $\mathfrak{h}$ contains all curvature transformations of $T_{x_0}(M)$ and in particular the endomorphisms $R(e_i, e_j)$, $R(e_i, Je_j)$ and $R(e_i, Je_i)$ for all i, j. Their matrices with respect to the above basis are respectively

$$\left(\lambda_i\lambda_j+\frac{\tilde{c}}{4}\right)K^i_j\,,\qquad -\left(\lambda_i\lambda_j-\frac{\tilde{c}}{4}\right)S^i_j\qquad\text{and}\qquad -\frac{\tilde{c}}{2}J+2\left(\lambda_i^2-\frac{\tilde{c}}{4}\right)S^i_i\,,$$

as may be verified by using (7). In the proofs which follow we make repeated use of the fact that these are elements of $\mathfrak{h}$.

LEMMA 4. *Let* $\tilde{c}>0$.

i) $K^k_l\in\mathfrak{h}$ *for all* k, l $(k\neq l)$.

ii) *If* $S^j_j\in\mathfrak{h}$ *for some* j, *then* $\mathfrak{h}=\mathfrak{u}(n)$.

iii) *If* $S^i_j\in\mathfrak{h}$ *and* $\lambda_i\neq\lambda_j$ *for some pair* (i, j), *then* $\mathfrak{h}=\mathfrak{u}(n)$.

PROOF. i) Since $\lambda_k\geqq 0$ for all k and $\tilde{c}>0$, $R(e_k, e_l)\in\mathfrak{h}$ implies $K^k_l\in\mathfrak{h}$ for every pair (k, l).

ii) For $k\neq j$, we have $[K^j_k, S^j_j]=-S^j_k\in\mathfrak{h}$ using (i) and the assumption. Thus $[K^j_k, S^j_k]=2(S^j_j-S^k_k)\in\mathfrak{h}$ and hence $S^k_k\in\mathfrak{h}$ for all k. In addition, $[K^k_l, S^k_k]=-S^k_l\in\mathfrak{h}$ when $k\neq l$. Since K^i_j for all $i\neq j$ and S^p_q for all p, q together span $\mathfrak{u}(n)$, we have $\mathfrak{h}=\mathfrak{u}(n)$.

iii) By (i) and by the assumption, we have $[K^i_j, S^i_j]=2(S^i_i-S^j_j)\in\mathfrak{h}$. Since

$$\begin{aligned} R(e_i, Je_i)-R(e_j, Je_j) &= -\frac{\tilde{c}}{2}(S^i_i-S^j_j)+2(\lambda_i^2S^i_i-\lambda_j^2S^j_j) \\ &= \left(2\lambda_i^2-\frac{\tilde{c}}{2}\right)(S^i_i-S^j_j)+2(\lambda_i^2-\lambda_j^2)S^j_j \end{aligned}$$

belongs to $\mathfrak{h}$, we infer that $(\lambda_i^2-\lambda_j^2)S^j_j\in\mathfrak{h}$ and hence $S^j_j\in\mathfrak{h}$ since $\lambda_i\neq\lambda_j$. By (ii), we have $\mathfrak{h}=\mathfrak{u}(n)$.

THEOREM 2. *Let M be a complex hypersurface of complex dimension $n\geqq 1$ in a space $\tilde{M}$ of constant holomorphic curvature $\tilde{c}$ ($\neq 0$) and let H be the restricted holonomy group of M (with respect to the induced Kählerian structure). Then*

i) *if $\tilde{c}<0$, H is always isomorphic to $U(n)$.*

ii) *if $\tilde{c}>0$, H is isomorphic either to $U(n)$ or to $SO(n)\times S^1$, where S^1 denotes the circle group, the second case arising only when M is locally holomorphically isometric to the complex quadric Q^n in $P^{n+1}(C)$.*

PROOF. i) Since $\tilde{c}<0$, the Ricci tensor is negative definite according to (8) and M is therefore nondegenerate (see [4]; actually it was proved there that $J\in H$ but the proof shows that $J\in\mathfrak{h}$). Since $R(e_i, Je_j)=\left(\frac{\tilde{c}}{4}-\lambda_i\lambda_j\right)S_j^i\in\mathfrak{h}$ and since $\lambda_k\geqq 0$ for all k and $\tilde{c}<0$, we have $S_j^i\in\mathfrak{h}$ for every pair (i,j). Since $R(e_i, Je_i)\in\mathfrak{h}$ and $J\in\mathfrak{h}$, we have $S_i^i\in\mathfrak{h}$. Thus $K_j^i=[S_j^i, S_i^i]\in\mathfrak{h}$ for all i, j. Hence $\mathfrak{h}=\mathfrak{u}(n)$.

ii) We first dispense with the case where M is an Einstein manifold, in which case $A^2=\lambda^2 I$. Since $\sum_{r=1}^{n} R(e_r, Je_r)=-\rho J\in\mathfrak{h}$, where ρ is the Ricci curvature of M, and since ρ is nonzero in view of Proposition 9 [8], we deduce that $J\in\mathfrak{h}$. From the curvature transformations $R(e_i, e_j)$, $R(e_i, Je_j)$ and $R(e_i, Je_i)$ we conclude that all K_j^i $(i\neq j)$ and S_j^i ($i=j$ included) are contained in $\mathfrak{h}$, that is, $H=U(n)$, unless $\lambda^2=\tilde{c}/4$ (i. e. $\rho=n\tilde{c}/2$). At any rate we know that M is locally symmetric so that the curvature transformations at any point x_0 generate the holonomy algebra $\mathfrak{h}$. If $\lambda^2=\tilde{c}/4$, we readily see that $\mathfrak{h}$ is generated by J and by all K_j^i, that is $H=SO(n)\times S^1$. On the other hand, the complex quadric $Q^n=SO(n+2)/SO(n)\times SO(2)$ imbedded in $P^{n+1}(C)$ with holomorphic curvature $\tilde{c}$ is Einstein and has holonomy group isomorphic to $SO(n)\times SO(2)$ (i. e. $SO(n)\times S^1$). Thus $\lambda^2=\tilde{c}/4$ for Q^n. Now if $\lambda^2=\tilde{c}/4$ for M, the same argument as was used in Proposition 11 of [8] can be applied locally to show that M is locally holomorphically isometric to Q^n. We have thus taken care of Theorem 2 in the case where M is Einstein (getting a more precise result than Proposition 10 of [8]).

If M is not an Einstein manifold we may assume that the characteristic roots of A^2 at x_0 are not all equal. By (i) of Lemma 4 we know that $K_l^k\in\mathfrak{h}$ for all k, l. If $4\lambda_i^2=\tilde{c}$ for some i, then $R(e_i, Je_i)=-\frac{\tilde{c}}{2}J\in\mathfrak{h}$. By the assumption on A^2 at x_0, we have $4\lambda_j^2\neq\tilde{c}$ for some j and consequently $S_j^j\in\mathfrak{h}$ from $R(e_j, Je_j)=-\frac{\tilde{c}}{2}J+2\left(\lambda_j^2-\frac{\tilde{c}}{4}\right)S_j^j\in\mathfrak{h}$. By (ii) of Lemma 4 we conclude that $\mathfrak{h}=\mathfrak{u}(n)$, that is, $H=U(n)$. We may therefore suppose $4\lambda_i^2\neq\tilde{c}$ for every i. If $4\lambda_1^2<\tilde{c}$, then $4\lambda_1\lambda_n<\tilde{c}$, since $\lambda_1>\lambda_n$; therefore $R(e_1, Je_n)\in\mathfrak{h}$ implies $S_n^1\in\mathfrak{h}$. By (iii) of Lemma

4 we have $\mathfrak{h}=\mathfrak{u}(n)$. Similarly, if $4\lambda_n^2>\tilde{c}$, we find $\mathfrak{h}=\mathfrak{u}(n)$ again. Thus we are led to suppose

$$\lambda_1^2\geqq\lambda_2^2\geqq\cdots\geqq\lambda_m^2>\frac{\tilde{c}}{4}>\lambda_{m+1}^2\geqq\cdots\geqq\lambda_n^2,\qquad 1\leqq m<n.$$

Taking the case $n\geqq 3$, we see that if $m\geqq 2$ then $\lambda_1\lambda_m\geqq\lambda_m^2>\frac{\tilde{c}}{4}$, so that $S_m^1\in\mathfrak{h}$; however $[K_n^1, S_m^1]=-S_m^n\in\mathfrak{h}$ and $\lambda_m\neq\lambda_n$. Thus $\mathfrak{h}=\mathfrak{u}(n)$ again, by (iii) of Lemma 4. If $n\geqq 3$ and $m=1$, then $\lambda_2\lambda_n\leqq\lambda_2^2<\frac{\tilde{c}}{4}$ and $S_n^2\in\mathfrak{h}$ so that $[K_1^2, S_n^2]=-S_n^1\in\mathfrak{h}$. Thus $\mathfrak{h}=\mathfrak{u}(n)$ again. Finally, we suppose $n=2$, in which case $m=1$. If $J\notin\mathfrak{h}$, then the Ricci tensor is singular everywhere [4], or what amounts to the same thing, $A^2-\frac{3\tilde{c}}{4}I$ is singular everywhere. Thus $\left(\lambda_1^2-\frac{3\tilde{c}}{4}\right)\left(\lambda_2^2-\frac{3\tilde{c}}{4}\right)=0$. Since $\lambda_2^2<\frac{\tilde{c}}{4}$, we must have $\lambda_1^2=\frac{3\tilde{c}}{4}$. Since $\lambda_1\lambda_2=\frac{\tilde{c}}{4}$, we have $\lambda_2^2=\frac{\tilde{c}}{12}$. We see then that $R(e_1, Je_1)$ and $R(e_2, Je_2)$ are linear combinations of S_1^1 and S_2^2, from which we can solve for S_1^1 and S_2^2. Thus $S_1^1, S_2^2\in\mathfrak{h}$ and hence $J=S_1^1+S_2^2\in\mathfrak{h}$. We have thus shown $J\in\mathfrak{h}$. Now $\lambda_1^2>\frac{\tilde{c}}{4}$ and $R(e_1, Je_1)\in\mathfrak{h}$ imply $S_1^1\in\mathfrak{h}$. By (ii) of Lemma 4 we have $\mathfrak{h}=\mathfrak{u}(2)$. This completes the proof of Theorem 2.

COROLLARY. *Let M be a complete complex hypersurface in $P^{n+1}(C)$ or in D^{n+1}. Then the largest connected group of affine transformations of M (with respect to the induced Kählerian connection) preserves the complex structure.*

PROOF. This follows from Theorem 2 and from Theorem 3 of [4].

The following theorem has been obtained by Kerbrat [3] using a different method.

THEOREM 3. *Let M be a complex n-dimensional hypersurface in a flat Kähler manifold $\tilde{M}$. If at some point the rank of M equals $2n$, then the restricted holonomy group of M is isomorphic to $U(n)$.*

PROOF. We may suppose that rank $A=2n$ at x_0. An examination of the curvature transformations reveals that $K_j^i, S_j^i, S_i^i\in\mathfrak{h}$ for all i, j. Thus H is isomorphic to $U(n)$.

§ 3. Hypersurfaces with parallel Ricci tensor.

On a neighborhood $U(x_0)$ of each point x_0 of a complex hypersurface M in a space $\tilde{M}$ of constant holomorphic curvature $\tilde{c}$, Codazzi's equation

$$(\nabla_X A)Y-(\nabla_Y A)X-s(X)JAY+s(Y)JAX=0$$

holds, where $X, Y\in T_x(M)$ and x is any point of $U(x_0)$. When the simpler equation $(\nabla_X A)Y=s(X)JAY$ is valid on a neighborhood of every point in M

we say that Codazzi's equation reduces. We have

LEMMA 5. *The following conditions are equivalent on M:*

i) *Codazzi's equation reduces.*

ii) *The Ricci tensor of M is parallel, that is $\nabla S=0$.*

iii) *M is locally symmetric.*

REMARK. This result has been obtained independently by T. Takahashi [**9**] using another method. In the case $\tilde{c}\neq 0$ we know by Theorem 2 in § 2 that either M is locally Q^n, which is Einstein, or the holonomy group of M is $U(n)$. In the second case, $\nabla S=0$ implies that M is Einstein because M is irreducible. Thus Lemma 5 generalizes Theorem 2 of [8] only in the case $\tilde{c}=0$. We shall, however, give a direct proof of (ii)$\rightarrow$(i).

PROOF. The proof of Theorem 2 [8] shows that (i) implies (iii). (iii) implies (ii) trivially. We now show that (ii) implies (i). $\nabla S=0$ is equivalent to $\nabla A^2=0$ and this in turn implies that the characteristic roots of A^2 together with their multiplicities are constant on M. Consequently, if $A^2=0$ at one point then A^2 vanishes identically and Codazzi's equation reduces trivially. Assuming that this is not the case, let $\lambda_1, \cdots, \lambda_r$ be the distinct positive characteristic roots of A on $U(x_0)$. Consider the distributions on $U(x_0)$ defined by

$$T_i^+(x)=\{X\in T_x(M)\,|\,AX=\lambda_i X\}\,,$$
$$T_i^-(x)=\{X\in T_x(M)\,|\,AX=-\lambda_i X\}\,,$$
$$T_i(x)=T_i^+(x)\oplus T_i^-(x)\,,$$
$$T^0(x)=\{X\in T_x(M)\,|\,AX=0\}\,.$$

Clearly J interchanges $T_i^+(x)$ and $T_i^-(x)$. When X is an arbitrary vector field and Y is a vector field in T^0 we deduce from

$$0=(\nabla_X A^2)(Y)=\nabla_X(A^2Y)-A^2(\nabla_X Y)=-A^2(\nabla_X Y)$$

that $\nabla_X Y\in T^0$. Hence T^0 is parallel. (A similar argument shows that each T_i is parallel.)

If $Y\in T^0$, we have $(\nabla_X A)Y=\nabla_X(AY)-A\nabla_X Y=0$. On the other hand, we have $s(X)JAY=0$ so that $(\nabla_X A)Y=s(X)JAY$. By Codazzi's equation we also obtain $(\nabla_Y A)X=s(Y)JAX$. In other words, the reduced Codazzi equation holds when X or Y is in T^0. Now $\nabla A^2=0$ being equivalent to $(\nabla_X A)A+A(\nabla_X A)=0$ (for all X), we see that $(\nabla_X A)T_i^+\subset T_i^-$ and $(\nabla_X A)T_i^-\subset T_i^+$. By virtue of Codazzi's equation the reduced Codazzi equation holds for vector fields $X\in T_i$ and $Y\in T_j$ $(i\neq j)$. We draw the same conclusion when $X\in T_i^+$ and $Y\in T_i^-$, or vice versa. Finally, if X, $Y\in T_i^+$ (or T_i^-), then using $J(\nabla_X A)=-(\nabla_X A)J$ and $JY\in T_i^-$ we get

$$(\nabla_X A)Y=-JJ(\nabla_X A)Y=J(\nabla_X A)JY=Js(X)JA(JY)=s(X)JAY\,.$$

In short, we have shown that the equation $(\nabla_X A)Y=s(X)JAY$ holds for all

X, Y.

THEOREM 4. *Let M be a complex hypersurface of complex dimension $n \geqq 1$ in a space $\tilde{M}$ of constant holomorphic curvature $\tilde{c}$. If the Ricci tensor of M is parallel, then either M is of constant holomorphic curvature $\tilde{c}$ and totally geodesic in $\tilde{M}$ or M is locally holomorphically isometric to the complex quadric Q^n in $P^{n+1}(C)$, the latter case arising only when $\tilde{c} > 0$.*

PROOF. When $n=1$ the condition $\nabla S=0$ simply means that M is of constant curvature and the classification obtained in § 6 will show that Theorem 4 is valid.

Assume $n \geqq 2$. Let $\tilde{c} \neq 0$. In view of Lemma 5, M is locally symmetric. Consequently, each $\tau \in H$, considered as parallel displacement of $T_{x_0}(M)$ along a closed curve through x_0, maps the curvature tensor R_{x_0} at x_0 into R_{x_0}. Thus if M has restricted holonomy group $U(n)$ then, since $U(n)$ acts transitively on the set of holomorphic planes at x_0, we conclude that all holomorphic planes at x_0 have the same sectional curvature; since x_0 is an arbitrary point, M has constant holomorphic sectional curvature and immerses totally geodesically in $\tilde{M}$ (see Theorem 1 [8]). If the restricted holonomy group of M is not $U(n)$, M is locally holomorphically isometric to Q^n and $\tilde{c} > 0$, by virtue of Theorem 2.

Let $\tilde{c}=0$. The roots of A^2 are constant in value and multiplicity on M, since $\nabla A^2=0$. Let us now suppose that $A^2 \neq 0$ and choose a basis $\{e_1, \cdots, e_n, Je_1 \cdots, Je_n\}$ of $T_{x_0}(M)$ diagonalizing A in the manner described in the previous section. Using the computations of § 2 and the fact that $\nabla R=0$ and $\tilde{c}=0$, we find

$$\begin{aligned} 0=(R(e_i, e_j)R)(e_i, Je_j) &= [R(e_i, e_j), R(e_i, Je_j)]-R(R(e_i, e_j)e_i, Je_j)-R(e_i, R(e_i, e_j)Je_j) \\ &= -\lambda_i^2\lambda_j^2[K_j^i, S_j^i]+\lambda_i\lambda_j R(e_j, Je_j)-\lambda_i\lambda_j R(e_i, Je_j) \\ &= -2\lambda_i^2\lambda_j^2(S_i^i-S_j^j)+2\lambda_i\lambda_j^3 S_j^j-2\lambda_i^3\lambda_j S_i^i \\ &= -2\lambda_i^2\lambda_j(\lambda_i+\lambda_j)S_i^i+2\lambda_i\lambda_j^2(\lambda_i+\lambda_j)S_j^j . \end{aligned}$$

Thus $\lambda_i\lambda_j=0$ or $\lambda_i+\lambda_j=0$. Since $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq 0$ and $\lambda_1 > 0$, A^2 has precisely one nonzero characteristic root λ_1^2 and its multiplicity is 2. We confine our attention to the distributions T_1^+, T_1^-, T_1 and T^0 on $U(x_0)$, as defined in Lemma 5. We have already seen that T_1 and T^0 are parallel on M and that the reduced Codazzi equation holds by virtue of Lemma 5. Thus if Z is an arbitrary vector and W is a unit vector field in T_1^+, then

$$s(Z)JAW=(\nabla_Z A)W=\nabla_Z(AW)-A\nabla_Z W=\lambda_1\nabla_Z W-A\nabla_Z W .$$

But since T_1 is parallel and (real) 2-dimensional and W is a unit vector in T_1^+, we see that $\nabla_Z W \in T_1^-$ and $\lambda_1\nabla_Z W-A\nabla_Z W=2\lambda_1\nabla_Z W$. Therefore, the equation above reduces to $\lambda_1 s(Z)JW=2\lambda_1\nabla_Z W$, that is, $\nabla_Z W=\frac{1}{2}s(Z)JW$. It is an easy matter to verify that $R(X, Y)W=ds(X, Y)JW$, for arbitrary vectors

X, Y, so that $S(W, W)=R(W, JW, W, JW)=ds(JW, W)$. By virtue of proposition 4 [8], $S(W, W)=-2ds(JW, W)$. Hence $0=S(W, W)=-2\lambda_1^2$ and this is a contradiction. Therefore $A^2=0$ and M is flat and totally geodesic in $\tilde{M}$. This completes the proof of Theorem 4.

With a view to obtaining a global version of this theorem, we suppose that M is a complete complex hypersurface in $\tilde{M}$ with parallel Ricci tensor. f will denote the Kählerian immersion of M in $\tilde{M}$. Let $\hat{M}$ be the universal covering manifold of M and let π be the covering map. On $\hat{M}$ we take the Kählerian structure which makes π a holomorphic isometric immersion; $\hat{M}$ is then simply-connected and complete and its Ricci tensor is parallel. Moreover $f\circ\pi$ is a holomorphic isometric immersion of $\hat{M}$ in $\tilde{M}$.

If $\tilde{M}=P^{n+1}(C)$ then, in view of Theorem 4, $\hat{M}$ is holomorphically isometric either to $P^n(C)$ or to Q^n and, by rigidity (Theorem 1), $\hat{M}$ immerses either onto a projective hyperplane or onto a complex quadric in $P^{n+1}(C)$. In either case $f\circ\pi(\hat{M})$ is a simply-connected manifold and since $f\circ\pi$ is a covering map (see Theorem 4.6 in [**5**, p. 176]), it is one-to-one. Hence π is one-to-one and therefore M is holomorphically isometric either to P^n or to Q^n. The same type of argument can be applied when $\tilde{M}=D^{n+1}$ or C^{n+1} (without assuming that M is simply connected). We thus obtain the following improved form of Theorem 3 of [8]:

THEOREM 5.

i) *$P^n(C)$ and the complex quadric Q^n are the only complete complex hypersurfaces in $P^{n+1}(C)$ which have parallel Ricci tensors*[1].

ii) *D^n (resp. C^n) is the only complete complex hypersurface in D^{n+1} (resp. C^{n+1}) which has parallel Ricci tensor.*

§ 4. Hypersurfaces of rank 2 in $P^{n+1}(C)$.

The main purpose of this section is to prove that in $P^{n+1}(C)$, $n\geqq 3$, there is no compact complex hypersurface M which has rank 2 everywhere. We must, however, develop a few preliminary results on the nullity space of a curvature-type tensor field, which are generalized adaptations of some results of Maltz [**6**].

In general, let M be a Riemannian manifold with metric g and let D be a tensor field of type (1, 3) on M. We shall say that D is curvature-type if it satisfies the following conditions:

i) $D(X, Y)$ is a skew-symmetric transformation for any pair of vectors X and Y,

1) After the completion of our work we learned of a further generalization of (i) by S. Kobayashi (Hypersurfaces of complex projective space with constant scalar curvature, to appear).

ii) $$D(Y, X)=-D(X, Y),$$

iii) $$\mathfrak{S}\{D(X, Y)Z\}=0,$$

where $\mathfrak{S}$ is the cyclic sum taken over X, Y and Z,

iv) $$\mathfrak{S}\{(\nabla_X D)(Y, Z)\}=0.$$

It is well known that the Riemannian curvature tensor field R of M satisfies these conditions. We also note that (i), (ii) and (iii) imply

v) $$g(D(X, Y)Z, W)=g(D(Z, W)X, Y),$$

as is the case for R (see [**5**], p. 198).

We define the *nullity space* T_x^0 of D at each point $x \in M$ to be the subspace $\{X \mid D(X, Y)=0 \text{ for all } Y \in T_x(M)\}$ of $T_x(M)$; its dimension is called the *index of nullity* of D. Let T_x^1 be the orthogonal complement of T_x^0. The following lemmas can be proved in exactly the same way as those in [**6**].

LEMMA 6.

i) *If* $X \in T_x^0$, *then* $D(Y, Z)X=0$ *for all* $Y, Z \in T_x(M)$.

ii) T_x^1 *coincides with the subspace spanned by all* $D(X, Y)Z$, *where* $X, Y, Z \in T_x(M)$.

LEMMA 7. *Assume that the index of nullity of a curvature-type tensor field* D *is constant on* M. *Then the distribution* T^0: $x \rightarrow T_x^0$ *is involutive and totally geodesic (that is,* $\nabla_X T^0 \subset T^0$ *for any vector* $X \in T^0$ *so that any integral manifold of* T^0 *is a totally geodesic submanifold of* M).

We shall apply the foregoing lemma to the situation where M is a complex hypersurface in a space $\tilde{M}$ of constant holomorphic curvature $\tilde{c}$. The curvature tensor R of M is given by Gauss' equation

$$R(X, Y)=\tilde{R}(X, Y)+D(X, Y),$$

the expressions for $\tilde{R}(X, Y)$ and $D(X, Y)$ being as in §1. Since both R and $\tilde{R}$ are curvature-type tensor fields on M, so is their difference D. We know (Lemma 1, §1) that the nullity space T_x^0 coincides with the kernel of A at x. Hence $\dim T_x^1$ equals the rank of M at x. Assume now that this is constant on M. The distribution T^0 is integrable and totally geodesic by Lemma 7; it is also invariant by the complex structure J, because $JA=-AJ$. If M^0 is a maximal integral manifold of T^0, we conclude that M^0 is a complex submanifold of M which is totally geodesic in M. The curvature tensor R^0 of M^0 (with respect to the metric induced from that of M) is given by $R^0(X, Y) = R(X, Y)$, where $X, Y \in T_x(M^0)$, which is equal to $\tilde{R}(X, Y)$, since $D(X, Y)=0$ for $X, Y \in T_x(M^0)=T_x^0$. Thus

$$R^0(X, Y)=\frac{\tilde{c}}{4}\{X \wedge Y+JX \wedge JY+2g(X, JY)J\},$$

which means that M^0 has constant holomorphic curvature $\tilde{c}$.

Considering M^0 as a complex submanifold of $\tilde{M}$, we may establish the formula

$$\tilde{K}(X)=K^0(X)+2\sum_{i=1}^{k}\{g(A_iX, X)^2+g(JA_iX, X)^2\}$$

(for a unit vector X tangent to M^0) relating the sectional curvatures $\tilde{K}(X)$ and $K^0(X)$ in $\tilde{M}$ and M^0, respectively, of the holomorphic plane generated by X. In this formula $A_1, \cdots, A_k$ are the second fundamental forms corresponding to a choice of an orthonormal family of vector fields $\xi_1, \cdots, \xi_k$ normal to M^0, and k is the complex codimension of M^0 in $\tilde{M}$. This formula generalizes that of Corollary 2 [8]. Since $\tilde{K}(X)=K^0(X)=\tilde{c}$ in our case, it follows that each A_i is identically zero, which means that M^0 is totally geodesic in $\tilde{M}$.

Let us now assume that M is a complete complex hypersurface in $P^{n+1}(C)$, C^{n+1} or D^{n+1} such that the rank of M is everywhere equal to $2r$. We show that M^0 is then complete. Let $\gamma(s)$ be a geodesic in M^0 defined on $a<s<b$. Since M is complete and M^0 is totally geodesic in M, $\gamma(s)$ can be extended as a geodesic $\gamma^*(s)$ in M, defined for all values of s. Let $(x^1, \cdots, x^{2m}, x^{2m+1}, \cdots, x^{2n})$, where $m=n-r$, be a system of local coordinates on M with origin $\gamma^*(b)$, such that $\left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{2m}}\right\}$ is a local basis for T^0. When s is in a certain neighborhood of b, say $(b-\varepsilon, b+\varepsilon)$, we may express $\gamma^*(s)$ by the set of equations $x^i(\gamma^*(s))=f^i(s)$, $1\leqq i\leqq 2n$. Since $\gamma^*(s)=\gamma(s)cM^0$ when $a<s<b$ we must then have $f^i(s)=c^i$ (a constant) for $2m+1\leqq i\leqq 2n$. Letting s approach b from below we find that $0=f^i(b)=c^i$, $2m+1\leqq i\leqq 2n$. Thus $\gamma^*(b)$ is in the maximal integral manifold which contains $\gamma(s)$, $a<s<b$. In other words $\gamma^*(b)\in M^0$ and it is possible to extend $\gamma(s)$ as a geodesic in M^0 for parameter values larger than b. Thus M^0 is complete.

Since we know that any complete totally geodesic complex n-dimensional submanifold of $P^{n+1}(C)$, C^{n+1}, or D^{n+1} is of the form $P^m(C)$, C^m, D^m, respectively, we obtain

PROPOSITION 1. *Let M be a complex hypersurface of $\tilde{M}=P^{n+1}(C)$, C^{n+1}, or D^{n+1}. If the rank (of the second fundamental form) of M is everywhere equal to a constant, $2r$, then M contains a complete totally geodesic complex $(n-r)$-dimensional submanifold of $\tilde{M}$, namely $P^{n-r}(C)$, C^{n-r}, D^{n-r}, respectively.*

We now prove the main theorem of this section

THEOREM 6. *Let M be a compact complex hypersurface of $P^{n+1}(C)$, $n\geqq 3$. The rank (of the second fundamental form) of M cannot be identically equal to 2.*

REMARK. For $n=1$, the quadrics are the only closed complex curves in $P^2(C)$ of rank identically equal to 2 (see (i) of Theorem 9 in § 6). The case $n=2$ remains unsettled.

PROOF. By virtue of Proposition 1, M contains a projective subspace P^{n-1}. Choose a system of homogeneous coordinates $(z_0, z_1, \cdots, z_{n+1})$ in $P^{n+1}(C)$ such that P^{n-1} is given by $z_0 = z_1 = 0$. By a theorem of Chow the compact complex hypersurface M can be defined by $f=0$, where f is a homogeneous polynomial in $z_0, z_1, \cdots, z_{n+1}$ such that the partial derivatives $\frac{\partial f}{\partial z_k}$ $(0 \leqq k \leqq n+1)$ are not all zero at any point of M. We write f in the form

$$f(z_0, \cdots, z_{n+1}) = F(z_2, \cdots, z_{n+1}) + z_0 f_0(z_2, \cdots, z_{n+1}) + z_1 f_1(z_2, \cdots, z_{n+1}) + \sum_{k+l \geqq 2} z_0^k z_1^l f_{kl}(z_2, \cdots, z_{n+1}),$$

where F, f_0, f_1 and f_{kl} are homogeneous polynomials in the variables $z_2, \cdots, z_{n+1}$. Since $P^{n-1} \subset M$, we have $f(0, 0, z_2, \cdots, z_{n+1}) = 0$ for all $z_2, \cdots, z_{n+1}$. Thus F is identically zero and

$$f = z_0 f_0 + z_1 f_1 + \sum_{k+l \geqq 2} z_0^k z_1^l f_{kl}.$$

Consequently

$$\frac{\partial f}{\partial z_0} = f_0 + \sum_{k+l \geqq 2} k z_0^{k-1} z_1^l f_{kl},$$

$$\frac{\partial f}{\partial z_1} = f_1 + \sum_{k+l \geqq 2} l z_0^k z_1^{l-1} f_{kl}$$

and

$$\frac{\partial f}{\partial z_j} = z_0 \frac{\partial f_0}{\partial z_j} + z_1 \frac{\partial f_1}{\partial z_j} + \sum_{k+l \geqq 2} z_0^k z_1^l \frac{\partial f_{kl}}{\partial z_j}$$

for $j \geqq 2$. At $(0, 0, z_2, \cdots, z_{n+1}) \in P^{n-1} \subset M$, we have $\frac{\partial f}{\partial z_j} = 0$ for $j \geqq 2$, $\frac{\partial f}{\partial z_0} = f_0$ and $\frac{\partial f}{\partial z_1} = f_1$. Lemma 8 will show, however, that unless f_0 and f_1 are constants there exist $z_2, \cdots, z_{n+1}$ (not all zero) for which $f_0 = f_1 = 0$. This would mean that there is a point $(0, 0, z_2, \cdots, z_{n+1}) \in M$ where all the partial derivatives $\frac{\partial f}{\partial z_k}$ are zero. Thus f_0 and f_1 are constants, so that f is of degree 1 and is given by $f = c_0 z_0 + c_1 z_1$, where c_0, c_1 are constants; therefore M is a projective hyperplane in P^{n+1} and thus M is of rank zero everywhere. This is a contradiction.

The following lemma occurs as a particular case of the main theorem of § 5 in Samuel's book [7], although it is easy to give a direct proof using the theory of resultants.

LEMMA 8. *For any two non-constant homogeneous polynomials* $g, h \in C[x_1, \cdots, x_n]$, $n \geqq 3$, *there is a non-trivial solution of* $g = h = 0$.

§ 5. Hypersurfaces in C^{n+1}.

To begin with, we suppose that M is a complex hypersurface in an arbitrary Kählerian manifold $\tilde{M}$. For any vector field X on M and for any field of vectors ξ normal to M in $\tilde{M}$, we define $\hat{\nabla}_X\xi$ to be the normal component of $\tilde{\nabla}_X\xi$, where $\tilde{\nabla}$ refers, as in [8], to covariant differentiation in $\tilde{M}$. We may easily verify that $\hat{\nabla}$ is a linear connection in the normal bundle over M, which we call the normal connection for the hypersurface M. The relative curvature tensor $\hat{R}$ of M (that is, the curvature tensor of the normal connection of M) is given by

$$\hat{R}(X, Y)\xi = [\hat{\nabla}_X, \hat{\nabla}_Y]\xi - \hat{\nabla}_{[X,Y]}\xi ,$$

where X and Y are vector fields tangent to M. If ξ is a field of unit normals, $\hat{\nabla}_X\xi$ is equal to $s(X)J\xi$ and, by an easy computation, we find

PROPOSITION 2. *The relative curvature tensor $\hat{R}$ of M is expressed by*

$$\hat{R}(X, Y)\xi = 2ds(X, Y)J\xi ,$$

where ξ is a field of unit normals to M.

Now assume that $\tilde{M}$ has constant holomorphic sectional curvature $\tilde{c}$. According to Proposition 4 of [8], we have

$$\tilde{S}(X, JY) = S(X, JY) + 2ds(X, Y) ,$$

where $\tilde{S}$ and S denote the Ricci tensors of $\tilde{M}$ and M, respectively. We shall prove

THEOREM 7. *Let M be a complex hypersurface of complex dimension $n \geqq 1$ in a space $\tilde{M}$ of constant holomorphic curvature $\tilde{c}$. The following conditions are equivalent:*

i) *The normal connection of M is trivial, that is, $\hat{R} = 0$.*

ii) $S = \tilde{S}$ *on M.*

iii) $S = 0$ *on M.*

iv) $\tilde{c} = 0$ *and M is totally geodesic in $\tilde{M}$.*

PROOF. It is clear that iv) implies each of the other conditions, while the equivalence of i) and ii) follows from Proposition 2 above. Assuming ii) we see that M is Einstein. By Theorem 4, M is then totally geodesic in $\tilde{M}$ or else $\tilde{c} > 0$ and M is locally holomorphically isometric to Q^n in $P^{n+1}(C)$. Thus $S = (n+1)\frac{\tilde{c}}{2}g$ or else $\tilde{c} > 0$ and $S = \frac{n\tilde{c}}{2}g$. However, $\tilde{S} = (n+2)\frac{\tilde{c}}{2}g$. Therefore $\tilde{c} = 0$ and $S = 0$ and consequently M is totally geodesic in $\tilde{M}$. In other words, ii) implies both iii) and iv). If $S = 0$, then M is Einstein and it is clear from the above that $\tilde{c} = 0$ and M is totally geodesic in $\tilde{M}$. Thus iii) implies iv) and the equivalence of all four conditions is proved.

The general object of the remainder of this section is to define the Gaussian mapping of a complex hypersurface in complex Euclidean space C^{n+1} into the complex projective space $P^n(C)$, and to give a geometric interpretation thereof. It is convenient to begin by establishing a relationship between the Riemannian connection on the sphere S^{2n+1} and the Kählerian connection on $P^n(C)$ (for the Fubini-Study metric, of course).

$P^n(C)$ can be regarded as the base of a principal fibre bundle S^{2n+1} (unit sphere in C^{n+1}) on which the structure group $S^1=\{e^{i\theta}\,|\,\theta\in R\}$ acts as follows: $S^{2n+1}\times S^1\ni(z, e^{i\theta})\rightarrow ze^{i\theta}\in S^{2n+1}$. π denotes the canonical projection of S^{2n+1} onto $P^n(C)$ and $g(z, w)=\mathrm{Re}\left(\sum_{k=0}^{n} z^k w^{-k}\right)$ for $z=(z^0, z^1, \cdots, z^n)$, $w=(w^0, w^1, \cdots, w^n)$ defines the Euclidean metric on C^{n+1}. With the natural identification between vectors tangent to S^{2n+1} and vectors C^{n+1}, we have

$$T_z(S^{2n+1})=\{w\in C^{n+1}\,|\,g(z, w)=0\}$$

for each $z\in S^{2n+1}$. The orthogonal complement of

$$T'_z=\{w\in C^{n+1}\,|\,g(z, w)=g(iz, w)=0\}$$

in $T_z(S^{2n+1})$ is the 1-dimensional subspace $\{iz\}$ which is spanned by the vector iz (in the sense of the above identification). The distribution T' defines a connection in the principal fibre bundle $S^{2n+1}(P^n(C), S^1)$, that is, T'_z is complementary to the subspace $\{iz\}$ tangent to the fibre through z, and T' is invariant by the action of S^1. Thus the projection π induces a linear isomorphism of T'_z onto $T_{\pi(z)}(P^n(C))$ and π maps $\{iz\}$ into zero for each $z\in S^{2n+1}$.

The classical Fubini-Study metric of holomorphic sectional curvature 1 is nothing but the metric $\tilde{g}$ defined by $\tilde{g}(\tilde{X}, \tilde{Y})=4g(X', Y')$, where $\tilde{X}, \tilde{Y}\in T_p(P^n(C))$ and X', Y' are their respective horizontal lifts at z ($\pi(z)=p$). Since g is invariant by S^1, the definition of $\tilde{g}(\tilde{X}, \tilde{Y})$ is independent of the choice of z. We might also observe that the complex structure in T'_z (defined by multiplication of vectors by i) induces the canonical complex structure J on $P^n(C)$, when transferred by means of π. (What we have said so far is more or less well known.)

The horizontal lift of a vector field $\tilde{X}$ on $P^n(C)$ will be denoted by X'. If $\tilde{X}$ and $\tilde{Y}$ are vector fields on $P^n(C)$, then the vector fields X' and Y' are invariant by S^1; since the Riemannian connection on S^{2n+1} is invariant by S^1, it follows that $\nabla'_{X'}Y'$ (where ∇' denotes covariant differentiation on S^{2n+1}) is also invariant by S^1 and hence projectable, that is, there exists a vector field $\tilde{Z}$ on $P^n(C)$ such that $\pi_*(\nabla'_{X'}Y')_z=\tilde{Z}_{\pi(z)}$ for all $z\in S^{2n+1}$.

PROPOSITION 3. *For every pair of vector fields $\tilde{X}, \tilde{Y}$ on $P^n(C)$ the vector field $\nabla'_{X'}Y'$ on S^{2n+1} is projectable and $\tilde{\nabla}_{\tilde{X}}\tilde{Y}=\pi_*(\nabla'_{X'}Y')$ defines the Kählerian connection on $P^n(C)$.*

PROOF. To prove this we verify the following:

i) *$\tilde{\nabla}$ is a linear connection.* Obviously $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ is bi-additive in X and Y.

For any differentiable function $\tilde{f}$ on $P^n(C)$ we let $f'=\tilde{f}\circ\pi$ be its lift to S^{2n+1}. Then $f'X'$ is the horizontal lift of $\tilde{f}\tilde{X}$ and $\nabla'_{f'X'}Y'=f'\nabla'_{X'}Y'$ is projectable. Thus

$$\tilde{\nabla}_{\tilde{f}\tilde{X}}\tilde{Y}=\pi_*(\nabla'_{f'X'}Y')=\pi_*(f'\nabla'_{X'}Y')=\tilde{f}\tilde{\nabla}_{\tilde{X}}\tilde{Y}.$$

Similarly, we can prove

$$\tilde{\nabla}_{\tilde{X}}(\tilde{f}\tilde{Y})=(\tilde{X}\tilde{f})\tilde{Y}+\tilde{f}\tilde{\nabla}_{\tilde{X}}\tilde{Y}.$$

ii) *The torsion tensor of* $\tilde{\nabla}$ *is zero.* If $\tilde{X}$ and $\tilde{Y}$ are vector fields on $P^n(C)$ then $[X', Y']$ is projectable and $\pi_*[X', Y']=[\tilde{X}, \tilde{Y}]$. Consequently

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y}-\tilde{\nabla}_{\tilde{Y}}\tilde{X}-[\tilde{X}, \tilde{Y}]=\pi_*(\nabla'_{X'}Y'-\nabla'_{Y'}X'-[X', Y'])=0.$$

iii) ∇ *is a metric connection for* $\tilde{g}$. Let $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ be vector fields on $P^n(C)$. On S^{2n+1} we have

$$X'g(Y', Z')=g(\nabla'_{X'}Y', Z')+g(Y', \nabla'_{X'}Z').$$

Denoting by h the horizontal component of vector fields on S^{2n+1}, we see that

$$\begin{aligned} g_z(\nabla'_{X'}Y', Z')&=g_z(h(\nabla'_{X'}Y'), Z') \\ &=\frac{1}{4}\tilde{g}_p(\pi_*h(\nabla'_{X'}Y'), \pi_*Z')=\frac{1}{4}\tilde{g}_p(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}), \end{aligned}$$

where $\pi(z)=p$. Similarly, we have $g_z(Y', \nabla'_{X'}Z')=\frac{1}{4}\tilde{g}_p(\tilde{Y}, \tilde{\nabla}_{\tilde{X}}\tilde{Z})$. On the other hand we have $g(Y', Z')=\tilde{f}\circ\pi$, where $\tilde{f}=\frac{1}{4}\tilde{g}(\tilde{Y}, \tilde{Z})$, so that

$$X'_z g(Y', Z')=X'_z(\tilde{f}\circ\pi)=(\pi_*X'_z)\tilde{f}=\tilde{X}_p\tilde{f}=\frac{1}{4}\tilde{X}_p\tilde{g}(\tilde{Y}, \tilde{Z}).$$

The metric condition for ∇' therefore gives rise to the same condition for $\tilde{\nabla}$, that is,

$$\tilde{X}\tilde{g}(\tilde{Y}, \tilde{Z})=\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z})+\tilde{g}(\tilde{Y}, \tilde{\nabla}_{\tilde{X}}\tilde{Z}).$$

REMARK. If z_t is a horizontal curve on S^{2n+1} and Y'_t is a family of horizontal vectors defined along z_t, then $\pi_*(\nabla'_{\vec{z}_t}Y'_t)=\tilde{\nabla}_{\pi_*(\vec{z}_t)}\pi_*(Y'_t)$ along z_t, where $\vec{z}_t$ is the velocity vector of the curve z_t at time t.

To verify this for each t_0 we extend $\vec{z}_t$ and Y'_t, respectively, to horizontal vector fields Z' and Y' in a neighborhood of z_{t_0}, as follows: extend $\pi_*(\vec{z}_t)$ (resp. $\pi_*(Y'_t)$) to a neighborhood of $\pi(z_{t_0})$ and let Z' (resp. Y') be its horizontal lift. We then have $\pi_*(\nabla'_{Z'}Y')=\tilde{\nabla}_{\tilde{Z}}\tilde{Y}$, which implies $\pi_*(\nabla'_{\vec{z}_t}Y'_t)=\tilde{\nabla}_{\pi_*(\vec{z}_t)}\pi_*(Y'_t)$ at z_{t_0}.

Turning our attention now to a complex hypersurface M in C^{n+1}, we shall first define a generalized Gaussian mapping of M into $P^n(C)$.

For each point $x\in M$ we can choose a unit vector ξ normal to M at x. As a vector in C^{n+1}, it is determined to within a multiple of the form $e^{i\theta}$.

Thus $\phi(x)=\pi(\xi)\in P^n(C)$ is well defined and the mapping $\phi: M\to P^n(C)$ is called the *Gaussian mapping* of M. We can relate ϕ to the second fundamental form A of M (in the formalism of [8]) as follows:

Let $X\in T_x(M)$ and take a curve x_t on M such that $x_0=x$ and $(\vec{x}_t)_{t=0}=X$. Choose a (differentiable) family of unit normals ξ_t along x_t. The differential ϕ_* of ϕ maps X upon

$$\left(\frac{d\pi(\xi_t)}{dt}\right)_{t=0}=\pi_*\left(\frac{d\xi_t}{dt}\right)_{t=0}\in T_{\phi(x)}(P^n(C)),$$

where $\left(\frac{d\xi_t}{dt}\right)_{t=0}$ is the tangent vector of the curve ξ_t on S^{2n+1} at ξ_0. On the other hand, the Weingarten formula for M as a complex hypersurface in C^{n+1} (with the flat connection D) gives

$$\left(\frac{d\xi_t}{dt}\right)_{t=0}=D_X\xi=-AX+s(X)J\xi\,,\quad \text{where}\quad J\xi=i\xi\,.$$

Since $J\xi$ is the initial tangent vector of the curve $e^{i\theta}\xi$ on S^{2n+1}, we have $\pi_*(J\xi)=0$. Hence

$$\phi_*(X)=-\pi_*(AX)\,.$$

The vector AX, considered by translation as a tangent vector to S^{2n+1} at ξ, belongs to T'_ξ because it is perpendicular to $J\xi$. Since $\pi_*: T'_\xi\to T_{\pi(\xi)}(P^n(C))$ is one-to-one, we conclude that

i) *$\phi_*(X)=0$ if and only if $AX=0$.*

ii) *The rank of ϕ_* is equal to the rank of A.*

Since $\phi_*(JX)=-\pi_*(AJX)=\pi_*(JAX)$ and since the complex structure J on T'_ξ corresponds to the complex structure $\tilde{J}$ on $T_{\pi(\xi)}(P^n(C))$, by means of π, we have

$$\phi_*(JX)=\tilde{J}\pi_*(AX)=-\tilde{J}\phi_*(X)\,,$$

namely,

iii) *the Gaussian mapping ϕ is anti-holomorphic.*

EXAMPLES.

i) If M is a hyperplane C^n in C^{n+1} we have a constant unit normal ξ over M, so that $\phi(M)$ is a single point in $P^n(C)$.

ii) If M is of the form $K\times C^{n-1}$, where K is a complex curve in a plane C^2 perpendicular to C^{n-1}, then the rank of ϕ is $\leqq 2$ everywhere and $\phi(M)$ lies in a projective line $P^1(C)$ in $P^n(C)$. It will be interesting to find an appropriate converse of this proposition.

In relating the Kählerian connection on M to that on $P^n(C)$, the following lemma will be useful.

LEMMA 9. *Let x_t be a differentiable curve on M. Then there is a family of unit normals ξ_t along x_t which, as a curve in S^{2n+1}, is horizontal.*

PROOF. For an arbitrary family of unit normals η_t along x_t we consider a family of unit normals given by $\xi_t = a\eta_t + bJ\eta_t$, where $a = a(t)$ and $b = b(t)$ are differentiable functions such that $a^2+b^2=1$. We show that by choosing a and b suitably we can make ξ_t horizontal, that is $g\left(\frac{d\xi_t}{dt}, J\xi_t\right)=0$ for all t. It is readily verified that

$$g\left(\frac{d\xi_t}{dt}, J\xi_t\right) = g\left(\frac{d\eta_t}{dt}, J\eta_t\right) + a\frac{db}{dt} - b\frac{da}{dt}.$$

Thus our purpose will be achieved if we can choose a and b such that

$$a\frac{db}{dt} - b\frac{da}{dt} = k(t) \quad \text{and} \quad a^2+b^2=1,$$

where $k(t) = -g\left(\frac{d\eta_t}{dt}, J\eta_t\right)$. Since $a^2+b^2=1$ implies $a\frac{da}{dt}+b\frac{db}{dt}=0$, we have

$$\frac{da}{dt} = -bk(t) \quad \text{and} \quad \frac{db}{dt} = ak(t).$$

Thus we may take

$$a(t) = \cos l(t) - \sin l(t), \qquad b(t) = \sin l(t) + \cos l(t),$$

where $l(t) = \int k(t)dt$.

THEOREM 8. *Let M be a complex hypersurface in C^{n+1} and let Y_t be a family of vectors tangent to M along a curve x_t. Choose a family of unit normals ξ_t along x_t as in Lemma 9 and let Y_t' be the vector tangent to S^{2n+1} at ξ_t which is parallel to Y_t in C^{n+1}. Let $\tilde{Y}_t = \pi_*(Y_t')$. Then Y_t is parallel along x_t on M if and only if $\tilde{Y}_t$ is parallel along $\phi(x_t)$ on $P^n(C)$.*

PROOF. For M we have

$$\frac{dY_t}{dt} = D_{\vec{x}_t} Y_t = \nabla_{\vec{x}_t} Y_t + h(\vec{x}_t, Y_t)\xi_t + k(\vec{x}_t, Y_t)J\xi_t, \tag{10}$$

where D is the flat connection in C^{n+1} and ∇ is the Kählerian connection on M. On the other hand, for S^{2n+1} (with the Riemannian connection ∇') we get

$$\frac{dY_t}{dt} = \frac{dY_t'}{dt} = D_{\vec{\xi}_t} Y_t' = \nabla'_{\vec{\xi}_t} Y_t' + h'(\vec{\xi}_t, Y_t')\xi_t, \tag{11}$$

where h' is the second fundamental form of S^{2n+1} with respect to the unit normals ξ_t. Equations (10) and (11) yield

$$\nabla_{\vec{x}_t} Y_t + \{h(\vec{x}, Y_t) - h'(\vec{\xi}_t, Y_t')\}\xi_t + k(\vec{x}_t, Y_t)J\xi_t = \nabla'_{\vec{\xi}_t} Y_t'$$

(considered as an identity among vectors in C^{n+1}). Therefore

$$\nabla_{\vec{x}_t} Y_t = \nabla'_{\vec{\xi}_t} Y_t' - k(\vec{x}_t, Y_t)J\xi_t.$$

Thus, if $\nabla_{\vec{x}_t} Y_t = 0$, the fact that both $\vec{\xi}_t$ and Y_t' are horizontal and that $J\xi_t$ is vertical in $T_{\xi_t}(S^{2n+1})$ implies

$$0=\pi_*(\nabla'_{\dot{\xi}_t} Y'_t - k(\dot{x}_t, Y_t)J\xi_t) = \pi_*(\nabla'_{\dot{\xi}_t} Y'_t) = \tilde{\nabla}_{\pi(\dot{\xi}_t)}\pi_*(Y'_t) = \tilde{\nabla}_{\overrightarrow{\phi(x_t)}} \tilde{Y}_t,$$

in view of the remark following Proposition 3. Conversely, suppose $\tilde{\nabla}_{\overrightarrow{\phi(x_t)}} \tilde{Y}_t = 0$. Then $\nabla'_{\vec{\xi}_t} Y'_t$ must be vertical, that is, in the direction of $J\xi_t$. From (11) we see that $\frac{dY_t}{dt}$ is a linear combination of ξ_t and $J\xi_t$. Therefore $\nabla_{\vec{x}_t} Y_t = 0$, by virtue of (10).

REMARK. For a complex n-dimensional submanifold M of C^{n+k} there is a naturally defined mapping $\phi: M \to U(n+k)/U(n)\times U(k)$ and an associated mapping of the bundle of unitary frames over M into $U(n+k)/U(k)$. This bundle mapping was studied by Kerbrat [3]. For an n-dimensional (real) orientable submanifold in real Euclidean space R^{n+k}, there is a naturally defined mapping $\phi: M \to SO(n+k)/SO(n)\times SO(k)$. If $k=2$, the latter space can be identified with the quadric Q^n in $P^{n+1}(C)$ and we may relate the Riemannian connection on M to the Kählerian connection on Q^n by means of ϕ in a geometric manner similar to that of Theorem 8.

§6. Complex curves.

We now turn to (nonsingular) complex curves in a complex 2-dimensional space $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$ and we derive a very convenient formula for their curvature. If M is such a curve, then $A^2 = \lambda_1^2 I$ on M. Since the curvature K of M is given by $K = -2\lambda_1^2 + \tilde{c}$ (Corollary 3, [8]), we have $K \leq \tilde{c}$ on M. We now suppose that $K(x_0) \neq \tilde{c}$, so that $\lambda_1^2 \neq 0$ in a neighborhood $U(x_0)$ of x_0; let λ_1 denote its positive square root. Consider the distributions T_1^+, T_1^- on $U(x_0)$ as defined in §3. Codazzi's equation may be written

$$\nabla_Y(AZ) - \nabla_Z(AY) - A\nabla_Y Z + A\nabla_Z Y - s(Y)JAZ + s(Z)JAY = 0$$

and, supposing that the vector fields Y and Z belong to T_1^- and T_1^+ respectively, we obtain

$$Y(\lambda_1)Z + \lambda_1\nabla_Y Z + Z(\lambda_1)Y + \lambda_1\nabla_Z Y - A\nabla_Y Z + A\nabla_Z Y - \lambda_1 s(Y)JZ - \lambda_1 s(Z)JY = 0.$$

If, in addition, Y and Z are unit vector fields, a consideration of the T_1^--component of this equation yields

$$-A\nabla_Y Z + \lambda_1\nabla_Y Z + Z(\lambda_1)Y - \lambda_1 s(Y)JZ = 0,$$

i. e.

$$2\lambda_1\nabla_Y Z + Z(\lambda_1)Y - \lambda_1 s(Y)JZ = 0,$$

i. e.

$$\nabla_Y Z = -\frac{1}{2}\left\{Z\left(\frac{1}{2}\ln \lambda_1^2\right)Y - s(Y)JZ\right\}$$

$$= -\frac{1}{2}\{Z(\mu)Y - s(Y)JZ\},$$

where $\mu=\frac{1}{2}\ln\lambda_1^2$. However, $Z(\mu)Y=-(JY)(\mu)JZ$ since $JY=\pm Z$. Hence

$$\nabla_Y Z=\frac{1}{2}[(JY)\mu+s(Y)\}JZ.$$

Note that this still holds if Y is an arbitrary vector field in T_1^-. Also, if Z is a unit vector field in T_1^- (instead of T_1^+), then JZ is a unit vector field in T_1^+ so that the formula above is valid when we replace Z by JZ. Using $\nabla_Y(JZ)=J(\nabla_Y Z)$ we obtain again

$$\nabla_Y Z=\frac{1}{2}\{(JY)\mu+s(Y)\}JZ,$$

where Z is a unit vector field in T_1^-.

Similarly, we obtain

$$\nabla_Z Y=\frac{1}{2}\{(JZ)\mu+s(Z)\}JY$$

on considering the T_1^+-component of Codazzi's equation. Note that this still holds if Z is an arbitrary vector field in T_1^+ and if Y is a unit vector field in T_1^+. Combining all cases we conclude that

$$\nabla_Y Z=\frac{1}{2}\{(JY)\mu+s(Y)\}JZ$$

when Z is a unit vector field in either T_1^+ or T_1^- and Y is an arbitrary vector.

It may be readily verified that

$$\nabla_X\nabla_Y Z=\frac{1}{2}\{X(JY)\mu+Xs(Y)\}JZ-\frac{1}{4}\{(JY)\mu+s(Y)\}\{(JX)\mu+s(X)\}Z,$$

where Z is a unit vector field in T_1^+. Therefore

$$\begin{aligned}R(JZ,Z)Z&=\frac{1}{2}\{(JZ)(JZ)\mu+ZZ\mu-J([JZ,Z])\mu\}JZ+ds(JZ,Z)JZ\\&=\frac{1}{2}\{ZZ\mu+(JZ)(JZ)\mu-(\nabla_Z Z+\nabla_{JZ}JZ)\mu\}JZ+ds(JZ,Z)JZ\\&=\left\{\frac{1}{2}\Delta\mu+ds(JZ,Z)\right\}JZ,\end{aligned}$$

where $\Delta\mu$ denotes the Laplacian of μ. Since $K=g(R(JZ,Z)Z,JZ)$, we obtain $K=\frac{1}{2}\Delta\mu+ds(JZ,Z)$ and, using (12), we have

PROPOSITION 4. *Let M be a complex curve in a complex 2-dimensional space $\tilde{M}$ of constant holomorphic curvature $\tilde{c}$. The curvature of M is given by*

$$K=\frac{1}{3}\Delta\mu+\frac{\tilde{c}}{2}$$

on the open set $\{x\in M\,|\,K(x)\neq\tilde{c}\}$, where $\mu=\frac{1}{2}\ln\lambda_1^2$ and λ_1^2 is defined by

$A^2=\lambda_1^2 I$.

It is now easy to prove Theorem 4 for the case $n=1$. Let M be of constant curvature but not totally geodesic in $\tilde{M}$, then λ_1^2 is a nonzero constant, so that $\Delta\mu=0$ on M. Thus $K=\tilde{c}/2$ by Proposition 4, and since $K=-2\lambda_1^2+\tilde{c}$, it follows that $\lambda_1^2=\tilde{c}/4$. In particular $\tilde{c}>0$. However the complex quadric Q^1 in $P^2(C)$ is of constant curvature and is not totally geodesic in $P^2(C)$; consequently $\lambda_1^2=\tilde{c}/4$ on Q^1 also. Thus if M is of constant curvature but not totally geodesic in $\tilde{M}$ then $\tilde{c}>0$ and M is locally holomorphically isometric to Q^1 in $P^2(C)$.

Consider $P^2(C)$ with the Fubini-Study metric of holomorphic curvature 1. Let M be a (nonsingular) closed complex curve in $P^2(C)$ and suppose $K<1$ at every point of M. Then $K=\frac{1}{3}\Delta\mu+\frac{1}{2}$ is a global formula for the curvature of M. Let dv denote the area element of the Riemannian manifold M. By virtue of the Gauss-Bonnet theorem and Green's theorem we obtain

$$2\pi\chi=4\pi(1-p)=\int_M K\,dv=\int_M\left(\frac{1}{3}\Delta\mu+\frac{1}{2}\right)dv=\frac{1}{2}\int_M dv>0,$$

where χ and p are the Euler characteristic and genus of M, respectively. The genus of M is therefore zero. However M, being a closed complex curve in $P^2(C)$, is algebraic and its genus is given by $p=\frac{(n-1)(n-2)}{2}$, where n is the order of the curve M (see p. 179, [10]). Thus M is of order 1 or 2, that is, M is either a projective line or a quadric. However, K is identically equal to 1 on a projective line. Thus M is a quadric, which is congruent to $Q^1: z_0^2+z_1^2+z_2^2=0$ by a projective transformation of $P^2(C)$ but not necessarily by a holomorphic motion of $P^2(C)$.

If further we assume either $K\leqq 1/2$ everywhere or $1/2\leqq K<1$ everywhere, then $\Delta\mu=3(K-1/2)$ is of constant sign on M and, by Green's Theorem, we must have $\Delta\mu=0$ on M, that is, $K\equiv 1/2$. In either case M is congruent to Q^1 by a holomorphic motion of $P^2(C)$.

We now show that M is a projective line if $K>1/2$ everywhere on M. If not there would be a point $x_0\in M$ of minimum curvature $K(x_0)<1$. Then $\mu(x_0)$ is a maximum for μ so that $\Delta\mu=3(K-1/2)\leqq 0$ at x_0. In other words $K(x_0)\leqq 1/2$ and this is a contradiction.

We summarize these results in the following theorem.

THEOREM 9. *On an arbitrary nonsingular complex curve M in the projective plane $P^2(C)$, the curvature K of M (considered with the induced Kähler structure) satisfies $K\leqq 1$ everywhere. If M is closed, the following results hold:*

i) *If $K<1$ everywhere, then M is (complex analytically) a quadric*[2].

2) Blaine Lawson has given us an example which shows that M need not be holomorphically isometric to the quadric Q^1.

ii) *If* $K \leqq 1/2$ *everywhere or if* $1/2 \leqq K < 1$ *everywhere, then* M *is congruent to the quadric* Q^1 *by a holomorphic motion of* $P^2(C)$, *and of course* $K = 1/2$ *everywhere.*

iii) *If* $K > 1/2$ *everywhere, then* M *is a projective line and* $K = 1$ *everywhere.*

From ii) and iii) we also obtain

COROLLARY. *Any closed nonsingular complex curve in* $P^2(C)$ *has a point where* $K \geqq 1/2$. *If* M *is not a projective line, it has a point where* $K = 1/2$.

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