J. Math. Soc. Japan
Vol. 20, No. 3, 1968

Note on cohomological dimension for non-compact spaces II

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(Received Feb. 13, 1967)

§ 1. Introduction.

Let (X, A) be a pair of a topological space X and a closed set A. For an abelian group G we shall denote by $H_f^*(X, A: G)$ and $H^*(X, A: G)$ the Čech cohomology groups based on all finite open coverings and all locally finite open coverings of X, respectively. As Dowker [**1**, Theorem 9.6] shows, even if X is a real line and G is the additive group of integers Z, the groups $H_f^1(X: Z)$ and $H^1(X: Z)$ are quite different. Define the following dimension functions:

$D(X: G)=$ the least integer n such that, for each $m \geqq n$ and each closed set A of X, the homomorphism $i^*: H^m(X: G) \rightarrow H^m(A: G)$ induced by the inclusion mapping $i: A \subset X$ is onto,

$d(X: G)=$ the largest integer n such that $H^n(X, A: G) \neq 0$ for some closed set A of X.

Similarly, the dimension functions $D_f(X: G)$ and $d_f(X: G)$ are defined by making use of the group H_f^* in place of H^*. Skljarenko [**10**] proved that, if X is paracompact, then $d(X: G)=D(X: G)$. We shall prove that, if X is a normal space with finite covering dimension and G is finitely generated, then $D(X: G)=D_f(X: G)=d(X: G)=d_f(X: G)$. As a consequence, we have the equality $D(X: Z)=D_f(X: Z)=d(X: Z)=d_f(X: Z)=\dim X$.

Next, let X be a normal space with finite covering dimension and let f be a closed continuous mapping of X onto a paracompact space Y. We shall show that, if G is finitely generated and $D(f^{-1}(y): G) \leqq k$ for each point y of Y, then $D(X: G) \leqq \operatorname{Ind} Y+k$. Moreover, if X is paracompact, then this relation holds for any abelian group G. As a consequence, we have Morita's theorem [**9**]. Finally, we shall discuss peripherial properties of the cohomological dimension.

Throughout this paper we assume that *all spaces are normal and mappings are continuous transformations.*

§ 2. Čech cohomology groups of normal spaces.

Let (M, N) be a pair of simplicial complexes with weak topology and let G be an abelian group. Take an element e of $H^n(M, N: G)$. Let $K(G, n)$ be an Eilenberg-MacLane space with base point k_0 which is a simplicial complex with metric topology. For a cocycle z which represents e, there is a mapping f_z of (M, N) into $(K(G, n), k_0)$ such that $f_z(M^{n-1} \cup N) = k_0$ and $f_z | \sigma$ represents the element $z(\sigma)$ of $\pi_n(K(G, n)) = G$ for each n-simplex σ of M. The homotopy class relative to N of a mapping f_z is uniquely determined by the element e. Thus, if we denote the set of homotopy classes relative to N of all mappings of (M, N) into $(K(G, n), k_0)$ by $\pi^n(M, N: G)$, then we have a transformation $\chi_{(M,N)}: H^n(M, N: G) \to \pi^n(M, N: G)$. It is well known that $\chi_{(M,N)}$ is 1:1 and onto. Moreover, $\chi_{(M,N)}$ is natural in the following sense. If f is a mapping of (M, N) into (M', N'), then $\chi_{(M,N)} f^* = f^* \chi_{(M',N')}$, where $f^*: H^n(M', N': G) \to H^n(M, N: G)$ and $f^*: \pi^n(M', N': G) \to \pi^n(M, N: G)$ are induced by f. For the sake of convenience we shall state these facts in the following lemma.

LEMMA 1. *Let (M, N) be a pair of simplicial complexes. Then a natural transformation $\chi_{(M,N)}: H^n(M, N: G) \to \pi^n(M, N: G)$ is* 1:1 *and onto.*

In case (X, A) is a pair of a paracompact space X and a closed set A, Goto [2] proved that $\chi_{(X,A)}: H^n(X, A: G) \to \pi^n(X, A: G)$ is a natural isomorphism under a group structure of $\pi^n(X, A: G)$ induced by an H-structure of $K(G, n)$. For the proof of Lemma 1, we have only to use the obstruction theory. (Cf. [4, Chap VI].)

By a *normal pair* we mean a pair of a normal space and its closed set. For a normal pair (X, A), we shall denote by $H_c^*(X, A: G)$ the Čech cohomology group based on all countable locally finite open coverings of X. Since $H_f^n(X, A: G)$ is the direct limit of the subsystem of the direct system which defines $H_c^n(X, A: G)$, there is a natural homomorphism $\mu: H_f^n(X, A: G) \to H_c^n(X, A: G)$ is defined. Similarly, a natural homomorphism $\nu: H_c^n(X, A: G) \to H^n(X, A: G)$ is defined. Then the following theorems hold.

THEOREM 1. *Let (X, A) be a normal pair with finite covering dimension. If G is finitely generated, then μ is onto.*

THEOREM 2. *Let (X, A) be a normal pair. If G is countable, then ν is onto.*

PROOF OF THEOREM 1. Let $\dim X < q$. For an element e of $H_c^n(X, A: G)$, take a countable locally finite open covering $\mathfrak{U}$ of X such that order of $\mathfrak{U} \leqq q$ and there is an element $e_{\mathfrak{u}}$ of $H^n(M_{\mathfrak{u}}, N_{\mathfrak{u}}: G)$ which represents e, where $M_{\mathfrak{u}}$ and $N_{\mathfrak{u}}$ are the nerves of $\mathfrak{U}$ and $\mathfrak{U} | A$. Put $K =$ the q-section of $K(G, n)$. There is a simplicial mapping $f_{\mathfrak{u}}$ of $(M_{\mathfrak{u}}, N_{\mathfrak{u}})$ into (K, k_0) such that $\chi_{(M_{\mathfrak{u}}, N_{\mathfrak{u}})}(e_{\mathfrak{u}}) = \{f_{\mathfrak{u}}\}$, where $\{f_{\mathfrak{u}}\}$ is the homotopy class relative to $N_{\mathfrak{u}}$ of $f_{\mathfrak{u}}$. Since G is finitely generated, we may assume that K is a finite simplicial complex. Let $\mathfrak{W}$ be the covering of K consisting of open stars. If $(M_{\mathfrak{w}}, N_{\mathfrak{w}})$ is the pair of the nerves of $f_{\mathfrak{u}}^{-1}\mathfrak{W}$ and $f_{\mathfrak{u}}^{-1}\mathfrak{W} | N_{\mathfrak{u}}$, then there are projections $\pi_{\mathfrak{w}}^{\mathfrak{u}} : (M_{\mathfrak{u}}, N_{\mathfrak{u}}) \to (M_{\mathfrak{w}}, N_{\mathfrak{w}})$ and $\pi : (M_{\mathfrak{w}}, N_{\mathfrak{w}}) \to (K, k_0)$ such that $\pi\pi_{\mathfrak{w}}^{\mathfrak{u}} \sim$(homotopic to) $f_{\mathfrak{u}}$ relative to $N_{\mathfrak{u}}$. Denote by $e_{\mathfrak{w}}$ the element of $H^n(M_{\mathfrak{w}}, N_{\mathfrak{w}} : G)$ such that $\chi_{(M_{\mathfrak{w}}, N_{\mathfrak{w}})}(e_{\mathfrak{w}}) = \{\pi\}$. By Lemma 1 we have $\pi_{\mathfrak{w}}^{\mathfrak{u}*} e_{\mathfrak{w}} = e_{\mathfrak{u}}$. Since $M_{\mathfrak{w}}$ is finite, there is an element $\bar{e}$ of $H_f^n(X, A : G)$ whose representative is $e_{\mathfrak{w}}$. It is obvious that $\mu(\bar{e}) = e$. This proves that μ is onto.

In case G is countable, there is an Eilenberg-MacLane space $K(G, n)$ which is a countable simplicial complex. By the same argument as in Theorem 1 we can prove Theorem 2.

§ 3. Cohomological dimension of normal spaces.

Throughout this section X is a normal space with finite covering dimension. We defined the cohomological dimension $D_f(X : G)$ and $D(X : G)$ in § 1. By $D_c(X : G)$ we mean the cohmological dimension defined by using the group H_c^*. Similarly we define $d_c(X : G)$.

THEOREM 3. *If G is finitely generated, then* $D_f(X : G) = D_c(X : G) = D(X : G) = d_f(X : G) = d_c(X : G) = d(X : G)$.

To prove Theorem 3 we need the following lemmas.

LEMMA 2. *Let K be a countable simplicial complex with metric topology. Then,*

(i) *K is an ANR (perfectly normal),*

(ii) *K is an ANR (normal) if K contains no infinite full subcomplexes, where a subcomplex N of K is called full if each finite subcollection of vertexes of N spans a simplex of K,*

(iii) *K is an AR (normal) if K is contractible and K contains no infinite full subcomplexes.*

The proof is found in Hanner [3].

LEMMA 3. *Let B be a closed subset of a normal space Y. If $\mathfrak{U}$ is a countable locally finite open covering of B, then there is a countable locally finite open covering $\mathfrak{V}$ of Y such that $\mathfrak{V}|B$ is a refinement of $\mathfrak{U}$.*

PROOF. Let N be the nerve of $\mathfrak{U}$ with metric topology. By Dowker [1, Lemma 3.1] we may assume that each open star of N has finite dimension. Since a cone $C(N)$ on N is an AR (normal) by Lemma 2, a canonical mapping $\phi : B \to N$ has an extension $\psi : Y \to C(N)$. Let $\mathfrak{W}$ be a countable locally finite open covering of $C(N)$ which refines the open covering of open stars of $C(N)$ It is obvious that $\mathfrak{V} = \psi^{-1}\mathfrak{W}$ satisfies the lemma.

Let $\dim X < q$. Consider the following four conditions:

$\boldsymbol{D}_1$: $D_f(X:G)\leqq n$.
$\boldsymbol{D}_2$: $D_c(X:G)\leqq n$.
$\boldsymbol{D}_3$: $D(X:G)\leqq n$
$\boldsymbol{D}_4$: *For each* $m\geqq n$ *and each closed set* A *of* X *every mapping of* A *into* K (= *the q-section of* $K(G, m)$) *is extendable over* X.

LEMMA 4. (i) *If* G *is finitely generated, then the conditions* $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{D}_3$ *and* $\boldsymbol{D}_4$ *are equivalent.*

(ii) *If* G *is countable, then the conditions* $\boldsymbol{D}_2$, $\boldsymbol{D}_3$ *and* $\boldsymbol{D}_4$ *are equivalent.*

PROOF. We shall prove only the first case. The implications $\boldsymbol{D}_1\Leftrightarrow\boldsymbol{D}_4$, $\boldsymbol{D}_2\Leftrightarrow\boldsymbol{D}_4$ and $\boldsymbol{D}_3\Rightarrow\boldsymbol{D}_4$ are showed by the same argument in the proof of [7, Theorem 1] by making use of Lemmas 2 and 3 in place of [7, Lemma 1] and [7, Lemma 2]. To prove the implication $\boldsymbol{D}_4\Rightarrow\boldsymbol{D}_3$ consider the following diagram:

$$\begin{array}{ccc} H_c^m(X:G) & \xrightarrow{i_c^*} & H_c^m(A:G) \\ \downarrow \nu & & \downarrow \nu \\ H^m(X:G) & \xrightarrow{i^*} & H^m(A:G) \end{array}$$

Here, i_c^* and i^* are the homomorphisms induced by the inclusion $i: A\subset X$, and ν is the homomorphism in Theorem 2. We already proved that $\boldsymbol{D}_4$ implies $\boldsymbol{D}_2$. Thus i_c^* is onto. Since ν is onto, i^* is onto. Thus $\boldsymbol{D}_4$ implies $\boldsymbol{D}_3$. This completes the proof.

Note that, if X is perfectly normal, then Lemma 4 (ii) holds without dimensional restriction for X. As a consequence of Lemma 4, we have the following corollaries.

COROLLARY 1. *If* G *is countable, then* $D_c(X:G)=D(X:G)$.

COROLLARY 2. *Let* X *be a normal space with finite covering dimension or a perfectly normal space, and let* G *be a countable abelian group.*

(i) *If* $\{A_i: i=1, 2, \cdots\}$ *is a closed covering of* X, *then* $D(X:G)=\mathrm{Max}\,\{D(A_i:G);\ i=1, 2, \cdots\}$.

(ii) *If* X *has weak topology with respect to* $\{A_\lambda | \lambda\in\Gamma\}$, *then* $D(X:G)=\mathrm{Max}\,\{D(A_\lambda:G);\ \lambda\in\Gamma\}$.

(iii) *If* A *is a closed subset of* X *such that the complement* $X-A$ *and* X *are both normal or perfectly normal, then* $D(X:G)\leqq\mathrm{Max}\,\{D(X-A:G), D(A:G)\}$. *Moreover, if* A *is* G_δ, *then the equality holds.*

PROOF OF THEOREM 3. Note that, by Lemma 3, the sequence of the cohomology groups H_f^* and H_c^* for a normal pair are well defined and exact. This shows that $D_f(X:G)\leqq d_f(X:G)$ and $D_c(X:G)\leqq d_c(X:G)$. Let us prove the relation $D(X:G)\leqq d(X:G)$. Let A be a closed set of X. By Ω denote the set of all locally finite open coverings of A such that, for each $\mathfrak{V}$ of Ω, there is a locally finite open covering $\mathfrak{U}$ of X whose restriction to A refines $\mathfrak{V}$. Let

$H^*_\Omega(A:G)$ be the Čech cohomology groups based on Ω. Consider the following commutative diagram:

$$\begin{array}{ccccccc} & & & H^m(A:G) & & & \\ & & {\scriptstyle i^*}\nearrow & \uparrow{\scriptstyle \nu_1} & & & \\ \cdots \longrightarrow & H^m(X:G) & \longrightarrow & H^m_\Omega(A:G) & \overset{\delta}{\longrightarrow} & H^{m+1}(X,A:G) & \longrightarrow \cdots \\ & & {\scriptstyle i^*_\Omega} & \uparrow{\scriptstyle \nu_2} & & & \\ & & & H^m_c(A:G) & & & \end{array}$$

Here, i^* and i^*_Ω are homomorphisms induced by the inclusion $i: A\subset X$, and ν_1 and ν_2 are homomorphisms defined by a similar way to the homomorphism ν of Theorem 2. Suppose that $d(X:G)\leqq n$. Since the sequence is exact and $H^{m+1}(X, A:G)=0$ for $m\geqq n$, i^*_Ω is onto. Since $\nu=\nu_1\nu_2$ is onto by Theorem 2, ν_1 is also onto. Thus $i^*=\nu_1 i^*_\Omega$ is onto for $m\geqq n$. This shows that $D(X:G)\leqq d(X:G)$. Next, we have the relation $d(X:G)\leqq d_c(X:G)\leqq d_f(X:G)$ by Theorem 1. From these facts and Lemma 4 it follows that $D_f(X:G)=D_c(X:G)=D(X:G)\leqq d(X:G)\leqq d_c(X:G)\leqq d_f(X:G)$. Thus, to prove Theorem 3, it is sufficient to show that $D_f(X:G)\geqq d_f(X:G)$. Let βX be the Čech compactification of X. Let us prove that $D_f(X:G)=D(\beta X:G)$ and $d_f(X:G)=d(\beta X:G)$. Since we know that $D(\beta X:G)=d(\beta X:G)$ by compactness of βX, the above equalities mean Theorem 3. By Morita [8], we have a natural isomorphism $H^n_f(X, A:G)\cong H^n(\beta X, \beta A:G)$ for a closed set A of X. This shows that $D_f(X:G)\leqq D(\beta X:G)$ and $d_f(X:G)\leqq d(\beta X:G)$. Suppose that $D_f(X:G)=n$ and let B be a closed set of βX. For an element e of $H^m(B:G)$, where $m\geqq n$, there is an open set V of βX containing B such that $i^*H^m(\bar{V}:G)\ni e$, where $\bar{V}$ is the closure of V in βX. Put $A=\bar{V}\cap X$. Then $i^*_f: H^m_f(X:G)\to H^m_f(A:G)$ is onto. Thus, if $j: B\subset X$, then $j^*: H^m(\beta X:G)\to H^m(B:G)$ is onto for $m\geqq n$. This shows that $D(\beta X:G)\leqq n$. The relation $d(\beta X:G)\leqq d_f(X:G)$ is proved similarly. This completes the proof.

Let Γ be any subset of the set of all locally finite open coverings of X which forms a directed set and contains the set of finite open coverings. From the proof of Theorem 3 we know that all the dimension functions $D_\Gamma(X:G)$ and $d_\Gamma(X:G)$ defined by groups based on Γ are equal to $D(X:G)$ if G is finitely generated. If we put $G=Z$, then we know that all the dimension functions $D_\Gamma(X:Z)$ and $d_\Gamma(X:Z)$ are equal to $\dim X$.

§ 4. Inductive properties of cohomological dimension.

Throughout this section we shall consider spaces with finite covering dimension.

LEMMA 5. *Let X be a normal space such that* $\dim X<q$, *and let G be a*

finitely generated abelian group. Suppose that there are closed sets A and C, mappings $f: X \to K$ (= the q-section of $K(G, m)$) and $g: C \cup A \to K$ such that $f|A = g|A$. If $H_f^m(C, C \cap A : G) = 0$, then the mapping g is extendable over X.

PROOF. Let $\mathfrak{W}_0$ be the covering of K consisting of open stars. Take a finite open covering $\mathfrak{V}$ of X such that order of $\mathfrak{V} \leqq q$, $\mathfrak{V}$ is a refinement of $f^{-1}\mathfrak{W}_0$ and $\mathfrak{V}|C \cup A$ is a refinement of $g^{-1}\mathfrak{W}_0$. Let ϕ be a canonical mapping of X into the nerve M of $\mathfrak{V}$ such that $\phi(C) \subset N_1$ and $\phi(A) \subset N_2$, where N_1 and N_2 are the nerves of $\mathfrak{V}|C$ and $\mathfrak{V}|A$. There are simplicial mappings $g_1: N_1 \to K$ and $f_1: M \to K$ such that (i) g and $g_1\phi|C$ are contiguous, (ii) f and $f_1\phi$ are contiguous and (iii) $g_1|N_1 \cap N_2 = f_1|N_1 \cap N_2$. The obstruction of the homotopy relative to $N_1 \cap N_2$ in connection with the pair $(f_1|N_1, g_1)$ belongs to $H^m(N_1, N_1 \cap N_2 : G)$. (See Hu [4, Chap. VI]). Since $H_f^m(C, C \cap A : G) = 0$, we can find a finite refinement $\mathfrak{V}'$ of $\mathfrak{V}$ satisfying the following condition: (iv) if M', N_1' and N_2' are the nerves of $\mathfrak{V}'$, $\mathfrak{V}'|C$ and $\mathfrak{V}'|A$ and $\pi: M' \to M$ is a projection, then $g_1\pi|N_1' \sim f_1\pi|N_1'$ relative to $N_1' \cap N_2'$. By homotopy extension theorem, $g_1\pi|N_1'$ has an extension $g_2: M' \to K$ such that $g_2|N_2' = f_1\pi|N_2'$. Let ϕ' be a canonical mapping of X into M'. Since $g_2\phi'|C \cup A$ and g are contiguous by (i), g is extendable over X by [7, Lemma 1]. This completes the proof.

THEOREM 4. *Let X be a normal space and let G be a finitely generated abelian group. If f is a closed mapping of X onto a paracompact space Y such that $D(f^{-1}(y): G) \leqq k$ for each point y of Y, then the relation $D(X: G) \leqq \text{Ind}\, Y + k$. Here* Ind *$Y$ is the large inductive dimension of Y.*

PROOF. We shall give the proof by an analogous argument as in Hurewicz-Wallman [5, Theorem VI 7]. Suppose that the theorem is true in case Ind $Y \leqq n-1$. Let Ind $Y = n$. Take a closed set A of X and a mapping g of A into K (= the q-section of $K(G, m)$), where $\dim X < q$ and $n + k \leqq m$. We shall show that g is extendable over X. Since $D(f^{-1}(y): G) \leqq k \leqq m$ for each point y of Y, by Lemma 4 and Lemma 2, we can find an open set W_y of X containing $f^{-1}(y)$ such that g is extendable over $A \cup W_y$. We denote its extension by g_y. Since f is closed and Y is a paracompact space with large inductive dimension n, there is a locally finite open covering $\mathfrak{V} = \{U_\alpha : \alpha < \Omega\}$ of Y such that (i) $\{f^{-1}(\bar{U}_\alpha) : \alpha < \Omega\}$ is a refinement of $\{W_y : y \in Y\}$ the (ii) Ind $(\bar{U}_\alpha - U) \leqq n-1$ for $\alpha < \Omega$, where Ω is some ordinal. For each $\beta < \Omega$, set $F_\beta = \bigcup_{\beta' < \beta} f^{-1}(\bar{U}_{\beta'})$ and $H_\beta = F_\beta \cup f^{-1}(\bar{U}_\beta)$. Suppose that there is an ordinal $\gamma < \Omega$ such that g has an extension g_β over $H_\beta \cup A$ for each $\beta < \gamma$ and $g_{\beta'} = g_\beta | H_{\beta'} \cup A$ for $\beta' \leqq \beta$. Define $h_\gamma: F_\gamma \cup A \to K$ by setting $h_\gamma | H_\beta = g_\beta$ for $\beta < \gamma$. Then h_γ is continuous. Set $B = f^{-1}(\bar{U}_\gamma) - \bigcup_{\beta < \gamma} f^{-1}(U_\beta)$, $C = B \cap \bigcup_{\beta < \gamma} f^{-1}(\bar{U}_\beta - U_\beta)$ and $D = C \cap A$. To prove the theorem it is sufficient to prove that h_γ is extendable over $H_\gamma \cup A$. Take an open set W_y containing $\bar{U}_\gamma$, and set $h = h_\gamma | C \cup (B \cap A)$ and $k_1 = g_y | B$. By induction hypothesis and (ii) we have $D(C: G) \leqq m-1$. From Theorem 3 it follows that $d_f(C:G)\leqq m-1$. Thus we have $H_f^m(C, D: G)=0$. Put $X=B$, $C=C$, $A=B\frown A$, $g=h$ and $f=k_1$ in Lemma 5 and apply Lemma 5. Then we know that h has an extension $h': B\rightarrow K$. Define $g_\gamma: H_\gamma\cup A\rightarrow K$ by setting $g_\gamma|F_\gamma\cup A=h_\gamma$ and $g_\gamma|B=h'$. Obviously g_γ is continuous. This completes the proof.

If we put $G=Z$ in Theorem 4, then we have the following corollary.

COROLLARY 3 (Morita [8]). *If f is a closed mapping of a normal space X onto a paracompact space Y such that* $\dim f^{-1}(y)\leqq k$ *for each point y of Y, then* $\dim Y\leqq \operatorname{Ind} Y+k$.

COROLLARY 4. *If X is paracompact, then Theorem 4 is true for any abelian group G.*

PROOF. Let us the notations in the proof of Theorem 4. Consider the closed subsets B, C and D of X in the proof of Theorem 4. Since $D(C:G)\leqq m-1$ by induction hypothesis, we know $D(C\times I: G)\leqq m$ by [7, Corollary 5]. Define a mapping $F:(C\times 0)\cup(C\times 1)\cup(D\times I)\rightarrow K$ by setting $F(x, 0)=h(x)$, $F(x, 1)=k(x)$ for $x\in C$ and $F(x, t)=h(x)$ for $(x, t)\in D\times I$. Since $D(C\times I: G)\leqq m$, F is extendable over $C\times I$. Denote its extension by F again. Next, define $F':\{(C\cup(A\frown B))\times I\}\cup(B\times 1)\rightarrow K$ by setting $F'|C\times I=F$, $F'|B\times 1=k_1$ and $F'(x, t)=k_1(x)$ for $(x, t)\in(A\frown B)\times I$. Since X is paracompact, F' has an extension F'' over $B\times I$ by homotopy extension theorem. Define $g_\gamma: H_\gamma\cup A\rightarrow K$ by setting $g_\gamma|F_\gamma=h_\gamma$ and $g_\gamma|B=F''|B\times 0$. It is obvious that g_γ is a continuous extension of h_γ over $H_\gamma\cup A$. This completes the proof.

THEOREM 5. *Let X be a normal space and let G be finitely generated. Suppose that, for each closed set A and each open set U containing A there is an open set V such that $A\subset V\subset \bar{V}\subset U$ and $D(\bar{V}-V:G)\leqq n-1$. Then $D(X:G)\leqq n$. If X is paracompact, then the theorem is true for any abelian group G.*

PROOF. Let $\dim X<q$. Take a closed set A and a mapping f of A into K (=the q-section of $K(G, n)$). Let g be an extension of f over some open set U. There is an open set V such that $A\subset V\subset \bar{V}\subset U$ and $D(\bar{V}-V:G)\leqq n-1$. By Theorem we know $d_f(\bar{V}-V:G)\leqq n-1$. Thus, $H_f^n(\bar{V}-V:G)=0$. An analogous argument as in the proof of Lemma 5 shows that $g|\bar{V}$ is extendable over X. Thus we know $D(X:G)\leqq n$.

Define the dimension function $D_B(X:G)$ as the least integer n such that, for each closed set A and each open set U containing A, there is an open set V which satisfies $A\subset V\subset \bar{V}\subset U$ and $D(\bar{V}-V:G)\leqq n-1$. Theorem 5 shows that $D(X:G)\leqq D_B(X:G)\leqq D(X:G)+1$ for a normal space X and a finitely generated abelian group G. As a consequence of this relation, we have Vedenissoff's relation $\dim X\leqq \operatorname{Ind} X$. However, the equality $D(X:G)=D_B(X:G)$ does not generally hold. For, let M_0 be a Cantor manifold constructed in [6, p. 44]. Then $D_B(M_0:G)=2$ for any abelian group G. On the other hand $D(M_0:G)=1$

for a finite group G. To determine a compact space X such that the equality $D(X:G)=D_{B}(X:G)$ for any abelian group G is an interesting problem.

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