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On the generators of non-negative contraction semi-groups in Banach lattices

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1. Introduction. Let $\mathfrak{B}$ be a Banach lattice. That is, $\mathfrak{B}$ is a Banach space with the real scalar field $\boldsymbol{R}$ and a lattice at the same time and the both structures are related by the axioms: (i) If $f \geqq g$, then $f+h \geqq g+h$; (ii) If $f \geqq g$ and $a \in \boldsymbol{R}^+$ (the set of non-negative real numbers), then $af \geqq ag$; (iii) If $f \geqq g$, then $-f \leqq -g$; (iv) If $|f| \geqq |g|$, then $\|f\| \geqq \|g\|$. We use the notations

$$f \vee g = \sup\{f, g\}, \qquad f \wedge g = \inf\{f, g\},$$

$$|f| = f \vee (-f), \qquad f^+ = f \vee 0, \qquad f^- = -(f \wedge 0).$$

An element $f \geqq 0$ is called non-negative and the cone of non-negative elements is denoted by $\mathfrak{B}^+$. We call a family of linear operators $\{T_t; t \geqq 0\}$ from $\mathfrak{B}$ into $\mathfrak{B}$ an s-continuous non-negative contraction semi-group if they satisfy (i) $T_t T_s = T_{t+s}$ and $T_0 = I$ (identity); (ii) T_t is strongly continuous, i.e., $\operatorname{s-lim}_{t \to 0+} T_t f = f$[1] for each $f \in \mathfrak{B}$; (iii) T_t is a contraction, i.e., $\|T_t\| \leqq 1$; (iv) T_t is non-negative in the sense that T_t maps $\mathfrak{B}^+$ into itself. R.S. Phillips [7] characterized the generators of such semi-groups, introducing the notion of *dispersiveness*. He used a special type of Lumer's semi-inner product, that is, a mapping $s(f, g)$[2] from $\mathfrak{B} \times \mathfrak{B}$ into $\boldsymbol{R}$ which satisfies $s(f, g+h) = s(f, g) + s(f, h)$, $s(f, ag) = as(f, g)$, $|s(f, g)| \leqq \|f\| \|g\|$, $s(f, f) = \|f\|^2$, $s(f^+, f) = \|f^+\|^2$ and carries $\mathfrak{B}^+ \times \mathfrak{B}^+$ into $\boldsymbol{R}^+$. He called an operator A dispersive if $s(f^+, Af) \leqq 0$ for each $f \in \mathfrak{D}(A)$[3] and proved the following theorem: A is the generator of an s-continuous semi-group if and only if A is linear dispersive, $\mathfrak{D}(A)$ is dense and $\mathfrak{R}(\lambda - A) = \mathfrak{B}$ for some $\lambda > 0$. M. Hasegawa [3] noticed that the functional $\tau(f, g)$ defined by

$$\tau(f, g) = \lim_{\varepsilon \to 0+} \varepsilon^{-1}(\|f + \varepsilon g\| - \|f\|) \tag{1.1}$$

is useful for the characterization of the same generators. $\|f\| \tau(f, g)$ shares some properties with $s(f, g)$. Making use of $\tau'(f, g)$ defined by

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1) s-lim denotes limit in the strong convergence.

2) The notation of Lumer and Phillips is $[g, f] = s(f, g)$.

3) The domain of A is denoted by $\mathfrak{D}(A)$ and the range by $\mathfrak{R}(A)$.

$$\tau'(f, g) = 2^{-1}(\tau(f, g) - \tau(f, -g)),$$

he called A to satisfy (d2) if $\tau'(f^+, Af) \leqq 0$ for each $f \in \mathfrak{D}(A)$ and proved that dispersiveness in Phillips' theorem can be replaced by (d2). Hasegawa's (d2)-condition has an advantage in that one can give concrete express the condition in many Banach lattices, as we see in examples in § 6 of this paper. But his $\tau'(f, g)$ does not possess subadditivity with respect to g, which causes inconvenience for us to deal with, for instance, sums of generators.

In this paper we introduce a new functional

$$\sigma(f, g) = \inf \tau(f, (g+k) \vee (-bf)) \qquad \text{for } f \in \mathfrak{B}^+ \tag{1.2}$$

where the infimum is taken for all $b \in \boldsymbol{R}^+$ and all k satisfying $f \wedge |k| = 0$.

DEFINITION. We call an operator A with domain $\mathfrak{D}(A)$ *dispersive in the strict sense* or *dispersive* (*s*) if $\sigma(f^+, Af) \leqq 0$ for each $f \in \mathfrak{D}(A)$, and *dispersive in the wide sense* or *dispersive* (*w*) if $\sigma(f^+, -Af) \geqq 0$ for each $f \in \mathfrak{D}(A)$.

The functional σ has some important properties which τ and τ' do not have. Dispersiveness (s) implies dispersiveness (w) and we prove that Phillips' theorem is valid if we replace his dispersiveness by either of our dispersiveness (w) and (s) (Theorems 1 and 2). Further we prove the existence of a closed extension of a linear dispersive (w) operator with dense domain (Theorem 3), and obtain a necessary and sufficient condition in order that its smallest closed extension generates an s-continuous non-negative contraction semi-group (Theorem 4).

In § 5, we apply these results and give some sufficient condition for the sum of a generator and a dispersive (w) operator to be again the generator of an s-continuous non-negative contraction semi-group (Theorem 6). This is a generalization of a result of K. Yosida [**12**]. We use the fact that the sum of a dispersive (s) operator and a dispersive (w) operator is dispersive (w).

We investigate, in § 6, concrete expressions of dispersivities (w) and (s) in various Banach lattices. Results in the case of $\mathfrak{B} = C(X)$, the space of continuous functions on a compact space X are as follows: Dispersiveness (s) of A is equivalent to that

(1.3) if f in $\mathfrak{D}(A)$ attains a positive maximum at x_0, then $Af(x_0) \leqq 0$.

Dispersiveness (*w*) of A is equivalent to that

(1.4) if f in $\mathfrak{D}(A)$ attains a positive maximum at x_0, then there exists a point x_1 where $f(x_1) = f(x_0)$ and $Af(x_1) \leqq 0$.

The both properties are familiar in the theory of Markov processes and investigated, e.g., in [**8**] and [**10**].

I would like to express my hearty thanks to Takesi Watanabe for his valuable advice. He called my attention to the fact of Lemma 4.1, and this

made Theorem 4 attain the present generality. The original version of Theorem 4 was restricted to a class of Banach lattices including C and L_p $(1<p\leqq\infty)$.

2. Functional τ. The limit in the right hand side of (1.1) exists for each pair $(f, g)\in\mathfrak{B}\times\mathfrak{B}$ and we can define $\tau(f, g)$. This functional satisfies

PROPOSITION 2.1.

(i) $$|\tau(f, g)|\leqq\|g\|,$$

(ii) $$\tau(f, ag)=a\tau(f, g) \qquad \textit{for } a\geqq 0,$$

(iii) $$\tau(f, af+g)=a\|f\|+\tau(f, g) \qquad \textit{for all } a,$$

(iv) $$\tau(f, g+h)\leqq\tau(f, g)+\tau(f, h),$$

(v) *if* $f\geqq 0$ *and* $af\leqq g\leqq h$ *for some* $a\in\boldsymbol{R}$, *then* $\tau(f, g)\leqq\tau(f, h)$.

All of these are easy consequences of the definition. (i)—(iv) are found in Dunford-Schwartz [2] Chapter V, 9, and (v) is clear since we have, for every sufficiently small $\varepsilon>0$, $\|f+\varepsilon h\|\geqq\|f+\varepsilon g\|$ by $f+\varepsilon h\geqq f+\varepsilon g\geqq f+\varepsilon af\geqq 0$.

Proposition 2.1 contains all the properties that we need in the following argument. In other words, any functional satisfying (i)—(v) can serve to make all the theorems hold, if we define σ and dispersiveness similarly.

REMARK. If $|f|\wedge|g|=0$, then $\tau(f, g)=\tau(|f|, |g|)$. This is a remarkable fact, though we do not need it below. The proof is based on the fact that $|f|\wedge|g|=0$ implies $|f+g|=|f|+|g|$. For, if $|f|\wedge|g|=0$, then we have $|f|\wedge|\varepsilon g|=0$, $|f+\varepsilon g|=|f|+|\varepsilon g|=|f|+\varepsilon|g|$ and $\|f+\varepsilon g\|-\|f\|=\||f|+\varepsilon|g|\|-\||f|\|$.

3. Functional σ. Let us examine properties of the functional σ from $\mathfrak{B}^+\times\mathfrak{B}$ into $\boldsymbol{R}$ defined by (1.2).

PROPOSITION 3.1. *Let* $f\geqq 0$. *Then,*

(i) $$-\|g^-\|\leqq\sigma(f, g)\leqq\|g^+\|,$$

(ii) $$\sigma(f, ag)=a\sigma(f, g) \qquad \textit{for } a\geqq 0,$$

(iii) $$\sigma(f, af+g)=a\|f\|+\sigma(f, g) \qquad \textit{for all } a,$$

(iv) $$\sigma(f, g+h)\leqq\sigma(f, g)+\sigma(f, h),$$

(v) *if* $g\leqq h$, *then* $\sigma(f, g)\leqq\sigma(f, h)$,

(vi) *if* $f\wedge|h|=0$, *then* $\sigma(f, g)=\sigma(f, g+h)$.

Among direct consequences of (i), (iv) and (vi), we have

$$\sigma(f, 0)=0, \tag{3.1}$$

$$(3.2)\qquad -\sigma(f,-g)\leqq\sigma(f,g),$$

$$(3.3)\qquad |\sigma(f,g)-\sigma(f,h)|\leqq\|g-h\|,$$

$$(3.4)\qquad \sigma(0,g)=0.$$

There are many Banach lattices in which $\sigma(f,g)=\tau(f,g)$ holds if $f\in\mathfrak{B}^+$ and $f\neq 0$. However, it is not true in some Banach lattices, as is seen from the example $\mathfrak{B}=L_1$ or $A(\mathscr{B})$ in § 6. The properties of σ are really stronger than those of τ.

PROOF OF PROPOSITION 3.1. Throughout the proof, we suppose $f\geqq 0$, $f\wedge|k|=0$ and $b\in\boldsymbol{R}^+$. We first note that, if $0\leqq b<b'$, then $\tau(f,g\vee(-bf))\geqq\tau(f,g\vee(-b'f))$ by (v) of Proposition 2.1. It follows that

$$(3.5)\qquad \begin{aligned}\sigma(f,g)&=\inf_k\lim_{b\to\infty}\tau(f,(g+k)\vee(-bf))\\&=\lim_{b\to\infty}\inf_k\tau(f,(g+k)\vee(-bf)).\end{aligned}$$

Note that we have not yet proved $\sigma(f,g)>-\infty$. The proof of (iii) is obtained from (3.5) and Proposition 2.1 (iii):

$$\begin{aligned}\tau(f,(af+g+k)\vee(-bf))&=\tau(f,af+(g+k)\vee(-(a+b)f))\\&=a\|f\|+\tau(f,(g+k)\vee(-(a+b)f)).\end{aligned}$$

(vi) is also clear from the definition of σ, since $f\wedge|h|=f\wedge|k|=0$ implies $f\wedge|h+k|=0$. In order to prove (iv), let us prove, for each $g',h'\in\mathfrak{B}$ and $b\in\boldsymbol{R}^+$,

$$(3.6)\qquad \tau(f,(g'+h')\vee(-bf))\leqq\tau(f,g'\vee(-b/2)f)+\tau(f,h'\vee(-b/2)f).$$

Let $u_1=(g'+h')\vee(-bf)$ and $u_2=g'\vee(-b/2)f+h'\vee(-b/2)f$. We have $u_2\geqq -bf$, and hence $u_2=u_2\vee(-bf)\geqq u_1\geqq -bf$. Therefore we have (3.6) by virtue of (iv) and (v) of Proposition 2.1. If we set $g'=g+k_1$ and $h'=h+k_2$ in (3.6), supposing $f\wedge|k_1|=f\wedge|k_2|=0$, then we get to (iv) by using (3.5). If $a>0$, then (ii) is seen from Proposition 2.1 (ii):

$$\tau(f,(ag+k)\vee(-bf))=a\tau(f,(g+a^{-1}k)\vee(-a^{-1}bf)).$$

In case $a=0$, (ii) is contained in the assertion (i).

Now it remains to prove (i) and (v). Among these, the second inequality in (i) is obtained from the definition of σ and Proposition 2.1 (i): $\sigma(f,g)\leqq\tau(f,g\vee 0)\leqq\|g^+\|$. It follows from this that $g\leqq 0$ implies $\sigma(f,g)\leqq 0$. From this and (iv) we can see (v): $\sigma(f,g)\leqq\sigma(f,g-h)+\sigma(f,h)\leqq\sigma(f,h)$ if $g\leqq h$. Let us see that

$$(3.7)\qquad g\geqq 0\quad\text{implies}\quad \sigma(f,g)\geqq 0.$$

(This does not follow from (v), since we did not yet prove $\sigma(f, 0)=0$.) $g \geqq 0$ implies $(g+k)\vee(-bf) \geqq k\vee(-bf)=-((-k)\wedge(bf)) \geqq -(|k|\wedge(bf))=0$, and hence, $\tau(f, (g+k)\vee(-bf)) \geqq \tau(f, 0)=0$ by Proposition 2.1 (i) and (v), which proves (3.7). Now we can verify $-\|g^-\| \leqq \sigma(f, g)$, because $\sigma(f, g)+\|g^-\| \geqq \sigma(f, g)+\sigma(f, g^-) \geqq \sigma(f, g^+) \geqq 0$. The proof of Proposition 3.1 is complete.

4. Characterization of generators. The generator A of an s-continuous non-negative contraction semi-group $\{T_t;\, t \geqq 0\}$ is defined by

$$Af = \operatorname*{s\text{-}lim}_{t\to 0+} t^{-1}(T_t f - f),$$

the domain $\mathfrak{D}(A)$ being the set of f for which the right side exists. The following two theorems combined with the well-known Hille-Yosida theorem form the counter-part of Phillips' theorem mentioned in § 1.

THEOREM 1. *If A is the generator of an s-continuous non-negative contraction semi-group, then A is dispersive* (*s*).

Note that any dispersive (s) operator is dispersive (w) by the property (3.2) of σ.

PROOF. Let $f \in \mathfrak{D}(A)$. Keeping in mind $f^+ \wedge f^- = 0$, $T_t f^- \geqq 0$ and $\|T_t\| \leqq 1$ and using the properties (vi), (iii), (v) and (i) of σ in turn, we have $\sigma(f^+, T_t f - f) = \sigma(f^+, T_t f - f^+ + f^-) = \sigma(f^+, T_t f - f^+) = -\|f^+\| + \sigma(f^+, T_t f)$ and $\sigma(f^+, T_t f) = \sigma(f^+, T_t f^+ - T_t f^-) \leqq \sigma(f^+, T_t f^+) \leqq \|T_t f^+\| \leqq \|f^+\|$, and hence,

$$\sigma(f^+, T_t f - f) \leqq 0.$$

Multiply this by t^{-1}, use (ii) of Proposition 3.1, and make t tend to zero. Then we get $\sigma(f^+, Af) \leqq 0$, since (3.3) asserts that $\sigma(f, g)$ varies continuously as g varies in the strong topology. Consequently, A is dispersive in the strict sense.

THEOREM 2. *If A is a linear dispersive* (*w*) *operator with dense domain and if $\Re(\lambda - A) = \mathfrak{B}$ for some $\lambda > 0$, then A is the generator of an s-continuous non-negative contraction semi-group.*

LEMMA 4.1. *Let A be dispersive* (*w*). *If $(\mu - A)f = g$ and $\mu > 0$, then $\|f^+\| \leqq \|g^+\|/\mu$ and $\|f^-\| \leqq \|g^-\|/\mu$.*

PROOF. Using (i), (iii) and (vi) of the properties of σ, we get the first inequality: $\|g^+\| \geqq \sigma(f^+, g) = \sigma(f^+, \mu f - Af) = \sigma(f^+, \mu f^+ - Af) = \mu\|f^+\| + \sigma(f^+, -Af) \geqq \mu\|f^+\|$. The second inequality is a consequence of the first, since $f^- = (-f)^+$ and $g^- = (-g)^+$.

PROOF OF THEOREM 2. By the above lemma, $(\mu - A)^{-1}$ exists for all $\mu > 0$, and it is non-negative since $g^- = 0$ implies $f^- = 0$. We write $G_\mu = (\mu - A)^{-1}$. In order to prove the theorem, it is sufficient to show

$$\mathfrak{D}(G_\mu)=\mathfrak{B}, \tag{4.1}$$

$$\|G_\mu\| \leqq 1/\mu, \tag{4.2}$$

by virtue of the Hille-Yosida theorem and, as for the non-negativity of the semi-group, by Yosida's or Hille's representation formula of semi-group operators by resolvents. (4.2) is a consequence of (4.1) if we use Lemma 4.1 and the relation: $|G_\mu g| \leqq |G_\mu g^+|+|G_\mu g^-|=G_\mu g^+ + G_\mu g^- = G_\mu |g|$. Let us prove (4.1). It is assumed to hold for $\mu=\lambda$. Suppose $0<\mu<2\lambda$. Then $f=(1-(\lambda-\mu)G_\lambda)^{-1}G_\lambda g$ exists for every $g\in\mathfrak{B}$, since $\|(\lambda-\mu)G_\lambda\|<1$. We have $(1-(\lambda-\mu)G_\lambda)f=G_\lambda g$, and hence, $f\in\mathfrak{D}(A)$ and $(\mu-A)f=g$. Repeating this procedure, we get to (4.1) for arbitrary $\mu>0$. The proof is complete.

The next two theorems are concerned with closed extensions of dispersive (w) operators.

THEOREM 3. *Every linear dispersive* (w) *operator with dense domain has a closed extension.*

PROOF. Let A be linear dispersive (w) with dense domain. It suffices to prove that if $\{f_n\}$ is a sequence in $\mathfrak{D}(A)$ strongly converging to 0 and if Af_n tends strongly to some g, then $g=0$. Suppose $g\neq 0$. We may and do suppose $g^+\neq 0$ and $\|g^+\|=1$. $\mathfrak{D}(A)$ being dense, there is an element $h\in\mathfrak{D}(A)$ such that $\|g-h\|<1/3$. Since $(h_1+h_2)^+\leqq h_1^+ + h_2^+$ holds in general, we have $1=\|g^+\| \leqq \|h^+ + (g-h)^+\| \leqq \|h^+\|+\|(g-h)^+\| < \|h^+\|+1/3$ and so $\|h^+\|>2/3$. We have, for arbitrary $\varepsilon>0$,

$$\begin{aligned}&\sigma((f_n+\varepsilon h)^+, -g-\varepsilon Ah)\\ &\qquad\leqq \sigma((f_n+\varepsilon h)^+, -\varepsilon^{-1}f_n-h)+\varepsilon^{-1}\|f_n^+\|+\|(h-g)^+\|+\varepsilon\|(Ah)^-\|\end{aligned}$$

by repeated use of (i) and (iv) of Proposition 3.1, and

$$\sigma((f_n+\varepsilon h)^+, -\varepsilon^{-1}f_n-h)=-\varepsilon^{-1}\|(f_n+\varepsilon h)^+\| \to -\|h^+\|, \qquad n\to\infty$$

by (iii), (vi) and a general inequality $|(h_1+h_2)^+-h_1^+| \leqq |h_2|$. Hence,

$$\begin{aligned}\limsup_{n\to\infty} \sigma((f_n+\varepsilon h)^+, -g-\varepsilon Ah) &\leqq -\|h^+\|+\|(h-g)^+\|+\varepsilon\|(Ah)^-\|\\ &< -1/3+\varepsilon\|(Ah)^-\|<0\end{aligned}$$

if we choose $\varepsilon>0$ appropriately small. On the other hand,

$$\liminf_{n\to\infty} \sigma((f_n+\varepsilon h)^+, -g-\varepsilon Ah)=\liminf_{n\to\infty} \sigma((f_n+\varepsilon h)^+, -(Af_n+\varepsilon Ah)) \geqq 0$$

by (3.3) and dispersiveness (w). This is a contradiction, and the theorem is proved.

REMARK. We do not know whether the smallest closed extention of linear dispersive (w) operator with dense domain is necessarily dispersive (w). But the answer is given affirmatively, in case $\mathfrak{B}$ has the property

(4.3) If $f_n \in \mathfrak{B}^+$ and $\underset{n\to\infty}{\text{s-lim}} f_n = f \neq 0$, then $\limsup_{n\to\infty} \sigma(f_n, g) \leqq \sigma(f, g)$ for all $g \in \mathfrak{B}$.

The condition (4.3) is satisfied for $B(X)$, $C(X)$, $C_0(X)$ and $L_p(X, \mathcal{B}, m)$, $1 < p \leqq \infty$, but is not satisfied for $L_1(X, \mathcal{B}, m)$ and $A(\mathcal{B})$ except trivial cases (see § 6 as for notations).

THEOREM 4. *If A is a linear dispersive* (*w*) *operator with dense domain and if $\mathfrak{R}(\lambda - A)$ is dense for some $\lambda > 0$, then the smallest closed extention $\bar{A}$ of A is the generator of an s-continuous non-negative contraction semi-group.*

PROOF. The existence of $\bar{A}$ is proved by Theorem 3. If $(\mu - \bar{A})f = g$ and $\mu > 0$, then $\|f^+\| \leqq \|g^+\|/\mu$ and $\|f^-\| \leqq \|g^-\|/\mu$. For, there is a sequence $\{f_n\}$ in $\mathfrak{D}(A)$ such that f_n tends to f strongly, and Af_n to $\bar{A}f$ and we have the corresponding inequalities for f_n by Lemma 4.1. The proof of Theorem 4 is, therefore, carried out in the same way as that of Theorem 2. Note that $(\mu - A)^{-1}$ is bounded by $2/\mu$, and hence, denseness of $\mathfrak{R}(\mu - A)$ implies $\mathfrak{R}(\mu - \bar{A}) = \mathfrak{B}$.

REMARK. In case A is one-to-one, the assumption of denseness of $\mathfrak{R}(\lambda - A)$ in Theorem 4 can be replaced by denseness of $\mathfrak{R}(\lambda V + I)$, where $V = -A^{-1}$. For, we have $\mathfrak{R}(\lambda V + I) = \mathfrak{R}((\lambda - A)V) = \mathfrak{R}(\lambda - A)$. Denseness of $\mathfrak{R}(\lambda V + I)$ was employed in Yosida [**13**].

5. Sums of generators and dispersive (w) **operators.** Concerning sums of dispersive (w or s) operators we have

THEOREM 5. *If A and B are both dispersive* (*s*) *operators, then $A + B$* (*defined on $\mathfrak{D}(A) \cap \mathfrak{D}(B)$*) *is dispersive* (*s*). *If A is dispersive* (*s*) *and B is dispersive* (*w*), *then $A + B$ is dispersive* (*w*).

PROOF. The both assertions are consequences of the subadditivity of σ. That is, $\sigma(f^+, (A+B)f) \leqq \sigma(f^+, Af) + \sigma(f^+, Bf) \leqq 0$ under the first assumption, and $\sigma(f^+, -(A+B)f) \geqq \sigma(f^+, -Bf) - \sigma(f^+, Af) \geqq 0$ under the second assumption.

Before proceeding to Theorem 6, we prepare some facts about Bochner's subordination of semi-groups [**1**]. Let $(\Omega, \mathcal{B}, P)$ be a probability space, and $\{x_t(\omega);\ t \geqq 0\}$ be a stochastic process with stationary independent increments starting at 0 and having right-continuous non-decreasing paths. It is known that the law F_t of x_t is uniquely expressed by a constant $c \geqq 0$ and a measure n on $(0, \infty)$ satisfying $\int_0^\infty \frac{r}{1+r}\, n\,(dr) < \infty$ in such a way that

$$E(e^{-\lambda x_t}) = \int_{0-}^{\infty} e^{-\lambda s} F_t(ds) = \exp\left[-t(c\lambda + \int_0^\infty (1 - e^{-\lambda r}) n(dr))\right], \quad \lambda > 0.$$

Given an s-continuous non-negative contraction semi-group $\{T_t;\ t \geqq 0\}$ in $\mathfrak{B}$, define

$$T'_t f = \int_{0-}^{\infty} T_s f F_t(ds), \qquad f \in \mathfrak{B}.$$

Then, $\{T'_t;\ t \geqq 0\}$ is again an s-continuous non-negative contraction semi-group. This operation to obtain a new semi-group $\{T'_t\}$ from $\{T_t\}$ is called subordination, and the process $\{x_t\}$ is called a subordinator. The relation of the generators A and A' of $\{T_t\}$ and $\{T'_t\}$ is that $\mathfrak{D}(A) \subset \mathfrak{D}(A')$ and

$$A'f = cAf + \int_0^{\infty} (T_s f - f) n(ds), \qquad f \in \mathfrak{D}(A). \tag{5.1}$$

A proof is found in [**6**]. If $c=0$, we call *subordination of the first kind,* and, if $c>0$, *subordination of the second kind.* Subordination by a one-sided stable process with exponent α $(0<\alpha<1)$, that is,

$$E(e^{-\lambda x_t}) = e^{-t\lambda^{\alpha}} = \exp\left[-\frac{\alpha t}{\Gamma(1-\alpha)}\int_0^{\infty}(1-e^{-\lambda r})\frac{dr}{r^{1+\alpha}}\right],$$

is an example of the first kind, and, in this case, A' is the so-called fractional power of A: $A' = -(-A)^{\alpha}$.

THEOREM 6. *Let A be the generator of an s-continuous non-negative contraction semi-group and B be a linear dispersive* (*w*) *operator. If $\mathfrak{D}(A') \subset \mathfrak{D}(B)$ holds for some A' obtained from A by subordination of the first kind, then $A+B$* (*with domain equal to $\mathfrak{D}(A)$*) *is the generator of an s-continuous non-negative contraction semi-group.*

The proof consists of the following two lemmas.

LEMMA 5.1. *Let $a>0$ be given. Under the assumption of the above theorem, we can find a constant $b \geqq 0$ such that*

$$\|Bf\| \leqq a\|Af\| + b\|f\|, \qquad f \in \mathfrak{D}(A). \tag{5.2}$$

PROOF. Let $f \in \mathfrak{D}(A)$. Then, $T_s f - f = \int_0^s T_t A f dt$, and hence, $\|T_s f - f\| \leqq s\|Af\|$. By (5.1) with $c=0$ and $\int_0^{\infty} \frac{s}{1+s} n(ds) < \infty$, we have

$$\|A'f\| \leqq \int_0^{\infty} \|T_s f - f\| n(ds) \leqq a'\|Af\| + b'\|f\|,$$

where a' can be chosen arbitrarily small. On the other hand, since A' is closed and B is closable (Theorem 3), there is a constant d such that

$$\|Bf\| \leqq d(\|A'f\| + \|f\|), \qquad f \in \mathfrak{D}(A')$$

by an application of the closed graph theorem ([**11**], Chapter II, 6), and we have (5.2).

The next lemma is essentially found in Trotter [**9**] (cf. Nelson [**5**], Theorem 7).

LEMMA 5.2. *Let A be the generator of an s-continuous non-negative contraction semi-group, and B be a linear dispersive (w) operator with $\mathfrak{D}(B)\supset\mathfrak{D}(A)$. If there are constants $1/2\geqq a\geqq 0$ and $b>0$ such that (5.2) holds, then $A+B$ is the generator of an s-continuous non-negative contraction semi-group*[1].

PROOF. By Theorems 1 and 5, $A+B$ is dispersive (w). Hence, by Theorem 2, it suffices to see $\mathfrak{R}(\lambda-A-B)=\mathfrak{B}$ for some $\lambda>0$. We have $\mathfrak{R}(\lambda-A-B)=\mathfrak{R}((\lambda-A-B)(\lambda-A)^{-1})=\mathfrak{R}(1-B(\lambda-A)^{-1})$, and

$$\|B(\lambda-A)^{-1}f\|\leqq a\|A(\lambda-A)^{-1}f\|+b\|(\lambda-A)^{-1}f\|\leqq(2a+b/\lambda)\|f\|$$

by (5.2) and by $\|A(\lambda-A)^{-1}\|=\|\lambda(\lambda-A)^{-1}-I\|\leqq 2$. If we choose λ large enough, then $\|B(\lambda-A)^{-1}\|<1$ and the lemma is obtained.

REMARK. Let $\mathfrak{B}$ be a Banach space with the real scalar field not assumed to have the lattice structure and let us call A *dissipative* (s) if $\tau(f, Af)\leqq 0$, $f\in\mathfrak{D}(A)$, and *dissipative* (w) if $\tau(f, -Af)\geqq 0$, $f\in\mathfrak{D}(A)$. Then, *all of our results remain valid if we replace 'dispersive' by 'dissipative' and remove 'non-negative' from 's-continuous non-negative contraction semi-group'*. (Theorems 1 and 3 changed in this way are due to Hasegawa [**3**], Remark 3 and Proposition 7.) As a corollary to the analogue to Theorem 6, we have the following result: *Let A' be the generator of an s-continuous contraction semi-group obtained by subordination from an s-continuous contraction semi-group with generator A. If the subordination is of the second kind, then $\mathfrak{D}(A')=\mathfrak{D}(A)$.* On the other hand, Daisuke Fujiwara (private communication) proved that if the subordination is of the first kind and A is unbounded, then $\mathfrak{D}(A')\supsetneqq\mathfrak{D}(A)$.

6. Examples. In the sequel the word *function* means a real-valued function.

6.1. Let $\mathfrak{B}$ be the Banach lattice $C(X)$ of continuous functions f on a compact space X with norm $\|f\|=\max_{x\in X}|f(x)|$. Then

$$\tau(f, g)=\max_{x\in X(f)}(\operatorname{sgn} f(x))g(x)\,, \qquad f\neq 0 \tag{6.1}$$

where $X(f)=\{x\,;\,|f(x)|=\|f\|\}$, and

$$\sigma(f, g)=\tau(f, g)\,, \qquad f\neq 0,\ f\geqq 0\,. \tag{6.2}$$

Hence, remembering (3.4), we see that dispersiveness (s) and (w) are equivalent to (1.3) and (1.4), respectively.

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1) (Added in proof.) This lemma can be extended to the case $a<1$. The proof will be published in

Karl Gustaffon and Ken-iti Sato, Some perturbation theorems for non-negative contraction semi-groups, to appear.

PROOF. Let $f \neq 0$. If $x \in X(f)$, then $\varepsilon^{-1}(\|f+\varepsilon g\|-\|f\|) \geqq \varepsilon^{-1}((\operatorname{sgn} f(x))(f+\varepsilon g)(x)-|f(x)|)=(\operatorname{sgn} f(x))g(x)$. Hence, $\tau(f, g)$ is not smaller than the right side of (6.1). Let $\varepsilon_n > 0$ be a sequence decreasing to zero, and let $x_n \in X(f+\varepsilon_n g)$. Choosing a subsequence if necessary, we may suppose either $(f+\varepsilon_n g)(x_n) \geqq 0$ for all n or $(f+\varepsilon_n g)(x_n) \leqq 0$ for all n. We do suppose the former. Then, $\varepsilon_n^{-1}(\|f+\varepsilon_n g\|-\|f\|) \leqq \varepsilon_n^{-1}((f+\varepsilon_n g)(x)-f(x_n))=g(x_n)$. Since X is compact, there is a point x_∞ such that every neighborhood of x_∞ contains x_n infinitely often. Thus we can choose a subsequence $\{x_{n(k)}\}$ such that $f(x_{n(k)})$ tends to $f(x_\infty)$ and $g(x_{n(k)})$ tends to $g(x_\infty)$. Using this subsequence, we see that $f(x_\infty) > 0$, $x_\infty \in X(f)$, and $\tau(f, g) \leqq g(x_\infty) \leqq \max_{x \in X(f)} (\operatorname{sgn} f(x))g(x)$. The case $(f+\varepsilon_n g)(x_n) \leqq 0$ for all n is similar, and we have proved (6.1). (6.2) is an easy consequence of (6.1) and (3.5).

6.2. Let X be a locally compact space which is not compact, and $\mathfrak{B}=C_0(X)$ be the Banach lattice of continuous functions f on X such that, for each $\varepsilon > 0$, $|f(x)| < \varepsilon$ holds outside of a compact closed set. Then we have the same results as above. Note that $X(f)$ is compact, and $\{x_n\}$ in the proof is contained in a compact set.

In the following two remarks we suppose that $\mathfrak{B}=C(X)$ in 6.1 or $\mathfrak{B}=C_0(X)$ in 6.2, and, in addition, that X is Hausdorff.

REMARK. If A is linear dispersive (w) with dense domain, then A is dispersive (s). For the proof, assume that the conclusion does not hold: there is an $f \in \mathfrak{D}(A)$ such that $\sigma(f^+, Af) > 0$. Then, $f^+ \neq 0$ by (3.4), and we have, by (6.1) and (6.2), $f(x_0)=\|f^+\|$ and $Af(x_0)=a_0 > 0$ at some x_0. There is an open neighborhood U of x_0 on which $Af(x) > a_0/2$. Since X is completely regular, there is a continuous function g such that $g(x_0)=1$ and $g(x)=0$, $x \in X-U$. It follows from denseness of $\mathfrak{D}(A)$ that we can choose $h \in \mathfrak{D}(A)$ such that $h(x) > 1/2$ and $h(x) < 1/2$, $x \in X-U$. We have

$$\sigma((f+\varepsilon h)^+, -Af-\varepsilon Ah) \leqq \sup_{x \in U} (-Af-\varepsilon Ah)(x) \leqq -a_0/2+\varepsilon\|Ah\|$$

for each $\varepsilon > 0$. The last member is negative for small ε, which contradicts to dispersiveness (w) of A, and the proof is complete.

In a similar way, we can prove dispersiveness (s) of $\bar{A}$. This is obtained also as a consequence of the present remark, if we use the remark immediately after Theorem 3.

REMARK. Let A be one-to-one and write $V=-A^{-1}$. (It can be proved that if A is linear dispersive (w) and has dense range, then A is one-to-one.) Dispersiveness of A has a close connection with the maximum principles for V considered, e.g., in [4]. Especially, if A is linear and if both $\mathfrak{D}(A)$ and $\mathfrak{R}(A)$ are dense, the following four conditions are equivalent to each other: (i) A is dispersive (s). (ii) A is dispersive (w). (iii) Let $f \in \mathfrak{D}(V)$. Then $f(x)$ is non-negative at every point where $Vf(x)$ achieves its maximum, provided the maximum is non-negative. (iv) Let $f\in\mathfrak{D}(V)$. If the supremum a of $Vf(x)$ is positive, then the supremum of $Vf(x)$ on the set $\{x; f(x)>0\}$ is equal to a.

6.3. Let X be an arbitrary set and $\mathfrak{B}$ be the Banach lattice $B(X)$ of bounded functions on X normed with the supremum of the absolute value. Then we have

$$\tau(f, g)=\lim_{\varepsilon\to 0+}\sup_{x\in X(f,\varepsilon)}(\operatorname{sgn} f(x))g(x), \qquad f\neq 0 \tag{6.3}$$

where $X(f, \varepsilon)=\{x; |f(x)|>\|f\|-\varepsilon\}$, and (6.2) holds true. The space of bounded measurable functions on a measurable space, the space of bounded continuous functions on a topological space and the space of bounded uniformly continuous functions on a uniform space are subspaces of the respective $B(X)$, and hence, have the same expression of τ and σ.

PROOF. As ε decreases to 0, $\sup_{x\in X(f,\varepsilon)}(\operatorname{sgn} f(x))g(x)$ is non-increasing to a limit. Fix a decreasing sequence $\varepsilon_n\downarrow 0$, such that $\|f\|>\varepsilon_n'$ where $\varepsilon_n'=2\varepsilon_n\|g\|+\varepsilon_n^2$. If $x\in X(f, \varepsilon_n^2)$, then

$$\begin{aligned}\varepsilon_n^{-1}(\|f+\varepsilon_n g\|-\|f\|)&\geqq\varepsilon_n^{-1}((\operatorname{sgn} f(x))(f+\varepsilon_n g)(x)-|f(x)|-\varepsilon_n^2)\\&=(\operatorname{sgn} f(x))g(x)-\varepsilon_n.\end{aligned}$$

Hence, $\tau(f, g)$ is not smaller than the right side of (6.3). To prove the reverse inequality, take $x_n\in X(f+\varepsilon_n g, \varepsilon_n^2)$. If $(f+\varepsilon_n g)(x_n)\geqq 0$, then

$$\varepsilon_n^{-1}(\|f+\varepsilon_n g\|-\|f\|)\leqq\varepsilon_n^{-1}((f+\varepsilon_n g)(x_n)+\varepsilon_n^2-f(x_n))=g(x_n)+\varepsilon_n$$

and $f(x_n)>\|f+\varepsilon_n g\|-\varepsilon_n^2-\varepsilon_n g(x_n)\geqq\|f\|-\varepsilon_n'>0$. Similarly, if $(f+\varepsilon_n g)(x_n)\leqq 0$, then $\varepsilon_n^{-1}(\|f+\varepsilon_n g\|-\|f\|)\leqq -g(x_n)+\varepsilon_n$ and $-f(x_n)>\|f\|-\varepsilon_n'>0$. In both cases we have $\varepsilon_n^{-1}(\|f+\varepsilon_n g\|-\|f\|)\leqq(\operatorname{sgn} f(x_n))g(x_n)+\varepsilon_n$ and $x_n\in X(f, \varepsilon_n')$, and hence, $\tau(f, g)$ does not exceed the right side of (6.3). (6.2) is a consequence of the expression (6.3).

6.4. If $(X, \mathcal{B}, m)$ is a measure space and $\mathfrak{B}$ is the real L_∞ space on it, then we have

$$\tau(f, g)=\lim_{\varepsilon\to 0+}\operatorname*{ess\,sup}_{x\in X(f,\varepsilon)}(\operatorname{sgn} f(x))g(x), \qquad f\neq 0, \tag{6.4}$$

and (6.2). Proof is essentially the same as in 6.3.

6.5. Let $\mathfrak{B}$ be the real L_1 space on a measure space $(X, \mathcal{B}, m)$. Write $X_0(f)=\{x; f(x)=0\}$ and $X_1(f)=\{x; f(x)\neq 0\}$. Then,

$$\tau(f, g)=\int_{X_1(f)}(\operatorname{sgn} f(x))g(x)m(dx)+\int_{X_0(f)}|g(x)|m(dx) \tag{6.5}$$

for all f and g, and

$$\sigma(f, g)=\int_{X_1(f)}g(x)m(dx) \tag{6.6}$$

for all $f \geqq 0$ and g.

PROOF. Integration below is relative to the measure m. We have

$$\varepsilon^{-1}(\|f+\varepsilon g\|-\|f\|)=\int_{X_0(f)}|g|+\varepsilon^{-1}\Big(\int_{X_1(f)}|f+\varepsilon g|-\int_{X_1(f)}|f|\Big),$$

and this equality becomes (6.5) as ε decreases to zero, since $\varepsilon^{-1}(|f(x)+\varepsilon g(x)| - |f(x)|)$ tends to $(\operatorname{sgn} f(x))g(x)$ on $X_1(f)$, being dominated by $|g(x)|$ in the absolute value. Thus we get (6.5). Let $f \geqq 0$. If $f\wedge|h|=0$, then $h(x)=0$ almost everywhere on $X_1(f)$ and hence

$$\tau(f, (g+h)\vee(-bf)) \geqq \int_{X_1(f)} \max\{g(x), -bf(x)\} \geqq \int_{X_1(f)} g .$$

On the other hand, if $h_0(x)=-g(x)$ on $X_0(f)$ and 0 on $X_1(f)$, then $f\wedge|h_0|=0$, and

$$\tau(f, (g+h_0)\vee(-bf))=\int_{X_1(f)} \max\{g(x), -bf(x)\} \to \int_{X_1(f)} g , \qquad b\to\infty .$$

Hence, (6.6) holds.

6.6. Let $\mathfrak{B}$ be the real L_p space, $1<p<\infty$, on a measure space $(X, \mathcal{B}, m)$. Then we have the expression

$$\tau(f, g)=\int_X (\operatorname{sgn} f(x))|f(x)|^{p-1}g(x)m(dx)\Big/\|f\|^{p-1}, \qquad f\neq 0 \tag{6.7}$$

and (6.2). Especially in the case of the Hilbert space L_2, $\tau(f, g)$ is the inner product of f and g divided by $\|f\|$, if $f\neq 0$.

PROOF. The right side of (6.7) is linear in g and, by Hölder's inequality, majorized by $\|g\|$ in the absolute value. Since $\tau(f, g)$ is also continuous in g, it suffices to prove (6.7) for dense g's. Hence we suppose that there are positive constants c and δ such that $|g(x)|\leqq c$ on X and $g(x)=0$ on the set X_1 of points x where $0<|f(x)|<\delta$. Let X_0 be the set of points where $f(x)$ vanishes and X_2 be the set of points where $|f(x)|\geqq\delta$. We have

$$\int_X |f+\varepsilon g|^p=\varepsilon^p\int_{X_0}|g|^p+\int_{X_1}|f|^p+\int_{X_2}|f+\varepsilon g|^p ,$$

$$\int_{X_2}|f+\varepsilon g|^p=\int_{X_2}|f|^p\Big(1+\varepsilon\frac{g}{f}\Big)^p=\int_{X_2}|f|^p+\varepsilon p\int_{X_2}|f|^p\frac{g}{f}+O(\varepsilon^2),\ \varepsilon\to 0+,$$

since $|\varepsilon g/f|$ is bounded by $\varepsilon c/\delta$ on X_2. It follows that

$$\int_X |f+\varepsilon g|^p=\int_X |f|^p+\varepsilon p\int_X (\operatorname{sgn} f)|f|^{p-1}g+o(\varepsilon),$$

$$\begin{aligned}\varepsilon^{-1}(\|f+\varepsilon g\|-\|f\|)&=\varepsilon^{-1}\|f\|\Big[\Big(1+\varepsilon p\int_X (\operatorname{sgn} f)|f|^{p-1}g\Big/\|f\|^p+o(\varepsilon)\Big)^{1/p}-1\Big]\\ &=\int_X (\operatorname{sgn} f)|f|^{p-1}g\Big/\|f\|^{p-1}+o(1),\end{aligned}$$

and the proof of (6.7) is complete. Let $f\geqq 0$. It follows from (6.7) that $\tau(f, g\vee(-bf))$ tends to $\tau(f, g)$ as $b\rightarrow\infty$ and that $\tau(f, g+h)=\tau(f, g)$ if $f\wedge|h|=0$. Hence we have (6.2).

6.7. Let $\mathfrak{B}$ be the Banach lattice $A(\mathscr{B})$ of bounded signed measures on a measurable space $(X, \mathscr{B})$ with the norm of total variation. Let f and $g\in A(\mathscr{B})$ and let g_f^c and g_f^s be, respectively, the absolutely continuous part and the singular part of g with respect to $|f|$. Then, we have

$$\tau(f, g)=g_f^c(X_f^+)-g_f^c(X_f^-)+\|g_f^s\|, \tag{6.8}$$

where X_f^+ and X_f^- are the positivity set and the negativity set in the Hahn decomposition of X relative to f, and

$$\sigma(f, g)=g_f^c(X), \qquad f\geqq 0. \tag{6.9}$$

PROOF. Let $\psi(x)$ be the Radon-Nikodym derivative of g_f^c with respect to $|f|$ and $\varphi(x)$ be the function which is 1 on X_f^+ and -1 on X_f^-. We have

$$\|f+\varepsilon g\|=\|f+\varepsilon g_f^c\|+\|\varepsilon g_f^s\|=\int_X|\varphi(x)+\varepsilon\psi(x)|\,|f|(dx)+\varepsilon\|g_f^s\|, \qquad \varepsilon>0,$$

$$\|f\|=\int_X|\varphi(x)|\,|f|(dx),$$

and hence, the proof of (6.8) and (6.9) is similar to 6.5.

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