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A remark on theorem A for Stein spaces

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1. Concerning Stein spaces the following fundamental theorems of K. Oka—H. Cartan—J. P. Serre are well-known: Theorem A. If $\mathcal{F}$ is a coherent analytic sheaf over a Stein space X with the structure sheaf $\mathcal{O}(X)$, then $\Gamma(X, \mathcal{F})$ generates $\mathcal{F}_x$ for every $x \in X$. Theorem B. In the same case. $\mathrm{H}^q(X, \mathcal{F})=0$ for any $q \geqq 1$.

It is known further that the validity of the latter is sufficient for X to be a Stein space. Namely, a reduced complex space X is a Stein space if $\mathrm{H}^1(X, \mathcal{J})=0$ for any coherent sheaf of ideals $\mathcal{J}$ in $\mathcal{O}(X)$ determined by a zero-dimensional analytic set in X.

In the present note we shall consider the problem if the former is sufficient for X to be a Stein space. Concerning a domain X in $\boldsymbol{C}^n$, Cartan ([1] p. 57) made a remark that, if a certain condition similar to theorem A is satisfied, then X would perhaps be a domain of holomorphy.

Our result is the following:

THEOREM. *Let $(X, \mathcal{O}(X))$ be an n-dimensional reduced connected normal complex space. Suppose it satisfies the following condition* (A): *For any coherent sheaf of ideals $\mathcal{J}$ in $\mathcal{O}(X)$ determined by a zero-dimensional analytic set in X, $\Gamma(X, \mathcal{J})$ generates $\mathcal{J}_x$ as an $\mathcal{O}(X)_x$-module at each point $x \in X$. Then X is* K-*complete and identical with its Kerner's* K-*hull* [4]. *If, in addition, $\Gamma(X, \mathcal{O}(X))$ is isomorphic as a $\boldsymbol{C}$-algebra to $\Gamma(X', \mathcal{O}(X'))$ of an n-dimensional reduced Stein space $(X', \mathcal{O}(X'))$, then X is a Stein space.*

For example, if an unramified covering manifold over a Stein manifold satisfies the condition (A), then it is a Stein manifold. (In general a K-hull of a normal complex space is not necessarily a Stein space [2].)

2. In the following a complex space should be understood to be reduced. A complex space $(X, \mathcal{O}(X))$ is said to be K-complete if, for each point $x \in X$, there exists a holomorphic mapping τ from X to a complex affine space $\boldsymbol{C}^{m_x}$ which is non-degenerate at x, i.e., x is an isolated point of $\tau^{-1}(\tau(x))$. We call a complex space $(X, \mathcal{O}(X))$ a Stein space if it is K-complete and holomorphically convex.

Let $\mathfrak{K}^n$ be a category whose objects are n-dimensional K-complete connected normal complex spaces and whose morphisms are non-degenerate holomorphic mappings. Kerner [4] showed, for any object $(X, \mathcal{O}(X))$ of $\mathfrak{K}^n$, the existence and uniqueness of its holomorphy hull $(H(X), \mathcal{O}(H(X))$ with the following conditions (we shall refer to it as *Kerner's* K-*hull*):

(i) $(H(X), \mathcal{O}(H(X)))$ is an object of $\mathfrak{K}^n$ and there exists a morphism $\alpha: X \to H(X)$ which induces the canonical isomorphism $\alpha^*: \Gamma(H(X), \mathcal{O}(H(X))) \to \Gamma(X, \mathcal{O}(X))$.

(ii) For any object $(Y, \mathcal{O}(Y))$ of $\mathfrak{K}^n$ which has a morphism $\beta: X \to Y$ such that $\beta^*: \Gamma(Y, \mathcal{O}(Y)) \to \Gamma(X, \mathcal{O}(X))$ is an isomorphism, there exists a morphism $\gamma: Y \to H(X)$ satisfying $\alpha = \gamma \circ \beta$.

3. Proof of the theorem.

Suppose $(X, \mathcal{O}(X))$ is an n-dimentional connected normal complex space and satisfies the condition (A). In the following we shall denote by $\mathcal{J}(V)$ the coherent sheaf of ideals in $\mathcal{O}(X)$ determined by an anaytic set V in X.

For a pair of distinct points $x_1, x_2 \in X$, the relation $\mathcal{J}(\{x_1\})_{x_2} = \mathcal{O}(X)_{x_2}$ holds. From the condition (A), there exists an element $f \in \Gamma(X, \mathcal{O}(X))$ with $f(x_2) \neq 0$, $f(x_1) = 0$. Hence $\Gamma(X, \mathcal{O}(X))$ separates points of X and, therefore, X is K-complete.

As $\Gamma(X, \mathcal{O}(X))$ separates points of X, we may consider X to be a subdomain of $H(X)$ (Kerner [4]). Now, assume $X \subsetneqq H(X)$, then there exist a point x_0 in ∂X and a sequence of points $\{x_\nu\}_{\nu=1}^\infty$ in X which converges to x_0. This sequence is a zero-dimensional analytic set in X. For any $f \in \Gamma(X, \mathcal{J}(\{x_\nu\}_{\nu=1}^\infty))$, there exists a holomorphic function $\hat{f}$ on $H(X)$ such that $f = \hat{f}$ on X by the definition of $H(X)$. As f vanishes on $\{x_\nu\}_{\nu=1}^\infty$, so does $\hat{f}$ on x_0. The set $V_f = \{\hat{f} = 0\}$ is an analytic set containing $\{x_\nu\}_{\nu=1}^\infty \cup \{x_0\}$ in $H(X)$, and so is $V = \bigcap_{f \in I} V_f$ ($I = \Gamma(X, \mathcal{J}(\{x_\nu\}_{\nu=1}^\infty))$). In a sufficiently small neighborhood of x_0 in $H(X)$, the number of irreducible components of V is finite. Hence $V \cap X$ contains a point $x \notin \{x_\nu\}_{\nu=1}^\infty$. By the condition (A), $\Gamma(X, \mathcal{J}(\{x_\nu\}_{\nu=1}^\infty))$ generates $\mathcal{J}(\{x_\nu\}_{\nu=1}^\infty)_x$ as an $\mathcal{O}(X)_x$-module. On the other hand, every element of $\Gamma(X, \mathcal{J}(\{x_\nu\}_{\nu=1}^\infty))$ vanishes at x as x is contained in V, and $\mathcal{J}(\{x_\nu\}_{\nu=1}^\infty)_x = \mathcal{O}(X)_x$ holds as x is not contained in $\{x_\nu\}_{\nu=1}^\infty$, a contradiction. Therefore $X = H(X)$. The proof of the first half of the theorem is hereby complete.

4. Proof of the theorem (continued).

Now we prove the second half under the assumption that $\Gamma(X, \mathcal{O}(X))$ is isomorphic as a $\boldsymbol{C}$-algebra to $\Gamma(X', \mathcal{O}(X'))$ of an n-dimensional Stein space $(X', \mathcal{O}(X'))$. Denote the isomorphism by $\tau: \Gamma(X', \mathcal{O}(X')) \to \Gamma(X, \mathcal{O}(X))$. The proof will be divided into five steps.

(i) Construction of $\psi: X \to X'$:

Since X' is a Stein space, by a theorem of Iwahashi [3], there exists a mapping $\psi: X \to X'$ with

$$\tau(f')(x) = f'(\psi(x)) \quad \cdots\cdots\cdots\cdots\cdots\cdots (*)$$

for every $f' \in \Gamma(X', \mathcal{O}(X'))$. It is injective because $\Gamma(X, \mathcal{O}(X))$ separates points of X and τ is an isomorphism.

(ii) ψ is continuous:

To show $\psi(x_\nu) \to \psi(x_0)$ for a sequence of points $x_\nu \in X$ with $x_\nu \to x_0 \in X$, it suffices to verify $f' \circ \psi(x_\nu) \to f' \circ \psi(x_0)$ for every $f' \in \Gamma(X', \mathcal{O}(X'))$, since X' is a Stein space ([3]). This is obtained from $(*)$.

(iii) ψ is holomorphic:

Let g' be any function holomorphic at $\psi(x)$. Since X' is a Stein space, there exists a sequence of holomorphic functions f'_ν on X' which converges uniformely to g' in a sufficiently small neighborhood of $\psi(x)$. Then $\tau(f'_\nu)$ converges uniformely to $g' \circ \psi$ in a neighborhood of x. Hence $g' \circ \psi$ is holomorphic at x, and so is ψ.

(iv) X' is irreducible:

$\psi(X)$ is contained completely in an irreducible component X'_1 of X', since X is a connected normal complex space. As $\psi: X \to X'$ is holomorphic, so is $\psi: X \to X'_1$. Hence, an arbitrary holomorphic function on X'_1 can be understood to be a holomorphic function on X. By assumption, it is extended to a unique holomorphic function on X'. This yields that $\Gamma(X'_1, \mathcal{O}(X'_1))$ is isomorphic as a $\boldsymbol{C}$-algebra to $\Gamma(X', \mathcal{O}(X'))$. As X' is a Stein space, $X' = X'_1$ holds by Iwahashi's theorem mentioned above. Consequently X' is irreducible.

(v) X is a Stein space:

For this purpose, let us show that X' is a normal complex space. To this end, assume the contrary. Let $(X'^*, \mathcal{O}(X'^*))$ be a normalization of $(X', \mathcal{O}(X'))$, and $\pi: X'^* \to X'$ be an associated holomorphic mapping. By assumption, $(X', \mathcal{O}(X'))$ is not isomorphic to $(X'^*, \mathcal{O}(X'^*))$, and $(X'^*, \mathcal{O}(X'^*))$ is a Stein space by a theorem of R. Narasimhan. Consequently, by Iwahashi's theorem, there exists an element f^* of $\Gamma(X'^*, \mathcal{O}(X'^*))$ such that, even though $f^* \circ \pi^{-1}$ is a holomorphic function in the weak sense on X', $f^* \circ \pi^{-1}$ is not an element of $\Gamma(X', \mathcal{O}(X'))$. The set of those points of X' where $f^* \circ \pi^{-1}$ is not holomorphic is included in a thin analytic set V' in X'. Since $\dim X = \dim X'$, $\psi^{-1}(V') \cap X$ is also included in a thin analytic set in X. On a normal complex space the Riemann's continuation theorem holds, hence $f^* \circ \pi^{-1}$ determines an element $f \in \Gamma(X, \mathcal{O}(X))$, and $\tau^{-1}(f)$ coincides with $f^* \circ \pi^{-1}$ on X' except for a thin analytic set in X'. Since $\tau^{-1}(f)$ and f^* are continuous, $f^* \circ \pi^{-1}$ is a function on X' satisfying $\tau^{-1}(f) = f^* \circ \pi^{-1}$. This is a contradiction to our assumption. Hence,

X' is a K-complete normal complex space with an injective holomorphic mapping $\psi: X \to X'$ with the property $\psi^*: \Gamma(X', \mathcal{O}(X')) \to \Gamma(X, \mathcal{O}(X))$ is an isomorphism. From the definition of Kerner's K-hull and from the first half $X = H(X)$ of our theorem, $(X, \mathcal{O}(X))$ is isomorphic to $(X', \mathcal{O}(X'))$. Therefore $(X, \mathcal{O}(X))$ is a Stein space.

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