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Distance, holomorphic mappings and the Schwarz lemma

By Shoshichi KOBAYASHI*)

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1. Introduction

According to Pick the classical Schwarz lemma can be stated in the following invariant manner. Every holomorphic map f of the open unit disk D into itself is distance-decreasing with respect to the Poincaré metric ds^2, i. e., $f^*(ds^2) \leqq ds^2$, and if the equality holds at one point of D, then f is biholomorphic. Bochner and Martin proved in their book [2] the following generalization of the Schwarz lemma to higher dimension. Let D_n be the open unit ball in $\boldsymbol{C}^n$,

$$D_n = \{z = (z^1, \cdots, z^n);\ \|z\|^2 = \sum |z^j|^2 < 1\}.$$

If f is a holomorphic mapping of D_m into D_n such that $f(0)=0$, then $\|f(z)\| \leqq \|z\|$ for every $z \in D_m$. Using the fact that D_m and D_n are homogeneous, we can formulate this in the following invariant manner. Every holomorphic mapping $f: D_m \to D_n$ is distance-decreasing with respect to the Bergman metrics ds^2_{Dm} and ds^2_{Dn} of D_m and D_n, i. e., $f^*(ds^2_{D_n}) \leqq ds^2_{Dm}$.

Recently Korányi [7] obtained the following generalization of the Schwarz lemma. If M is a hermitian symmetric space of non-compact type with the Bergman metric ds^2, then every holomorphic map $f: M \to M$ satisfies $f^*(ds^2) \leqq l \cdot ds^2$, where l is the rank of M.

On the other hand, Ahlfors exposed in his generalization of the Schwarz lemma the essential rôle played by the curvature. Let M be a Riemann surface with hermitian metric ds^2_M whose Gaussian curvature is bounded above by a negative constant $-B$. Let D be the unit disk in $\boldsymbol{C}$ with an invariant metric ds^2_D whose Gaussian curvature is a negative constant $-A$. (If we take $dzd\bar{z}/(1-|z|^2)^2$ for ds^2_D, then its curvature is equal to -4.) Then the generalized Schwarz lemma by Ahlfors says that every holomorphic mapping $f: D \to M$ satisfies $f^*(ds^2_M) \leqq \frac{A}{B} ds^2_D$.

The main purpose of this paper is to generalize the results above in the following form:

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THEOREM. *Let D be a bounded symmetric domain with an invariant Kähler metric ds_D^2 whose holomorphic sectional curvature is bounded below by a negative constant $-A$. Let M be a Kähler manifold with metric ds_M^2 whose holomorphic sectional curvature is bounded above by a negative constant $-B$. Then every holomorphic mapping $f: D \rightarrow M$ satisfies $f^*(ds_M^2) \leqq \frac{A}{B} ds_D^2$.*

Although the theorem above can be generalized to the case when M is a hermitian manifold (with a suitable definition of holomorphic sectional curvature) we shall restrict ourselves to the Kähler case in this paper.

2. The case $\dim D = 1$.

Let D_a be the open disk of radius a in $\boldsymbol{C}$, $D_a = \{z \in \boldsymbol{C};\ |z| < a\}$. Then the metric

$$ds_a^2 = \frac{4a^2 dz d\bar{z}}{A(a^2 - z\bar{z})^2}$$

on D_a has the curvature $-A$. Let M be a Kähler manifold with metric ds_M^2 whose holomorphic sectional curvature is bounded above by $-B$. Let u be the non-negative function on D_a defined by

$$f^*(ds_M^2) = u \cdot ds_a^2 .$$

We want to prove that $u \leqq \frac{A}{B}$ on D_a. Although u may not attain its maximum in (the interior of) D_a in general, we shall show that we have only to consider the case when u attains its maximum in D_a. Let r be a positive number smaller than a. Let z_0 be an arbitrary point of D_a. Taking r sufficiently close to a, we may assume that $z_0 \in D_r$. From the explicit expression for ds_a^2 given above, we see that $(ds_r^2)_{z_0} \rightarrow (ds_a^2)_{z_0}$ as $r \rightarrow a$. If we define a non-negative function u_r on D_r by $f^*(ds_M^2) = u_r \cdot ds_r^2$, then $u_r(z_0) \rightarrow u(z_0)$ as $r \rightarrow a$. Hence it suffices to prove that $u_r \leqq \frac{A}{B}$ on D_r. If we write $f^*(ds_M^2) = h dz d\bar{z}$ on D_a, then h is bounded on D_r. On the othen hand, the coefficient of ds_r^2 approaches infinity at the boundary of D_r. Hence, the function u_r defined on D_r goes to zero at the boundary of D_r. In particular, u_r attains its maximum in D_r. The problem is thus reduced to the case where u attains its maximum in D_a.

We shall now prove that $u \leqq \frac{A}{B}$ on D_a under the assumption that u attains its maximum in D_a, say at $z_0 \in D_a$. If $u(z_0) = 0$, then $u \equiv 0$ and there is nothing to prove. Assume that $u(z_0) > 0$. Then the mapping $f: D_a \rightarrow M$ is non-degenerate in a neighborhood of z_0 so that f gives a holomorphic imbedding of a neighborhood U of z_0 into M.

We claim that the curvature of the (1-dimensional) complex submanifold $f(U)$ of M is bounded above by $-B$. This is a consequence of the following general fact. Let S be a complex submanifold of a Kaehler manifold M. Let R_M and R_S denote the Riemannian curvature tensors of M and S respectively. Let α denote the second fundamental form of S; it is a symmetric bilinear map of the tangent space $T_p(S)$ into the normal space at p. From the equations of Gauss-Codazzi we obtain

$$R_S(X, JX, X, JX) = R_M(X, JX, X, JX) - 2\|\alpha(X, X)\|^2 .$$

See O'Neill [8] for the detail of calculation leading to the formula above. The formula implies that the holomorphic sectional curvature of S does not exceed that of M. (This fact is true for a hermitian manifold M and a complex submanifold S of M. But the proof is more technical and will be given in a forthcoming paper.)

Since u attains its maximum at z_0, $\partial^2 \log u/\partial z\partial\bar{z}$ is non-positive at z_0. We shall now express $\partial^2 \log u/\partial z\partial\bar{z}$ in terms of the curvatures of D_a and $f(U)$. Since $f: U \to f(U)$ is a biholomorphic mapping, we define the coordinate system w in $f(U)$ by $w \circ f = z$. Identifying $f(U)$ with U by the mapping f, we shall identify w with z. Then we can consider $f^*(ds_M^2) = h dz d\bar{z}$ as the induced metric on $f(U)$ as well as on U. If we write $ds_a^2 = g dz d\bar{z}$, then

$$u = h/g .$$

Hence

$$\partial^2 \log u/\partial z\, \partial\bar{z} = \partial^2 \log h/\partial z\, \partial\bar{z} - \partial^2 \log g/\partial z\, \partial\bar{z} .$$

If we denote by k the curvature of the metric $h dz d\bar{z}$, then

$$k = -\frac{1}{2h}(\partial^2 \log h/\partial z\, \partial\bar{z}) .$$

Since the curvature of the metric $g dz d\bar{z}$ is equal to $-A$, we have

$$-A = -\frac{1}{2g}(\partial^2 \log g/\partial z\, \partial\bar{z}) .$$

Since $k \leqq -B$ as we have seen above, we have

$$\partial^2 \log u/\partial z\, \partial\bar{z} = -2kh - 2Ag \geqq 2Bh - 2Ag .$$

Since the left hand side is non-positive at z_0, so is the right hand side. Hence, $A/B \geqq h/g$ at z_0. Since $u = h/g$ attains its maximum at z_0, it follows that $A/B \geqq u$ everywhere. This completes the proof of Theorem for the case $\dim D = 1$.

This case is closely related with Aussage 3 in Grauert-Reckziegel [4]. Instead of assuming that the holomorphic sectional curvature of M is bounded by $-B$, they assume that the curvature of every 1-dimensional complex submanifold of M is bounded by $-B$.

3. The case where $D=D_a^l=D_a\times\cdots\times D_a$.

Let $\alpha=(\alpha_1, \cdots, \alpha_l)$ be an l-tuple of complex numbers such that $\sum_{i=1}^{l}|\alpha_i|^2=1$. Let $j: D_a\to D_a^l$ be the imbedding defined by

$$j(z)=(\alpha_1 z, \cdots, \alpha_l z).$$

Let ds_D^2 be the product metric in $D=D_a^l$. From the explicit expression of ds_a^2 given in Section 2, we see that $j: D_a\to D_a^l$ is isometric at the origin of D_a, i.e., $(ds_a^2)_0=(j^*ds_D^2)_0$.

Let X be a tangent vector of D_a^l at the origin. For a suitable $\alpha=(\alpha_1, \cdots, \alpha_l)$, we can find a tangent vector Y of D_a at the origin such that $j_*(Y)=X$. Then, for any holomorphic mapping $f: D_a^l\to M$, we have

$$\|f_*X\|^2=\|f_*j_*Y\|^2\leqq\frac{A}{B}\|Y\|^2=\frac{A}{B}\|X\|^2,$$

where the inequality follows from the special case of Theorem proved in Section 2 (applied to $f\circ j: D_a\to M$) and the last equality follows from the fact that j is isometric at the origin. Since D_a^l is homogeneous, the inequality $\|f_*X\|^2\leqq\frac{A}{B}\|X\|^2$ holds for all tangent vectors X of D_a^l. This completes the proof of Theorem in the case $D=D_a^l$.

4. The case where D is a symmetric bounded domain of rank l

Let D be a symmetric bounded domain of rank l. With respect to a canonical metric, its holomorphic sectional curvature lies between $-A$ and $-A/l$. For every tangent vector X of D, there is a (totally geodesic) complex submanifold D_a^l of D such that X is tangent to D_a^l. (It is a complex submanifold of D. More precisely, write $D=G/H$ and $\mathfrak{g}=\mathfrak{h}+\mathfrak{p}$ in the usual manner. Let $\mathfrak{a}$ be a maximal abelian subalgebra contained in $\mathfrak{p}$ so that $\dim\mathfrak{a}=\operatorname{rank} D=l$. We may assume that X is an element of $\mathfrak{a}$ under the usual identification. Let $J:\mathfrak{p}\to\mathfrak{p}$ be the complex structure tensor. Then the manifold generated by $\mathfrak{a}+J\mathfrak{a}$ is the desired submanifold D_a^l). Now our theorem in its full generality follows from the special case considered in Section 3.

COROLLARY. *Let D be a symmetric bounded domain with holomorphic sectional curvature $\geqq -A$. Let M be a symmetric bounded domain of rank l so that its holomorphic sectional curvature lies between $-lB$ and $-B$. Then every holomorphic mapping $f: D\to M$ satisfies $f^*(ds_M^2)\leqq\frac{A}{B}ds_D^2$.*

This corollary is in Korányi [7].

5. Concluding remarks

In the case where $\dim D = \dim M = 1$, a holomorphic mapping $f: D \to M$ is distance-decreasing if and only if it is volume decreasing. Under a suitable assumption on the Ricci tensor of M every holomorphic mapping f of the unit ball D in $\boldsymbol{C}^n$ into an n-dimensional complex manifold M is volume-decreasing. See Dinghas [5] for the case where M is a Kähler-Einstein, Chern [3] for the case M is a hermitian-Einstein and Kobayashi [6] for a further generalization.

University of California, Berkeley

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