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Invariant distances on complex manifolds and holomorphic mappings

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1. Introduction

The classical Schwarz lemma, in its invariant form formulated by Pick, states that every holomorphic mapping of the unit disk into itself is distance-decreasing with respect to the Poincaré-Bergman metric. It has been since generalized to higher dimensions in various forms, [**1, 2, 10, 12, 16, 17, 20, 21** etc.]. Most of these generalizations originate from Ahlfors's generalization of Schwarz lemma [**1**]. The essence of these generalizations is that, given complex manifolds M and N endowed with either metrics or volume elements, every holomorphic mapping $f: M \to N$ is distance- or volume-decreasing under the conditions that M is a ball or a symmetric domain in $\boldsymbol{C}^m$ and that N has negative curvature in one sense or other. In this way we get some control over the family of holomorphic mappings $f: M \to N$.

In [**18**] I announced a canonical way of constructing a new pseudo-distance d_M on each complex manifold M. In this paper we prove basic properties of these pseudo-distances and apply them to the study of holomorphic mappings. The two most important properties of d_M are that every holomorphic mapping $f: M \to N$ is distance-decreasing with respect to d_M and d_N and that these pseudo-distances are often true distances. For instance, if M is a Riemann surface covered by a disk or more generally a complex manifold covered by a bounded domain, then d_M is a true distance. We call a complex manifold *hyperbolic* if its invariant pseudo-distance is a distance. If N is hyperbolic, we can therefore draw useful conclusion on the value distribution of a holomorphic mapping $f: M \to N$. The first basic question in the study of holomorphic mappings from this view point is therefore to decide whether the image manifold is hyperbolic and also complete with respect to its invariant distance. From Ahlfor's generalized Schwarz lemma we can prove that if a complex manifold admits a (complete) hermitian metric of strongly negative curvature or a (complete) differential metric of strongly negative curvature in the sense of Grauert-Reckziegel, then it is (complete) hyperbolic. But there

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are instances where it is easier to verify directly that a manifold is (complete) hyperbolic than to check whether it admits a (complete) hermitian metric of strongly negative curvature. For example, if M is (complete) hyperbolic, then its quotient manifold by a properly discontinuous group is also (complete) hyperbolic. But a similar statement for a hermitian manifold of strongly negative curvature is not clear. We shall show that the class of (complete) hyperbolic manifolds is closed under many basic operations on manifolds.

In a recent paper [26] Wu introduced the notion of taut manifold. A hermitian manifold N is said to be taut if the set of holomorphic mappings $f: M \to N$ is a normal family for every complex manifold M. (Actually it suffices to consider the case where M is the unit disk in $\boldsymbol{C}$.) This notion is essentially equivalent to our notion of complete hyperbolic manifold whereas the condition that the set of holomorphic mappings $f: M \to N$ is equi-continuous for every M (or the unit disk M) is essentially equivalent to the condition that N is hyperbolic. But the use of the pseudo-distances d_M and d_N in proving results on the family of holomorphic mappings $f: M \to N$ makes proofs more transparent in general.

In Sections 1 through 8 only two theorems (Theorems 3.8 and 5.8) are differential geometric and their proofs are given in the last section of the paper. The reader unfamiliar with hermitian differential geometry may completely ignore these two theorems.

In preparing this paper I had many valuable conversations with H. Wu who has obtained some of the results in Sections 5 and 6 using the notion of normal family [26]. It is a pleasure to express my thanks to him.

2. Invariant distances

Let D denote the open unit disk in the complex plane $\boldsymbol{C}$, i.e.,

$$D = \{z \in \boldsymbol{C};\ |z| < 1\}.$$

The Poincaré-Bergman metric of D defines the distance ρ given by

$$\tanh \frac{1}{2}\rho(z, z') = |z - z'|/|1 - \bar{z}z'| \qquad \text{for } z, z' \in D.$$

Let M be a complex manifold of complex dimension n. The *Carathéodory pseudo-distance* $c = c_M$ of M is defined by [5, 6, 7]

$$c(p, p') = \sup \rho(f(p), f(p')) \qquad \text{for } p, p' \in M,$$

where the supremum is taken with respect to the family of holomorphic mappings of M into the unit disk D. (It follows from Proposition 2.3 below that $c(p, p') < \infty$.) If we admit temporarily that $c(p, p') < \infty$, then it follows easily that c is continuous on $M \times M$ and satisfies the axioms for pseudo-distance:

$$c(p, q) \geqq 0, \quad c(p, q) = c(q, p), \quad c(p, q) + c(q, r) \geqq c(p, r).$$

A necessary and sufficient condition for c to be a (true) distance is that there are sufficiently many bounded holomorphic functions on M to separate the point of M, that is, given two distinct points $p, q \in M$, there is a bounded holomorphic function f on M such that $f(p) \neq f(q)$. For instance, if M is a bounded domain in $\boldsymbol{C}^n$, then c is a distance on M. On the other hand, if M is compact or is the complex Euclidean space $\boldsymbol{C}^n$, then $c(p, q) = 0$ for all points $p, q \in M$.

We shall now define a new pseudo-distance d on a complex manifold M. Given two points $p, q \in M$ we choose points $p = p_0, p_1, \cdots, p_{k-1}, p_k = q$ of M, points $a_1, \cdots, a_k, b_1, \cdots, b_k$ of the unit disk D and holomorphic mappings $f_1, \cdots, f_k$ of D into M such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for $i = 1, \cdots, k$. For each choice of points and mappings thus made, we consider the number

$$\rho(a_1, b_1) + \cdots + \rho(a_k, b_k).$$

Let $d(p, q)$ be the infimum of the numbers obtained in this manner for all possible choices. It is also easy to verify that d is a pseudo-distance on M. As we shall see (cf. Proposition 2.3), d is a distance whenever the Carathéodory pseudo-distance c is a distance. Even for some compact manifolds, d is a distance. A differential geometric condition on M which makes d a distance will be given later. On the other hand, we have $d(p, q) = 0$ for all p and q in $\boldsymbol{C}^n$.

PROPOSITION 2.1. *Let M and N be two complex manifolds, c_M and c_N the Carathéodory pseudo-distances of M and N, and d_M and d_N the new pseudo-distances of M and N defined above. Then every holomorphic mapping $f: M \to N$ is distance-decreasing in the sense that*

$$c_M(p, q) \geqq c_N(f(p), f(q)) \quad \textit{and} \quad d_M(p, q) \geqq d_N(f(p), f(q)) \quad \textit{for } p, q \in M.$$

COROLLARY. *Every biholomorphic mapping $f: M \to N$ is an isometry, i.e., $c_M(p, q) = c_N(f(p), f(q))$ and $d_M(p, q) = d_N(f(p), f(q))$ for $p, q \in M$.*

The proof is trivial.

PROPOSITION 2.2. *For the open unit disk D, both c and d coincide with the distance ρ defined by the Poincaré-Bergman metric of D.*

PROOF. The classical Schwarz lemma, invariantly formulated by Pick, states that every holomorpoic mapping $f: D \to D$ is distance-decreasing with respect to ρ. From the very definitions of c and d, we obtain

$$d(p, q) \geqq \rho(p, q) \geqq c(p, q) \qquad \text{for } p, q \in D.$$

Considering the identity transformation of D, we obtain the equalities $d(p, q) = \rho(p, q) = c(p, q)$. QED.

REMARK. Similarly, it can be shown that for the unit open ball in C^n both c and d coincide with the distance defined by the Bergman metric.

PROPOSITION 2.3. *For any complex manifold M, we have $d \geqq c$, i.e.,*

$$d(p, q) \geqq c(p, q) \qquad \text{for } p, q \in M.$$

PROOF. Let p and q be points of M. As in the definition of d, choose points $p=p_0, p_1, \cdots, p_{k-1}, p_k=q$ of M and points $a_1, \cdots, a_k, b_1, \cdots, b_k$ of the unit disk D and also holomorphic mappings $f_1, \cdots, f_k$ of D into M such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i)=p_i$. Let f be a holomorphic mapping of M into D. Then

$$\begin{aligned}\rho(a_1, b_1)+\cdots+\rho(a_k, b_k) &\geqq \rho(ff_1(a_1), ff_1(b_1))+\cdots+\rho(ff_k(a_k), ff_k(b_k))\\ &\geqq \rho(ff_1(a_1), ff_k(b_k))\\ &= \rho(f(p), f(q)),\end{aligned}$$

where the first inequality follows from the Schwarz lemma and the second inequality is a consequence of the triangular axiom. Hence

$$d(p, q) = \inf(\rho(a_1, b_1)+\cdots+\rho(a_k, b_k)) \geqq \sup \rho(f(p), f(q)) = c(p, q).$$

QED.

More generally, we have

PROPOSITION 2.4. *Let M be a complex manifold. Let c' be any pseudo-distance on M such that*

$$c'(p, q) \geqq \rho(f(p), f(q)) \qquad p, q \in M$$

for every holomorphic mapping f of M into the unit disk D. Let d' be any pseudo-distance on M such that

$$d'(h(a), h(b)) \leqq \rho(a, b) \qquad a, b \in D$$

for every holomorphic mapping h of D into M. Then

$$c(p, q) \leqq c'(p, q) \quad \text{and} \quad d(p, q) \geqq d'(p, q) \qquad \text{for } p, q \in M.$$

PROOF. The first assertion is trivial. To prove the second assertion, let $p_0, p_1, \cdots, p_{k-1}, p_k, a_1, \cdots, a_k, b_1, \cdots, b_k, f_1, \cdots, f_k$ be as in the proof of Proposition 2.3. Then

$$\begin{aligned}d'(p, q) &\leqq \sum_{i=1}^{k} d'(p_{i-1}, p_i) = \sum_{i=1}^{k} d'(f_i(a_i), f_i(b_i))\\ &\leqq \sum_{i=1}^{k} \rho(a_i, b_i).\end{aligned}$$

Hence, $d'(p, q) \leqq \inf \sum_{i=1}^{k} \rho(a_i, b_i) = d(p, q)$. QED.

Note that Proposition 2.4 together with Propositions 2.1 and 2.2 implies Proposition 2.3.

PROPOSITION 2.5. *Let M and M' be two complex manifolds. Then*

$$c_M(p, q)+c_{M'}(p', q')\geqq c_{M\times M'}((p, p'), (q, q'))\geqq \mathrm{Max}\,(c_M(p, q), c_{M'}(p', q'))$$

and

$$d_M(p, q)+d_{M'}(p', q')\geqq d_{M\times M'}((p, p'), (q, q'))\geqq \mathrm{Max}\,(d_M(p, q), d_{M'}(p', q'))$$

for $(p, p'), (q, q')\in M\times M'$.

PROOF. We have

$$\begin{aligned} c_M(p, q)+c_{M'}(p', q') &\geqq c_{M\times M'}((p, p'), (q, p'))+c_{M\times M'}((q, p'), (q, q')) \\ &\geqq c_{M\times M'}((p, p'), (q, q'))\,, \end{aligned}$$

where the first inequality follows from the fact that the mappings $f: M\to M\times M'$ and $f': M'\to M\times M'$ defined by $f(x)=(x, p')$ and $f'(x')=(q, x')$ are distance-decreasing by Proposition 2.1 and the second inequality is a consequence of the triangular axiom. The inequality $c_{M\times M'}((p, p'), (q, q'))\geqq \mathrm{Max}\,(c_M(p, q), c_{M'}(p', q'))$ follows from the fact that the projections $M\times M'\to M$ and $M\times M'\to M'$ are both distance-decreasing by Proposition 2.1. The proof for d is similar. QED.

PROPOSITION 2.6. *Let M be a complex manifold and $\tilde{M}$ a covering manifold of M with covering projection π. Let $p, q\in M$ and $\tilde{p}, \tilde{q}\in \tilde{M}$ such that $\pi(\tilde{p})=p$ and $\pi(\tilde{q})=q$. Then*

$$d_M(p, q)=\inf_{\tilde{q}} d_{\tilde{M}}(\tilde{p}, \tilde{q})\,,$$

where the infimum is taken over all $\tilde{q}\in \tilde{M}$ such that $q=\pi(\tilde{q})$.

PROOF. *Since* $\pi: \tilde{M}\to M$ is distance-decreasing by Proposition 2.1, we have $d_M(p, q)\leqq \inf_{\tilde{q}} d_{\tilde{M}}(\tilde{p}, \tilde{q})$. Assuming the strict inequality, let $d_M(p, q)+\varepsilon<\inf_{\tilde{q}} d_{\tilde{M}}(\tilde{p}, \tilde{q})$ where ε is a positive number. By the very definition of d_M there exist points $a_1, \cdots, a_k, b_1, \cdots, b_k$ of the unit disk D and holomorphic mappings $f_1, \cdots, f_k$ of D into M such that

$$p=f_1(a_1),\ f_1(b_1)=f_2(a_2),\ \cdots,\ f_{k-1}(b_{k-1})=f_k(a_k),\ f_k(b_k)=q$$

and

$$d_M(p, q)+\varepsilon>\sum_{i=1}^{k}\rho(a_i, b_i)\,.$$

Then we can lift $f_1, \cdots, f_k$ to holomorphic mappings $\tilde{f}_1, \cdots, \tilde{f}_k$ of D into $\tilde{M}$ in such a way that $\tilde{p}=\tilde{f}_1(a_1)$ and $\tilde{f}_i=(b_i)=\tilde{f}_{i+1}(a_{i+1})$ for $i=1, \cdots, k-1$ and that $\pi\circ\tilde{f}_i=f_i$ for $i=1, \cdots, k$. If we set $\tilde{q}=\tilde{f}_k(b_k)$, then $\pi(\tilde{q})=q$ and $d_{\tilde{M}}(\tilde{p}, \tilde{q})\leqq\sum_{i=1}^{k}\rho(a_i, b_i)$. Hence $d_{\tilde{M}}(\tilde{p}, \tilde{q})<d_M(p, q)+\varepsilon$, which contradicts our assumption. QED.

It is not clear whether $\inf_{\tilde{q}} d_{\tilde{M}}(\tilde{p}, \tilde{q})$ is really attained by some $\tilde{q}$.

Proposition 2.6 does not hold for the Carathéodory distance. In fact, let $\tilde{M}$ be the unit disk and M a compact Riemann surface of genus greater than

1. Then $c_{\tilde{M}}$ coincides with the Poincaré-Bergman distance of D by Proposition 2.2. But $c_M(p, q)=0$ for $p, q \in M$ since M is compact. Proposition 2.5 makes d_M more useful than c_M.

3. Hyperbolic manifolds

Let M be a complex manifold and $d=d_M$ the new pseudo-distance on M defined in Section 2. If d is a distance, i.e., $d(p, q)>0$ for $p \neq q$, then we shall call M a *hyperbolic manifold.*

PROPOSITION 3.1. *A complex manifold M is hyperbolic if its Carathéodory pseudo-distance c_M is a distance.*

COROLLARY. *Every bounded domain in C^n is hyperbolic.*

PROOF. Proposition 3.1 follows from Proposition 2.3. QED.

PROPOSITION 3.2. *If a complex manifold M admits a distance d' for which every holomorphic mapping f of the unit disk D into M is distance-decreasing (i.e., $d'(f(a), f(b)) \leqq \rho(a, b)$ for $a, b \in D$), then it is hyperbolic.*

PROOF. This follows from Proposition 2.4. QED.

PROPOSITION 3.3. *If M and M' are hyperbolic manifolds, then their direct product $M \times M'$ is also hyperbolic.*

PROOF. This follows from Proposition 2.5. QED.

THEOREM 3.4. *Let M be a complex manifold and $\tilde{M}$ a covering manifold of M. Then $\tilde{M}$ is hyperbolic if and only if M is hyperbolic.*

COROLLARY. *If M is a complex manifold having a bounded domain of C^n as a covering manifold, then M is hyperbolic.*

PROOF. Assume that M is hyperbolic. Let $\tilde{p}, \tilde{q} \in \tilde{M}$ and assume $d_{\tilde{M}}(\tilde{p}, \tilde{q})=0$. Since the projection $\pi: \tilde{M} \rightarrow M$ is distance-decreasing, $d_M(\pi(\tilde{p}), \pi(\tilde{q}))=0$ and hence $\pi(\tilde{p})=\pi(\tilde{q})$. Let $\tilde{U}$ be a neighborhood of $\tilde{p}$ in $\tilde{M}$ such that $\pi: \tilde{U} \rightarrow \pi(\tilde{U})$ is a diffeomorphism and $\pi(\tilde{U})$ is an ε-neighborhood of $\pi(\tilde{p})$ with respect to d_M. In particular, $\tilde{U}$ does not contain $\tilde{q}$ unless $\tilde{p}=\tilde{q}$. Since $d_{\tilde{M}}(\tilde{p}, \tilde{q})=0$ by assumption, there exist points $a_1, \cdots, a_k, b_1, \cdots, b_k$ of D and holomorphic mappings $f_1, \cdots, f_k$ of D into $\tilde{M}$ such that $\tilde{p}=f_1(a_1)$, $f_i(b_i)=f_{i+1}(a_{i+1})$ for $i=1, \cdots, k-1$ and $f_k(b_k)=\tilde{q}$ and that $\sum_{i=1}^{k} \rho(a_i, b_i)<\varepsilon$. Let $\widehat{a_i b_i}$ denote the geodesic arc from a_i to b_i in D. Joining the curves $f_1(\widehat{a_1 b_1}), \cdots, f_k(\widehat{a_k b_k})$ in $\tilde{M}$, we obtain a curve, say $\tilde{C}$, from $\tilde{p}$ to $\tilde{q}$ in $\tilde{M}$. Since $\pi \circ f_1, \cdots, \pi \circ f_k$ are distance-decreasing mappings of D into M and $\widehat{a_1 b_1}, \cdots, \widehat{a_k b_k}$ are geodesics in D, every point of the curve $\pi(\tilde{C})$ remains in the ε-neighborhood $\pi(\tilde{U})$ of $\pi(\tilde{p})$. Hence the end point $\tilde{q}$ must coincide with $\tilde{p}$.

Conversely assume that $\tilde{M}$ is hyperbolic. Let $p, q \in M$ and assume $d_M(p, q)=0$. Let $\tilde{p}$ be a point of $\tilde{M}$ such that $\pi(\tilde{p})=p$. By Proposition 2.6, there exists a sequence of points $\tilde{q}_1, \cdots, \tilde{q}_i, \cdots$ of $\tilde{M}$ such that $\pi(\tilde{q}_i) = q$ and $\lim d_{\tilde{M}}(\tilde{p}, \tilde{q}_i) = 0$. Then the sequence $\{\tilde{q}_i\}$ converges to $\tilde{p}$. Hence $\pi(\tilde{q}_i)$ converges to p. Since $\pi(\tilde{q}_i) = q$, we obtain $p = q$. QED.

A complex manifold $\tilde{M}$ is called a *spread* (domaine etalé in French) over a complex manifold M with projection π if every point $\tilde{p} \in \tilde{M}$ has a neighborhood $\tilde{U}$ such that π is a holomorphic diffeomorphism of $\tilde{U}$ onto the open set $\pi(\tilde{U})$ of M. The first half of the proof of Theorem 3.4 gives the following

THEOREM 3.5. *A spread $\tilde{M}$ over a hyperbolic manifold M is also hyperbolic.*

The following theorem is immediate from Proposition 2.1.

THEOREM 3.6. *If a complex manifold M' is immersed in a hyperbolic manifold M, then M' is also hyperbolic.*

The proof of the following theorem will be given in Section 6.

THEOREM 3.7. *Let E be a complex analytic fibre bundle over M with fibre F. If M is hyperbolic and F is compact hyperbolic, then E is hyperbolic.*

The proof of the following theorem will be given in Section 9.

THEOREM 3.8. *A hermitian manifold M whose holomorphic sectional curvature is bounded above by a negative constant is hyperbolic.*

4. Completeness with respect to Carathéodory distance

In general we say that a metric space M is *complete* if for each point $p \in M$ and each positive number r the closed ball of radius r around p is a compact subset of M. If M is complete in this sense, then every Cauchy sequence of M converges, but not conversely in general. It will be shown in Section 8 that for a hyperbolic manifold the usual completeness with respect to d_M implies this strong completeness defined here.

THEOREM 4.1. *If M is a complex manifold with complete Carathéodory distance c_M, then M is a complete hyperbolic manifold.*

This is immediate from Proposition 2.3 and from the following trivial

LEMMA. *Let c and d be two distances on a topological space M such that $c(p, q) \leqq d(p, q)$ for $p, q \in M$. If M is c-complete, it is d-complete.*

THEOREM 4.2. *Let M and M_i, $i \in I$, be a family of complex submanifolds of a complex manifold N such that $M = \bigcap_i M_i$.*

(1) *If each M_i is complete with respect to its Carathéodory distance, so is M;*

(2) *If each M_i is complete hyperbolic, so is M.*

This follows from the fact that each injection $M \to M_i$ is distance-decreasing by Proposition 2.1 and from the following trivial

LEMMA. *Let M and M_i, $i \in I$, be subsets of a topological space N such that $M = \bigcap M_i$. Let d and d_i be distances on M and M_i. If $d(p, q) \geqq d_i(p, q)$ for*

$p, q \in M$ and if each M_i is complete with respect to d_i, then M is complete with respect to d.

Horstmann [14] has shown that a domain in $\boldsymbol{C}^n$ which is complete with respect to its Carathéodory distance is a domain of holomorphy. We shall give a generalization of Horstmann's result. Let F be a family of holomorphic functions on a complex manifold M. Let K be a subset of M. Denote by $\sup|f(K)|$ the supremum of $|f(q)|$ for $q \in K$. We set

$$\hat{K}_F = \{p \in M;\ |f(p)| \leqq \sup|f(K)| \qquad \text{for all } f \in F\}.$$

In general, whether K is closed or not, $\hat{K}_F$ is a closed subset of M containing K and is called the *convex hull* of K with respect to F. The convex hull of $\hat{K}_F$ with respect to F coincides with $\hat{K}_F$ itself. If $\hat{K}_F$ is compact for every compact subset K of M, then M is said to be *convex with respect to F*. If $F' \subset F$, then $\hat{K}_{F'} \supset \hat{K}_F$. Hence if M is convex with respect to F' and if $F' \subset F$, then M is convex with respect to F. If M is convex with respect to the family of all holomorphic functions, then M is said to be *holomorphically convex*.

THEOREM 4.3. *Let M be a complex manifold with Carathéodory distance c_M. Fix a point o of M and let F be the set of holomorphic mappings f of M into the open unit disk D such that $f(o)=0$. If M is complete with respect to c_M, then M is convex with respect to F and hence holomorphically convex.*

PROOF. Let a be a positive number and B the closed ball of radius a around o, i.e.,

$$B = \{p \in M;\ c_M(o, p) \leqq a\}.$$

Since M is complete, B is compact. Since every compact subset K of M is contained in B for a sufficiently large a, it suffices to show that $\hat{B}_F$ is compact. We shall actually prove that $\hat{B}_F = B$. In fact,

$$\begin{aligned} \hat{B}_F &= \{p \in M;\ |f(p)| \leqq \sup|f(B)| \qquad \text{for } f \in F\} \\ &= \{p \in M;\ \rho(0, f(p)) \leqq \sup_{q \in B} \rho(0, f(q)) \qquad \text{for } f \in F\} \\ &\subset \{p \in M;\ \rho(0, f(p)) \leqq \sup_{q \in B} c_M(o, q)\} \\ &= \{p \in M;\ \rho(0, f(p)) \leqq a\} \\ &= B. \end{aligned}$$

Since $\hat{B}_F$ contains B, we conclude that $\hat{B}_F = B$. QED.

It is not clear if the converse to Theorem 4.3 holds. Let M be a complex manifold of complex dimension n and A an analytic subset of dimension $\leqq n-1$. Since every bounded holomorphic function on $M-A$ can be extended to a bounded holomorphic function on M, it follows that $M-A$ can not be convex with respect to the family of bounded holomorphic functions if A is non-empty and that c_M coincides with c_{M-A} on $M-A$. In this way we obtain many examples of holomorphically convex manifolds which are not convex with respect to the family of bounded holomorphic functions and not complete with respect to their Carathéodory distances. The punctured disk $D-\{0\}$ is the simplest example. In connection with this example, we have the following theorem whose proof is more or less direct.

THEOREM 4.4. *Let M be a complex manifold with Carathéodory distance c_M. If M is complete with respect to c_M, then every holomorphic mapping f from the punctured disk $D-\{0\}$ into M can be extended to a holomorphic mapping of D into M.*

We shall now give a large class of bounded domains which are complete with respect to their Carathéodory distances. Let M be a domain in C^n and $f_1, \cdots, f_k$ holomorphic functions defined in M. Let P be a connected component of the open subset of M defined by

$$|f_1(z)|<1, \cdots, |f_k(z)|<1.$$

Assume that the closure of P is compact and is contained in M. Then P is called an *analytic polyhedron.*

THEOREM 4.5. *An analytic polyhedron P is complete with respect to its Carathéodory distance c_P.*

PROOF. Let F be the set of holomorphic mappings of P into the unit disk D. Let o be a point of P. Given a positive number a, choose a positive number b, $0<b<1$, such that

$$\{z \in D\,;\ \rho(f_i(o), z) \leqq a \quad \text{for} \quad i=1, \cdots, k\} \subset \{z \in D\,;\ |z| \leqq b\}.$$

Then

$$\begin{aligned}\{p \in P\,;\ c_P(o, p) \leqq a\} &= \{p \in P\,;\ \rho(f(o), f(p)) \leqq a \quad \text{for} \quad f \in F\}\\ &\subset \{p \in P\,;\ \rho(f_i(o), f_i(p)) \leqq a \quad \text{for} \quad i=1, \cdots, k\}\\ &\subset \{p \in P\,;\ |f_i(p)| \leqq b\}.\end{aligned}$$

Since $\{p \in P\,;\ |f_i(p)| \leqq b\}$ is compact, $\{p \in P\,;\ c_P(o, p) \leqq a\}$ is compact. QED,

COROLLARY. *Let M be a domain in C^n which can be written as an intersection of a family of analytic polyhedrons $\{P\}$. Then M is complete with respect to its Carathéodory distance c_M.*

This is a direct consequence of Theorems 4.2 and 4.5.

The following proposition is immediate from Proposition 2.5.

PROPOSITION 4.6. *If M and M' are complex manifolds with complete Carathéodory distance, then $M \times M'$ is also complete with respect to its Carathéodory distance.*

5. Complete hyperbolic manifolds

Theorem of Van Dantzig and Van der Waerden [**11** and **19**; p. 46] states: *The group G of isometries of a connected, locally compact metric space M is locally compact with respect to the compact-open topology and its isotropy subgroup G_p is compact for every $p \in M$. If M is moreover compact, then G is compact.*

In fact, it is shown [**19**; p. 49] that, for any point $p \in M$ and for any compact subset K of M, the set $\{f \in G; f(p) \in K\}$ is a compact subset of G. Postponing applications of this theorem to the following section, we shall consider here the following lemma.

LEMMA. *Let M be a concected, locally compact space with pseudo-distance d_M and N a connected, locally compact, complete metric space with distance d_N. Then the set F of distance-decreasing mappings $f: M \to N$ is locally compact with respect to the compact open topology. In fact, if p is a point of M and K is a compact subset of N, then the subset $F(p, K) = \{f \in F; f(p) \in K\}$ of F is compact.*

The proof of this lemma is very similar to that of Theorem of Van Dantzig and Van der Waerden but is easier since a closed ball of any radius in N is compact by definition of completeness. Leaving the details to the proof of Theorem Van Dantzig and Van der Waerden as given in [**19**; pp. 46–49] we shall sketch the proof of Lemma.

Let $\{f_n\}$ be a sequence of mappings belonging to $F(p, K)$. We shall show that a suitable subsequence converges to an element of $F(p, K)$. We take a countable set $\{p_i\}$ of points which is dense in M. We set $K_i = \{q \in N; d_N(q, K) \leqq d_M(p, p_i)\}$; K_i is a closed $(d_M(p, p_i))$-neighborhood of K and hence is compact. Since each f_n is distance-decreasing, we have $d_N(f_n(p), f_n(p_i)) \leqq d_M(p, p_i)$. Since $f_n(p) \in K$, this shows that $f_n(p_i)$ is contained in the compact set K_i. By the standard argument, we can choose a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(p_i)\}$ converges to some point of K_i for each p_i as k tends to infinity. Then the mapping f defined by $f(p_i) = \lim_k f_{n_k}(p_i)$ is the desired element of $F(p, k)$.

As an application of Lemma we have

THEOREM 5.1. *Let M be a complex manifold and N a complete hyperbolic manifold. Then the set F of holomorphic mappings $f: M \to N$ is locally compact with respect to the compact-open topology. For a point p of M and a compact subset K of N, the subset $F(p, K) = \{f \in F; f(p) \in K\}$ of F is compact.*

This theorem follows from Lemma, Proposition 2.1 and the fact that if a sequence of holomorphic mappings f_n converges to a continuous mapping f (with respect to the compact-open topology), then f is holomorphic.

Theorem 5.1 is essentially equivalent to Theorem of Grauert and Reckziegel [13] which says that if N admits a differential metric of strongly negative curvature for which N is complete, then F is a normal family, (see [13] for undefined terms).

Let M be a domain in $\boldsymbol{C}^n$ (or more generally in another complex mnaifold) and $\bar{D}$ the closed unit disk in $\boldsymbol{C}$. According to Oka [22], M is said to be *pseudo-convex* if every continuous mapping $f: \bar{D} \times [0, 1] \to \boldsymbol{C}^n$ such that

1) for each $t \in [0, 1]$ the mapping f_t defined by $f_t(z) = f(z, t)$ is holomorphic and

2) $f(z, t) \in M$ unless $|z| < 1$ and $t = 1$,

maps $\bar{D} \times [0, 1]$ necessarily into M. Let D_a denote the open disk of radius a in $\boldsymbol{C}$. Then for a suitable $a < 1$, f maps $(\bar{D} - D_a) \times [0, 1]$ into M and the image $f((\bar{D} - D_a) \times [0, 1])$ is compact. Hence $f_t(\bar{D} - D_a)$ is contained in a compact subset K of M which is independent of t. Applying Theorem 5.1 to the family of holomorphic mappings f_t, $t \in [0, 1)$, of D into M, we obtain

THEOREM 5.2. *If a domain M is a complete hyperbolic manifold, then M is pseudo-convex.*

In contrast to Theorem 4.4, the punctured disk $D - \{0\}$ is a complete hyperbolic manifold. In fact, the universal covering space of the punctured disk is a disk and hence is a complete hyperbolic manifold[1]. From Proposition 2.6, it follows that the punctured disk is also a complete hyperbolic manifold. More generally we have

THEOREM 5.3. *Let M be a complete hyperbolic manifold and f a bounded holomorphic function on M. Then the open submanifold $M' = \{p \in M; f(p) \neq 0\}$ of M is also a complete hyperbolic manifold.*

PROOF. We may assume that f is a holomorphic mapping of M into the unit disk D. We denote by D' the punctured disk $D - \{0\}$. Let o be a point of M' and a and b positive numbers. Since D' is a complete hyperbolic manifold, for a given positive number a we can choose a small positive number b such that

$$\{z \in D;\ |z| \geqq b\} \supset \{z \in D';\ d_{D'}(f(o), z) \leqq a\}.$$

We set

$$A = \{p \in M;\ d_M(o, p) \leqq a\}, \quad A' = \{p \in M';\ d_{M'}(o, p) \leqq a\},$$

$$B = \{p \in M;\ |f(p)| \geqq b\}, \qquad B' = \{p \in M';\ |f(p)| \geqq b\}.$$

Since $d_{M'}(o, p) \geqq d_M(o, p)$ by Proposition 2.1, we have

$$A \supset A'.$$

Since b is positive and $M' = \{p \in M; f(p) \neq 0\}$, we have

1) Either by Theorem 3.8 or by Theorem 5.5 below.

$$B = B'.$$

Since $f: M' \to D'$ is distance-decreasing by Proposition 2.1, we have

$$A' \subset \{p \in M' ; d_{D'}(f(o), f(p)) \leqq a\} \subset \{p \in M' ; |f(p)| \geqq b\} = B' = B.$$

Since A is a compact subset of M by the completeness of M and B is closed in M, the intersection $A \cap B$ is compact. Since $B = B'$, $A \cap B$ is in M'. Since $A \cap B$ is a compact subset of M' and A' is closed in M', the intersection $A' \cap (A \cap B)$ is a compact subset of M'. Since both A and B contain A', A' coincides with $A' \cap (A \cap B)$ and hence is a compact subset of M'. This proves the completeness of M'. QED.

From Proposition 2.5 and 3.3 we obtain

PROPOSITION 5.4. *If M and M' are complete hyperbolic, so is $M \times M'$.*

THEOREM 5.5. *Let $\tilde{M}$ be a covering manifold of M. Then $\tilde{M}$ is complete hyperbolic if and only if M is so.*

PROOF. Assume that $\tilde{M}$ is complete hyperbolic. Let $\tilde{o} \in \tilde{M}$ and $o = \pi(\tilde{o}) \in M$. Let $\tilde{U}_r$ and $\tilde{B}_r$ be the open and the closed balls of radius r around $\tilde{o}$. Similarly, let U_r and B_r be the open and the closed balls of radius r around o. By Proposition 2.6, we have $\tilde{U}_r = \pi^{-1}(U_r)$. Hence,

$$\pi(\tilde{B}_r) = \pi(\bigcap_{\delta>0} \tilde{U}_{r+\delta}) = \bigcap_{\delta>0} U_{r+\delta} = B_r.$$

Since $\tilde{B}_r$ is compact by assumption, its image $B_r = \pi(\tilde{B}_r)$ is compact.

Conversely, assume that M is complete hyperbolic. It seems to be difficult to prove directly that $\tilde{M}$ is complete in the strong sense we defined in Section 4. We shall prove here that $\tilde{M}$ is complete in the usual sense. The equivalence between the two definitions of completeness will be proved in Section 8. Let $\{\tilde{p}_i\}$ be a Cauchy sequence in $\tilde{M}$. Since $\pi: \tilde{M} \to M$ is distance-decreasing, $\pi(\tilde{p}_i)$ is a Cauchy sequence in M and hence converges to a point $p \in M$. Let ε be a positive number and U the 2ε-neighborhood of p in M. Taking ε small we may assume that π induces a homeomorphism of each connected component of $\pi^{-1}(U)$ onto U. Let N be a large integer such that $\pi(p_i)$ is within the ε-neighborhood of p for $i > N$. Then every point outside U is of distance at least ε from $\pi(\tilde{p}_i)$. Let $\tilde{U}_i$ be the connected component of $\pi^{-1}(U)$ containing $\pi(\tilde{p}_i)$. We shall show that the ε-neighborhood of $\tilde{p}_i$ lies in $\tilde{U}_i$ for $i > N$. The proof is similar to that of Theorem 3.4. Let $\tilde{q}$ be a point of $\tilde{M}$ with $d_{\tilde{M}}(\tilde{p}_i, \tilde{q}) < \varepsilon$. We choose points $a_1, \cdots, a_k, b_1, \cdots, b_k$ of the unit disk D and holomorphic mappings $f_1, \cdots, f_k$ of D into M in the usual manner so that $\sum_{j=1}^{k} \rho(a_j, b_j) < \varepsilon$. Let $\widehat{a_j b_j}$ be the geodesic arc from a_j to b_j in D. Joining the curves $f_1(\widehat{a_1 b_1}), \cdots, f_k(\widehat{a_k b_k})$ in M, we obtain a curve, say $\tilde{C}$, from $\tilde{p}_i$ to $\tilde{q}$. Let $C = \pi(\tilde{C})$. From the construction of C we see that every point of C is in the

ε-neighborhood of $\pi(p_i)$ and hence in U. It follows that $\tilde{C}$ lies in $\tilde{U}_i$. Let $\tilde{p}$ be the point of $\tilde{U}_i$ defined by $p=\pi(\tilde{p})$. Then $\{\tilde{p}_j\}$ converges to $\tilde{p}$. QED.

PROPOSITION 5.6. *A closed complex submanifold M' of a complete hyperbolic manifold M is complete hyperbolic.*

PROOF. This is immediate from the fact that the injection $M'\to M$ is distance-decreasing by Proposition 2.1. QED.

THEOREM 5.7. *Let E be a complex analytic fibre bundle over M with fibre F. If M is complete hyperbolic and F is compact hyperbolic, then E is complete hyperbolic.*

PROOF. By Theorem 3.7 whose proof will be given later, E is hyperbolic. Let p be a point of E and B the closed ball of radius r around p. Let p' be the image of p under the projection $\pi: E\to M$ and B' the closed ball of radius r around p'. Since π is distance-decreasing, $\pi^{-1}(B')$ contains B. Since B' is compact and the fibre is also compact, $\pi^{-1}(B')$ is compact. Hence B is compact. QED.

THEOREM 5.8. *A complete hermitian manifold M whose holomorphic sectional curvature is bounded above by a negative constant is complete hyperbolic.*

PROOF. By Theorem 3.8 whose proof will be given later, M is hyperbolic. In the proof of Theorem 3.8 it will be shown that, when multiplied by a suitable positive constant, the hermitian metric defines a distance function d'_M such that $d'_M(p, q)\leqq d_M(p, q)$ for $p, q\in M$. Since M is complete with respect to d'_M by assumption, M is complete with respect to d_M. (Note that the equivalence of the two definitions of completeness in the Riemannian case is well known [**25**], [**19**; p. 172].) QED.

6. The automorphism group of a hyperbolic manifold

First we prove the following theorem which is of independent interest.

THEOREM 6.1. *Let M be a hyperbolic manifold. Then every holomorphic mapping f of $\boldsymbol{C}^m$ into M is a constant map.*

PROOF. This is an immediate consequence of Proposition 2.1. The only property of $\boldsymbol{C}^m$ used here is that the distance $d_{\boldsymbol{C}^m}$ of $\boldsymbol{C}^m$ is trivial, i.e., $d_{\boldsymbol{C}^m}(p, q)=0$ for all $p, q\in\boldsymbol{C}^m$. QED.

Theorem 6.1 is related to Picard Theorem, which states that every holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{C}-\{0, 1\}$ is a constant map. The universal covering space of $\boldsymbol{C}-\{0, 1\}$ is biholomorphic to the unit disk and hence is hyperbolic by Proposition 3.4. To prove that the universal covering space of $\boldsymbol{C}-\{0, 1\}$ is biholomorphic with the disk D or rather with the upper half plane, one usually makes use of the modular function. In [**13**] Grauert and Reckziegel construct a Kahler metric of strongly negative curvature on $\boldsymbol{C}-\{0, 1\}$. This, combined with Theorems 3.7 and 6.1, yields Picard Theorem.

THEOREM 6.2. *Let M be a hyperbolic manifold. Then the group $H(M)$ of holomorphic transformations of M is a Lie transformation group and its isotropy subgroup $H_p(M)$ at $p \in M$ is compact. If M is moreover compact, then $H(M)$ is finite.*

PROOF. Let $I(M)$ be the group of isometries of M with respect to the distance d_M. Since $H(M)$ is a closed subgroups of $I(M)$, Theorem of Van Dantzig and Van der Waerden quoted at the beginning of Section 5 implies that $H(M)$ is a Lie transformation group with compact $H_p(M)$ and that $H(M)$ is compact if M is compact. (Of course, one makes use of Theorem of Bochner-Montgomery which states that a locally compact group of differentiable transformations of a manifold is a Lie transformation group, [3]). Another theorem of Bochner-Montgomery [4] says that the group of holomorphic transformations of a compact complex manifold is a complex Lie group. The last assertion of Theorem 6.2 follows from the following theorem:

THEOREM 6.3. *A connected complex Lie group of holomorphic transformations acting effectively on a hyperbolic manifold M reduces to the identity element only.*

PROOF. Assume the contrary. Then a complex one-parameter subgroup acts effectively and holomorphically on M. Its universal covering group $\boldsymbol{C}$ then acts (not necessarily effectively but essentially effectively) on M. For each point p we obtain a holomorphic mapping $z \in \boldsymbol{C} \rightarrow z(p) \in M$ from this action. By Theorem 6.1 this holomorphic mapping must be a constant map. Since the identity element $1 \in \boldsymbol{C}$ maps p into p, every element z of $\boldsymbol{C}$ maps p into p. Since p is an arbitrary point of M, the action of $\boldsymbol{C}$ on M is trivial. This is absurd. QED.

Theorem 6.2 has been proved by Wu [**26**] under the assumption which amounts to saying that M is hyperbolic complete. His method relies on the notion of normal family of holomorphic mappings. Theorem 6.2 generalizes Theorem of H. Cartan [8] on the group of holomorphic transformations of a bounded domain in $\boldsymbol{C}^n$. A similar theorem has been obtained by the author for a complex manifold admitting Bergman metric, [**15**].

Making use of the last assertion of Theorem 6.2 we shall give

PROOF OF THEOREM 3.7. Let $H(F)$ be the group of holomorphic transformations of the compact fibre F. By Theorem 6.2 $H(F)$ is finite. Let P be the principal fibre bundle over M with group $H(F)$ associated with the bundle E. Since $H(F)$ is finite, P is a (not necessarily connected) covering space of M. By Theorem 5.5 P is complete hyperbolic. By Proposition 5.4 $P \times F$ is complete hyperbolic. Since E is the quotient space of $P \times F$ by the action of $H(F)$, $P \times F$ is a covering space of E. By Theorem 5.5 E is complete hyperbolic.

QED.

In Theorem 6.1 the only property of C^m needed is that the pseudo-distance d_{C^m} is trivial. If V is a complex manifold with $d_V=0$ and M is a hyperbolic manifold, then every holomorphic mapping $f: V \rightarrow M$ is a constant map. It is therefore of some interest to find other manifolds V with $d_V=0$.

THEOREM 6.4. *For a homogeneous complex manifold G/H of a complex Lie group G, the pseudo-distance $d_{G/H}$ is trivial.*

PROOF. Let $p \in G/H$ and U a small neighborhood of p such that every element q of U lies on the orbit of a complex 1-parameter subgroup of G through p. Hence, for every point $q \in U$ we have a holomorphic mapping $f: C \rightarrow G/H$ whose image contains both p and q. Since f is distance-decreasing and d_C is trivial, it follows that $d_{G/H}(p, q)=0$. To prove that the pseudo-distance between any two points of G/H is zero, we connect the two points by a chain of small open sets U and apply the triangular axiom. QED.

The reasoning in the proof of Theorem 6.1 may be applied to a holomorphic mapping of one fibre space into another. By a complex fibre space we shall mean here a triple (E, π, M) consisting of a holomorphic mapping π from a complex manifold E onto a complex manifold M such that each fibre $\pi^{-1}(p)$, $p \in M$, is a complex submanifold of E.

THEOREM 6.5. *Let (E, π, M) and (E', π', M') be two complex fibre spaces in the sense defined above. If each fibre $\pi^{-1}(p)$, $p \in M$, of E is connected and its pseudo-distance $d_{\pi^{-1}(p)}$ is trivial and if M' is hyperbolic, then every holomorphic mapping $f: E \rightarrow E'$ is fibre-preserving.*

PROOF. The restriction of $\pi' \circ f$ to each fibre $\pi^{-1}(p)$ is a holomorphic mapping of a complex manifold with trivial pseudo-distance into a hyperbolic manifold and hence is a constant map. QED.

7. Cross-sections in a family of complete hyperbolic manifolds

Let V and T be complex manifolds and let π be a holomorphic mapping of V onto T which is regular in the sense that the rank of the differential of π is equal to the dimension of T at every point of V. For each $t \in T$, let M_t denote the regularly imbedded complex submanifold $\pi^{-1}(t)$ of V. Let $\tilde{M}$ denote a simply connected complex manifold. Assum that every point t of T has a neighborhood U biholomorphic to the open unit ball in C^k, $(k=\dim T)$, such that $U \times \tilde{M}$ is the universal covering space of $\pi^{-1}(U)$ in such a way that the diagram

$$\begin{tikzcd} U \times \tilde{M} \arrow[rr] \arrow[dr, "\tilde{\pi}"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

is commutative, where the horizontal arrow is the covering projection and

$\tilde{\pi}: U \times \tilde{M} \to U$ is the obvious projection. Then we say that V is a family of complex manifolds $\{M_t; t \in T\}$ uniformisable by $\tilde{M}$.

THEOREM 7.1. *Let $V = \{M_t; t \in T\}$ be a family of complex manifolds uniformisable by a complete hyperbolic manifold $\tilde{M}$. Then for any domain G in T the set of holomorphic cross-sections $s: G \to V$ (i.e., mappings s satisfying $\pi \circ s(t) = t$ for $t \in G$) is locally compact with respect to the compact-open topology. If M_t is compact, then the set of holomorphic cross-section $s: G \to V$ is compact.*

PROOF. It suffices to prove the theorem in the case where the domain G is a small open ball U which is biholomorphic to the open unit ball in $\boldsymbol{C}^k$ and $U \times \tilde{M}$ is the universal covering space of $\pi^{-1}(U)$ as explained above. Since both U and $\tilde{M}$ are complete hyperbolic manifolds, their product $U \times \tilde{M}$ is also complete hyperbolic by Proposition 5.4. By Theorem 5.5 $\pi^{-1}(U)$ is complete hyperbolic. Now our assertion follows from Theorem 5.1. QED.

The case where T is a Riemann surface and each M_t is a compact Riemann surface of genus greater than 1 so that $\tilde{M}$ is a disk in $\boldsymbol{C}$ has been already considered by Grauert and Reckziegel [13].

8. Invariant metric and completeness

In Section 2 we defined the pseudo-distance $d_M(p, q)$ from p to q. We shall now define an infinitesimal analogue of d_M.

Let D be the open unit disk in $\boldsymbol{C}$ and ds^2 the Poincaré-Bergman metric in D. Let M be a complex manifold, p a point of M and X a tangent vector at p. Choose a point $a \in D$ and a tangent vector A at a together with a holomorphic mapping $f: D \to M$ such that $f(a) = p$ and $f_*(A) = X$. Let $F_M(X)$ denote the infimum of the length of A with respect to ds^2 for all possible choices of a, A and f. Then F_M is a function defined on the tangent bundle $T(M)$ of M and satisfies

$$F_M(0) = 0,$$

$$F_M(X) \geqq 0 \qquad \text{for } X \in T(M),$$

$$F_M(cX) = cF_M(X) \qquad \text{for } X \in T(M) \text{ and for a positive number } c.$$

We shall call F_M the *invariant pseudo-metric* on M. If $F_M(X) > 0$ for every nonzero vector X, then we call it the *invariant metric* on M. I do not know which of the conditions for Finsler metric are satisfied by the invariant metric F_M. Since every point of M has a neighborhood which is biholomorphic with the unit ball of $\boldsymbol{C}^n$, it follows easily that F_M is bounded on each compact subset of $T(M)$. But I do not know if F_M is continuous or not. Let C be a piecewise differentiable curve in M and C' the tangent vector field along C defined by $C'(t) = dC(t)/dt$. I do not know if the integral

$$L(C)=\int_C F_M(C'(t))dt$$

exists or not. Let p and q be points of M and consider all piecewise differentiable curves C from p to q for which $L(C)$ exist. Then it is easy to verify $\inf L(C) \leqq d_M(p, q)$. I do not know if the equality holds or not. An infinitesimal analogue of Carathéodory distance has been systematically investigated by Reiffen [24].

Our pseudo-distance d_M enjoys the following property which seems to be peculiar to the pseudo-distance defined from an infinitesimal pseudo-metric by integration, for example, the distance defined by a Riemannian metric. Given a subset A of M and a positive number r, we set

$U(A; r)=\{p\in M; d_M(p, a)<r\}$ for some point $a\in A\}$. Then for any point $o\in M$ and any positive numbers r and r', we have

$$U(U(o; r); r')=U(o; r+r').$$

The proof is straightforward. The inclusion in one direction is true for any pseudo-distance and makes use of the triangular axiom. But the inclusion in the other direction comes from the fact that d_M is essentially defined by a method of integration.

We shall now show that the completeness of a hyperbolic manifold in the strong sense defined in Section 4 is equivalent to the completeness in the usual sense as a metric space. More generally, we prove

THEOREM 8.1. *Let M be a locally compact metric space with distance d satisfying the equality $U(U(o; r); r')=U(o; r+r')$ as above for every $o\in M$ and all positive numbers r and r'. Then M is complete in the sense that every Cauchy sequence converges if and only if the closure $\bar{U}(o; r)$ of $U(o; r)$ is compact for all $o\in M$ and all positive numbers r.*

PROOF. We have only to prove that if M is complete then $\bar{U}(o; r)$ is compact, since the implication in the opposite direction is trivial.

LEMMA. *$\bar{U}(o; r)$ is compact if there exists a positive number b such that $\bar{U}(p; b)$ is compact for every $p\in U(o; r)$.*

PROOF OF LEMMA. Let $s<r$ be a positive constant such that $\bar{U}(o; s)$ is compact. It suffices to show that $\bar{U}\left(o; s+\frac{b}{2}\right)$ is compact. Let $p_1, p_2, \cdots$ be points of $\bar{U}\left(o; s+\frac{b}{2}\right)$. Choose points $q_1, q_2, \cdots$ of $\bar{U}(o; s)$ such that $d(p_i, q_i) <\frac{3}{4}b$. Since $\bar{U}(o; s)$ is compact, we may assume (by choosing a subsequence if necessary) that $q_1, q_2, \cdots$ converges to some point, say q, of $\bar{U}\left(o; s+\frac{b}{2}\right)$. Then $\bar{U}(q; b)$ contains all p_i for large i. Since it is compact, a suitable subsequence of $p_1, p_2, \cdots$ converges to some point p in $\bar{U}(q\,;b)$. Since $\bar{U}\left(o\,;s+\frac{b}{2}\right)$ is closed, p is in $\bar{U}\left(o\,;s+\frac{b}{2}\right)$. This completes the proof of Lemma.

To complete the proof of Theorem 8.1 we shall show that there exists a positive number b such that $\bar{U}(p;b)$ is compact for every point $p\in M$. Assume the contrary. Then there exists a point $p_1\in M$ such that $\bar{U}\left(p_1\,;\frac{1}{2}\right)$ is non-compact. Apply Lemma to $\bar{U}\left(p_1\,;\frac{1}{2}\right)$ and we see that there exists a point $p_2\in\bar{U}\left(p_1\,;\frac{1}{2}\right)$ such that $\bar{U}\left(p_2\,;\frac{1}{2^2}\right)$ is non-compact. Apply Lemma to $\bar{U}\left(p_2\,;\frac{1}{2^2}\right)$ and we see that there exists a point $p_3\in\bar{U}\left(p_2\,;\frac{1}{2^2}\right)$ such that $\bar{U}\left(p_3\,;\frac{1}{2^3}\right)$ is non-compact. In this way we obtain a Cauchy sequence $p_1, p_2, p_3, \cdots$. Let p be its limit point. Since M is locally compact, there exists a positive number c such that $\bar{U}(p;c)$ is compact. For a sufficiently large i, $\bar{U}\left(p_i\,;\frac{1}{2^i}\right)$ is contained in $\bar{U}(p;c)$ and hence must be compact. This is a contradiction. QED.

Let M be a hyperbolic manifold and M^* its completion with respect to distance d_M. I do not know if M^* is complete in the strong sense that every closed ball of radius r in M^* is compact. If it is so, M^* is necessarily locally compact. Conversely, if M^* is locally compact, the proof above implies that M^* is complete in the strong sense. The question whether M^* is locally compact or not seems to be very delicate in view of the following example due to D. Epstein of a Riemannian manifold M whose completion M^* (with respect to the distance defined by the Riemannian metric) is not locally compact. Let M be the open subset of the xy-plane obtained by deletion of the set $\left\{\left(x,\frac{1}{n}\right); x\geqq 0, n=1,2,\cdots\right\}$ and the set $\{(x,y); x\geqq 0, y\leqq 0\}$ and make it into a Riemannian manifold by taking the induced flat metric.

9. Curvature of a hermitian manifold

Let M be a hermitian manifold. The *hermitian connection* of M is a unique affine connection such that both the metric tensor and the complex structure tensor are parallel and that the torsion tensor is pure in the sense described in the structure equations below, cf. [9].

$$d\theta^A=-\Sigma\omega^A{}_B\wedge\theta^B+\Theta^A,\qquad \Theta^A=\frac{1}{2}\,\Sigma T^A{}_{BC}\theta^B\wedge\theta^C,$$

$$d\omega^A{}_B=-\Sigma\omega^A{}_C\wedge\omega^C{}_B+\Omega^A{}_B,\ \Omega^A{}_B=\Sigma R^A{}_{BC\bar{D}}\theta^C\wedge\bar{\theta}^D\,.$$

As in the Riemannian case or the Kählerian case, we define the hermitian curvature tensor which is a covariant tensor of degree 4 and denote it by R. The sectional curvature $K(\sigma)$ of a plane σ with orthonormal basis X, Y is given by $K(\sigma)=R(X, Y, X, Y)$. If $JX=Y$ (where J denotes the complex structure tensor) so that σ is a complex line, then $K(\sigma)$ is called the holomorphic sectional curvature of σ.

Let V be a complex submanifold of M. Letting $n=\dim M$ and $k=\dim V$, we shall use the following convention on the ranges of indices:

$$1\leqq A, B, C, \cdots \leqq n\,; \quad 1\leqq a, b, c, \cdots \leqq k\,; \quad k+1\leqq p, q, r, \cdots \leqq n\,.$$

With respect to frames adapted to V, we then have

$$\begin{aligned}&\theta^p=0\,,\\ &d\theta^A=-\Sigma\omega^A{}_b\wedge\theta^b+\Theta^A, \qquad \Theta^A=\frac{1}{2}\Sigma T^A{}_{bc}\theta^b\wedge\theta^c\,.\end{aligned}$$

Since $d\theta^p=0$ and Θ^p is pure (i. e., does not contain $\bar{\theta}^B$), we may write

$$\omega^p{}_b=\Sigma A^p{}_{bc}\theta^c\,.$$

Then

$$\begin{aligned}d\omega^a{}_b&=-\Sigma\omega^a{}_C\wedge\omega^C{}_b+\Omega^a{}_b\\ &=-\Sigma\omega^a{}_c\wedge\omega^c{}_b+\Sigma\omega^p{}_a\wedge\omega^p{}_b+\Omega^a{}_b\\ &=-\Sigma\omega^a{}_c\wedge\omega^c{}_b+\Sigma(R^a{}_{bc\bar{d}}-\Sigma_pA^p{}_{bc}\bar{A}^p{}_{ad})\theta^c\wedge\bar{\theta}^d\,.\end{aligned}$$

Hence,

THEOREM 9.1. *Let V be a complex submanifold of a hermitian manifold M. Then the holomorphic sectional curvature $K_V(\sigma)$ of V is less than or equal to the holomorphic sectional curvature $K_M(\sigma)$ of M.*

It is also clear from the formula above that $K_V(\sigma)=K_M(\sigma)$ if and only if $A^p{}_{bc}=0$. In the Kählerian case, Theorem 9.1 has been obtained by O'Neill [23].

THEOREM 9.2. *Let D be the open unit disk in $\boldsymbol{C}$ with the invariant metric ds^2 with sectional curvature $-A$. Let M be a hermitian manifold with metric ds^2_M and holomorphic sectional curvature bounded above by a negative constant $-B$. Then every holomorphic mapping $f:D\to M$ satisfies $f^*(ds^2_M)\leqq\frac{A}{B}ds^2$.*

Because of Theorem 9.1, the proof of Lemma 9 in my previous paper [17] (i. e., the proof in the Kählerian case) is valid in the hermitian case also and gives Theorem 9.2. Theorem 9.2 may be also derived with the help of Theorem 9.1 from Ahlfors's generalized Schwarz lemma [1], cf. also the paper of Grauert-Reckziegel [13].

Multiplying the metric ds^2_M by a suitable constant, we may assume that $A=B$ in Theorem 9.2. Then Theorem 9.2 states that every holomorphic mapping $f:D\to M$ is distance-decreasing. If we denote by d' the distance function on M defined by ds_M, then every holomorphic mapping $f:(D, \rho)\rightarrow(M, d')$ is distance-decreasing (where ρ is the distance function defined by ds^2). By Proposition 2.4, $d_M(p, q)\geqq d'(p, q)$ for $p, q\in M$, which proves that M is hyperbolic (Theorem 3.8).

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