J. Math. Soc. Japan
Vol. 19, No. 4, 1967

Completely faithful modules and quasi-Frobenius algebras

By Shizuo ENDO

(Received Oct. 1, 1966)
(Revised Jan. 17, 1967)

Introduction.

A generator in the category (cf. [12]) of left (right) modules over a ring Λ will be called a completely faithful left (right) Λ-module according to [4]. The complete faithfulness of modules is a Morita invariant property, which plays an essential part for the categorical theory of rings.

B. Müller introduced in [19] the notion of a quasi-Frobeninus extension Λ of a ring Ω. In case Ω is in the center of Λ, i.e., in case Λ is a Ω-algebra, this coincides with that of a semi-Frobenius algebra in [15], and, in this paper, we shall call this a quasi-Frobenius algebra. An algebra Λ over a commutative ring R, which is a finitely generated projective R-module, will be called a quasi-Frobenius R-algebra, if $\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$ is a completely faithful left (and right) Λ-module. The purpose of this paper is to show some basic properties in quasi-Frobenius algebras.

As is well known, any completely faithful Λ-module is faithful, but a faithful Λ-module is not always completely faithful, and if Λ is commutative, then any finitely generated, faithful, projective Λ-module is completely faithful. It is also known (cf. [3] or [9]) that, in case Λ is a quasi-Frobenius ring, any faithful Λ-module is completely faithful, and it was proved in [1] that, if Λ is a maximal order over a Dedekind domain in a central simple algebra, then any finitely generated projective Λ-module is completely faithful. However, it seems that such facts have not been treated systematically. Recently, G. Azumaya gave in [4] a characterization of a ring Λ with the property: (G) Any faithful Λ-module is completely faithful. Another purpose of this paper is to examine the structure of a ring Λ with each of the following properties:

(FG) Any finitely generated, faithful Λ-module is completely faithful.

(PFG) Any finitely generated, faithful, projective Λ-module is completely faithful.

This work was supported by the Matsunaga Science Foundation.

In § 3 we prove a fundamental theorem for a quasi-Frobenius algebra, which shows the strict connection with the classical one, and, in § 4, we give a characterization of a quasi-Frobenius algebra over a Noetherian ring which is hereditary. As applications of these results we determine, in § 5, the structure of algebras with (FG) over a Noetherian ring, and, in § 6, we give a sufficient condition for a ring to have (PFG). Furthermore, § 7 is devoted to establishing the commutor theory of quasi-Frobenius subalgebras of a central separable algebra, and, finally, § 8 is devoted to giving an affirmative answer to a problem on semi-simple algebras in [15], in a special case.

Throughout this paper we shall only consider rings with a unit element 1 and modules over such rings on which 1 operates as identity. We shall denote by R a commutative ring and by Λ a ring which is not always commutative. An R-algebra will mean an algebra over a commutative ring R which is a faithful R-module.

§ 1. Preliminaries.

First we refer to some well known facts, which will be freely used throughout this paper (cf. [1], [4], [6], [11], [13], [18]).

For a left Λ-module M we denote by $\mathfrak{T}_\Lambda(M)$ the trace ideal of M in Λ (for definition see [1]).

PROPOSITION 1.1. *For any left Λ-module M the following statements are equivalent:*

(1) *M is Λ-completely faithful.*

(2) $\mathfrak{T}_\Lambda(M)=\Lambda$.

(3) *For any maximal two-sided (left) ideal $\mathfrak{M}$ of Λ, there exists a $\varphi \in \mathrm{Hom}_\Lambda(M, \Lambda)$ such that $\mathrm{Im}\,\varphi \not\subseteq \mathfrak{M}$.*

(4) *Λ is a direct summand of the direct sum of some copies of M.*

PROPOSITION 1.2. *Let P be a projective left Λ-module. Then the following statements are equivalent for P:*

(1) *P is Λ-completely faithful.*

(2) *For any maximal two-sided (left) ideal $\mathfrak{M}$ of Λ, we have $\mathrm{Hom}_\Lambda(P, \Lambda/\mathfrak{M}) \neq 0$.*

More generally, we have $P=\mathfrak{T}_\Lambda(P)P$ and $\mathfrak{T}_\Lambda(P)$ is an idempotent ideal of Λ.

PROPOSITION 1.3. *Let Λ be a ring with the Jacobson radical $\mathfrak{N}$ which is the unique maximal two-sided ideal in Λ. Then any projective, non-zero left Λ-module is Λ-completely faithful.*

PROOF. Let P be a projective non-zero left Λ-module. Then, by (1.2), we have $P=\mathfrak{T}_\Lambda(P)P$. If $\mathfrak{T}_\Lambda(P)\subsetneqq \Lambda$, then we have $\mathfrak{T}_\Lambda(P)\subseteqq \mathfrak{N}$, hence $P=\mathfrak{N}P$. Thus we have $P=0$. This is obviously a contradiction. Thus we must have $\mathfrak{T}_\Lambda(P)=\Lambda$.

PROPOSITION 1.4. *Let $\mathfrak{A}$ be a two-sided ideal of a ring Λ.*

(1) *If $\mathfrak{A}^2=\mathfrak{A}$, then $\mathfrak{T}_\Lambda(\mathfrak{A})=\mathfrak{A}$.*

(2) *If $\mathfrak{T}_\Lambda(\mathfrak{A})=\mathfrak{A}$ and $\mathfrak{A}$ is left Λ-projective, tnen $\mathfrak{A}^2=\mathfrak{A}$.*

PROPOSITION 1.5. *Any finitely generated faithful, projective module over a commutative ring is completely faithful.*

PROPOSITION 1.6. *Let Λ, Γ be rings such that $\Lambda \subseteqq \Gamma$. If Γ is a completely faithful left Λ-module, then Λ is the direct summand of Γ as a left Λ-module.*

§ 2. Faithful and completely faithful modules over an algebra.

Almost all of our results in this section may also be known. However, as these are not given anywhere explicitely, we shall give them here in the explicit form.

Let R be a commutative ring, Λ an R-algebra and M a left Λ-module. For any multiplicative system S of R which does not contain 0, we put $\mathfrak{c}_S(R) =\{r\in R/sr=0$ for some $s\in S\}$, $\mathfrak{c}_S(\Lambda)=\{\lambda\in\Lambda/s\lambda=0$ for some $s\in S\}$ and $\mathfrak{c}_S(M)=\{u\in M/su=0$ for some $s\in S\}$. Then $\mathfrak{c}_S(R)$ is an ideal of R, and, putting $\bar{S}=S+\mathfrak{c}_S(R)/\mathfrak{c}_S(R)$, $\bar{S}$ is a multiplicative system of $R/\mathfrak{c}_S(R)$ consisting of non-zero divisors in $R/\mathfrak{c}_S(R)$. Therefore R_S is contained in the total quotient ring of $R/\mathfrak{c}_S(R)$. Also, $\mathfrak{c}_S(\Lambda)$ is a two-sided ideal of Λ and $\mathfrak{c}_S(M)$ is a left Λ-submodule of M. As is well known, we have $\Lambda_S=(\Lambda/\mathfrak{c}_S(\Lambda))_{\bar{S}}\cong R_S\otimes_R \Lambda$ as R_S-algebras and $M_S\cong(M/\mathfrak{c}_S(M))_{\bar{S}}\cong R_S\otimes_R M$ as left Λ_S-modules. Especially, if S is the complementary set of a prime ideal $\mathfrak{p}$ in R, we use $R_\mathfrak{p}$, $\Lambda_\mathfrak{p}$, $M_\mathfrak{p}$, $\mathfrak{c}_\mathfrak{p}(R)$, $\mathfrak{c}_\mathfrak{p}(\Lambda)$, $\mathfrak{c}_\mathfrak{p}(M)$ instead of R_S, Λ_S, M_S, $\mathfrak{c}_S(R)$, $\mathfrak{c}_S(\Lambda)$, $\mathfrak{c}_S(M)$, respectively. Throughout this paper we shall use these notations.

We begin with the well known

LEMMA 2.1. *Let Λ be an R-algebra which is a finitely generated R-module. Then, for any maximal two-sided ideal $\mathfrak{M}$ of Λ, $\mathfrak{M}\cap R$ is a maximal ideal of R. Furthermore, for any maximal ideal $\mathfrak{m}$ of R, we have $\mathfrak{m}\Lambda\cap R=\mathfrak{m}$ and there exists only a finite number of maximal two-sided ideals of Λ in which $\mathfrak{m}$ is contained.*

PROOF. For example, see [**11**], V, § 6.

PROPOSITION 2.2. *Let Λ be an R-algebra which is a finitely generated R-module and M a finitely generated left Λ-module. If M is Λ-faithful, then M_S is Λ_S-faithful for any multiplicative system $S(\not\ni 0)$ of R. Conversely, if $M_\mathfrak{m}$ is $\Lambda_\mathfrak{m}$-faithful for any maximal ideal $\mathfrak{m}$ of R, then M is Λ-faithful.*

PROOF. As this is easy, we omit it.

PROPOSITION 2.3. *Let Λ be an R-algebra which is a finitely generated R-module and M a finitely generated left Λ-module. Let $\mathfrak{a}$ be a proper ideal of R. If M is Λ-faithful, then $\mathrm{Ann}_{\Lambda/\mathfrak{a}\Lambda}M/\mathfrak{a}M$ is a nil ideal of $\Lambda/\mathfrak{a}\Lambda$.*

PROOF. As M is finitely generated over R, we can put $M=\sum_{i=1}^{t} Ru_i$. When $\lambda M \subseteqq \mathfrak{a}M$ for some $\lambda \in \Lambda$, we have $\lambda u_i = \sum_{j=1}^{t} a_{ij}u_j$ for some $a_{ij} \in \mathfrak{a}$. If we put $\Delta = |\delta_{ij}\lambda - a_{ij}|$, then we have $\Delta u_i = 0$ for each i. Since M is Λ-faithful, we must have $\Delta = 0$. From this we can easily see $\lambda^t \in \mathfrak{a}\Lambda$.

The following proposition is given in [1] in a special case (see the proof of (3.9) in [1]).

PROPOSITION 2.4. *Let Λ be an R-algebra which is a finitely generated R-module and M a finitely generated left Λ-module. If M is Λ-completely faithful, then M_S is Λ_S-completely faithful for any multiplicative system S ($\not\ni 0$) of R. Further assume that, for any maximal ideal $\mathfrak{m}$ of R, $\mathfrak{c}_{\mathfrak{m}}(\Lambda)$ is a finitely generated R-module. Then, if $M_{\mathfrak{m}}$ is $\Lambda_{\mathfrak{m}}$-completely faithful for any maximal ideal $\mathfrak{m}$ of R, M is Λ-completely faithful.*

PROOF. Since the first part of this proposition is obvious, we have only to show the second part. By (1.1) it suffices to show that, for any maximal two-sided ideal $\mathfrak{M}$ of Λ, there is a $\varphi \in \mathrm{Hom}_{\Lambda}(M, \Lambda)$ such that $\mathrm{Im}\,\varphi \not\subseteq \mathfrak{M}$. If we put $\mathfrak{m} = \mathfrak{M} \cap R$, then $\mathfrak{m}$ is a maximal ideal of R according to (2.1), and $\mathfrak{M}_{\mathfrak{m}}$ is a maximal two-sided ideal of $\Lambda_{\mathfrak{m}}$. As $M_{\mathfrak{m}}$ is $\Lambda_{\mathfrak{m}}$-completely faithful, there exists, again by (1.1), a $\tilde{\varphi} \in \mathrm{Hom}_{\Lambda_{\mathfrak{m}}}(M_{\mathfrak{m}}, \Lambda_{\mathfrak{m}})$ such that $\mathrm{Im}\,\tilde{\varphi} \not\subseteq \mathfrak{M}_{\mathfrak{m}}$. If we put $M = \sum_{i=1}^{t} Ru_i$ and we denote by $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_t$ the residues of $u_1, u_2, \cdots, u_t$ in $M/\mathfrak{c}_{\mathfrak{m}}(M)$, then we have $M_{\mathfrak{m}} = \sum_{i=1}^{t} R_{\mathfrak{m}}\bar{u}_i$. Let s be an element of $R-\mathfrak{m}$ such that $s\tilde{\varphi}(\bar{u}_i) \in \Lambda/\mathfrak{c}_{\mathfrak{m}}(\Lambda)$ for all $1 \leqq i \leqq t$, and put $\varphi^*(\overline{u_i}) = s\tilde{\varphi}(\overline{u_i})$ for any i. Then φ^* can be considered as an element of $\mathrm{Hom}_{\Lambda/\mathfrak{c}_{\mathfrak{m}}(\Lambda)}(M/\mathfrak{c}_{\mathfrak{m}}(M), \Lambda/\mathfrak{c}_{\mathfrak{m}}(\Lambda))$ and we have $\mathrm{Im}\,\varphi^* \not\subseteq \mathfrak{M}/\mathfrak{c}_{\mathfrak{m}}(\Lambda)$. Since $\mathfrak{c}_{\mathfrak{m}}(\Lambda)$ is finitely generated over R by our assumption, we can find some $t \in R-\mathfrak{m}$ such that $t \cdot \mathfrak{c}_{\mathfrak{m}}(\Lambda) = 0$. Let $\lambda_1, \lambda_2, \cdots, \lambda_t$ be the representatives of $s\tilde{\varphi}(\bar{u}_1), s\tilde{\varphi}(\bar{u}_2), \cdots, s\tilde{\varphi}(\bar{u}_t)$ in Λ, respectively. Then, for some $\mu_1, \mu_2, \cdots, \mu_t \in \Lambda$, we have $\sum_{i=1}^{t} \mu_i\tilde{\varphi}(\bar{u}_i) = \bar{0}$ if and only if $t\sum_{i=1}^{t} \mu_i\lambda_i = \sum_{i=1}^{t} \mu_i(t\lambda_i) = 0$. Therefore, putting $\hat{\varphi}(\bar{u}_i) = t\lambda_i$ for each i, $\hat{\varphi}$ can be considered as an element of $\mathrm{Hom}\,(M/\mathfrak{c}_{\mathfrak{m}}(M), \Lambda)$ and we have $\mathrm{Im}\,\hat{\varphi} \not\subseteq \mathfrak{M}$. Let ψ be a natural epimorphism of M onto $M/\mathfrak{c}_{\mathfrak{m}}(M)$ and put $\varphi = \hat{\varphi} \circ \psi$. Then φ is as is required. This completes our proof.

We remark that the assumption in the second part of (2.4) is satisfied in case R is Noetherian or in case R is an integral domain and Λ is R-torsion-free.

PROPOSITION 2.5. *Let Λ be an R-algebra which is a finitely generated R-module. Then, for any projective left Λ-module P, the following statements are equivalent:*

(1) *P is Λ-completely faithful.*

(2) *For any maximal ideal* $\mathfrak{m}$ *of* R, $P_{\mathfrak{m}}$ *is* $\Lambda_{\mathfrak{m}}$*-completely faithful.*

(3) *For any maximal ideal* $\mathfrak{m}$ *of* R, $P/\mathfrak{m}P$ *is* $\Lambda/\mathfrak{m}\Lambda$*-completely faithful.*

PROOF. The implications $(1)\Rightarrow(2)\Rightarrow(3)$ are obvious. Hence it suffices to show $(3)\Rightarrow(1)$. Suppose that P satisfies (3). Let $\mathfrak{M}$ be a maximal two-sided ideal of Λ and put $\mathfrak{m}=\mathfrak{M}\cap R$. Then, by (2.1), $\mathfrak{m}$ is a maximal ideal of R. Since $P/\mathfrak{m}P$ is $\Lambda/\mathfrak{m}\Lambda$-completely faithful by our assumption, we have $\mathrm{Hom}_{\Lambda/\mathfrak{m}\Lambda}(P/\mathfrak{m}P, \Lambda/\mathfrak{M})\neq 0$ according to (1.2). As P is Λ-projective, we easily obtain $\mathrm{Hom}_{\Lambda}(P, \Lambda/\mathfrak{M})\neq 0$. Again by (1.2) P must be Λ-completely faithful. This implies $(3)\Rightarrow(1)$.

PROPOSITION 2.6. *Let* Λ *be an* R*-algebra which is a finitely generated projective* R*-module. Then, for any finitely generated* R*-module* M,

(1) M *is* R*-faithful if and only if* $\Lambda\otimes_R M$ *is* Λ*-faithful.*

(2) M *is* R*-completely faithful if and only if* $\Lambda\otimes_R M$ *is* Λ*-completely faithful.*

(3) M *is* R*-projective if and only if* $\Lambda\otimes_R M$ *is* Λ*-projective.*

PROOF. (1) By (2.2) we may assume that R is local. Hence we may also suppose that Λ is R-free. Then we can easily show $\mathrm{Ann}_{\Lambda}(\Lambda\otimes_R M)=\Lambda\cdot(\mathrm{Ann}_R M)$. From this we obtain (1). (2) Since the only if part is obvious, we have only to show the if part. This can be proved by using (1.1), (1.5) and (1.6). (3) is also trivial.

§3. Quasi-Frobenius algebras over a commutative ring.

Let Λ be an R-algebra and put ${}_{\Lambda}\Lambda^*=\mathrm{Hom}_R(\Lambda_{\Lambda}, R)$. Then Λ is said to be a left quasi-Frobenius R-algebra if (1) Λ is a finitely generated projective R-module and $(2)^l$ ${}_{\Lambda}\Lambda^*$ is Λ-completely faithful (cf. [**19**]). The condition $(2)^l$ in this definition can be replaced by $(2')^l$: ${}_{\Lambda}\Lambda$ is (Λ, R)-injective or $(2'')^r$: Λ^*_{Λ} is Λ-projective. Similarly we define a right quasi-Frobenius R-algebra, and an R-algebra is called a quasi-Frobenius R-algebra if it is left and right quasi-Frobenius. It is reported in [**19**] that Rosenberg and Chase proved the equivalence $(2)^l$ and $(2)^r$ under the assumption (1). In fact, in our main theorem (3.3) in this section, we shall also see this. A quasi-Frobenius R-algebra Λ is said to be a locally Frobenius R-algebra, if, for any maximal ideal $\mathfrak{m}$ of R, $\Lambda_{\mathfrak{m}}$ is a Frobenius $R_{\mathfrak{m}}$-algebra. If an R-algebra with (1) is separable ([**2**]), semi-simple ([**14**]), Frobenius, or symmetric ([**10**]), it is quasi-Frobenius.

In this section we shall prove some basic results in quasi-Frobenius algebras.

We begin with the following general

LEMMA 3.1. *Let* Λ *be an* R*-algebra which is a finitely generated projective* R*-module and* M *a finitely generated left* Λ*-module which is* R*-projective.*

(1) *For any commutative R-algebra S, we have* $\mathrm{dh}_{S\otimes_R\Lambda} S\otimes_R M \leqq \mathrm{dh}_\Lambda M$.

(2) *We have* $\mathrm{dh}_\Lambda M = \sup_{\mathfrak{m}} \mathrm{dh}_{\Lambda_\mathfrak{m}} M_\mathfrak{m} = \sup_{\mathfrak{m}} \mathrm{dh}_{\Lambda/\mathfrak{m}\Lambda} M/\mathfrak{m}M$ *where* $\mathfrak{m}$ *runs over all maximal ideals of* R.

PROOF. By our assumption on M, we can find an R-split projective resolution of a Λ-module M:

$$\to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

where each P_n is Λ-finitely generated. From this we obtain the projective resolution of a $S\otimes_R \Lambda$-module $S\otimes_R M$:

$$\to S\otimes_R P_n \to \cdots \to S\otimes_R P_1 \to S\otimes_R P_0 \to S\otimes_R M \to 0 .$$

Hence we have $\mathrm{dh}_{S\otimes_R\Lambda} S\otimes_R M \leqq \mathrm{dh}_\Lambda M$, which proves (1). By (1) we have $\mathrm{dh}_\Lambda M \geqq \mathrm{dh}_{\Lambda_\mathfrak{m}} M_\mathfrak{m}$, $\mathrm{dh}_{\Lambda/\mathfrak{m}\Lambda} M/\mathfrak{m}M$ for any maximal ideal $\mathfrak{m}$ of R. On the other hand, since each P_n is a finitely generated projective Λ-module and any $R_\mathfrak{m}$ is R-flat, we have $R_\mathfrak{m} \otimes_R \mathrm{Ext}^n_\Lambda(M, N) \cong \mathrm{Ext}^n_{\Lambda_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m})$ for any left Λ-module N, and so we obtain $\mathrm{dh}_\Lambda M \leqq \sup_\mathfrak{m} \mathrm{dh}_{\Lambda_\mathfrak{m}} M_\mathfrak{m}$. Thus $\mathrm{dh}_\Lambda M = \sup_\mathfrak{m} \mathrm{dh}_{\Lambda_\mathfrak{m}} M_\mathfrak{m}$. In order to complete the proof of (2), it suffices to show $\mathrm{dh}_{\Lambda_\mathfrak{m}} M_\mathfrak{m} \leqq \mathrm{dh}_{\Lambda/\mathfrak{m}\Lambda} M/\mathfrak{m}M$ for any maximal ideal $\mathfrak{m}$ of R. Therefore we may assume that R is a local ring with a maximal ideal $\mathfrak{m}$. Let $\hat{R}$ be the Henselization of R (cf. [21]). Then we also have $\hat{R} \otimes_R \mathrm{Ext}^n_\Lambda(M, N) \cong \mathrm{Ext}^n_{\hat{R}\otimes_R\Lambda}(\hat{R}\otimes_R M, \hat{R}\otimes_R N)$ for any left Λ-module N, and so, as $\hat{R}$ is R-faithfully flat, we obtain $\mathrm{dh}_\Lambda M = \mathrm{dh}_{\hat{R}\otimes_R\Lambda} \hat{R}\otimes_R M$. Since $\hat{R}/\mathfrak{m}\hat{R} \cong R/\mathfrak{m}$, $\hat{R}\otimes_R \Lambda/\mathfrak{m}(\hat{R}\otimes_R\Lambda) \cong \Lambda/\mathfrak{m}\Lambda$ and $\hat{R}\otimes_R M/\mathfrak{m}(\hat{R}\otimes_R M) \cong M/\mathfrak{m}M$, we may further suppose that R is Henselian. Then, for a finitely generated projective left $\Lambda/\mathfrak{m}\Lambda$-module $\bar{P}$, there is a finitely generated projective left Λ-module P such that $P/\mathfrak{m}P \cong \bar{P}$. Now let

$$\to \bar{P}_n \to \cdots \to \bar{P}_1 \to \bar{P}_0 \to M/\mathfrak{m}M \to 0$$

be a projective resolution of $M/\mathfrak{m}M$ where each $\bar{P}_n$ is $\Lambda/\mathfrak{m}\Lambda$-finitely generated. Then we can find the projective resolution of M:

$$\to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

such that $P_n/\mathfrak{m}P_n \cong \bar{P}_n$ for any n. From this we can easily see $\mathrm{dh}_\Lambda M \leqq \mathrm{dh}_{\Lambda/\mathfrak{m}\Lambda} M/\mathfrak{m}M$, which completes our proof.

PROPOSITION 3.2. *Let* Λ *be a quasi-(locally) Frobenius R-algebra. Then, for any commutative (not always faithful) R-algebra S,* $S\otimes_R \Lambda$ *is a quasi-(locally) Frobenius S-algebra.*

PROOF. As it is easy, we omit it.

Now we give

THEOREM 3.3. *Let Λ be an R-algebra which is a finitely generated projective R-module. Then the following statements are equivalent:*

(1) *Λ is a left (right) quasi-Frobenius R-algebra.*

(2) *For any maximal ideal $\mathfrak{m}$ of R, $\Lambda_{\mathfrak{m}}$ is a left (right) quasi-Frobenius $R_{\mathfrak{m}}$-algebra.*

(3) *For any maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is a quasi-Frobenius $R/\mathfrak{m}$-algebra.*

PROOF. This can be proved by applying (3.1), (2) to $M=\Lambda^*$.

PROPOSITION 3.4. *Let Λ be an R-algebra which is a finitely generated projective R-module. Then Λ is a locally Frobenius R-algebra if and only if, for any maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is a Frobenius $R/\mathfrak{m}$-algebra.*

PROOF. Suppose that $\Lambda/\mathfrak{m}\Lambda$ is a Frobenius $R/\mathfrak{m}$-algebra for any maximal ideal $\mathfrak{m}$ of R. Then we have $\Lambda_{\mathfrak{m}}/\mathfrak{m}\Lambda_{\mathfrak{m}} \cong (\Lambda_{\mathfrak{m}}/\mathfrak{m}\Lambda_{\mathfrak{m}})^* \cong \Lambda_{\mathfrak{m}}^*/\mathfrak{m}\Lambda_{\mathfrak{m}}^*$ as left $\Lambda_{\mathfrak{m}}$-modules. Since $\Lambda_{\mathfrak{m}}$ and $\Lambda_{\mathfrak{m}}^*$ are $\Lambda_{\mathfrak{m}}$-finitely generated projective, we obtain $\Lambda_{\mathfrak{m}} \cong \Lambda_{\mathfrak{m}}^*$, and so $\Lambda_{\mathfrak{m}}$ is a Frobenius $R_{\mathfrak{m}}$-algebra. Then, according to (3.3), Λ is a locally Frobenius R-algebra.

PROPOSITION 3.5. *Let Λ, Γ be R-algebras which are finitely generated projective R-modules. Then $\Lambda \otimes_R \Gamma$ is a quasi-(localy) Frobenius R-algebra if and only if both Λ and Γ are quasi-(locally) Frobenius R-algebras.*

PROOF. The if part is evident and the only if part can be shown, for example, by reducing this to the classical case by (3.3) and (3.4).

PROPOSITION 3.6. *Let Λ be an R-algebra which is a finitely generated projective R-module and S a commutative R-algebra such that $\mathfrak{m}S \cap R=\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R. Then Λ is a quasi-(locally) Frobenius R-algebra if and only if $S\otimes_R \Lambda$ is a quasi-(locally) Frobenius S-algebra.*

PROOF. We have only to show the if part. Assume that $S\otimes_R \Lambda$ is a quasi-Frobenius S-algebra, and let $\mathfrak{m}$ be a maximal ideal of R. Then, by our assumption, there exists a maximal ideal $\mathfrak{M}$ in S containing $\mathfrak{m}$, such that $\mathfrak{M}\cap R =\mathfrak{m}$. Now we have $S\otimes_R \Lambda/\mathfrak{M}(S\otimes_R \Lambda) \cong S/\mathfrak{M}\otimes_R \Lambda \cong S/\mathfrak{M}\otimes_{R/\mathfrak{m}} \Lambda/\mathfrak{m}\Lambda$. Since $S/\mathfrak{M}$ is the extension field of $R/\mathfrak{m}$ and $S/\mathfrak{M}\otimes_{R/\mathfrak{m}} \Lambda/\mathfrak{m}\Lambda$ is a quasi-Frobenius $S/\mathfrak{M}$-algebra, $\Lambda/\mathfrak{m}\Lambda$ is a quasi-Frobenius $R/\mathfrak{m}$-algebra, as is well known in the classical theory. Hence, again by (3.3), Λ must be a quasi-Frobenius R-algebra.

The assumption on S in (3.6) can not be omitted.

PROPOSITION 3.7. *Let Λ be an R-algebra which is a finitely generated projective R-module. Then the following statements are equivalent:*

(1) *R is a quasi-Frobenius ring and Λ is a quasi-Frobenius R-algebra.*

(2) *Λ is a quasi-Frobenius ring.*

PROOF. (1)⇒(2) was proved in [**19**], Satz 3 in a more general form. Suppose that Λ is a quasi-Frobenius ring. Then, as Λ is left Λ-injective, it is

(Λ, R)-injective, and so Λ is a quasi-Frobenius R-algebra. Since R is an R-direct summand of Λ, R is obviously Artinian. Let M be a faithful left R-module. By (2.6), $\Lambda \otimes_R M$ is Λ-faithful, and then $\Lambda \otimes_R M$ is Λ-completely faithful (cf. [4]). Again, by (2.6), M is R-completely faithful. Therefore, according to [4], Th. 6, R is also a quasi-Frobenius ring. This implies (2)⇒(1).

§ 4. Quasi-Frobenius algebras and hereditary rings.

In this section we shall determine the structure of a quasi-Frobenius algebra which is a hereditary ring.

LEMMA 4.1. *Let R be a ring with the total quotient ring K and Λ an R-finitely generated, torsion-free R-algebra which is semi-hereditary. Let M be a finitely generated left Λ-module which is R-torsion-free. If $K \otimes_R M$ is $K \otimes_R \Lambda$-projective, then M is Λ-projective.*

PROOF. Since Λ is semi-hereditary, it suffices to prove that M is a submodule of some free left Λ-module F. As M is finitely generated over R, we can put $M = \sum_{i=1}^{l} Ru_i$, and, as M is R-torsion-free, M can be regarded as a submodule of $K \otimes_R M$. Since $K \otimes_R M$ is $K \otimes_R \Lambda$-projective, $K \otimes_R M$ is imbedded in a free $K \otimes_R \Lambda$-module F' with a free basis $v_1, v_2, \cdots, v_m$. Then we have $u_i = \sum_{j=1}^{m} \left(\frac{1}{t_{ij}} \otimes \lambda_{ij}\right) v_j$ for some non-zero divisors t_{ij} of R and some element λ_{ij} of Λ. If we put $t = \prod_{i=1}^{l} \prod_{j=1}^{m} t_{ij}$ and $w_j = \left(\frac{1}{t} \otimes 1\right) v_j$ for any j, then $F = \sum_{j=1}^{m} \Lambda w_j$ is as is required. This completes our proof.

PROPOSITION 4.2. *Let R be a ring with the total quotient ring K, and Λ an R-algebra which is a finitely generated projective R-module. If Λ is a semi-hereditary ring and $K \otimes_R \Lambda$ is a quasi-Frobenius K-algebra, then R is a semi-hereditary ring and Λ is a quasi-Frobenius R-algebra.*

PROOF. From (2.6) it follows that R is also semi-hereditary. As Λ is R-projective, Λ^* is also R-projective. By applying (4.1) to Λ^* we can show that Λ is a quasi-Frobenius R-algebra.

COROLLARY 4.3. *Let R be a Noetherian integral domain with the quotient field K. Let Σ be a semi-simple K-algebra and Λ an R-projective R-order in Σ. If Λ is a hereditary ring, then R is a Dedekind domain and Λ is a quasi-Frobenius R-algebra.*

PROOF. This is an immediate consequence of (4.2).

LEMMA 4.4. *Let R be a Noetherian complete local integral domain which is not a field, and Λ an R-finitely generated torsion-free R-algebra. If Λ is a hereditary ring, then any finitely generated Λ-module which is R-torsion-free is*

Λ-*projective.*

PROOF. See [13], (3.6).

PROPOSITION 4.5. *Let R be a Noetherian ring, and Λ an R-finitely generated projective R-algebra which is a hereditary ring. Then*

(1) *R is expressible as a direct sum of non-trivial Dedekind domains D_i, $1 \leqq i \leqq l$, with the quotient fields K_i and fields F_j, $1 \leqq j \leqq m$.*

(2) *Any $K_i \otimes_R \Lambda$ is a semi-simple K_i-algebra.*

(3) *Any $D_i \otimes_R \Lambda$ is a hereditary D_i-order in $K_i \otimes_R \Lambda$.*

(4) *Any $F_j \otimes_R \Lambda$ is a F_j-algebra which is hereditary.*

(5) *Λ is expressible as a direct sum of $D_i \otimes_R \Lambda$, $1 \leqq i \leqq l$ and $F_j \otimes_R \Lambda$, $1 \leqq j \leqq m$.*

PROOF. (1) follows from (2.6) and (3), (4), (5) follows directly from (1), (2). Hence we have only to show (2). Without loss of generality we may assume that R is a non-trivial discrete valuation ring. Let $\hat{R}$ be the completion of R and put $\hat{\Lambda} = \hat{R} \otimes_R \Lambda$. Then $\hat{R}$, $\hat{\Lambda}$ satisfy the assumptions in (4.4). As $(\hat{\Lambda})^*$ is $\hat{R}$-projective, $(\hat{\Lambda})^*$ is $\hat{\Lambda}$-projective by (4.4). Therefore, Λ^* is Λ-projective, i.e., Λ is a quasi-Frobenius R-algebra. Let K be the quotient field of R. By virtue of (3.2), $K \otimes_R \Lambda$ is also a quasi-Frobenius K-algebra. On the other hand, $K \otimes_R \Lambda$ is hereditary, as Λ is hereditary. Then $K \otimes_R \Lambda$ must be a semi-simple K-algebra by [10], Th. 16. This proves (2). Thus our proof is completed.

Our main result in this section is given in the following

THEOREM 4.6. *Let R be a Noetherian ring and Λ an R-algebra which is a finitely generated projective R-module. Then the following statements are equivalent:*

(1) *Λ is a quasi-Frobenius R-algebra which is an hereditary ring.*

(2) *R is expressible as the direct sum of Dedekind domains D_i, $1 \leqq i \leqq l$ and Λ is expressible as the direct sum of $D_i \otimes_R \Lambda$, $1 \leqq i \leqq l$, each of which is a hereditary D_i-order in a semi-simple algebra over the quotient field of D_i.*

PROOF. (1)$\Rightarrow$(2) follows from (4.5) and (2)$\Rightarrow$(1) was proved in (4.3).

In (4.6), replacing the words "a hereditary ring" and "Dedekind domains" by "a regular ring" and "regular domains", (1)$\Rightarrow$(2) can be shown similarly. However, in this case, we did not succeed in proving (2)$\Rightarrow$(1).

PROPOSITION 4.7. *Let R be a Dedekind domain and K the quotient field of R. Let Σ be a semi-simple K-algebra and Λ an R-order in Σ. If Λ is a maximal order, then it is locally Frobenius R-algebra.*

PROOF. By (3.4) we may assume that R is a discrete valuation ring. As is easily seen, a maximal order over a discrete valuation ring is a principal ideal ring (cf. [1]). Hence, for a maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is a uni-serial

$R/\mathfrak{m}$-algebra. Again by (3.4), Λ is a Frobenius R-algebra.

§5. Rings with (FG).

In this section we shall determine the structure of an algebra with (FG) over a Noetherian ring.

First we shall prove, as a special case,

THEOREM 5.1. *A commutative Noetherian ring has* (FG) *if and only if it is expressible as the direct sum of Dedekind domains and quasi-Frobenius local rings.*

It is well known that any finitely generated module over a Dedekind domain can be expressed as the direct sum of a projective module and a torsion module. From this fact and [4] the if part of (5.1) follows immediately.

Now we shall prove the only if part of (5.1), step by step, in the following lemmas and proposition.

LEMMA 5.2. *Let R be a commutative Noetherian ring, and suppose that there is a prime divisor of* 0 *with height* >1 *in R. Then R has not* (FG),

PROOF. Denote by K the total quotient ring of R and let $\mathfrak{p}$ be a maximal prime divisor of 0 such that $\mathrm{ht}_R\,\mathfrak{p}>1$. Then $\mathfrak{p}K$ is a maximal ideal of K and we have $R_\mathfrak{p}=K_{\mathfrak{p}K}$. As $\mathrm{Ann}_{R_\mathfrak{p}}\mathfrak{p}R_\mathfrak{p}\neq 0$, there is a $a\in \mathrm{Ann}_R\,\mathfrak{p}$ such that $\bar{0}\neq\bar{a}$ in $R_\mathfrak{p}$ where $\bar{0}$, $\bar{a}$ are the residues of 0, a in $R_\mathfrak{p}$. It is easily seen that Ka is a minimal ideal in K. Since $\bigcap_{i=1}^{\infty}\mathfrak{p}^iK_{\mathfrak{p}K}=0$ and $\mathrm{ht}_K\,\mathfrak{p}K\geqq 1$, we find some $b\in\mathfrak{p}$ such that $\bar{a}\notin\bar{b}K_{\mathfrak{p}K}$, $\bar{b}\neq\bar{0}$, where $\bar{b}$ is the residue of b in $R_\mathfrak{p}$. From the minimality of Ka we obtain $Ka\cap Kb=0$. Now let $F=Ru+Rv$ be a free R-module with a free basis $\{u, v\}$, and put $M=F/R(au-bv)$. As $Ra\cap Rb=0$, M is clearly a finitely generated faithful R-module. We shall show that M is not R-completely faithful. In order to prove this, it suffices, by (2.4), to show that M is not R-completely faithful. As $R_\mathfrak{p}=K_{\mathfrak{p}K}$, we may assume $R=K$. Let $\bar{u}$, $\bar{v}$ be the residues of u, v and u^*, v^* the residues of $\bar{u}$, $\bar{v}$ in M, respectively. Then we have $M_\mathfrak{p}=F_\mathfrak{p}/R_\mathfrak{p}(\overline{au}-\overline{bv})=R_\mathfrak{p}u^*+R_\mathfrak{p}v^*$ and $\bar{a}u^*=\bar{b}v^*$. Let φ be an element of $\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p})$, and put $\varphi(u^*)=\bar{c}$, $\varphi(v^*)=\bar{d}$, $\bar{c}$, $\bar{d}\in R_\mathfrak{p}$. Since $\bar{a}u^*=\bar{b}v^*$, we have $\bar{a}\bar{c}=\bar{b}\bar{d}$. If $\bar{c}\in\mathfrak{p}R_\mathfrak{p}$. then $\bar{a}\in\bar{b}R_\mathfrak{p}$ which is obviously a contradiction, and so we have $\bar{c}\in\mathfrak{p}R_\mathfrak{p}$. From the fact that $\bar{a}\in\mathrm{Ann}_R\mathfrak{p}R_\mathfrak{p}$, we obtain $\bar{a}\bar{c}=\bar{b}\bar{d}=\bar{0}$ and, then, as $\bar{b}\neq\bar{0}$, we have $\bar{d}\in\mathfrak{p}R_\mathfrak{p}$. Therefore we have $\mathrm{Im}\,\varphi\subseteqq\mathfrak{p}R_\mathfrak{p}$. Thus $M_\mathfrak{p}$ is not $R_\mathfrak{p}$-completely faithful. This proves that R has not (FG).

LEMMA 5.3. *Let R be a commutative Noetherian ring. Suppose that the zero ideal of R is unmixed and that there is a prime ideal $\mathfrak{p}$ of R with* $\mathrm{ht}_R\,\mathfrak{p}\geqq 1$ *such that $R_\mathfrak{p}$ is not a discrete valuation ring. Then R has not* (FG).

PROOF. By our assumptions, $\mathfrak{p}$ is a finitely generated faithful R-module.

Now we shall show that $\mathfrak{p}$ is not R-completely faithful. In order to prove this, it suffices to show that $\mathfrak{p}R_{\mathfrak{p}}$ is not $R_{\mathfrak{p}}$-completely faithful. Hence we may suppose that R is a local ring with a maximal ideal $\mathfrak{p}$. Since R is not a discrete valuation ring, a minimal basis of $\mathfrak{p}$ contains at least two elements, and we denote it by $\{p_1, p_2, \cdots, p_t\}$, $t \geqq 2$. Let φ be an element of $\mathrm{Hom}_R(\mathfrak{p}, R)$ and put $\varphi(p_i) = a_i \in R$ for any i. If, for some i_0, a_{i_0} is a unit of R, then we have $p_i = a_{i_0}^{-1} a_i p_{i_0} \in R p_{i_0}$ for any i. This is obviously a contradiction. Thus we must have $\mathrm{Im}\,\varphi \subseteqq \mathfrak{p}$, which shows, by (1.1), that $\mathfrak{p}$ is not R-completely faithful. This proves that R has not (FG).

From lemmas (5.2) and (5.3) it follows directly that a ring R with (FG) is expressible as the direct sum of Dedekind domains and Artinian local rings. Therefore the proof of the only if part of (5.1) is completed if the following proposition is proved.

PROPOSITION 5.4. *A commutative Artinian ring R has* (FG) *if and only if it is a quasi-Frobenius ring.*

PROOF. The if part of this proposition is obvious ([4]). Hence we have only to prove the only if part. Since a commutative Artinian ring can be expressed as the direct sum of local rings, we may assume that R is a local ring with a maximal ideal $\mathfrak{m}$. Suppose that R is not quasi-Frobenius. Then $\mathrm{Ann}_R \mathfrak{m}$ is not a principal ideal of R, and therefore there exist some $a, b \in \mathrm{Ann}_R \mathfrak{m}$ such that $a \notin Rb$ and $b \notin Ra$. As Ra and Rb are minimal ideals of R, we have $Ra \cap Rb = 0$. Let $F = Ru + Rv$ be a free R-module with a free basis u, v and put $M = F/R(au - bv)$. By using the same method as in the proof of (5.2), we can show that M is R-faithful but not R-completely faithful. Hence R has not (FG). This proves the only if part of our proposition.

A ring Λ is said to have $(\mathrm{FG})^l$ ($(\mathrm{FG})^r$) if any finitely generated, faithful left (right) Λ-module is completely faithful.

Now we can conjecture that a left and right Artinian ring with $(\mathrm{FG})^l$ ($(\mathrm{FG})^r$) is a quasi-Frobenius ring. However we did not succeed in proving this in the general case. We remark that the only if part of (5.4) can also be proved by applying [17], (3.11) and (4.1).

Secondly we shall concern with orders in a semi-simple algebra. The essential part of the proof of the following theorem was shown in [1] and [13], though this is not given in [1] and [13] explicitely.

THEOREM 5.5. *Let R be a Noetherian integral domain and K the quotient field of R. Let Σ be a semi-simple K-algebra and Λ an R-projective R-order in Σ. Then Λ is a hereditary, maximal R-order if and only if it has* $(\mathrm{FG})^l$ ($(\mathrm{FG})^r$).

PROOF. The only if part: Suppose that Λ is a hereditary, maximal R-order. Then, by [13], (2.2), the center of Λ is a hereditary ring which contains all central idempotents in Σ. Therefore we may assume that R is a Dedekind domain and that Σ is a central simple K-algebra. By (2.2), (2.4) and [**13**], (1.5), we may further assume that R is a discrete valuation ring with a maximal ideal $\mathfrak{m}$. As Λ is a maximal R-order, the Jacobson radical of Λ is a unique maximal two-sided ideal according to [**1**], (2.1). Hence, by (1.3), any non-zero projective left Λ-module is completely faithful. Since Λ is hereditary and Σ is simple, we see easily that any finitely generated, faithful left Λ-module is a direct sum of a non-zero projective left Λ-module and a left Λ-module which is a torsion R-module. Consequently any finitely generated faithful left Λ-module is completely faithful, and so Λ has $(\mathrm{FG})^l$. This proves the only if part of our theorem.

The if part: Suppose that Λ has $(\mathrm{FG})^l$. Then, by (2.6), R has also (FG), and so R is a Dedekind domain by (5.1). Further assume that Λ is not a hereditary, maximal order. Since a maximal order over a Dedekind domain is hereditary by [**1**], (2.3) and [**13**], (2.1), Λ is not maximal under our assumption, and therefore there exists an R-order Γ which contains Λ strictly. Let $C(\Gamma)$ be the conductor of Γ with respect to Λ. Then $C(\Gamma)$ is a two-sided ideal of Λ which is a faithful left Λ-module, and, by [**13**], (1.6), we have $\mathfrak{T}_\Lambda(C(\Gamma)) = C(\Gamma)$. As $C(\Gamma) \neq \Lambda$, $C(\Gamma)$ is not Λ-completely faithful. This contradicts the assumption that Λ has $(\mathrm{FG})^l$. Thus Λ must be a hereditary, maximal R-order. This completes our proof.

Finally, combining (5.5) with (5.1), we obtain

THEOREM 5.6. *Let R be a Noetherian ring and Λ an R-algebra which is a finitely generated projective R-module. Then Λ has* $(\mathrm{FG})^l$ $((\mathrm{FG})^r)$ *if and only if it is the direct sum of a finite number of hereditary, maximal orders over Dedekind domains in semi-simple algebras and quasi-Frobenius rings which are finitely generated modules over commutative quasi-Frobenius local rings.*

PROOF. The if part of our theorem follows from (5.5). Therefore we have only to show the only if part. Suppose that Λ has $(\mathrm{FG})^l$. Since Λ is a finitely generated projective R-module, R has (FG) by (2.6). According to (5.1) we can put $R = D_1 \oplus D_2 \oplus \cdots \oplus D_s \oplus E_1 \oplus E_2 \oplus \cdots \oplus E_t$ where any D_i is a non-trivial Dedekind domain and any E_j is a quasi-Frobenius local ring. If we put $\Lambda_i = D_i \underset{R}{\otimes} \Lambda$ for any i and $\Gamma_j = E_j \underset{R}{\otimes} \Lambda$ for any j, then any $\Lambda_i(\Gamma_j)$ is a $D_i(E_j)$-algebra which is a finitely generated projective $D_i(E_j)$-module. Since $\Lambda_i^*(\Gamma_j^*)$ is $\Lambda_i(\Gamma_j)$-finitely generated faithful, it is $\Lambda_i(\Gamma_j)$-completely faithful, and so $\Lambda_i(\Gamma_j)$ is a quasi-Frobenius $D_i(E_j)$-algebra. From (3.7) it follows directly that any Γ_j is a quasi-Frobenius ring. Hence we have only to show that any Λ_i is a hereditary, maximal D_i-order in a semi-simple algebra. In order to simplify our notation, we put $D = D_i$ and $\Lambda = \Lambda_i$ and denote by K the quotient field of D. By (5.5) it suffices to show that $K\underset{D}{\otimes}\Lambda$ is a semi-simple K-algebra. If we assume that $K\underset{D}{\otimes}\Lambda$ is not semi-simple, then the nil radical $\mathfrak{N}$ of Λ is not 0. Without loss of generality we may assume that D is a discrete valuation ring with a maximal ideal Dp ($p\neq 0$). Since we have $\mathfrak{N}\neq 0$ and $\bigcap_{l=1}^{\infty}\Lambda_p^l=0$, there exists an integer l_0 such that $\mathfrak{N}\subsetneqq\Lambda_{p^{l_0}}$. Now put $\mathfrak{A}=\Lambda_{p^{l_0}}+\mathfrak{N}$. Then $\mathfrak{A}$ is a two-sided ideal of Λ which is a faithful left Λ-module, and so, as Λ has $(\mathrm{FG})^l$, $\mathfrak{A}$ is also Λ-completely faithful. Therefore there exist some $\varphi_1, \varphi_2, \cdots, \varphi_m \in \mathrm{Hom}_\Lambda(\mathfrak{A}, \Lambda)$, some $u_1, u_2, \cdots, u_m\in\mathfrak{N}$ and some $\lambda_1, \lambda_2, \cdots, \lambda_m\in\Lambda$ such that $\sum_{i=1}^{m}\varphi_i(\lambda_i p^{l_0}+u_i)=\sum_{i=1}^{m}\varphi_i(\lambda_i p^{l_0})+\sum_{i=1}^{m}\varphi_i(u_i)=1$. As Λ is a quasi-Frobenius D-algebra, $K\underset{K}{\otimes}\Lambda$ is also a quasi-Frobenius K-algebra (in the classical sense) by (3.2), and so it is left self-injective. Then we have $\mathfrak{T}_{K\underset{D}{\otimes}\Lambda}(K\underset{D}{\otimes}\mathfrak{N})=K\underset{D}{\otimes}\mathfrak{N}$. From this we obtain $\mathfrak{T}_\Lambda(\mathfrak{N})=\mathfrak{N}$, as $\mathfrak{N}$ is a nil ideal of Λ. Hence we have $\varphi_i(\mathfrak{N})\subseteqq\mathfrak{N}$ for any i, i.e., $\sum_{i=1}^{m}\varphi_i(u_i)\in\mathfrak{N}$. Therefore $\sum_{i=1}^{m}\varphi_i(\lambda_i p^{l_0})$ is a unit of Λ. Let u be an element of $\mathfrak{N}$ which is not contained in $\Lambda_{p^{l_0}}$. However we have $\varphi_i(u\lambda_i p^{l_0})=u\varphi_i(\lambda_i p^{l_0})$ $=p^{l_0}\varphi_i(u\lambda_i)\in\Lambda p^{l_0}$ for any i, and so $u\sum_{i=1}^{m}\varphi_i(\lambda_i p^{l_0})\in\Lambda p^{l_0}$. Hence $u\in\Lambda p^{l_0}$, which is obviously a contradiction. Thus $K\underset{D}{\otimes}\Lambda$ must be a semi-simple K-algebra. This completes our proof.

§ 6. Rings with (PFG).

A fairly general sufficient condition for a ring to have (PFG) is given in the following

THEOREM 6.1. *Let Λ be a ring and C the center of Λ. Suppose that there exists a subring R of C satisfying the following two conditions:*

(1) *Λ is a finitely generated R-module.*

(2) *For any maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is the direct sum of primary rings.*

Then Λ has (PFG).

PROOF. Let P be a finitely generated, faithful, projective left (right) Λ-module. By (2.5) it suffices to prove that, for any maximal ideal $\mathfrak{m}$ of R, $P/\mathfrak{m}P$ is $\Lambda/\mathfrak{m}\Lambda$-completely faithful. Put $\bar{\Lambda}=\Lambda/\mathfrak{m}\Lambda$, $P=P/\mathfrak{m}P$ and $\bar{\mathfrak{c}}=\mathrm{Ann}_{\bar{\Lambda}}\bar{P}$. By virtue of (2.3), $\bar{\mathfrak{c}}$ is contained in the Jacobson radical of $\bar{\Lambda}$. By our assumption we have $\Lambda=\bar{\Lambda}_1\oplus\bar{\Lambda}_2\oplus\cdots\oplus\bar{\Lambda}_l$ where any $\bar{\Lambda}_i$ is a primary ring. If we put $\bar{P}_i=\bar{\Lambda}_i\bar{P}$ and $\bar{\mathfrak{c}}_i=\bar{\Lambda}_i\bar{\mathfrak{c}}$ for $1\leqq i\leqq l$, then we have $\bar{P}=\bar{P}_1\oplus\bar{P}_2\oplus\cdots\oplus\bar{P}_l$ as $\bar{\Lambda}$-modules and $\bar{\mathfrak{c}}=\bar{\mathfrak{c}}_1\oplus\bar{\mathfrak{c}}_2\oplus\cdots\oplus\bar{\mathfrak{c}}_l$ as two-sided ideals of Λ. Here any $\bar{\mathfrak{c}}_i$ is contained in the Jacobson radical of $\bar{\Lambda}_i$ which is a unique maximal two-sided ideal of $\bar{\Lambda}_i$, and we have also $\mathfrak{c}_i = \mathrm{Ann}_{\bar{\Lambda}_i}\bar{P}_i$ for any i. Therefore any $\bar{P}_i$ is a non-zero projective $\bar{\Lambda}_i$-module. Hence any $\bar{P}_i$ is $\bar{\Lambda}_i$-completely faithful by (1.3). From this we see easily that $\bar{P}$ is $\bar{\Lambda}$-completely faithful. This completes our proof.

The assumption in (6.1) is satisfied in each of the following cases:

(1) Λ is a commutative ring (cf. (1.5)).

(2) Λ is a semi-simple R-algebra which is a finitely generated R-module (cf. [14]).

(3) Λ is a hereditary, maximal order over a Dedekind domain R in a semi-simple algebra (5.5).

(4) Λ is an R-algebra which is a finitely generated R-module such that, for any maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is a uni-serial ring. (1), (2), (3) are special cases of (4).

From (1.2) we obtain directly

PROPOSITION 6.2. *Let Λ be a ring and let P be a finitely generated, faithful, projective left Λ-module which is not Λ-completely faithful. Then $\mathfrak{T}_\Lambda(P)$ is a proper idempotent two-sided ideal of Λ which is Λ-faithful. Especially, if Λ is left Noetherian or if Λ is a finitely generated module over the center C of it, then $\mathfrak{T}_\Lambda(P)$ is a finitely generated left Λ-module.*

PROOF. We have only to show the final assertion, under the assumption that Λ is C-finitely generated. Since P and $\mathrm{Hom}_\Lambda(P, \Lambda)$ are Λ-finitely generated, they are also C-finitely generated, and so $\mathfrak{T}_\Lambda(P)$ is C-finitely generated. Hence $\mathfrak{T}_\Lambda(P)$ is Λ-finitely generated and this completes our proof.

COROLLARY 6.3. *A left Noetherian, left hereditary ring Λ has* (PFG) *if and only if there is no proper idempotent two-sided ideal in Λ which is Λ-faithful.*

PROOF. This can easily be derived from (1.4) and (6.2).

COROLLARY 6.4. *Let R be a Dedekind domain and K the quotient field of R. Let Σ be a semi-simple K-algebra and Λ a hereditary R-order in Σ. Then Λ has* (PFG) *if and only if it is a maximal R-order.*

It was shown in (4.3) that a hereditary order over a Dedekind domain R in a semi-simple algebra is a quasi-Frobenius R-algebra. On the other hand, by (6.4), a non-maximal hereditary R-order has not (PFG). Hence we have

PROPOSITION 6.5. *A quasi-Frobenius algebra over a Dedekind domain has not always* (PFG).

§7. Quasi-Frobenius subalgebras of a central separable algebra.

Let Γ be an R-algebra and Λ an R-subalgebra of Γ. Now we put $V_\Gamma(\Lambda) = \{\gamma \in \Gamma \mid \gamma\lambda = \lambda\gamma \text{ for any } \lambda \in \Lambda\}$.

A. Hattori proved in [14] the following basic results: Let Γ be a central separable R-algebra and Λ an R-subalgebra of Γ.

(I) If Γ is Λ-completely faithful, then there hold

(i) Γ is $\Lambda \underset{R}{\otimes} \Gamma^0$-completely faithful.

(ii) Γ is $V_\Gamma(\Lambda)$-projective and $\Lambda \underset{R}{\otimes} \Gamma^0 \cong \mathrm{Hom}_{V_\Gamma(\Lambda)}(\Gamma, \Gamma)$.

(iii) $V_\Gamma(V_\Gamma(\Lambda)) = \Lambda$.

(II) If Γ is Λ-projective and Λ-completely faithful, then Γ is $\Lambda \underset{R}{\otimes} \Gamma^0$-projective and $\Lambda \underset{R}{\otimes} \Gamma^0$-completely faithful, and so $V_\Gamma(\Lambda)^0$ is Morita equivalent to $\Lambda \underset{R}{\otimes} \Gamma^0$.

In this section, as an application of the above mentioned results, we shall establish the commutor theory of quasi-Frobenius subalgebras of a central separable algebra.

First we give

PROPOSITION 7.1. *Let Λ, Ω be R-algebras which are finitely generated projective R-modules. Suppose that Ω is Morita equivalent to Λ. Then Ω is a quasi-Frobenius R-algebra if and only if Λ is a quasi-Frobenius R-algebra.*

PROOF. By (3.3) it suffices to prove our proposition in case R is a field. However, as the property (G) is obviously Morita invariant, our proposition holds in this case.

Now we have

THEOREM 7.2. *Let Γ be a central separable R-algebra and Λ a quasi-Frobenius R-subalgebra of Γ which is an R-direct summand of Γ.*

(1) *We have $V_\Gamma(V_\Gamma(\Lambda)) = \Lambda$.*

(2) *$V_\Gamma(\Lambda)$ is a quasi-Frobenius R-subalgebra of Γ which is an R-direct summand of Γ if and only if Γ is Λ-projective.*

PROOF. Since Λ is (Λ, R)-injective, Λ is an R-direct summand of Γ if and only if Γ is Λ-completely faithful. From (I) it follows $V_\Gamma(V_\Gamma(\Lambda)) = \Lambda$. Now assume that Γ is Λ-projective. Then, according to (II), $V_\Gamma(\Lambda)^0$ is Morita equivalent to $\Lambda \underset{R}{\otimes} \Gamma^0$. As $\Lambda \underset{R}{\otimes} \Gamma^0$ is quasi-Frobenius by (3.5), $V_\Gamma(\Lambda)$ is quasi-Frobenius by (3.3) and (7.1). Since Γ is Λ-projective, Γ is also $\Lambda \underset{R}{\otimes} \Gamma^0$-projective, and so, by virtue of the Morita theorem, Γ is $V_\Gamma(\Lambda)$-completely faithful. Consequently $V_\Gamma(\Lambda)$ is an R-direct summand of Γ. Conversely, suppose that $V_\Gamma(\Lambda)$ is a quasi-Frobenius R-subalgebra which is an R-direct summand of Γ. Then Γ is $V_\Gamma(\Lambda)$-projective and $V_\Gamma(\Lambda)$-completely faithful, and therefore $V_\Gamma(\Lambda)^0$ is Morita equivalent to $\Lambda \underset{R}{\otimes} \Gamma^0 \cong \mathrm{Hom}_{V_\Gamma(\Lambda)}(\Gamma, \Gamma)$. Again, by the Morita theorem, Γ is $\Lambda \underset{R}{\otimes} \Gamma^0$-projective, and so Γ is Λ-projective. This completes our proof.

For a commutative ring R, we put f. gl. dim $R = \mathrm{Sup}\,\{\mathrm{dh}_R M \mid M$ is a finitely generated R-module with $\mathrm{dh}_R M < \infty\}$.

COROLLARY 7.3. *Let R be a commutative ring with* $\mathrm{f.gl.dim}\, R = 0$. *Let Γ be a central separable R-algebra and Λ a quasi-Frobenius R-subalgebra of Γ.*

(1) *We have $V_\Gamma(V_\Gamma(\Lambda)) = \Lambda$.*

(2) *$V_\Gamma(\Lambda)$ is a quasi-Frobenius R-subalgebra of Γ if and only if Γ is Λ-projective.*

PROOF. As Λ and Γ are R-projective, we have $\mathrm{dh}_R \Gamma/\Lambda \leqq 1$. By our assumption on R, Γ/Λ is R-projective, and therefore Λ is an R-direct summand of Γ. Thus our corollary is an immediate consequence of (7.2).

The classical case is obviously included in (7.3). However it seems that (2) has not been given in the classical theory.

COROLLARY 7.4. *Let Γ be a central separable R-algebra and Λ a quasi-Frobenius R-subalgebra with* (PFG) *of Γ which is an R-direct summand of Γ If Γ is Λ-projective, then $V_\Gamma(\Lambda)$ is a quasi-Frobenius R-subalgebra with* (PFG).

PROOF. The property (PFG) is evidently Morita invariant. Hence it suffices to prove that $\Lambda \otimes_R \Gamma^0$ has (PFG). Let P be a finitely generated, faithful, projective $\Lambda \otimes_R \Gamma^0$-module. Then P is obviously Λ-finitely generated, faithful, projective. As Λ has (PFG), P is Λ-completely faithful. Since P is $\Lambda \otimes_R \Gamma^0$-projective, it suffices, according to (1.2), to show that, for any maximal two-sided ideal $\mathfrak{M}'$ of $\Lambda \otimes_R \Gamma^0$, we have $P/\mathfrak{M}'P \neq 0$. Now we can find a maximal two-sided ideal $\mathfrak{M}$ of Λ such that $\mathfrak{M}' = \mathfrak{M} \otimes_R \Gamma^0$, as Γ^0 is a central separable R-algebra. However, since P is Λ-completely faithful, we have $P/\mathfrak{M}'P = P/\mathfrak{M}P \neq 0$. Consequently P is $\Lambda \otimes_R \Gamma^0$-completely faithful.

LEMMA 7.5. *Let R be a commutative ring with the total quotient ring K and Γ an R-algebra which is a torsion-free R-module. Let Λ be an R-subalgebra of Γ such that $V_{K \otimes_R \Gamma}(V_{K \otimes_R \Gamma}(K \otimes_R \Lambda)) = K \otimes_R \Lambda$. Then we have $V_\Gamma(V_\Gamma(\Lambda)) = \Lambda$ if and only if $K \otimes_R \Lambda \cap \Gamma = \Lambda$.*

PROOF. As it is easy, we omit it.

PROPOSITION 7.6. *Let R be a commutative ring with the total quotient ring K such that* $\mathrm{f.gl.dim}\, R \leqq 1$ *and* $\mathrm{f.gl.dim}\, K = 0$. *Let Γ be a central separable R-algebra and Λ a quasi-Frobenius R-subalgebra of Γ. Then the following statements are equivalent:*

(1) *We have $V_\Gamma(V_\Gamma(\Lambda)) = \Lambda$.*

(2) *Γ is Λ-completely faithful.*

(3) *Λ is an R-direct summand of Γ.*

If Λ satisfies these conditions, $V_\Gamma(\Lambda)$ is a quasi-Frobenius R-subalgebra of Γ when and only when Γ is Λ-projective.

PROOF. For the first part, it suffices to show (1)$\Rightarrow$(3). Assume $V_\Gamma(V_\Gamma(\Lambda)) = \Lambda$. Then we have $V_{K\otimes_R\Gamma}(V_{K\otimes_R\Gamma}(K\otimes_R\Lambda)) = K\otimes_R\Lambda$, and therefcre, by (7.5), we obtain $K\otimes_R\Lambda\cap\Gamma=\Lambda$. Hence Γ/Λ is R-torsion-free. Since both $K\otimes_R\Lambda$ and $K\otimes_R\Gamma$ is K-projective and f. gl. dim $K=0$, $K\otimes_R\Gamma/\Lambda$ is K-projective. So Γ/Λ is an R-submodule of a finitely generated free R-module. While, Λ and Γ are R-projective, and so we have $\mathrm{dh}_R\,\Gamma/\Lambda\leqq 1$. As f. gl. dim $R\leqq 1$, this shows that Γ/Λ is R-projective. Thus Λ must be an R-direct summand of Γ. The second part of our proposition follows from (7.2), as we have $V_\Gamma(V_\Gamma(V_\Gamma(\Lambda)))=V_\Gamma(\Lambda)$.

It is to be noted that there exists a quasi-Frobenius R-subalgebra Λ of a central separable R-algebra Γ which is not an R-direct summand of Γ.

THEOREM 7.7. *Let R be a Dedekind domain which is not a field and Γ a central separable R-algebra. Let Λ be an R-subalgebra of Γ which is a hereditary ring.*

(1) *We have $V_\Gamma(V_\Gamma(\Lambda))=\Lambda$ if and only if Γ is Λ-completely faithful.*

(2) *$V_\Gamma(\Lambda)$ is an R-subalgebra of Γ which is a hereditary ring.*

PROOF. According to (4.5), Λ is a quasi-Frobenius R-algebra and $K\otimes_R\Lambda$ is a semi-simple K-algebra, where K denotes the quotient field of R. Therefore, by (7.6), we have (1). If we put $\Omega=K\otimes_R\Lambda\cap\Gamma$, then we have $V_\Gamma(\Lambda)=V_\Gamma(\Omega)$ and $V_\Gamma(V_\Gamma(\Omega))=\Omega$. Since Λ is hereditary and $\Omega\supseteqq\Lambda$, Ω is also hereditary by [13], (1, 4). Hence, in order to prove (2), we may assume that Γ is Λ-completely faithful. As $K\otimes_R\Lambda$ is semi-simple, $K\otimes_R\Gamma$ is $K\otimes_R\Lambda$-projective, and so Γ can be regarded as a Λ-submodule of a finitely generated free Λ-module. Since Λ is hereditary, this shows that Γ is Λ-projective. Consequently $V_\Gamma(\Lambda)$ is Morita equivalent to $\Lambda\otimes_R\Gamma^0$. By [2], (1, 8) we have gl. dim $\Lambda\otimes_R\Gamma^0\leqq$gl. dim Λ, and so $\Lambda\otimes_R\Gamma^0$ is hereditary. Hence $V_\Gamma(\Lambda)$ is also hereditary.

COROLLARY 7.8. *Let R, Γ be as in Theorem 7.7 and Λ an R-subalgebra with* (PFG) *of Γ which is hereditary. Then Λ is a maximal R-order in a semi-simple K-algebra $K\otimes_R\Lambda$, and $V_\Gamma(\Lambda)$ is a maximal R-order in $V_{K\otimes_R\Gamma}(K\otimes_R\Lambda)$.*

PROOF. This follows directly from (6.4), (7.4) and (7.7).

§ 8. A remark on semi-simple algebras over Dedekind domains.

Finally we give, as a supplement to Hattori's paper [14], the following

THEOREM 8.1. *Let R be a Dedekind domain and Λ an R-algebra which is a finitely generated projective R-module. Then Λ is a semi-simple R-algebra if and only if, for any maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is a semi-simple $R/\mathfrak{m}$-algebra.*

This is also an affirmative answer to Problem 7 in [15] in a special case. The only if part of our theorem was proved in [14], (2.7). Hence we have only to show the if part. Before proving this we shall give some lemmas.

Let Λ, Γ be rings with the common unit element such that $\Lambda \subset \Gamma$ and assume that Γ is a finitely generated left (right) Λ-module. Then Γ is said to be a left (right) semi-simple extension of Λ if any finitely generated left (right) Γ-module is (Γ, Λ)-projective. This is a slight generalization of the notion of semi-simple algebras in [14].

LEMMA 8.2.[1] *Let Λ, Γ, Ω be rings with the common unit element such that $\Lambda \subset \Gamma \subset \Omega$.*

(1) *If Ω is a left (right) semi-simple extension of Λ, then Ω is also a left (right) semi-simple extension of Γ.*

(2) *If Γ is a left (right) semi-simple extension of Λ, and Ω is left (right) semi-simple extension of Γ, then Ω is also a left (right) semi-simple extension of Λ.*

PROOF. For any left Ω-module M, we define a Γ-homomorphism $\Phi^{\Gamma}_{\Lambda}(M)$: $\Gamma \otimes_{\Lambda} M \to M$ by putting $\Phi^{\Gamma}_{\Lambda}(M)(\gamma \otimes u) = \gamma u$, $\gamma \in \Gamma$, $u \in M$, and similarly, we define Ω-homomorphisms $\Phi^{\Omega}_{\Lambda}(M)$ and $\Phi^{\Omega}_{\Gamma}(M)$. Then we have the following commutative diagram as left Ω-modules:

$$\begin{array}{ccc} \Omega \otimes_{\Gamma} \Gamma \otimes_{\Lambda} M & \cong & \Omega \otimes_{\Lambda} M \\ {\scriptstyle I_{\Omega} \otimes_{\Gamma} \Phi^{\Gamma}_{\Lambda}(M)} \downarrow & & \downarrow {\scriptstyle \Phi^{\Omega}_{\Lambda}(M)} \\ \Omega \otimes_{\Gamma} M & \xrightarrow[\Phi^{\Omega}_{\Gamma}(M)]{} & M \end{array}$$

From this we can easily obtain our lemma.

LEMMA 8.3. *The full matrix algebra $M_n(\Lambda)$ of degree n over a ring Λ is a left and right semi-simple extension of Λ.*

PROOF. By using the same method as in the proof of [9], IX, (7.3), we can show that $M_n(\Lambda)$ is $M_n(\Lambda) \otimes_{\Lambda} M_n(\Lambda)^0$-projective. Then we can prove, along the same line as in the proof of [14], (2.3), that $M_n(\Lambda)$ is a left and right semi-simple extension of Λ.

LEMMA 8.4. *Let Λ be an R-algebra which is a finitely generated projective R-module and C be a subring of the center of Λ such that $R \subseteqq C$. If Λ is C-projective and, for any maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is a semi-simple $R/\mathfrak{m}$-algebra, then C is R-projective and, for any maximal ideal $\mathfrak{m}$ of R, $C/\mathfrak{m}C$ is a semi-simple $R/\mathfrak{m}$-algebra.*

PROOF. Since Λ is a finitely generated projective C-module, C is a C-direct summand of Λ. As Λ is R-projective, C is also R-projective and we have

1) K. Hirata and K. Sugano gave also this in the following paper: On semi-simple extensions and separable extensions over non-commutative rings, J. Math. Soc. Japan, 18 (1966), 360–373.

$\mathfrak{m}\Lambda \cap C = \mathfrak{m}C$ for any maximal ideal $\mathfrak{m}$ of R. Therefore $C/\mathfrak{m}C$ can be considered as a subring of the center of a semi-simple ring $\Lambda/\mathfrak{m}\Lambda$, and so $C/\mathfrak{m}C$ is also semi-simple.

LEMMA 8.5. *Let Λ be an R-algebra and $\mathfrak{a}$ an ideal of R. Then $\Lambda/\mathfrak{a}\Lambda$ is (Λ, R)-projective as a left (right) Λ-module.*

PROOF. Since $\Lambda/\mathfrak{a}\Lambda \cong \Lambda \otimes_R R/\mathfrak{a}$, this is obvious.

The proof of the if part of (8.1). By [14], (2,14), we may assume that R is a discrete valuation ring with a maximal ideal pR. Suppose that $\Lambda/p\Lambda$ is a semi-simple R/pR-algebra. Then Λ is a quasi-Frobenius R-algebra by (3.3), and it is a hereditary ring by [13], (3.6), as $p\Lambda$ is the Jacobson radical of Λ. Let K be the quotient field of R and put $\Sigma = K \otimes_R \Lambda$. By (4.5), (6.1) and (6.2), Σ is a semi-simple K-algebra and Λ is a hereditary maximal R-order in Σ. By virtue of [13], (2.2) and (2.3), the center C of Λ is a direct sum of Dedekind domains $D_1, D_2, \cdots, D_l$, and, denoting by F_i the quotient field of D_i and putting $\Lambda_i = D_i \otimes_R \Lambda$, $\Sigma_i = F_i \otimes_R \Lambda$, we have $\Lambda = \sum_{i=1}^{l} \oplus \Lambda_i$, $\Sigma = \sum_{i=1}^{l} \oplus \Sigma_i$ and any Λ_i is an hereditary maximal D_i-order in a central simple F_i-algebra Σ_i. As Λ is C-projective, C/pC is a semi-simple R/pR-algebra by (8.4), and then C is a semi-simple R-algebra by [14], (4.4). Therefore it suffices, by (8.2), to prove that Λ is a semi-simple C-algebra, i. e., that any Λ_i is a semi-simple D_i-algebra. Since any Λ_i is D_i-projective and any $\Lambda_i/p\Lambda_i$ is a semi-simple D_i/pD_i-algebra, it suffices to prove, under the assumption that Σ is a central simple K-algebra, that Λ is a semi-simple R-algebra.

Now, by [1], (3.8), Λ is a full matrix algebra over a maximal R-order Ω in a central division K-algebra. Then Ω is obviously an R-algebra with the Jacobson radical $p\Omega$. Let $\hat{R}$ be the completion of R and put $\hat{\Lambda} = \hat{R} \otimes_R \Lambda$. Then we have $\hat{R}/p\hat{R} \cong R/pR$ and $\hat{\Lambda}/p\hat{\Lambda} \cong \Lambda/p\Lambda$, and Λ is a semi-simple R-algebra if and only if $\hat{\Lambda}$ is a semi-simple $\hat{R}$-algebra. Therefore we may further assume that R is complete. By [1], (3.11) and its corollary, then, any left (right) ideal of Ω coincides with $p^k\Omega$ for some integer $k \geqq 0$. Since Ω is a principal ideal ring (cf. [1], Corollary to (3.3)), $\Omega/p^k\Omega$ is a uniserial ring for $k > 0$. Then we can easily show that any finitely generated left (right) Ω-module is expressible as the direct sum of cyclic Ω-modules, each of which is isomorphic to $\Omega/p^k\Omega$ for some $k \geqq 0$. Therefore Ω is a semi-simple R-algebra by (8.5). On the other hand, Λ is a left and right semi-simple extension of Ω according to (8.3). Finally, by applying (8.2) to R, Ω, Λ, we can show that Λ is a semi-simple R-algebra. This completes our proof.

Tokyo University of Education

References

[1] M. Auslander and O. Goldman, Maximal orders, Trans. Amer. Math. Soc., **97** (1960), 1-24.

[2] M. Auslander and O. Goldman, The Brauer group of a commutative ring, Trans. Amer. Math. Soc., **97** (1960), 367-409.

[3] G. Azumaya and T. Nakayama, Algebra II, Iwanami, Tokyo, 1954 (in Japanese).

[4] G. Azumaya, Completely faithful modules and self-injective rings, Nagoya Math. J., **27** (1966), 697-708.

[5] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc., **95** (1960), 466-488.

[6] H. Bass, The Morita theorems, Lecture note at Univ. of Oregon, 1962 (mineographed note).

[7] H. Bass, K-theory and stable algebras, Publ. de L'Institut des Haute Etudes Sci., 1964, N 22, 489-544.

[8] H. Cartan and S. Eilenberg, Homological algebra, Princeton Univ. Press, 1956.

[9] C. Curtis and I. Reiner, Representation theory of finite groups and algebras, Interscience, 1962.

[10] S. Eilenberg and T. Nakayama, On the dimensions of modules and algebras, II, Nagoya Math. J., **9** (1955), 1-16.

[11] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France, **90** (1962), 323-448.

[12] A. Grothendieck, Sur quelques points d'algebre homologique, Tôhoku Math. J., **9** (1957), 119-220.

[13] M. Harada, Hereditary orders, Trans. Amer. Math. Soc., **107** (1963), 273-290.

[14] A. Hattori, Semi-simple algebras over a commutative ring, J. Math. Soc. Japan, **15** (1963), 404-419.

[15] A. Hattori, Semi-simple algebras over a commutative ring, Proc. Symp. Algebra, Math. Soc. Japan, **6** (1965), 37-40, (in Japanese).

[16] A. Hattori, Simple algebras over a commutative ring, Nagoya Math. J., **27** (1966), 611-616.

[17] E. Matlis, Injective modules over Noetherian rings, Pacific J. Math., **8** (1958), 511-528.

[18] K. Morita, Duality for modules and its application to the theory of rings with minimum condition, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, **6** (1958), 83-142.

[19] B. Müller. Quasi-Frobenius Erweiterungen, Math. Zeit., **85** (1964), 345-368.

[20] B. Müller, Quasi-Frobenius Erweiterungen II, Math. Zeit., **88** (1965), 380-409.

[21] M. Nagata, Local rings, Interscience, 1962.

[22] T. Nakayama and T. Tsuzuku, On Frobenius extensions I, Nagoya Math. J., **17** (1960), 89-110.