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Projective and injective limits of weakly compact sequences of locally convex spaces

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Silva [15] and Raikov [12] [13] studied projective and injective limits of compact sequences of locally convex spaces and revealed remarkable properties of the locally convex spaces expressed as those limits. However, they do not seem to have noticed at first that those spaces are exactly the Fréchet Schwartz spaces and their strong dual spaces discussed by Grothendieck [5].

We extend their results to the limit spaces of weakly compact sequences of locally convex spaces and show that almost all important properties are preserved. We presuppose only the text of Bourbaki [1] except for the closed range theorem and the definition of (DF) spaces.

A projective (injective) sequence of locally convex spaces with (one-one) continuous linear mappings:

$$X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_n \longleftarrow \cdots$$

$$(X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots)$$

is said to be weakly compact or compact if all mappings are weakly compact or compact respectively. The limit space $\varprojlim X_j$ ($\varinjlim X_j$) of a weakly compact or compact projective (injective) sequence is said to be (FS*) or (FS) ((DFS*) or (DFS)) respectively. (FS*) spaces are totally reflexive and Fréchet and (FS) spaces are also separable and Montel. (DFS*) spaces are Hausdorff, totally reflexive, fully complete, bornologic and (DF), and (DFS) spaces are moreover separable and Montel.

Closed subspaces, quotient spaces and projective limits of sequences of (FS*) spaces ((FS) spaces) are (FS*) ((FS)). Closed subspaces, quotient spaces and injective limits of sequences of (DFS) spaces are (DFS). Quotient spaces and direct sums of sequences of (DFS*) spaces are (DFS*). Closed subspaces of (DFS*) spaces are not always (DFS*). However, the bornologic topology and the Mackey topology associated with the induced topology are the same on any closed subspace and they make the subspace into a (DFS*) space.

The strong dual spaces of (FS*) spaces ((FS) spaces) are (DFS*) ((DFS)) and conversely the strong dual spaces of (DFS*) spaces ((DFS) spaces) are

(FS*) ((FS)). More explicitly we have the isomorphisms $(\varprojlim X_j)' = \varinjlim X'_j$ and $(\varinjlim X_j)' = \varprojlim X'_j$ when the sequence satisfies certain conditions. The permanency of those classes of locally convex spaces is proved also through similar representations as limit spaces of subspaces, quotient spaces etc. If X is an (FS*) space and Y its closed subspace, then the strong dual spaces of Y and X/Y are X'/Y^0 and Y^0 equipped with the bornologic topology respectively. If X is a (DFS*) space and Y its closed subspace, then the strong dual spaces of Y and X/Y are X'/Y^0 and Y^0 respectively.

A locally convex space X is (FS*) ((FS)) if and only if it is Fréchet and for each absolutely convex neighborhood V of zero there is another neighborhood $U \subset V$ such that $\hat{X}_U \to \hat{X}_V$ is weakly compact (compact). There is a similar characterization of (DFS*) and (DFS) spaces in terms of bounded sets. Lastly two lemmas in Serre's paper [**14**] on his duality are discussed in our setting.

The author wishes to thank Professor J. Wloka who informed him of the works by Silva and Raikov. Since Raikov's papers [**12**] and [**13**] were not available when this paper was prepared, the author owes much to Wloka's lecture [**16**] as regards Raikov's theory.

Compact and weakly compact mappings. Let X and Y be locally convex spaces. A linear mapping $u: X \to Y$ is said to be weakly compact (compact) if there is a neighborhood V of zero in X such that $u(V)$ is relatively weakly compact (relatively compact) in Y.

The composition of a weakly compact (compact) mapping and a continuous linear mapping on either side is weakly compact (compact). If a weakly compact (compact) mapping $u: X \to Y$ maps a closed subspace X_1 of X into a closed subspace Y_1 of Y, then the restriction $\bar{u}: X_1 \to Y_1$ and the induced mapping $u^*: X/X_1 \to Y/Y_1$ are weakly compact (compact). If $v: Z \to W$ is another weakly compact (compact) mapping, then the direct product $u \times v: X \times Z \to Y \times W$ is weakly compact (compact). The proofs are trivial.

LEMMA 1. *Let X and Y be Banach spaces and let $u: X \to Y$ be a continuous linear mapping. Denote by X', Y', X'' and Y'' the strong dual spaces and the strong bidual spaces of X and Y, and by u' and u'' the dual and bidual mappings of u. Then the following are equivalent:*

(a) *u is weakly compact;*

(b) *u' is weakly compact;*

(c) *u'' maps X'' into Y.*

PROOF. The following proof is a little shorter than that given in Dunford-Schwartz [**3**].

(a)$\Rightarrow$(b). u maps each bounded set in X into a relatively weakly compact set. Therefore by the duality u' is continuous on Y'_τ with the Mackey topology into X' with the strong topology. Since $(Y'_\tau)'=Y$, u' is also continuous with respect to the weak* topology $\sigma(Y', Y)$ and the weak topology $\sigma(X', X'')$. Thus the unit ball in Y', which is compact in $\sigma(Y', Y)$, is mapped by u' to a weakly compact set in X'.

(b)$\Rightarrow$(c). It is enough to show that the unit ball B'' in X'' is mapped into Y. From the above proof it follows that u'' is continuous with respect to the weak* topology $\sigma(X'', X')$ and the weak topology $\sigma(Y'', Y''')$. Since B'' is the $\sigma(X'', X')$-closure of $B=B''\cap X$, $u(B'')$ is contained in the $\sigma(Y'', Y''')$-closure of $u(B)$ which is in Y. The weak closure coincides with the strong closure for convex sets. Therefore $u(B'')$ is contained in Y.

(c)$\Rightarrow$(a). $u'': X''\rightarrow Y$ is continuous with respect to the weak* topology $\sigma(X'', X')$ and the weak topology $\sigma(Y, Y')$. Since the unit ball B in X is relatively weakly* compact in X'', $u(B)$ is relatively weakly compact. This completes the proof.

In particular, if either X or Y is reflexive, any continuous linear mapping $u: X\rightarrow Y$ is weakly compact.

Projective and injective limits. A projective (injective) sequence of locally convex spaces is by definition a system of a sequence of locally convex spaces X_j, $j=1, 2, \cdots$, and (one-one) continuous linear mappings $u_{jk}: X_k\rightarrow X_j$ defined for any pair $j<k$ $(j>k)$ and satisfying the chain condition $u_{ij}\circ u_{jk}=u_{ik}$ for $i<j<k$ $(i>j>k)$. (The mappings u_{jk} are assumed to be one-one for injective sequences in order to avoid some difficulty in proving Theorem 6.) Any u_{jk} are written as compositions of mappings of the form $u_{j\,j+1}$ $(u_{j+1\,j})$.

Two projective (injective) sequences $\{X_j, u_{jk}\}$ and $\{Y_p, v_{pq}\}$ are said to be equivalent if for each j there are an index $p\geqq j$ and a (one-one) continuous linear mapping $s_{jp}: Y_p\rightarrow X_j$ $(s_{pj}: X_j\rightarrow Y_p)$ and for each p there are an index $j>p$ and a (one-one) continuous linear mapping $t_{pj}: X_j\rightarrow Y_p$ $(t_{jp}: Y_p\rightarrow X_j)$ such that $t_{pj}\circ s_{jq}=v_{pq}$ and $s_{jp}\circ t_{pk}=u_{jk}$ $(t_{jp}\circ s_{pk}=u_{jk}$ and $s_{pj}\circ t_{jq}=v_{pq})$. It is easy to prove that this is an equivalence relation. Subsequences are equivalent to the initial sequence.

The projective (injective) limit $X=\varprojlim X_j$ $(X=\varinjlim X_j)$ of a sequence $\{X_j, u_{jk}\}$ is defined to be the subspace of the direct product $\prod X_j$ composed of the elements (x_j) satisfying $x_j=u_{jk}(x_k)$ (the quotient space of the direct sum $\sum X_j$ obtained by identifying those elements (x_j) satisfying $\sum_j u_{kj}(x_j)=0$ for sufficiently large k with zero). There are natural linear mappings $u_j: X\rightarrow X_j$ (one-one linear mappings $u_j: X_j\rightarrow X$) which satisfy $u_j=u_{jk}\circ u_k$ $(u_k=u_j\circ u_{jk})$. The topology of the projective (injective) limit is defined to be the weakest (strongest) locally convex topology which makes the mappings u_j continuous. Equivalent sequences have the same and isomorphic limits.

The sets of the form $u_j^{-1}(V_j)$, where V_j is a convex neighborhood of zero in X_j, form a fundamental system of neighborhoods of zero in the projective limit X. Therefore the projective limit is always Hausdorff. On the other hand, a *convex* set V in the injective limit X is a neighborhood of zero if and only if $u_j^{-1}(V)$ is a neighborhood of zero in X_j for all j. The injective limit topology is not necessarily Hausdorff.

Let $\{X_j, u_{jk}\}$ and $\{Y_j, v_{jk}\}$ be projective (injective) sequences and let X and Y be their limits. If $h_j: X_j \to Y_j$ are continuious linear mappings such that $v_{jk} \circ h_k = h_j \circ u_{jk}$ for any j and k, then there is a unique continuous linear mapping $h: X \to Y$ such that $v_j \circ h = h_j \circ u_j$ ($v_j \circ h_j = h \circ u_j$). If all h_j are one-one, then so is h. If all h_j are onto in the injective case, then so is h. However, this is not necessarily true in the projective case.

Weakly compact sequences. A projective (injective) sequence $\{X_j, u_{jk}\}$ is said to be weakly compact if for each j there is some k such that $u_{jk}(u_{kj})$ is weakly compact. A projective (injective) sequence of Banach spaces $\{X_j, u_{jk}\}$ is said to be strictly weakly compact if the image by $u_{j-1\,j}$ ($u_{j+1\,j}$) of the unit ball in X_j is weakly compact in $X_{j-1}(X_{j+1})$.

LEMMA 2. *Any weakly compact projective (injective) sequence of locally convex spaces $\{X_j, u_{jk}\}$ is equivalent to a strictly weakly compact projective (injective) sequence of Banach spaces.*

PROOF. We consider only the injective case. The proof is the same in the projective case. Choosing a subsequence, we may assume that $u_{j+1\,j}$ are all weakly compact. If a linear mapping $u: X \to Y$ is weakly compact and maps an absolutely convex neighborhood V of zero in X into an absolutely convex weakly compact set A in Y, then it is decomposed as the composition of two linear mappings:

$$X \xrightarrow{\hat{u}} Y_A \xrightarrow{i} Y,$$

where Y_A is the subspace of Y generated by A. Y_A normed with the gauge of A is a Banach space ([1] Chap. III, Lemma 1, p. 21). $\hat{u}$ is continuous, and one-one if u is one-one. Applying this decomposition to our case, we get a sequence of Banach spaces Y_j, one-one continuous linear mappings $\hat{u}_{j+1\,j}$: $X_j \to Y_j$ and injections $i_j: Y_j \to X_{j+1}$ which map unit balls to weakly compact sets. Let $v_{j+1\,j} = \hat{u}_{j+2\,j+1} \circ i_j$ and define v_{jk} by their compositions. Then $\{Y_j, v_{jk}\}$ forms a strictly weakly compact sequence of Banach spaces which is equivalent to $\{X_j, u_{jk}\}$.

Projective limits of weakly compact sequences.

THEOREM 1. *The projective limit $\varprojlim X_j$ of a weakly compact sequence of locally convex spaces is a reflexive Fréchet space.*

PROOF. By Lemma 2 we may assume that $\{X_j, u_{jk}\}$ is a strictly weakly compact sequence of Banach spaces. Let B_j be the unit ball in X_j. Then the countable system $\{n^{-1}u_j^{-1}(B_j)\}$ forms a fundamental system of neighborhoods of zero in $X=\varprojlim X_j$. Thus the limit X is a metrizable locally convex space. To prove the completeness let x_n be a Cauchy sequence in X. For each j $u_j: X\to X_j$ is continuous and therefore $u_j(x_n)$ is a Cauchy sequence in X_j. Let y_j be its limit. Then by continuity of u_{jk} we have $u_{jk}(y_k)=y_j$ for any $j<k$. Hence there is an element x in X such that $y_j=u_j(x)$. We have $u_j(x_n-x)\to 0$ for any j. Thus x_n converges to x in X.

Later we will prove that $X=X''$ explicitly. However, Mackey's criterion of reflexivity is also easy to check. We want to show that any bounded set B in X is relatively weakly compact. By Eberlein's theorem ([8] p. 316) it is enough to show that any sequence $x_n\in B$ has a weakly convergent subsequence. Since B is bounded, $u_j(B)=u_{j\,j+1}\circ u_{j+1}(B)$ is relatively weakly compact for any j. By Smulian's theorem ([8] p. 316) $u_1(x_n)$ has a weakly convergent subsequence $u_1(x_n^{(1)})$ in X_1. Extract a weakly convergent subsequence $u_2(x_n^{(2)})$ of $u_2(x_n^{(1)})$ and so on. Then the diagonal sequence $y_n=x_n^{(n)}$ has the property that $u_j(y_n)$ converges weakly for all j. By the same argument as above we see that $u_j(y_n)$ converges weakly to $u_j(x)$ for some $x\in X$.

Now let $f: X\to \boldsymbol{C}$ be a continuous linear functional. In view of the definition of the topology of X, f can be decomposed as the composition of continuous linear mappings:

$$X \xrightarrow{u_j} X_j \xrightarrow{f_j} \boldsymbol{C} \tag{1}$$

for some j. Thus $f(y_n)=f_j(u_j(y_n))$ converges to $f(x)$.

THEOREM 2. *Let Y be a closed subspace of the projective limit $X=\varprojlim X_j$ of a weakly compact sequence, and let Y_j be the closure of $u_j(Y)$ in X_j. Then the sequence $\{Y_j, \bar{u}_{jk}\}$ is weakly compact and the subspace Y is isomorphic to the projective limit $\varprojlim Y_j$.*

PROOF. Since the continuous mapping u_{jk} maps $u_k(Y)$ onto $u_j(Y)$, it maps Y_k into Y_j. If $u_{jk}: X_k\to X_j$ is weakly compact, its restriction $\bar{u}_{jk}: Y_k\to Y_j$ is also weakly compact. Thus the system $\{Y_j, \bar{u}_{jk}\}$ forms a weakly compact projective sequence of locally convex spaces. Let Z be its projective limit. The continuous injections $i_j: Y_j\to X_j$ induce a continuous injection $i: Z\to X$. Clearly $i(Z)$ contains Y.

To prove the converse, let z be an element of X which is not in Y. Then by the Hahn-Banach theorem there is a continuous linear functional f on X such that $f(y)=0$ for any $y\in Y$ and $f(z)\neq 0$. Decompose f as (1). We have $f_j(u_j(z))=f(z)\neq 0$. On the other hand, $f_j(y_j)=0$ for any $y_j\in Y_j$. Thus $u_j(z)$ is not in Y_j. This means that z is not in $i(Z)$. Therefore i is a one-one onto continuous linear mapping.

The mapping i is also open. In fact, let V be a neighborhood of zero in

Z of the form $\bar{u}_j^{-1}(V_j)$ with an absolutely convex neighborhood V_j of zero in Y_j. Since $V_j = Y_j \cap U_j$ for a neighborhood U_j of zero in X_j, $i(V) = Y \cap u_j^{-1}(Y_j \cap U_j) = Y \cap u_j^{-1}(U_j)$ is a neighborhood of zero in Y. This is also proved by the open mapping theorem.

REMARK 1. Let Y be X. Then we see that any projective limit X of a weakly compact sequence is the projective limit of a weakly compact sequence $\{X_j, u_{jk}\}$ such that $u_j(X)$ is dense in X_j for any j.

THEOREM 3. *Let X be the projective limit of locally convex spaces such that $u_j(X)$ is dense in X_j for any j and let Y and Y_j be as in Theorem 2. Then the sequence $\{X_j/Y_j\}$ is weakly compact and the quotient space X/Y is isomorphic to the projective limit $\varprojlim X_j/Y_j$.*

PROOF. Clearly the sequence $\{X_j/Y_j\}$ with the induced continuous linear mappings u_{jk}^* forms a weakly compact projective sequence. Let Z be its projective limit. The continuous projections $p_j : X_j \to X_j/Y_j$ induce a continuous linear mapping $p : X \to Z$. The kernel is Y by Theorem 2. Therefore we have a continuous injection $i : X/Y \to Z$.

The image $i(X/Y)$ is dense in Z. For, let z be an arbitrary element in Z and let W be a neighborhood of zero in Z. W contains a set of the form $u_j^{*-1}(W_j)$, where W_j is a neighborhood of zero in X_j/Y_j. Since $u_j(X)$ is dense in X_j, we can find an element $x \in X$ such that $u_j^*(z) - p_j \circ u_j(x) \in W_j$. Then $z - p(x)$ is in W.

The injection i is also an isomorphism. Let V be a neighborhood of zero in X/Y of the form $(u_j^{-1}(2U_j)+Y)/Y$ with a neighborhood U_j of zero in X_j and let W be the neighborhood $u_j^{*-1}((U_j+Y_j)/Y_j)$ of zero in Z. If z is in $W \cap i(X/Y)$, there is an $x \in X$ such that $z = p(x)$ and $u_j(x) \in U_j + Y_j \subset 2U_j + u_j(Y)$. Thus $i^{-1}(z) = (x+Y)/Y$ belongs to V. Since both X/Y and Z are complete, the injection i is an onto isomorphism.

THEOREM 4. *Let $X = \varprojlim X_j$ and $Y = \varprojlim Y_j$ be projective limits of weakly compact sequences of locally convex spaces. Then the sequence $\{X_j \times Y_j\}$ is weakly compact and the product space $X \times Y$ is isomorphic to the projective limit $\varprojlim (X_j \times Y_j)$.*

THEOREM 5. *Let $X^{(k)} = \varprojlim X_j^{(k)}$, $k = 1, 2, \cdots$, be projective limits of weakly compact sequences of locally convex spaces. Then the product space $\prod_{k=1}^{\infty} X^{(k)}$ is isomorphic to the projective limit $\varprojlim Z_j$ of the weakly compact sequence of locally convex spaces $\{Z_j\}$ defined by*

$$Z_j = X_j^{(1)} \times X_{j-1}^{(2)} \times \cdots \times X_1^{(j)} .$$

Proofs of Theorems 5 and 6 are easy and omitted.

REMARK 2. In the proofs of Theorems 1-5 we have not made any essential use of the fact that the sequences are weakly compact except for the proof of the reflexivity. Thus projective limits of sequences of Banach spaces are Fréchet and we can prove the projective representations of their closed subspaces, quotient spaces and product spaces with countable factors.

Injective limits of weakly compact sequences.

LEMMA 3. *The inductive limit $X=\varinjlim X_j$ of a weakly compact sequence of locally convex spaces $\{X_j, u_{kj}\}$ is Hausdorff. Any bounded set B in X is the image $u_j(B_j)$ of a bounded set B_j in X_j for some j.*

PROOF. we may assume, without loss of generality, that $\{X_j, u_{kj}\}$ is a strictly weakly compact sequence of Banach spaces. Suppose that x is an element in X different from zero. x is written as $x=u_p(x_p)$ for a p. We construct a sequence of absolutely convex neighborhoods V_j of zero in X_j for $j=p, p+1, \cdots$ such that

(i) $u_{kj}(V_j)\subset V_k$ for $k>j$;

(ii) $x_j=u_{jp}(x_p)\notin V_j$;

(iii) $u_{kj}(V_j)$ is weakly compact in X_k for $k>j$.

Then $V=\bigcup_{j\geqq p} u_j(V_j)$ is a neighborhood of zero in X in which x is not contained. In fact, $u_j^{-1}(V)$ contains either $u_{jp}^{-1}(V_p)$ or V_j. Therefore V is a neighborhood of zero in X. Because of (ii) x is not contained in V.

Choose for V_p any closed ball which does not contain x_p. Since the sequence is strictly weakly compact, condition (iii) is satisfied for V_p. Suppose that $V_p, V_{p+1}, \cdots, V_j$ have been chosen. $u_{j+1\,j}(V_j)$ is weakly compact and hence is strongly closed in X_{j+1}. Since x_{j+1} is not contained in $u_{j+1\,j}(V_j)$, there is a small closed ball B_{j+1} such that the convex hull $\mathrm{Conv}\,(B_{j+1}, u_{j+1\,j}(V_j))$ does not contain x_{j+1}. Let V_{j+1} be the convex hull. Then (i) and (ii) are clearly satisfied. If $k>j+1$, $u_{k\,j+1}(V_{j+1})=\mathrm{Conv}\,(u_{k\,j+1}(B_{j+1}), u_{kj}(V_j))$ is weakly compact because both $u_{k\,j+1}(B_{j+1})$ and $u_{kj}(V_j)$ are weakly compact.

The second statement is proved similarly. Let B be a bounded set in X and assume contrarily that for each j either B is not contained in $u_j(X_j)$ or $u_j^{-1}(B)$ is unbounded in X_j. Then we can construct, in the same way as above, a sequence of absolutely convex neighborhoods V_j of zero in X_j and a sequence of elements x_j in B such that

(i) $u_{kj}(V_j)\subset V_k$ for $k>j$;

(ii) $x_1, \dfrac{1}{2}x_2, \cdots, \dfrac{1}{j}x_j \notin u_j(V_j)$;

(iii) $u_{kj}(V_j)$ is weakly compact in X_k for $k>j$.

$V=\bigcup u_j(V_j)$ is a neighborhood of zero in X in which x_j/j are not contained. This is a contradiction because the sequence x_j/j converges to zero.

THEOREM 6. *The injective limit $X=\varinjlim X_j$ of a weakly compact sequence of locally convex spaces is a complete reflexive and bornologic* (DF) *space. For* *each bounded set B in X there is an index k such that B is the image $u_k(B_k)$ of a bounded set B_k in X_k and u_k is a weak homeomorphism of B_k onto B. In particular, a sequence x_n in X converges weakly to zero if and only if there is a sequence y_n in some X_k with $x_n = u_k(y_n)$ which converges weakly to zero in X_k.*

PROOF. Any bounded set B in X is of the form $B = u_j(B_j)$ with a bounded set B_j in X_j. Therefore $B = u_k(u_{kj}(B_j))$ is relatively weakly compact in X, where k is an index such that u_{kj} is weakly compact. u_k gives a weak homeomorphism of $B_k = u_{kj}(B_j)$ onto B. Thus X is semi-reflexive and consequently quasi-complete. X is bornologic. For, an absolutely convex set V in X is a neighborhood of zero if and only if $u_j^{-1}(V)$ absorbs a neighborhood of zero in X_j for any j and hence if and only if V absorbs all bounded sets in X. Thus X is barrelled. Clearly X has a countable fundamental system of bounded sets. Therefore X is a reflexive (DF) space. In particular, X is complete as the strong dual space of a Fréchet space.

THEOREM 7. *Let Y be a closed subsapace of the injective limit $X = \varinjlim X_j$ of a weakly compact sequence and let Y_j be the inverse image $u_j^{-1}(Y)$. Then the sequence $\{Y_j\}$ is weakly compact. Its limit $Z = \varinjlim Y_j$ is the same as Y as a set and the natural mapping $Z \to Y$ is continuous. Moreover Y and Z have the same strong dual space. If Y is barrelled, then Y is isomorphic to Z.*

PROOF. Y_j is a closed subspace of X_j and u_{kj} maps Y_j into Y_k. Thus the subspaces Y_j with the restrictions $\bar{u}_{kj}$ form a weakly compact injective sequence. The injections $i_j: Y_j \to X_j$ induce a continuous injection $i: Z \to X$ and clearly the image $i(Z)$ coincides with Y. Both in Y and in Z bounded sets are images of bounded sets in some Y_j. Thus both spaces have the same bounded sets. Therefore the dual mapping i' is an isomorphism of the strong dual space Y' into the strong dual space Z'. Later we will prove that X'/Y^0 is the strong dual space of the reflexive space Z. On the other hand, by the Hahn-Banach theorem Y' is identified with X'/Y^0 as a set. Thus i' is an onto mapping. In particular, the strong bidual space Y'' is isomorphic to $Z'' = Z$. If Y is barrelled, the topology of Y is the same as the induced topology in Y as a subspace of Y'' and therefore Y is isomorphic to Z. The last statement follows also from Pták's open mapping theorem [**11**].

REMARK 3. Z is bornologic and barrelled. Thus the topology of Z is the bornologic and at the same time the Mackey topology associated with the induced topology in Y.

The topology of Z can be different from that of Y. Grothendieck ([**5**], p. 97) constructed a Montel (DF) space X which can be expressed as the inductive limit of a weakly compact sequence and a closed subspace Y of X which is neither (DF) nor quasi-barrelled. Later we will prove that two topologies coincide if the sequence $\{X_j\}$ is compact.

THEOREM 8. *Let X, Y and Y_j be as in Theorem 7. Then the sequence $\{X_j/Y_j\}$ is weakly compact and the quotient space X/Y is isomorphic to the inductive limit $\varinjlim X_j/Y_j$. Bounded sets in X/Y are the images of bounded sets in X.*

PROOF. If $k>j$, Y_j is the inverse image $u_{kj}^{-1}(Y_k)$. Thus the induced mapping $u_{kj}^*: X_j/Y_j \to X_k/Y_k$ is one-one and weakly compact. Let Z be the inductive limit of the weakly compact sequence $\{X_j/Y_j, u_{kj}^*\}$. The projections $p_j: X_j \to X_j/Y_j$ induce a continuous projection $p: X \to Z$. The kernel is clearly equal to Y. Thus we have a one-one onto and continuous mapping $i: X/Y \to Z$. i is also open. For, if U is an open convex set in X, then $U+Y$ is open in X and hence $p_j \circ u_j^{-1}(U+Y) = u_j^{*-1}(p(U))$ is open in X_j/Y_j. This means that $p(U) = \bigcup u_j^* \circ u_j^{*-1}(p(U))$ is open.

To prove the last statement we assume without loss of generality that $\{X_j, u_{kj}\}$ is a sequence of Banach spaces. Any bounded set in X/Y is the image $u_j^*(B_j^*)$ of a bounded set B_j^* in X_j/Y_j. Since X_j is Banach, B_j^* is the image of a bounded set B_j in X_j. $u_j(B_j)$ is bounded in X and $u_j^*(B_j^*)$ is equal to its image under the projection $p: X \to X/Y$.

THEOREM 9. *Let $X = \varinjlim X_j$ and $Y = \varinjlim Y_j$ be two inductive limits of weakly compact sequences of locally convex spaces. Then the product space $X \times Y$ is isomorphic to the inductive limit $\varinjlim (X_j \times Y_j)$ of the weakly compact sequence $\{X_j \times Y_j\}$.*

THEOREM 10. *Let $X^{(k)} = \varinjlim X_j^{(k)}$, $k=1, 2, \cdots$, be inductive limits of weakly compact sequences. Then the direct sum $\sum_{k=1}^{\infty} X^{(k)}$ is isomorphic to the inductive limit $\varinjlim Z_j$ of the weakly compact sequence*

$$Z_j = X_j^{(1)} \times X_{j-1}^{(2)} \times \cdots X_1^{(k)} .$$

Proofs of Theorems 9 and 10 are omitted.

Limits of compact sequences. A projective (injective) sequence $\{X_j, u_{jk}\}$ of locally convex spaces is said to be compact if for each j there is some k such that the mapping $u_{jk}(u_{kj})$ is compact. The projective and injective limits of compact sequences have been discussed by Silva [**15**] and Raikov [**12**] and [**13**]. Lemmas 2-3, Theorems 1-10 and their proofs remain true if we replace 'weakly compact' by 'compact'. The counterpart of Lemma 1 is the famous Schauder theorem. Of course, we have some improved results caused by compactness particularly for injective limits.

THEOREM 1′. *The projective limit $X = \varprojlim X_j$ of a compact sequence of locally convex spaces is a separable Fréchet Montel space.*

PROOF. The diagonal argument in Theorem 1 shows, in this case, that any bounded set is relatively compact. By Dieudonné's theorem ([8] p. 373) X is separable as a Fréchet Montel space. We can prove it, however, also in the following way. We may assume that X_j are Banach and $u_j(X)$ is dense in X_j for all j. Let u_{jk} be compact. Then the image $u_{jk}(B_k)$ of the unit ball B_k is separable. Thus each space X_j is separable. Choose a countable dense set of the form $u_j(D_j)$ in each X_j. Then the union $\cup D_j$ is a countable dense set in X.

THEOREM 6′. *The inductive limit $X=\varinjlim X_j$ of a compact sequence of locally convex spaces is a separable complete bornologic* (DF) *Montel space. On each bounded set B in X the inductive topology coincides with the weak topology and there is a bounded set B_k in some X_k such that $u_k: B_k \to B$ is a homeomorphism. A sequence x_n in X converges to zero if and only if there is a sequence y_n in some X_k with $x_n = u_k(y_n)$ which converges (weakly) to zero in X_k.*

The topology of X is the inductive limit topology of topological spaces X_j, i.e. a not necessarily convex set S is open (closed) in X if and only if $u_j^{-1}(S)$ are all open (closed) in X_j.

PROOF. We may assume that X_j are Banach. To prove the separablilty it is enough to show that the image $u_j(B_j)$ of the unit ball B_j in X_j is separable in X. If u_{kj} is compact, $u_{kj}(B_j)$ is relatively compact and therefore separable in X_k. Since $u_k: X_k \to X$ is a homeomorphism on the relatively compact set $u_{kj}(B_j)$, $u_j(B_j) = u_k(u_{kj}(B_j))$ is separable.

It is well known that the initial topology coincides with the weak topology on each bounded set in a Montel space.

Silva [**15**] gives a direct proof of the last statement. Let us prove it as a consequence of the Banach-Dieudonné theorem ([**1**] pp. 73-74). Necessity is clear. Suppose that $u_j^{-1}(S)$ is open in X_j for any j. Since X is the strong dual space of a Fréchet Montel space Y, the topology in X is the uniform convergence topology on each compact set in Y. Thus by the Banach-Dieudonné theorem it is enough to show that for any absolutely convex closed bounded set B, $S \cap B$ is open relative to the topology on B induced by the weak topology $\sigma(X, Y)$. B is the image $u_k(B_k)$ of a compact set B_k in some X_k and B and B_k are homeomorphic. By assumption $u_k^{-1}(S) \cap B_k$ is open in B_k. Since the weak topology is the same as the initial topology on B, it follows that $S \cap B = u_k(u_k^{-1}(S) \cap B_k)$ is weakly open in B.

THEOREM 7′. *Let Y be a closed subspace of the injective limit $X=\varinjlim X_j$ of a compact sequence and let $Y_j = u_j^{-1}(Y)$. Then the subspace Y is isomorphic to the injective limit $\varinjlim Y_j$ of the compact sequence $\{Y_j\}$.*

PROOF. Let Z be the injective limit $\varinjlim Y_j$ and let $i: Z \to Y$ be the natural mapping. We have to prove that i is open. Since equivalent sequences for X induce equivalent sequences $\{Y_j\}$, we may assume that the sequence $\{X_j, u_{kj}\}$ is a strictly compact sequence of Banach spaces. Let U be an absolutely convex open neighborhood of zero in Z. We construct a sequence of absolutely convex neighborhoods V_j, $j=1, 2, \cdots$, of zero in X_j such that

(i) $u_{kj}(V_k) \subset V_k$ for $k > j$;

(ii) $V_j \cap Y_j \subset u_j^{-1}(i(U))$;

(iii) $u_{kj}(V_j)$ is compact in X_k for $k > j$.

Then $V = \bigcup_{j=1}^{\infty} u_j(V_j)$ is an absolutely convex neighborhood of zero in X which satisfies $V \cap Y \subset i(U)$. Thus $i(U)$ is a neighborhood of zero in Y.

First note that $U_j = u_j^{-1}(i(U))$ is an open neighborhood of zero in Y_j, so that the complement of U_j relative to Y_j is closed in X_j. Choose for V_1 a closed ball so small that $V_1 \cap Y_1 \subset U_1$. (iii) is satisfied because the sequence is strictly compact. Suppose that $V_1, \cdots, V_j$ have been chosen. Then $u_{j+1\,j}(V_j)$ is compact in X_{j+1} and is disjoint with the complement of U_{j+1} relative to Y_{j+1}. Therefore the distance between these sets is positive. Take a closed ball W_{j+1} with radius less than the distance and let $V_{j+1} = \mathrm{Conv}\,(W_{j+1}, u_{j+1\,j}(V_j))$. Properties (i), (ii) and (iii) are easily checked.

Duality.

THEOREM 11. *Let $\{X_j, u_{jk}\}$ be a weakly compact projective sequence of Banach spaces such that $u_j(X)$ is dense in X_j for each j. Then the dual sequence $\{X_j', u_{jk}'\}$ is a weakly compact inductive sequence and the strong dual space of the projective limit $\varprojlim X_j$ is isomorphic to the injective limit $\varinjlim X_j'$.*

PROOF. Since u_{jk} has dense range, the dual mapping $u_{jk}': X_j' \to X_k'$ is one-one. If u_{jk} is weakly compact, then so is u_{jk}'. Thus $\{X_j', u_{jk}'\}$ is a weakly compact inductive sequence. Denote by X and Y the projective limit $\varprojlim X_j$ and the injective limit $\varinjlim X_j'$ respectively. There is a natural duality between X and Y. In fact, if $x \in X$ and $y = u_j'(y_j) \in Y$, then $\langle x, y\rangle = \langle u_j(x), y_j\rangle$ defines a bilinear form independent of the representation $y = u_j'(y_j)$. If $\langle x, y\rangle = 0$ for all y, then $u_j(x) = 0$ for all j and hence $x = 0$. Since $u_j(X)$ are dense, it follows that if $\langle x, y\rangle = 0$ for all x, then $y = 0$. Since $u_j: X \to X_j$ is continuous, any element $y \in Y$ is continuous on X. Conversely if x' is a continuous linear functional on X, it is decomposed as $x' = y_j \circ u_j$ with a continuous linear functional y_j on X_j for some j. Thus x' coincides with a $y \in Y$.

To prove that the topologies are the same, let B be an absolutely convex bounded set in X. Then its polar set B^0 in Y is the union $\cup u_j'((u_j(B))^0)$. Since $u_j(B)$ is bounded in X_j, $(u_j(B))^0$ is an absolutely convex neighborhood of zero in X_j'. Hence B^0 is a neighborhood of zero in Y. Conversely let V be an absolutely convex closed neighborhood of zero in Y. Let B be the polar set V^0 in X. Then $u_j(B)$ is bounded for any j because $u_j(B)$ is contained in $(u_j'^{-1}(V))^0$. If we prove that X is the dual space of Y, we have $V = B^0$ showing that V is a neighborhood of zero in X'.

Any element $x \in X$ belongs to Y', because $\langle x, u_j'(y_j)\rangle = \langle u_j(x), y_j\rangle$ is continuous on each X_j'. Conversely let f be a continuous linear functional on Y. Since $f\circ u_j'$ is continuous on X_j', there is an element $x_j\in X_j''$ such that $f(y)=\langle x_j, y_j\rangle$ for any $y=u_j'(y_j)$ with $y_j\in X_j'$. We have $\langle x_k, u_{jk}'y_j\rangle=\langle x_j, y_j\rangle$ for any $y_j\in X_j'$ and $k>j$. Hence it follows that $x_j=u_{jk}''x_k$ for $k>j$. Lemma 1 shows that x_j is in X_j and therefore there is an element $x\in X$ such that $x_j=u_j(x)$.

REMARK 4. Incidentally we have obtained a direct proof of the reflexivity of projective limits. Conversely we can prove the theorem starting with the reflexivity. Suppose in general that X is the projective limit of a (not necessarily weakly compact) sequence of Banach spaces such that $u_j(X)$ is dense in X_j. (Note that any Fréchet space can be expressed in this way.) If X is distinguished or in particular reflexive, then the strong dual space X' is isomorphic to the injective limit $\varinjlim X_j'$. In fact, the proof of the above theorem is valid except for the last step. Now suppose that V is an absolutely convex neighborhood of zero in Y. Then V absorbs every equicontinuous set B in X'. To prove this, it is enough to show that B is contained in the image $u_j'(B_j)$ of a bounded set B_j in X_j for some j. Any neighborhood of zero in X contains the inverse image $u_j^{-1}(U)$ of a ball U in X_j. Since $U\cap u_j(X)$ is dense in U, any linear functional on X which is bounded on $u_j^{-1}(U)$ can be written as u_j times a linear functional on X_j which is bounded on U. Thus the polar set of $u_j^{-1}(U)$ coincides with $u_j'(U^0)$. Since X is barrelled and X' is bornologic (Grothendieck [5] Théorème 7), V is a neighborhood of zero in X'.

The condition that X is distinguished is also necessary in order that the strong dual space X' is isomorphic to the injective limit Y. For, Y is bornologic as an injective limit of Banach spaces, and X' is bornologic if and only if X is distinguished.

THEOREM 12. *Let X be the inductive limit of a weakly compact sequence $\{X_j, u_{kj}\}$ of Banach spaces. Then $\{X_j', u_{kj}'\}$ is a weakly compact projective sequence and the strong dual space X' is isomorphic to the projective limit $\varprojlim X_j'$.*

PROOF. Clearly $\{X_j', u_{kj}'\}$ forms a weakly compact projective sequence. Let Y be the projective limit. The inner product between X and Y is defined by $\langle x, y\rangle=\langle u_j^{-1}(x), u_j'(y)\rangle$ as before. If $\langle x, y\rangle=0$ for all x, then we have $u_j'(y)=0$ for any j and hence $y=0$. Thus each $y\in Y$ corresponds to a unique linear functional on X. Y coincides with the dual space X' by this correspondence. In fact, for each j, $\langle u_j(x_j), y\rangle=\langle x_j, u_j'(y)\rangle$ is continuous on X_j. Therefore Y is contained in X'. On the other hand, let f be a continuous linear functional on X. Then for each j there is an element $y_j\in X_j'$ such that $f(u_j(x_j))=\langle x_j, y_j\rangle$ for all $x_j\in X_j$. From the identity $\langle u_{kj}(x_j), y_k\rangle=\langle x_j, y_j\rangle$, $j<k$, it follows that $u_{kj}'(y_k)=y_j$. Thus there is an element $y\in Y$ such that $f(u_j(x_j))=\langle x_j, u_j'(y)\rangle=\langle u_j(x), y\rangle$. Hence f coincides with $y\in Y$.

To prove that the topologies are the same, it suffices to show that the identity mapping $Y \to X'$ is continuous because both spaces are Fréchet. Any neighborhood of zero in X' contains the polar set B^0 of a bounded set B in X. B is the image $u_j(B_j)$ of a bounded set B_j in X_j. Thus $B^0 = u_j'^{-1}(B_j^0)$ is a neighborhood of zero in Y.

REMARK 5. To prove the isomorphism of X' and Y we used only the fact that every bounded set in X is the image of a bounded set in some X_j and that X is a (DF) space. Thus the isomorphism between $(\varinjlim X_j)'$ and $\varprojlim X_j'$ holds also for strict inductive limits of normed spaces ([1] p. 8) and complete inductive limits of Banach spaces ([6] p. 17). It should be noted that a locally convex space is complete and the inductive limit of a sequence of Banach spaces if and only if it is a (quasi-) complete bornologic (DF) space.

END OF THE PROOF OF THEOREM 7. We have to prove that the quotient space X'/Y^0 is the strong dual space of Z. It follows from Theorem 12 that $Z' = \varprojlim Y_j' = \varprojlim X_j'/Y_j^0$, and from Theorem 3 that $X'/Y^0 = \varprojlim X_j'/\overline{u_j'(Y^0)}$. We want to show that two sequences are equivalent. Since $Y = Y^{00}$ in the duality between X and X', $x_j \in X_j$ is in $Y_j = u_j^{-1}(Y)$ if and only if $\langle x_j, u_j'(Y^0)\rangle = 0$. In other words, we have $Y_j = (u_j'(Y^0))^0$. Thus Y_j^0 is the weak* closure of $u_j'(Y^0)$ in X_j'. Consequently there is a natural mapping $s_{jj}: X_j'/\overline{u_j'(Y^0)} \to X_j'/Y_j^0$. On the other hand, let u_{kj} be weakly compact. Then by Lemma 1 u_{kj}' is continuous on X_k' with the weak* topology into X_j' with the weak topology. Therefore u_{kj}' maps Y_k^0 into $\overline{u_j'(Y^0)}$. Thus the mapping $u_{jk}^*: X_k'/Y_k^0 \to X_j'/Y_j^0$ is decomposed as $u_{jk}^* = s_{jj} \circ t_{jk}$, where $t_{jk}: X_k'/Y_k^0 \to X_j'/\overline{u_j(Y^0)}$. The composition $t_{jk} \circ s_{kk}$ is clearly the initial mapping: $X_k'/\overline{u_k(Y^0)} \to X_j'/\overline{u_j(Y^0)}$.

THEOREM 13. *Let Y be a closed subspace of the projective limit $\varprojlim X_j$ of a weakly compact sequence of locally convex spaces. Then the strong dual space Y' is isomorphic to the quotient space X'/Y^0 of the strong dual space X' modulo the orthogonal subspace Y^0.*

PROOF. We may assume that $\{X_j, u_{jk}\}$ is a weakly compact sequence of Banach spaces such that $u_j(X)$ is dense in X_j for all j. Let $Y_j = \overline{u_j(Y)}$. Then by Theorem 2, $Y = \varprojlim Y_j$. Clearly $\bar{u}_j(Y)$ is dense in Y_j. Thus the strong dual space Y' is identified with the injective limit $\varinjlim Y_j'$ by Theorem 11. On the other hand, the quotient space X'/Y^0 is identified with the injective limit $\varinjlim X_j'/u_j'^{-1}(Y^0)$ by Theorem 8. Since X_j is Banach, the strong dual space of Y_j is the quotient space $X_j'/(u_j(Y))^0$, which is clearly the same as $X_j'/u_j'^{-1}(Y^0)$.

THEOREM 14. *Let Y be a closed subspace of the projective limit $X = \varprojlim X_j$ of a weakly compact sequence of locally convex spaces. Then the strong dual space $(X/Y)'$ of the quotient space is isomorphic to the subspace Y^0 of X' equipped with the bornologic or the Mackey topology associated with the induced* *topology. If the sequence is compact, $(X/Y)'$ is isomorphic to Y^0 with the induced topology.*

PROOF. We may assume that X_j are Banach and that $u_j(X)$ are dense in X_j. Then it follows from Theorems 3 and 11 that $(X/Y)' = \varinjlim (X_j/\overline{u_j(Y)})'$ $= \varinjlim u_j'^{-1}(Y^0)$. The last limit is exactly Y^0 with the bornologic or the Mackey topology associated with Y^0 by Theorem 7. If the sequence is compact, Y^0 itself is bornologic and barrelled by Theorem 7'.

THEOREM 15. *Let Y be a closed subspace of the injective limit $X = \varinjlim X_j$ of a weakly compact sequence of locally convex spaces. Then the strong dual space Y' is isomorphic to the quotient space X'/Y^0.*

PROOF. This follows from Theorems 7 and 14 easily.

THEOREM 16. *Let X and Y be as in Theorem 15. Then the strong dual space $(X/Y)'$ of the quotient space is isomorphic to the orthogonal subspace Y^0 of X'.*

PROOF. This is an immediate consequence of Theorems 1, 2 and 13.

Characterization of weakly compact and compact limits.

THEOREM 17. *The following conditions are equivalent for a Fréchet space X:*

(a) *X is the projective limit of a weakly compact (compact) sequence of locally convex spaces;*

(b) *Any continuous linear mapping on X into a Banach space Y is weakly compact (compact);*

(c) *For each absolutely convex neighborhood V of zero there is an absolutely convex neighborhood $U \subset V$ of zero such that the natural linear mapping: $\hat{X}_U \to \hat{X}_V$ is weakly compact (compact);*

(d) *The strong dual space X' is the injective limit of a weakly compact (compact) sequence of locally convex spaces.*

PROOF. (a)$\Rightarrow$(b). Let X be the projective limit $\varprojlim X_j$ of a weakly compact (compact) sequence of locally convex spaces. If a linear mapping $f: X \to Y$ is continuous, it is decomposed as the composition of continuous linear mappings:

$$X \xrightarrow{u_j} X_j \xrightarrow{f_j} Y$$

for some j. If u_{jk} is weakly compact (compact), $f_j \circ u_{jk}: X_k \to Y$ is weakly compact (compact). Thus there is a neighborhood V of zero in X of the form $u_k^{-1}(V_k)$ with a neighborhood V_k of zero in X_k which is mapped to a relatively weakly compact (compact) set in Y.

(b)$\Rightarrow$(c). Apply (b) when Y is $\hat{X}_V$.

(c)$\Rightarrow$(a). Let $\{V_j\}$, $j = 1, 2, \cdots$, be a fundamental system of absolutely convex neighborhoods of zero in X. We may assume that $V_j \supset V_{j+1}$. Let $X_j = \hat{X}_{V_j}$

and let $u_{jk}: X_k \to X_j$ be the natural linear mapping. Clearly $\{X_j, u_{jk}\}$ forms a weakly compact (compact) projective sequence. It is easy to prove that X is isomorphic to the projective limit.

(a)$\Rightarrow$(d) by Theorem 11.

(d)$\Rightarrow$(a). It follows from Theorem 12 that the strong bidual space X'' satisfies (a). Therefore the closed subspace X of X'' satisfies (a).

REMARK 6. Grothendieck [5] calls a locally convex space X a Schwartz space if it satisfies equivalent conditions (b) and (c) in the parenthesized form. Thus a locally convex space is Fréchet Schwartz if and only if it is the projective limit of a compact sequence of locally convex spaces.

We refer to Fréchet Schwartz spaces as (FS) spaces and to the projective limits of weakly compact sequences as (FS*) spaces for short.

THEOREM 18. *The following are equivalent for a complete* (DF) *space* X:

(a) *X is the inductive limit of a weakly compact* (*compact*) *sequence of locally convex spaces;*

(b) *For each absolutely convex closed bounded set B there is an absolutely closed bounded set $A \supset B$ such that the natural mapping $X_B \to X_A$ is weakly compact* (*compact*) *and X is barrelled* (*this is not necessary in the compact case*)*;*

(c) *The strong dual space X' is the projective limit of a weakly compact* (*compact*) *sequence of locally convex spaces and X is barrelled* (*the last condition is not necessary in the compact case*).

PROOF. (a)$\Rightarrow$(b). Suppose that $X = \varinjlim X_j$ with a weakly compact (compact) sequence of Banach spaces X_j. Then X is barrelled. Any absolutely convex closed bounded set B is the image $u_j(B_j)$ of an absolutely convex closed bounded set B_j in X_j. If k is sufficiently large, $u_{kj}(B_j)$ is relatively weakly compact (compact) in X_k. Thus if we choose a closed ball A_k which contains $u_{kj}(B_j)$, then the closed bounded set $A = \overline{u_k(A_k)}$ satisfies condition (b).

(b)$\Rightarrow$(c). Let $B_1 \subset B_2 \subset \cdots$ be a fundamental system of absolutely convex closed bounded sets in X and let $X_j = X_{B_j}$. Then the sequence $\{X_j\}$ with the natural injections $u_{kj}: X_j \to X_k$ forms a weakly compact (compact) injective sequence. The limit $Y = \varinjlim X_j$ coincides with X as a set and the natural mapping $i: Y \to X$ is clearly continuous. X and Y have the same class of bounded sets and both spaces are semi-reflexive. In fact, any bounded set is relatively weakly compact in Y and hence in X. Thus the dual mapping i' on the strong dual space X' into the strong dual space Y' is a homeomorphism with dense range. Since both X' and Y' are Fréchet, X' must coincide with Y' which is the projective limit of a weakly compact (compact) sequence.

We can also prove the isomorphism of X and Y directly by Pták's open mapping theorem [11].

(c)$\Rightarrow$(a). Suppose that X is barrelled. Then X is a closed barrelled subspace of the bidual space X''. Thus X satisfies (a) by Theorem 7. If X' is (FS), then X' is separable. Therefore any strongly bounded set in X' is equicontinuous. Since X is complete, thus X is barrelled.

We call the strong dual spaces of (FS) spaces and (FS*) spaces (DFS) spaces and (DFS*) spaces respectively.

Serre's lemma. The following theorem has been employed often to prove various duality theorems and existence theorems of partial differential equations ([**14**], [**10**], [**9**] etc.).

THEOREM 19. *Let X_1, X_2 and X_3 be Fréchet spaces and let $u_1 : X_1 \to X_2$ and $u_2 : X_2 \to X_3$ be densely defined closed linear mappings such that $u_2 \circ u_1 = 0$. Denote by X'_j and u'_j the strong dual spaces of X_j and the dual mappings of u_j respectively:*

$$X_1 \xrightarrow{u_1} X_2 \xrightarrow{u_2} X_3$$

$$X'_1 \xleftarrow{u'_1} X'_2 \xleftarrow{u'_2} X'_3 .$$

Then,

(i) *The image* $\mathrm{im}\, u_j$ *is closed in X_{j+1} if and only if the image* $\mathrm{im}\, u'_j$ *is (weakly*) closed in X'_j.*

(ii) *Suppose that both* $\mathrm{im}\, u_1$ *and* $\mathrm{im}\, u_2$ *are closed. Let $Z = \ker u_2$, $B = \mathrm{im}\, u_1$, $Z^* = \ker u'_1$ and $B^* = \mathrm{im}\, u'_2$. Then the quotient space $H = Z/B$ is Fréchet and its dual space H' is identified with the quotient space $H^* = Z^*/B^*$ as a set. If X_2 is* (FS*), *then H is* (FS*) *and the strong dual space H' is isomorphic to H^* equipped with the bornologic or the Mackey topology associated with the quotient topology in H^*. If X_2 is* (FS), *then so is H and H' is isomorphic to H^*.*

PROOF. (i) is exactly the closed range theorem of Dieudonné-Schwartz [**3**] supplemented by Browder [**2**] and Grothendieck ([**7**] p. 296).

(ii). Z and B are Fréchet spaces as closed subspaces of the Fréchet space X_2. Therefore $H = Z/B$ is a Fréchet space. Since Z^* and B^* are weakly* closed, it is easy to prove that Z^* and B^* are orthogonal subspaces B^0 and Z^0 of B and Z, respectively, relative to the duality between X_2 and X'_2. The dual space Z' is identified with X'_2/B^* in the weak sense by the Hahn-Banach theorem and topologically if X_2 is (FS*). In the same way the dual space H' is identified with the orthogonal subspace B^0 relative to the duality between Z and X'_2/B^*. Since the polar set of B in X'_2 is Z^*, $B^0 = Z^*/B^* = H^*$. If X_2 is (FS*) or (FS), then so are Z and B, and therefore so is the quotient space H. By Theorem 14 the strong dual space H' is H^* with the bornologic (or the Mackey) topology.

Schwartz's lemma. The assumption that $\mathrm{im}\, u_1$ and $\mathrm{im}\, u_2$ are closed is usually difficult to prove. The Banach-Dieudonné theorem and the following are the only general theorems available.

THEOREM 20. (i) *Suppose that* $H=Z/B$ *has a Fréchet cross-section, i.e. there is a Fréchet space* Y *with a continuous linear mapping* $f: Y \to Z$ *such that the composition* $Y \to Z \to Z/B$ *is one-one and onto. Then* $B=\operatorname{im} u_1$ *is closed and* H *is isomorphic to* Y. *The cross-section exists if* H *is of finite dimension.*

(ii) *Suppose that* X_2 *and* X_3 *are* (FS*). *If there is a* (DFS*) *cross-section* Y^* *of* $H^*=Z^*/B^*$, *then* $B^*=\operatorname{im} u_2'$ *is closed and* H^* *with the bornologic topology associated with the quotient topology is isomorphic to* Y^*. *The cross-section exists if the algebraic dimension of* H^* *is countable.*

PROOF. (i). Let $G(u_1)$ be the graph of u_1 in $X_1 \times X_2$. The mapping u_1 is realized as the continuous projection on $G(u_1)$ to the second component X_2. Thus u_1 is decomposed as the composition of continuous linear mappings:

$$G(u_1) \xrightarrow{p} G(u_1)/\ker u_1 \times \{0\} \xrightarrow{v} Z.$$

Since $\ker u_1$ is closed, $X=G(u_1)/\ker u_1 \times \{0\}$ is a Fréchet space. Consider the linear mapping

$$v \times f: X \times Y \to Z$$

defined by $v \times f(x, y)=v(x)+f(y)$. By assumtion $v \times f$ is a one-one continuous linear mapping on the Fréchet space $X \times Y$ onto the Fréchet space Z. Thus Z is isomorphic to the product $X \times Y$ by Banach's open mapping theorem. Therefore the subspace B which corresponds to $X \times \{0\}$ is closed in Z and $H=Z/B$ is isomorphic to Y. The cross-section clearly exists if H is of finite dimension.

(ii). In this case u_2' is decomposed as

$$G(u_2') \xrightarrow{p} G(u_2')/\ker u_2' \times \{0\} \xrightarrow{v} Z^*.$$

$X^*=G(u_2')/\ker u_2' \times \{0\}$ and Z^* become (DFS*) spaces if we introduce the bornologic topologies. v is also continuous under these topologies because it maps bounded sets in X^* to bounded sets in Z^*. Now $X^* \times Y^*$ is (DFS*) as a product of (DFS*) spaces and the mapping $v \times f: X^* \times Y^* \to Z^*$ is a one-one continuous linear mapping on the (DFS*) space $X^* \times Y^*$ onto the (DFS*) space Z^*. Apply Theorem 19 to the exact sequence $0 \to X^* \times Y^* \to Z^* \to 0$ and its dual. Then it follows that $(v \times f)'$ is an isomorphism of the Fréchet space $(Z^*)'$ onto the Fréchet space $(X^* \times Y^*)'$. Since $X^* \times Y^*$ and Z^* are reflexive, $v \times f$ is an isomorphism. Of course, Pták's open mapping theorem shows directly that $v \times f$ is open. The rest is the same as above. Since the bornologic topology is determined only by the family of bounded sets, the bornologic topology on B^* as a closed subspace of Z^* is the same as that on B^* as a closed subspace of X_2'.

If H^* is of countable dimension, choose a basis $\{e_j\}$ and then a representative f_j in Z^* from each class e_j. Let Y^* be the direct sum $\sum_{j=1}^{\infty}\boldsymbol{C}$ or $\sum_{j=1}^{\infty}\boldsymbol{R}$ and define $f(y)=\sum_{j=1}^{\infty}y_jf_j$. Then Y^* is (DFS) and f gives a cross-section of H^*.

Added in proof. Professon Raikov kindly informed the writer that some of the results of the paper had been obtained also by Makarov [**17**] and Raikov [**18**].

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