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Synthesis of asynchronous circuits

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In a series of papers [5, 6, 7] we have reorganized the theory of asynchronous circuits originated by the staffs of Digital Computer Laboratory, Univ. of Illinois [1, 2, 3, 4]. The aim of our previous papers was not to examine appropriateness of the fundamental concepts of the theory but to settle the theory on the security base by considering a mathematical system constructed over relations.

In the theory of asynchronous circuits the synthesis is one of the fundamental problems. Using our new formulation of the theory, we supply in this paper a synthesis procedure for binary, finite charts. Terminology of the paper relies heavily on the aforementioned series of papers [5, 6, 7].

In § 1 a congruence or equivalence relation, to be called a synthetic relation, is defined for given chart. There may be many synthetic relations for given chart, so that the definition is made implicitly by specifying their properties rather than explicitly stating the relation itself. In fact, it turns out in 1.6 Theorem that to define a synthetic relation for a chart is equivalent to give a synthesis procedure for that chart. Let (V, h) be a finite chart with a set J of nodes. A synthetic relation for (V, h) may be obtained from the synthetic relations for all $(V, h)|\{i, j\}$ and $(V, h)|\{i\}$, where i, j are distinct nodes. Therefore it is enough if we give a synthesis procedure for each (V, h) having at most two nodes. We distinguish in 1.9 four possible situations A, B, C and D for charts with at most two nodes.

In § 2 Lemmas are established for characterizing the situations. The characterization is of the type that it makes possible to define some congruences which turn out later in § 3 to be synthetic relations.

The synthesis procedure described in § 3 for each situation consists in adding new nodes in such a way that a congruence thereby introduced into the chart constitutes a synthetic relation for the chart. Except for the situation D the synthesis procedures are rather trivial, although proofs for the situations B and C are not so simple. In the last part § 4 of the paper an example is given.

For the further development of the theory the following problems would be important.

A. Find digital extensions for a binary finite chart which is minimal in some sense.

B. Develop a synthesis procedure which covers semi-modular rather than distributive state charts, or at least physical, finite state charts of the type.

C. Extend the theory so as to cover digital circuits whose signals are not integers but real numbers.

We are grateful to the referee for his efforts to make the paper more tangible.

§ 1. Synthetic relation

DEFINITION 1.1. A semi-modular state chart (V, h) is said to be *realizable* if there is a semi-modular extension (V^e, h^e) of (V, h), which is digital. If (V, h) is realizable, then $\boldsymbol{h}^e(\boldsymbol{V}^e)$ is a digital graph which is semi-modular with respect to $h^e(0^{J^e})$, by 4.3 of [7]. Then, by 5.10 of [7], the atlas A^e of the semi-modular circuit $\mathfrak{C}^e$ containing $\boldsymbol{h}^e(\boldsymbol{V}^e)$ subsumes the atlas A of $\mathfrak{C}$ containing $\boldsymbol{h}(\boldsymbol{V})$.

In particular, if the digital extension (V^e, h^e) is finite, then (V, h) is said to be *finitely realizable*. In this situation, (V, h) must be finite, by 5.12 of [7]. Since $\mathfrak{C}^e$ is finite, $\mathfrak{C}^e$ is speed independent with respect to $h^e(0^{J^e})$ by 4.8 of [6]. Then, by 8.6 of [5], the atlas A on $\mathfrak{C}$ is speed independent with respect to $h(0^J)$.

A procedure for constructing a digital, finite state chart forming an extension of a given semi-modular state chart is, in general, called a *synthesis procedure*. In this paper, however, we shall confine our attention to distributive states charts, and accordingly, as seen from 9.9 of [7], to charts.

DEFINITION 1.2. Let (V, h) be a finite chart with J as nodes. By $\simeq$ we shall denote a congruence or equivalence relation over V satisfying;

(1) if $A \simeq B$ then $A \sim B$, and

(2) the set of congruence classes $\{\pi, \cdots, \varphi\}$, called the $\simeq$ classes, is finite. If $A \simeq B$ then $h(A) = h(B)$ by (1). Therefore $h(\pi)$ is well-defined for each class π by taking $h(\pi) = h(A)$, where $A \in \pi$. A class π may be designated by $\pi(M)$ if π has a minimum state M. By K, which may be empty, we shall denote the set of unordered pairs (π, φ) of distinct classes such that $h(\pi) = h(\varphi)$. For an element $k = (\pi, \varphi)$ of the set K a finite extension (V^k, h^k) with J^k as nodes over (V, h) is said to be a *k-extension* with respect to $\simeq$ whenever

(3) if A^k, B^k are states of V^k such that $A^k|J \in \pi$ and $B^k|J \in \varphi$, then $h^k(A^k) \neq h^k(B^k)$, and

(4) if A^k, B^k are states of V^k such that $A^k|J \simeq B^k|J$ and $h^k(A^k) = h^k(B^k)$,

then $A^k \sim B^k$.

We say that the congruence $\simeq$ is a *synthetic relation* for the chart (V, h) if and only if either the set K is empty or there is a k-extension for each k of K. We refer to the $\simeq$ classes $\pi, \cdots, \varphi$, as *synthetic classes* with respect to $\simeq$. Furthermore, if an extension (V^e, h^e) of (V, h) is a p-extension for each p running through $m, \cdots, n$ then we say that (V^e, h^e) is an $(m, \cdots, n)$-extension of (V, h).

LEMMA 1.3. *Let (V, h) be a finite chart with J as nodes. If a synthetic relation is defined for (V, h) such that (V^l, h^l) and (V^k, h^k) are $(m, \cdots, n)$ and k-extensions respectively of (V, h) for $m, \cdots, n$, and k of K, then the amalgamation $(V^l, h^l) \otimes (V^k, h^k)$ is an $(m, \cdots, n, k)$-extension of (V, h).*

PROOF. Let (V^e, h^e) mean the amalgamation $(V^l, h^l) \otimes (V^k, h^k)$. Then (V^e, h^e) is a finite extension of (V, h) by 6.7 of [7]. Let A^e and B^e be states of V^e. If $A^e|J \in \pi$ and $B^e|J \in \varphi$ where $(\pi, \varphi) = p \in \{m, \cdots, n, k\}$, then $h^q(A^q) \neq h^q(B^q)$ by (3) where q is either l or k according as $p \in \{m, \cdots, n\}$ or $p = k$, and where $h^q = h^e|J^q$, $A^q = A^e|J^q$ and $B^q = B^e|J^q$. Then $h^e(A^e) \neq h^e(B^e)$, verifying (3). Suppose that $A^e|J \simeq B^e|J$ and $h^e(A^e) = h^e(B^e)$. It will be seen below that $A^e \sim B^e$. If so, the condition (4) is verified for (V^e, h^e) and (V^e, h^e) is an $(m, \cdots, n, k)$-extension of (V, h).

Since $h^e(A^e) = h^e(B^e)$, $h^q(A^q) = h^q(B^q)$ for $q = l, k$. Since $A^e|J \simeq B^e|J$, $A^q|J \simeq B^q|J$ and hence we have $A^q \sim B^q$ by (4). Let L^e be a state of $V^e_{A^e}$. Since $A^q \sim B^q$, $L^q + A^q$ is in V^q whenever $L^q + B^q$ is in V^q. Therefore $L^e + B^e \in V^e = V^l \otimes V^k$, which means that $L^e \in V^e_{B^e}$. Similarly we have that $L^e + B^e \in V^e$ implies $L^e + A^e \in V^e$, and hence $V^e_{A^e} = V^e_{B^e}$. Since $h^e_{A^e}(L^e)_j = h^q_{A^q}(L^q)_j = h^q_{B^q}(L^q)_j = h^e_{B^e}(L^e)_j$ for $j \in J^q$, $h^e_{A^e} = h^e_{B^e}$. Hence $A^e \sim B^e$.

THEOREM 1.4. *Let (V, h) be a finite chart with J as nodes. If a synthetic relation is defined for (V, h), then there is a finite digital extension (V^e, h^e) of (V, h). In particular, if the set K is empty, then $(V^e, h^e) = (V, h)$. Otherwise (V^e, h^e) is a k-extension of (V, h) for every k of the set K. Moreover, a chart (V^e, h^e) is a digital extension of (V, h) if (V^e, h^e) is a k-extension of (V, h) for each k of K.*

PROOF. Suppose that K is empty. Let A, B be states of V. Since there is no distinct classes π, φ such that $h(\pi) = h(\varphi)$, A and B belong to the same $\simeq$ class if $h(A) = h(B)$. Hence $A \sim B$ by (1) and (V, h) itself is digital.

Suppose that K is not empty. Then there is a k-extension (V^k, h^k) for each k of K, although some of the k-extensions may not be mutually distinct. Let $\{(V^m, h^m), (V^p, h^p), \cdots, (V^n, h^n)\}$ be the set of distinct (V^k, h^k)'s. Then by 1.3 the amalgamation $((\cdots((V^m, h^m) \otimes (V^p, h^p)) \otimes \cdots) \otimes (V^n, h^n))$, which we shall denote by (V^e, h^e), is a k-extension for each k of K. It is clear that (V^e, h^e) is a finite extension over (V, h).

Let us see that any extension (V^e, h^e) over (V, h) is digital if (V^e, h^e) is a k-extension for each k of K. Suppose that (V^e, h^e) is a k-extension for each k. Let A^e, B^e be states of V^e such that $h^e(A^e)=h^e(B^e)$. If $A^e|J \not\simeq B^e|J$, then $h^e(A^e) \neq h^e(B^e)$ by (3), entailing a contradiction. Therefore $A^e|J \simeq B^e|J$. Then by (4) we have that $A^e \sim B^e$. Therefore (V^e, h^e) is digital, completing the proof.

THEOREM 1.5. *Let a finite extension (V^e, h^e) of a chart (V, h) be digital. Define $A \overset{e}{\simeq} B$ for states A, B of V if and only if there are extensions A^e, B^e over A, B respectively such that $A^e \sim B^e$. Then $\overset{e}{\simeq}$ is a synthetic relation for (V, h). In fact, (V^e, h^e) is a k-extension of (V, h) for every k of the set K with respect to $\overset{e}{\simeq}$.*

We call $\overset{e}{\simeq}$ the induced synthetic relation.

PROOF. The reflexive and symmetric laws for $\overset{e}{\simeq}$ are trivial. Suppose that $A \overset{e}{\simeq} B$ and $B \overset{e}{\simeq} C$. Then there are states A^e, $B(0)^e$, $B(1)^e$ and C^e such that $A^e \sim B(0)^e$, $B(1)^e \sim C^e$ where $A^e|J=A$, $B(0)^e|J=B(1)^e|J=B$ and $C^e|J=C$. Let $B^e=B(1)^e \vee B(0)^e$. Then $B(0)^e \sim B(1)^e \sim B^e$ by 2.9 of [7]. Since $B(1)^e \sim C^e$ and $B^e-B(1)^e \in V^e_{B(1)^e}=V^e_{C^e}$, $B^e \sim C^e+B^e-B(1)^e$ by 2.2 of [7]. Similary we have that $B^e \sim A^e+B^e-B(0)^e$. Hence $C^e+B^e-B(1)^e \sim A^e+B^e-B(0)^e$. Moreover, $(C^e+B^e-B(1)^e)|J=C$ and $(A^e+B^e-B(0)^e)|J=A$. Therefore $A \overset{e}{\simeq} C$, proving the transitivity law. If $A \overset{e}{\simeq} B$ in V, then there are extensions A^e and B^e over A and B, respectively, such that $A^e \sim B^e$. Then by 5.11 of [7], $A \sim B$, proving (1) of 1.2.

Since (V^e, h^e) is finite and each similarity class corresponds to a class defined by $\overset{e}{\simeq}$, (2) of 1.2 is satisfied. If the set K is empty, then there is nothing to prove. Suppose that K is not empty. Let $A^e|J \in \pi$ and $B^e|J \in \varphi$ where $k=(\pi, \varphi) \in K$. If $h^e(A^e)=h^e(B^e)$, then $A^e \sim B^e$ because (V^e, h^e) is digital, a contradiction. Thus $h^e(A^e) \neq h^e(B^e)$, proving (3) of 1.2.

If $h^e(A^e)=h^e(B^e)$, then $A^e \sim B^e$, proving (4) of 1.2. Then $\overset{e}{\simeq}$ is a synthetic relation for (V, h) and also (V^e, h^e) is a k-extension for every k of K.

THEOREM 1.6. *A finite chart (V, h) is finitely realizable if and only if (V, h) admits a synthetic relation.*

This follows from 1.4 and 1.5.

LEMMA 1.7. *Let (V, h) be a finite chart with J as nodes admitting a synthetic relation $\simeq$. For non-empty subset J^p of J define a relation on $V^p(=V|J^p)$, say $A^p \overset{p}{\simeq} B^p$, if and only if there are extensions A, B of A^p, B^p such that $A \simeq B$. Then $\overset{p}{\simeq}$ is a synthetic relation for $(V, h)|J^p$.*

PROOF. Clearly $\overset{p}{\simeq}$ is a congruence. The conditions (1), (2) of 1.2 are trivially satisfied. Let (V^e, h^e) be a k-extension of (V, h) for every $k=(\pi, \varphi)$ of K, see 1.3, and let K^p be a set of pairs $K^p=(\pi^p, \varphi^p)$ of distinct $\overset{p}{\simeq}$ classes such that $h^p(\pi^p)=h^p(\varphi^p)$. Let us verify that (V^e, h^e) is a k^p-extension of (V^p, h^p) for each k. Then $\overset{p}{\simeq}$ is a synthetic relation.

If $A^p\in\pi^p$ and $B^p\in\pi^q$ where $k^p=(\pi^p, \varphi^p)$, then A and B are not in the same synthetic class where A, B are extensions, respectively of A^p, B^p over V. If $h(A)\neq h(B)$ then $h^e(A^e)\neq h^e(B^e)$ (where A^e, B^e are extensions respectively of A, B over V^e) proving (3) of 1.2. Otherwise $(\pi, \varphi)\in K$ and $h^e(A^e)\neq h^e(B^e)$ by (3) of 1.2 where $A\in\pi$, $B\in\varphi$. It proves (3) of 1.2. If $A^p\overset{p}{\simeq}B^p$ then there are extensions A, B such that $A\simeq B$. Thus $A^p\overset{p}{\simeq}B^p$ and $h^e(A^e)=h^e(B^e)$ implies $A^e\sim B^e$ by (4) of 1.2, which proves (4) of 1.2.

THEOREM 1.8. *Let (V, h) be a finite chart with J as nodes where J consists of at least two elements. Suppose that there is a synthetic relation $\overset{q}{\simeq}$ for each $(V, h)|q$ where $q=\{i, j\}$ is an unordered pair of distinct elements of J. Define $A\simeq B$ by taking $A^q\overset{q}{\simeq}B^q$ for every q, then $\cong$ is a synthetic relation for (V, h), where A^q, B^q are the grounds of A, B in q respectively. (For the definition of grounds, see* 5.1 *in* [7].)

PROOF. Clearly the relation $\simeq$ on V is a congruence and the condition (2) of 1.2 is satisfied. By Q we denote the set of $q=\{i, j\}$.

If $A\simeq B$ then $A^q\simeq B^q$ and $A^q\sim B^q$ for each q by (1) of 1.2. Suppose that for some $[\theta, p]$, $[\phi, c]\in\sum A+L\in V$, $[\theta, p]\leqq[\phi, c]$ and $(B+L)_p<\theta$. Since $A^d\sim B^d$, $(A+L)^d\in V^d$ and $(B+L)^d\in V^d$ where $d=\{p, c\}\in Q$, $(B+L)_c<\phi$. From this we conclude that $B+L\in V$, since this holds for any choice of $[\theta, p]$ and $[\phi, c]$, by 9.3 in [6]. It follows that $V_A\subset V_B$ and $V_B\subset V_A$ by the similar argument, and that $V_A=V_B$. Since $h_A(L)_j=h^q_{A^q}(L^q)_j=h^q_{B^q}(L^q)_j=h_B(L)_j$ where $j\in q$, $h_A=h_B$ and $A\sim B$, verifying (1) of 1.2.

By $(V^{f(q)}, h^{f(q)})$ we denote the digital extension with the nodes $J^{f(q)}$ of (V^q, h^q) obtained in 1.3. Then it is assumed without loss of generality that $J^{f(q)}\cap J=q$ and $J^{f(q)}\cap J^{f(d)}=q\cap d$ for any distinct q, d of Q. Then we may form the amalgamation $(V, h)\otimes(V^{f(q)}, h^{f(q)})$ for each q so that $(V, h)\otimes(V^{f(q)}, h^{f(q)})$, $q\in Q$, is a co-intersectional system of charts where $(V, h)\otimes(V^{f(q)}, h^{f(q)})|J=(V, h)$ for each q. Let (V^e, h^e) be the amalgamation $\otimes_q((V, h)\otimes(V^{f(q)}, h^{f(q)}))$, $q\in Q$. Then we shall see that (V^e, h^e) is a k-extension of (V, h) for every $k=(\pi, \varphi)$ of the set K with respect to the relation $\simeq$.

Suppose that $A^e|J\in\pi$ and $B^e|J\in\varphi$ where $(\pi, \varphi)\in K$. Then there is a q such that A^q and B^q are not in the $\overset{q}{\simeq}$ class where $A^q\in\pi^q$, and $B^q\in\varphi^q$ and π^q, φ^q are the synthetic classes with respect to $\overset{q}{\simeq}$. Since $(V^{f(q)}, h^{f(q)})$ is a (π^q, φ^q)-extension over (V^q, h^q) by 1.3, $h^{f(q)}(A^{f(q)})\neq h^{f(q)}(B^{f(q)})$ by (3) of 1.2.

Then $h^e(A^e) \neq h^e(B^e)$, proving (3) of 1.2.

Suppose that $A^e|J \simeq B^e|J$ and $h^e(A^e) = h^e(B^e)$. We shall prove that $A^e \sim B^e$ and verify (4) of 1.2, so that we may complete the proof. Suppose that $A^e + L^e \in V^e$, $[\theta, p] \leqq [\phi, c]$ and $(B^e + L^e)_p < \theta$. If $p, c \in J$, then $(B^e + L^e)_c < \phi$ since $A^e|J \simeq B^e|J$ implies $A^e|J \sim B^e|J$. If $p, c \in J^{f(q)} - J$, then $h^{f(q)}(A^{f(q)}) = h^{f(q)}(B^{f(q)})$. Since $(V^{f(q)}, h^{f(q)})$ is digital, $A^{f(q)} \sim B^{f(q)}$. Thus $A^{f(q)} + L^{f(q)} \in V^{f(q)}$ implies $B^{f(q)} + L^{f(q)}$, and hence $(B^e + L^e)_c < \phi$. If $p \in J^{f(q)} - J$ and $c \in J$ (or $p \in J$ and $c \in J^{f(q)} - J$), then there is a $[\rho, r]$ of $\sigma(V)$ where r is in J such that $[\theta, p] \leqq [\rho, r] \leqq [\phi, c]$ by 9.4 in [7]. Since $h^{f(q)}(A^{f(q)}) = h^{f(q)}(B^{f(q)})$ and $(V^{f(q)}, h^{f(q)})$ is digital, $(B^e + L^e)_r < \rho$. By the similar argument applied for (V, h) we have $(B^e + L^e)_c < \phi$. If $p \in J^{f(q)} - J$ and $e \in J^{f(d)} - J$ where $q \neq d$, then there is an $r \in J$ such that $[\theta, p] \leqq [\rho, r] \leqq [\phi, c]$. Then $(A^e + L^e)_p < \theta$ implies $(B^e + L^e)_c < \phi$ by an argument similar to the above one. Hence we conclude that if $A^e + L^e \in V^e$ then $B^e + L^e \in V^e$, and conversely. Hence $V^e_{A^e} = V^e_{B^e}$.

Let us prove that $h^e_{A^e} = h^e_{B^e}$. For any $J \in J^e$, there exists q such that $j \in J^q$ and $A^{f(q)} \sim B^{f(q)}$ because $(V^{f(q)}, h^{f(q)})$ is digital. Therefore $h^e_{A^e}(L^e)_j = h^e_{B^e}(L^e)_j$. Hence we have that $h^e_{A^e} = h^e_{B^e}$. Summing up we conclude that $A^e \sim B^e$.

REMARK. There are semi-modular state charts which violate the Theorem 1.8.

THEOREM 1.9. *A finite, binary chart is finitely realizable. In fact, it has a finite, binary digital extension.*

PROOF. Let (V, h) be a given finite, binary chart with J as nodes. By 1.6 and 1.8 it is enough to prove that each $(V, h)|\{i, j\}$ and each $(V, h)|\{i\}$ admit synthetic relations for $i, j \in J$ where $i \neq j$.

For $(V, h)|\{i, j\}$ there are the following possibilities, by the orthogonality as seen from 7.8 of [7]. For $(V, h)|\{i\}$, the possibilities are A and B.

A. It has no cycle.

B. It has a unique cycle z such that $z_j = 0$ for a j. (j may be empty.)

C. It has two cycles $z(i)$ and $z(j)$.

D. It has a cycle z such that $z_i \neq 0 \neq z_j$.

In § 2 the preliminary Lemmas PA, PB, PC and PD are provided for each of the cases A, B, C, D. Then in § 3 the synthesis procedures are established in the Lemmas SA, SB, SC and SD for the respective cases by giving synthetic relations for each of $(V, h)|\{i, j\}$ and $(V, h)|\{i\}$, thus completing the proof of this Theorem at the end of § 3.

§ 2. Preliminaries for the systhesis procedure

In the remainder of the paper, charts will be assumed as being finite and binary.

LEMMA 2.1. (PA) *Let (V, h) be a chart. If a similarity class T has no cycle, then T consists of just one state. In particular, if (V, h) has no cycle, then V is finite.*

PROOF. If T consists of at least two distinct states M and N, there is a state L covering M in T because $M < M \vee N$ and $M \vee N \in T$ by 2.9 of [7]. Then $z = L - M$ is a cycle of T, entailing a contradiction. Hence T consists of one state. Suppose that V is not finite. Since there is a finite number of similarity classes, one of them, say T, has infinitely many states. Then by the previous argument (V, h) has a cycle, which is a contradiction. Thus we conclude that (V, h) has no cycle whenever V is finite.

LEMMA 2.2. (PB) *Let (V, h) have a unique cycle z such that for some i $z_i \neq 0$ and $z_j = 0$ for each j where $j \neq i$. Then $z_i = 2$. Futhermore there is a change $[\theta, i]$ cyclicly spanned by z such that there is no change $[\phi, j]$ satisfying that $[\theta, i] < [\phi, i]$ for some j not equal to i where j may be empty.*

PROOF. By 10.8 of [7] there is a change $[\theta, i]$ cyclicly spanned by z such that there is no change $[\phi, j]$ satisfying $[\theta, i] < [\phi, j]$. Let us show that $z = 2\delta^i$. Since z is a cycle, $z \geqq 2\delta^i$. Let $M \leftrightarrow [\theta, i]$. (For $\leftrightarrow$ see 8.4 of [6].) Since $M \sim M+z$ and $M \leqq M+2\delta^i < M+z$, $N = M+2\delta^i \in V$ by 3.3 of [6]. Let us see that $V_M = V_N$. If so, since $h_M = h_N$, $M \sim N$. Therefore $2\delta^i \geqq z$ and $z = 2\delta^i$ by the previous result. If $M+L \in V$ then $M+z+L \in V$ and $M+L+2\delta^i = N+L \in V$. Conversely if $N+L \in V$, then $N+L+z \in V$, because $M < N$, and $M+L+z \in V$, because $N+L \leqq M+L+z \leqq N+L+z$. Therefore $M+L \in V$. Hence $V_M = V_N$, completing the proof.

LEMMA 2.3. (PC) *Let a chart (V, h) have just two nodes i and j. Suppose that there are two cycles $z(i)$ and $z(j)$ such that $z(i)_i \neq 0 \neq z(j)_j$. Then $z(i) = 2\delta^i$ and $z(j) = 2\delta^j$ and there are changes $[X_i, i]$ and $[X_j, j]$ cyclicly spanned by $z(i)$ and $z(j)$ respectively such that neither $[X_i, i] < [X_j, j]$ nor $[X_j, j] < [X_i, i]$.*

PROOF. By the orthogonality, see 7.8 of [7], we have that $z(i)_j = 0 = z(j)_i$. Let $T(i)^*$ and $T(j)^*$ be the unique minimal classes with respect to $z(i)$ and $z(j)$ of which existence is assured by 7.10 of [7], and M, N be the minimal states respectively of $T(i)^*$, $T(j)^*$, see 7.7 of [7]. Since $T(i)^*$ is the minimal class with respect to $z(i)$, there is a state Y such that for each integer $m \geqq M_i$, $Y_i = m$. Therefore we can conclude that $V_{M \vee N} = W^2$ where W is the set of non-negative integers. Since $M + az(i) \in V$ for each $a \in W$ and $V|\{i\}$ is a semi-modular subset, there is a state R of V such that $R_i = (M \vee N)_i$ and $R_j = M_j$. If $R = M \vee N$, then $V_R = W^2$. Suppose that $R \neq M \vee N$. Since there is a covering sequence $\{R = R(0), \cdots, R(r) = M \vee N\}$ in V and $V_{M \vee N} = W^2$, $V_R = W^2$. Since $V \ni M + az(i) \geqq R$ for some a, $V_{M+az(i)} = W^2$. Since $M \sim M + az(i)$, $V_M = V_{M+az(i)} = W^2$. Hence $z(j)$ is a cycle of M. Since $T(j)^*$ is minimal with respect to $z(j)$, $T(j)^* \succcurlyeq T(i)^*$. By the similar argument we have that $T(i)^* \succcurlyeq T(j)^*$.

and that $T(i)^* = T(j)^*$ and $M = N$.

Since $V_M = W^2$ and (V, h) is binary, $z(i) = 2\delta^i$ and $z(j) = 2\delta^j$.

By 10.8 of [7] there are changes $[X_i, i]$ and $[X_j, j]$ cyclicly spanned by $z(i)$ and $z(j)$ respectively such that neither $[X_i, i] < [X_j, j]$ nor $[X_j, j] < [X_i, i]$.

LEMMA 2.4. (PD) *Let a chart (V, h) have a unique cycle z such that $z_j \neq 0$ for each j. Then there is a change $[\theta, i]$ satisfying what follows. If $[\theta, i] \leftrightarrow X$, then $[X, j]$ is cyclicly spanned by z for each j. Let $M(p) \geqq X$ be the minimum state of a similarity class $T(p)$ having z as its cycle. Then for any p and q there is an integer k such that $M(p)_j < M(q)_j + kz_j$, for each j.*

Let $Z = kz$. Then states $\geqq X$ are classified into classes such that two states P and Q are in a same class if and only if $P \equiv Q \bmod Z$. Then the number of classes is finite and each class π has a minimum state and $h(\pi)$ is well defined by taking $h(\pi) = h(P)$ for any $P \in \pi$. Let M, N be the minimum states of distinct classes π, φ such that $h(\pi) = h(\varphi)$, where each class has a minimum state.

If $M_i < N_i$ then $M_i + Z_i > N_i$, and if $M_i = N_i$ and $M_j < N_j$ for some j then $M_j + Z_j > N_j$.

PROOF. By 10.8 of [7] a change $[\theta, i] \leftrightarrow X$ exists such that each $[X_j, j]$ is cyclicly spanned by z. Since (V, h) is finite, the existence of k is trivial. The relation $\equiv$ is a congruence which classifies states $\geqq X$ into finite classes. Also it is trivially seen that each π has a minimum state. Let us show that the final part of the Lemma is true.

Suppose that $M_i < N_i$ and $M_i + Z_i \leqq N_i$. Since $[N_i - Z_i, i]$ and $[N_i, i]$ are cyclicly spanned by z, $N(i) + Z \leftrightarrow [N_i, i]$ by 10.8 of [7] and $N \geqq N(i) + Z$ by 8.4 of [6], where $[N_i - Z_i, i] \leftrightarrow N(i)$. Since $N(i) + Z \sim N(i)$, $N - (N(i) + Z) \in V_{N(i)}$ and hence $N - Z \in V$. Since $N - Z \geqq N(i) \geqq X$, $N - Z$ and N are in the class $\varphi(N)$, contradicting that N is the minimum state in $\varphi(N)$. This proves that $M_i + Z_i > N_i$.

Next suppose that $M_i = N_i$ and $M_j < N_j$ for some j. Assume that $Q \geqq X$ be the minimum state in the similarity class containing N and that $N = Q + qz$. Since N is the minimum state, we have $0 \leqq q < k$. Let $[Q_j, j] \leftrightarrow Q(j)$, $[Q_j + qz_j, j] \leftrightarrow N(j)$ and $[M_j, j] \leftrightarrow M(j)$ for each $j \in J$. Then by 8.2 of [6] Q, M and N are joins of respectively $Q(j)$, $M(j)$, $N(j)$ where j runs through J. Since $M_i = N_i = Q_i + qz_i$, $M \geqq M(i) = Q(i) + qz \leftrightarrow [Q_i + qz_i, i] > [Q_i, i] \leftrightarrow Q(i)$. Let $R \geqq X$ be the minimum state of the similarity class containing $Q(i)$. Because Q, R are the minimum states $\geqq X$ of the similarity classes containing N and $Q(i)$ respectively, $Q < R + Z$ by the property of Z. Since $R \leqq Q(i)$, $R + Z \leqq Q(i) + Z$. Hence $Q < R + Z \leqq Q(i) + Z$. Therefore $N_j = Q_j + qz_j < Q(i)_j + Z_j + qz_j$. Since $M \geqq Q(i) + qz$, $Q(i)_j + Z_j + qz_j \leqq M_j + Z_j$. Therefore $N_j < M_j + Z_j$, completing the proof.

§ 3. Systhesis procedure

In this section a synthesis procedure, consisting in adding new nodes to a chart $[\Sigma, H]$ having J as nodes will be described. As in § 2 only finite, binary charts are considered.

DEFINITION 3.1. For the sake of convenience, it is assumed that the set J of nodes does not contain 0.

We shall, in fact, use two synthesis procedures of types α and β according as the cases.

(Type α) Suppose that a congruence is defined on V, which satisfies (1), (2) of 1.2. By K we shall denote the set of pairs (π, φ) of distinct classes such that $h(\pi)=h(\varphi)$.

For a $(\pi, \varphi)\in K$, choose a particular state M of π and a node i of J (the choice is dependent on the cases) and add the new node 0 by taking $\Sigma^k=\Sigma\cup[1, 0]$ where $\Sigma^k\supset\Sigma$ and $[M_i+1, i]\ll[1, 0]<[\theta, j]$ in Σ^k if and only if $[M_i+1, i]<[\theta, j]$ in Σ, and H^k where $H^k|J=H, H^k[0, 0]=0$ and $H^k[1, 0]=1$. It is easily seen that $[\Sigma^k, H^k]$ is binary change chart having nodes $J^k=J\cup\{0\}$ which is a binary extension of $[\Sigma, H]$, this extension of $[\Sigma, H]$ is said to be of type α with respect to M and i.

(Type β) The type β of the synthesis procedure is only needed for the case D. In the situation of 2.4, the states $\geqq X$ are classified into classes. Let $k=(\pi(M), \varphi(N))$ be a pair of the distinct classes such that $h(\pi)=h(\varphi)$ and M, N be the minimum states of π, φ respectively. Since $h(M)=h(N)=h(M+Z)$ and the chart is binary, we may assume that there exists a node r of J such that $M_r+1<N_r$ and $M_r+Z_r>N_r+1$, by 2.4. Let us add new node 0 by taking

$\Sigma^k=\Sigma\cup\{[\eta, 0]|\eta>0\}$ where $\Sigma\subset\Sigma^k$ and

(i) $[\theta, p]<[2m+1, 0]\ll[N_r+mZ_r, r]$ if and only if $[\theta, p]<[N_r+mZ_r, r]$ in Σ where $m\geqq 0$, and

(ii) $[N_r+mZ_r+1, r]\ll[2m+2, 0]<[\phi, j]$ if and only if $[N_r+mZ_r+1, r]<[\phi, j]$ in Σ where $m\geqq 0$ (for $\ll$, see 3.1 of [6]) and H^k where $H^k|J=H$ and $H^k[\eta, 0]=\eta \bmod 2$.

Then $[\Sigma^k, H^k]$ is a binary chart with the nodes $J^k=J\cup\{0\}$, which is an extension over $[\Sigma, H]$, called the extension of type β.

LEMMA 3.2. *Let $[\Sigma^k, H^k]$ be an extension over $[\Sigma, H]$ of type β. Define Z^k by taking $Z^k|J=Z$ and $Z_0^k=2$ and X^k by taking $X^k|J=X$ and $X_0^k=0$. Then*

(iii) *$[\theta, p]\leqq[\phi, q]$ in Σ^k where $[X_p, p]\leqq[\theta, p]$ and $[X_p, q]\leqq[\phi, q]$ in Σ^k if and only if $[\theta+Z_p^k, p]\leqq[\phi+Z_p^k, q]$.*

PROOF. If $[X_p, p]<[\theta, p]<[2m+1, 0]$ where $p\neq 0$ then $[\theta, p]<[N_r+mZ_r^k, r]$ by (i). Hence $[\theta+Z_p^k, p]<[N_r+(m+1)Z_r^k, r]$ by 10.8 of [7]. Therefore $[\theta+Z_p^k, p]<[2m+1+Z_0^k, 0]$ by (i), since $Z_0^k=2$. Conversely if $\theta\geqq X_p$ and $[\theta+Z_p^k, p]<[2m+1+Z_0^k, 0]$ then $[X_p, p]<[\theta, p]<[2m+1, 0]$.

If $[X_p, p] < [\theta, p] < [2m+2, 0]$ where $p \neq 0$ then $[\theta, p] \leqq [N_r+mZ_r^k+1, r] \ll [2m+2, 0]$ by (ii) and $[\theta+Z_p^k, p] \leqq [N_r+(m+1)Z_r^k+1, r] < [2m+2+Z_0^k, 0]$ by (ii) and 10.8 of [7]. Therefore $[\theta+Z_p^k, p] < [2m+2+Z_0^k, 0]$. The converse is also true. Hence $[\theta, p] \leqq [\eta, 0]$ if and only if $[\theta+Z_p^k, p] \leqq [\eta+Z_0^k, 0]$ where $p \in J^k$. By the similar argument we see that $[\eta, 0] \leqq [\phi, q]$ in Σ^k if and only if $[\eta+Z_0^k, 0] \leqq [\phi+Z_q^k, q]$. This result and 10.8 of [7] prove (iii).

LEMMA 3.3. *The extension* $[\Sigma^k, H^k]$ *of* $[\Sigma, H]$ *of type* α *and* β *is finite.*

PROOF. For each similarity class T of the extension (V^k, h^k), let us correspond a $\simeq$ class π by taking if there is a state M^k of T satisfying that $M^k|J \in \pi$. (The correspondence may not be single valued.)

Suppose that M^k, N^k are states of V^k such that $M^k|J$, $N^k|J$ are contained in the same class π. If $h^k(M^k) = h^k(N^k)$, then $M^k \sim N^k$ by (4) of 1.2. However, since $h(M^k|J) = h(N^k|J)$ and (V^k, h^k) is binary, either $h^k(M^k)_0 = h^k(N^k)_0$ or $|h^k(M^k)_0 - h^k(N^k)_0| = 1$.

Therefore there are at most two similarity classes corresponding to the $\simeq$ class. Since the number of the $\simeq$ classes is finite, (V^k, h^k) is finite.

LEMMA 3.4. (SA) *Let a chart* $[\Sigma, H]$ *have no cycle. Define* $M \simeq N$ *if and only if* $M = N$. *Then* $\simeq$ *is the synthetic relation.*

PROOF. By 2.1, (1) and (2) of 1.2 are satisfied. Each $\simeq$ class consists of one state in this case. For each $k = (M, N)$ of K it may be assumed that $M_i+1 < N_i$ for some i of J, because $h(M) = h(N)$ and $M \neq N$. Let $[\Sigma^k, H^k]$ be the extension of type α with respect to M and i. Let (V^k, h^k) be the induced chart. The condition (4) of 1.2 is trivially satisfied because $M^k = N^k$ whenever $M^k|J \simeq N^k|J$ and $h^k(M^k) = h^k(N^k)$. Suppose that $M^k, N^k \in V^k$ such that $M^k|J = M$, $N^k|J = N$. Since $[M_i+1, i] \ll [1, 0]$, $M_0^k < 1$, that is $M_0^k = 0$. Since $[1, 0] < [N_i, i]$, $N_0^k \geqq 1$ and $N_0^k = 1$. Therefore $h^k(M^k) \neq h^k(N^k)$, verifying the condition (3) of 1.2.

Thus $[\Sigma^k, H^k]$ is the required k-extension.

LEMMA 3.5. (SB) *Let a chart* $[\Sigma, H]$ *or* (V, h) *have a single cycle* z *such that* $z_i \neq 0$ *and* $z_j = 0$ *for an* i *and* $j \neq i$. *Using* 2.2 (PB), *define* $M \simeq N$ *if and only if*

$$M = N \quad \text{if either } M \ngeqq X \text{ or } N \ngeqq X,$$

$$M \sim N \quad \text{if } M, N \geqq X, \text{ where } X \leftrightarrow [\theta, i].$$

Then $\simeq$ *is the synthetic relation.*

PROOF. The conditions (1) and (2) of 1.2 are satisfied, because (V, h) is finite and the number of states $M \ngeqq X$ is finite.

It is also seen that there is the minimum state M in each $\simeq$ class π.

For $k = (\pi(M), \varphi(N))$ of the set K there are three cases,

(1) $$\pi(M) = \{M\}, \qquad \varphi(N) = \{N\}$$

(2) $\pi(M)=\{M\}, \quad \varphi(N)\neq\{N\}$

(3) $\pi(M)\neq\{M\}, \quad \varphi(N)\neq\{N\}$.

For (1) and (2) we may assume that there is a node r such that $M_r+1<N_r$. However, for (3) there is a node $r\neq i$ such that $M_r+1<N_r$. In any case, let $[\Sigma^k, H^k]$ be the extension of type α with respect to M and r. Then the condition (3) of 1.2 is verified by the corresponding argument in 3.5 (SA). Let us verify (4) of 1.2. Suppose that $A^k|J\simeq B^k|J$ in V and $h^k(A^k)=h^k(B^k)$. Then it is shown below $V^k_{A^k}=V^k_{B^k}$ and hence $A^k\sim B^k$, because $h^k(A^k)=h^k(B^k)$ and the chart is binary.

Since $h^k(A^k)=h^k(B^k)$, $A^k_0=B^k_0$. Hence, if $A^k|J=B^k|J$ then $A^k=B^k$ and $A^k\sim B^k$. Then it is assumed that $A^k|J\neq B^k|J$, and then $A^k|J$, $B^k|J\geqq X$. Suppose that $A^k+L^k\in V^k$, $[\eta, s]\leqq[\phi, j]$ and $(B^k+L^k)_s<\eta$. Then $(B^k+L^k)|J\in V$, because $A^k|J\simeq B^k|J$ implies $A^k|J\sim B^k|J$. If $[\eta, s], [\phi, j]\in\Sigma$ then $(B^k+L^k)_j<\phi$, since $(B^k+L^k)_s<\eta$ where $[(B^k+L^k), j]\in\Omega$. Then suppose that $[1, 0]\leqq[\phi, j]$ and $(B^k+L^k)_0<1$. If $j\neq i$, then $A^k_j=B^k_j$ by 2.2, because $A^k|J\sim B^k|J$. Since $(A^k+L^k)_0=(B^k+L^k)_0=0<1$, $(A^k+L^k)_j=(B^k+L^k)_j<\phi$. Assume that $[1, 0]\leqq[\phi, i]$, then it is shown that $r=i$. And then $[1, 0]\leqq[\phi, i]$ may not be happen as shown below. Suppose that $r\neq i$. Since $[M_r+1, r]\ll[1, 0]\leqq[\phi, i]$. $M_r+1\leqq P_r$ where $P\leftrightarrow[\phi, i]$. Since $B^k|J\geqq X$, $X_i\leqq B^k_i$. Since $[1, 0]\leqq[\phi, i]$ and $B^k_0=0<1$, $B^k_i<\phi$. Hence $[X_i, i]<[\phi, i]$, and $X<P$ by 8.4 of [6]. Since $[X_i, i]$ is cyclicly spanned by z and $r\neq i$, $X_r=P_r$. Let $[X_r, r]\leftrightarrow X(r)$, then $X(r)\leqq X$ by 8.1 of [6]. Let $[M_r+1, r]\leftrightarrow M(r)$. Since $M_r+1\leqq P_r=X_r$, $M(r)\leqq X(r)$. Since $M(r)\leqq X(r)\leqq X$ and $X\leftrightarrow[X_i, i]$, $[M_r+1, r]\ll[1, 0]<[X_i, i]$. Since $[1, 0]<[X_i, i]$ and $B^k_0=0$, $B^k_i<X_i$, contradicting $B^k_i\geqq X_i$. It proves that $r=i$. Then suppose that $r=i$ and then $[M_i+1, i]\ll[1, 0]\leqq[\phi, i]$. Since $r=i$, this is not the case (3) and it is assumed that $\pi(M)=\{M\}$, that is, $M_i<X_i$. Hence $[M_i+1, i]\ll[1, 0]<[X_i+1, i]$. From $[1, 0]<[X_i+1, i]$ and $A^k_0=B^k_0=0$ it follows $A^k_i, B^k_i<X_i+1$. However since $X\leqq A^k$ and B^k, $X_i\leqq A^k_i, B^k_i$ and $A^k_i=B^k_i=X_i$. Hence $A^k|J=B^k|J$, contradicting the assumption $A^k|J\neq B^k|J$.

It remains to prove that if $[\eta, s]<[1, 0]$ and $(B^k+L^k)_s<\eta$, then $(B^k+L^k)_0<1$. Suppose that $s\neq i$. Then $A^k_s=B^k_s$ and $(B^k+L^k)_s=(A^k+L^k)_s<\eta$. Since $h^k(A^k)_0=h^k(B^k)_0$, $(B^k+L^k)_0=(A^k+L^k)_0<1$. On the other hand it is seen below that if $[\eta, s]<[1, 0]$ then $s\neq i$, and then it verifies that if $A^k+L^k\in V^k$ then $B^k+L^k\in V^k$ and conversely. Hence $V^k_{A^k}=V^k_{B^k}$, completing the proof of the Lemma. Assume $s=i$. Since $[M_r+1, r]\ll[1, 0]$, $[\eta, i]\leqq[M_r+1, r]\ll[1, 0]$. Since $X\leqq B^k|J$ and $(B^k+L^k)_i<\eta$, $X_i\leqq B^k_i<\eta$. It follows that $[X_i, i]<[\eta, i]\leqq[M_r+1, r]$. Hence $r=i$, by 2.2. Then $[\eta, i]\leqq[M_i+1, i]\ll[1, 0]$. Hence this is not the case (3) and it is assumed that $\pi(M)=\{M\}$ and then $M_i<X_i$. Then $[M_i+1, i]\leqq[X_i, i]$ contradicting $[X_i, i]<[\eta, i]\leqq[M_i+1, i]$.

LEMMA 3.6. (SC) *Let a chart* (V, h) *have just two nodes* i *and* j, *and two cycles* $z(i)$, $z(j)$ *such that* $z(i)_i \neq 0 \neq z(j)_j$. *Let* $X = X(i) \vee X(j)$ *where* $X(i) \leftrightarrow [X_i, i]$ *and* $X(j) = [X_j, j]$, *using* 2.3(PC). *Define* $M \cong N$ *if and only if*

$M \sim N$ *if* (1) *either* $M, N \geqq X$ *or* (2) $M_i, N_i \geqq X_i$ *and* $M_j = N_j < X_j$ *or* (3) $M_j, N_j \geqq X_j$ *and* $M_i = N_i < X_i$,

(4) $M = N$, *otherwise.*

Then $\simeq$ *is the synthetic relation.*

PROOF. The conditions (1) and (2) of 1.2 are trivially satisfied and for each π, there is the minimum state. Then for each $k = (\pi(M), \varphi(N))$ of the set K it is assumed that there is an r such that $M_r + 1 < N_r$.

Let $[\Sigma^k, H^k]$ be the extension of $[\Sigma, H]$ with respect to M and r. Let us verify that, $[\Sigma^k, H^k]$ is the $(\pi(M), \pi(N))$-extension. Since the condition (4) of 1.2 is easily verified, it is enough to verify (3) of 1.2.

Assume that $A^k|J \simeq B^k|J$, $h^k(A^k) = h^k(B^k)$ and $A^k + L^k \in V^k$, where it is assumed that $A^k|J \neq B^k|J$ otherwise the condition (3) of 1.2 becomes trivial. Under the assumption, however it is also enough to prove $B^k + L^k \in V^k$ in the case $A^k|J$, $B^k|J \geqq X$, because other cases may be treated by the similar argument in 3.5 (SB). Then assume that $A^k|J$, $B^k|J \geqq X$. If $[\eta, s] \leqq [\phi, t]$ in Σ, then $(B^k + L^k)_t < \phi$ follows from the argument in 3.5 (SB), which treated similar situation. Suppose that $[1, 0] < [\phi, t]$ and $(B^k + L^k)_0 < 1$, where it may be assumed that $M_r < X_r$ otherwise $M \geqq X$ and $N \geqq X$ and $h(M) = h(N)$ implies $M \sim N$ and $\pi(M) = \pi(N)$. Then we may assume that $[M_r + 1, r] \ll [1, 0] < [X_r + 1, r]$. From $A^k, B^k \in V^k$, $[1, 0] < [\phi, t]$, $(B^k + L^k)_0 < 1$, $h^k(A^k) = h^k(B^k)$, it follows that $B_0^k = A_0^k = 0$ and $B_r^k, A_r^k < X_r + 1$. Since $A^k|J$, $B^k|J \geqq X$, $A_r^k = B_r^k = X_r$. If $t = r$, $[M_r + 1, r] \ll [1, 0] < [\phi, r]$ and $(A^k + L^k)_r = (B^k + L^k)_r < \phi$. If $t \neq r$, then by the last argument in 3.5 (SB) we may prove that $[M_r + 1, r] \ll [1, 0] < [X_t, t]$. Since $A_0^k = 0$, $A_t^k < X_t$, a contradiction. Then $[1, 0] < [\phi, t]$ may not appear. Finally suppose that $[\eta, t] < [1, 0]$ and $(B^k + L^k)_t < \eta$. Since $X_t \leqq (B^k + L^k)_t < \eta$, $[\eta, t]$ is cyclicly spanned by $z(t)$ and $r = t$. Since $M_r < X_r$ and $(B^k + L^k)_r < \eta$ and $[M_r + 1, r] \ll [1, 0] < [X_r, r]$, $(B^k + L^k)_r < \eta \leqq M_r + 1 \leqq X_r \leqq B_r$, a contradiction. Then $[\eta, t] < [1, 0]$ and $(B^k + L^k)_t < \eta$ may not appear. It proves $B^k + L^k \in V^k$ and then $A^k \sim B^k$, verifying (3) of 1.2.

LEMMA 3.7. (SD) *Let a chart* $[\Sigma, H]$ *have a unique cycle* z *such that* $z_j \neq 0$ *for each* j. *Using* 2.4, (PD), *define* $M \simeq N$ *if and only if*

$$M \equiv N \text{ if } M, N \geqq X, \text{ and}$$

$$M = N, \text{ otherwise.}$$

Then $\simeq$ *is the synthetic relation.*

PROOF. Since (V, h) is finite and each $[X_j, j]$ is cyclicly spanned by z, the condition (1) and (2) of 1.2 are easily verified. Now let us show each $k=(\pi(M), \varphi(N))$ of the set K has the k-extension $[\Sigma^k, H^k]$. In fact, two synthesis procedures SDα and SDβ will be used to get $[\Sigma^k, H^k]$ according as the cases.

(SDα) If $M \geqq X$ is not true, then $M=\pi(M) \neq \varphi(N)$ and $h(M)=h(N)$. Then there is an $r \in J$ such that $M_r+1<N_r$ where $r=i$ if $N>X$. Let $[\Sigma^k, H^k]$ be the extension of $[\Sigma, H]$ of type α with respect to M and r. Let us see that $[\Sigma^k, H^k]$ is the $(M, \varphi(N))$-extension. Since $[M_r+1, r] \ll [1, 0]$, $A^k|J=M$ implies $A_0^k=0$. Since $[1, 0]<[N_r, r]$, $B^k|J \in \varphi(N)$ implies $B_0^k=1$. Hence $h^k(A^k) \neq h^k(B^k)$, proving (3) of 1.2.

Let $A^k, B^k \in V^k$ such that $A^k|J \simeq B^k|J$ and $h^k(A^k)=h^k(B^k)$. Since $h^k(A^k)=h^k(B^k)$, $A_0^k=B_0^k$.

Then $A^k=B^k$ and $A^k \sim B^k$ if $A^k|J=B^k|J$. Then it is assumed that $B^k|J=(A^k|J)+mZ$, $m>0$, where $A^k|J \geqq X$. At first, let us see that $A_0^k=1$. Suppose that $A_0^k=0$. Then $B_0^k=0$. Since $[1, 0]<[N_r, r]$ and $B^k|J \geqq A^k|J \geqq X$, $X_r \leqq A_r \leqq B_r \leqq N_r$ by (2)$'$ of 9.3 of [**6**]. If $N \geqq X$ does not hold then $N_i<\theta=X_i$ by 8.1 of [**6**], because $[\theta, i] \leftrightarrow X$. Then $(N \vee X)_i=X_i$ and $X_r \leqq (N \vee X)_r$. If $N \geqq X$, then $N \vee X=N$. In any case, $N_r=(N \vee X)_r$ where $r=i$ if $N>X$, and X and $N \vee X$ are the minimum states in their synthetic classes whose states are $\geqq X$ whenever $N \neq X$. Then $N_r=(N \vee X)_r<X_r+Z_r$ by 2.4. Since $X_r \leqq A_r^k \leqq B_r^k \leqq N_r$, $B_r^k<X_r+Z_r<A_r^k+Z_r$, contradicting $(B^k|J)=(A^k|J)+mZ_r$, $m>0$. Hence $A_0^k=1$.

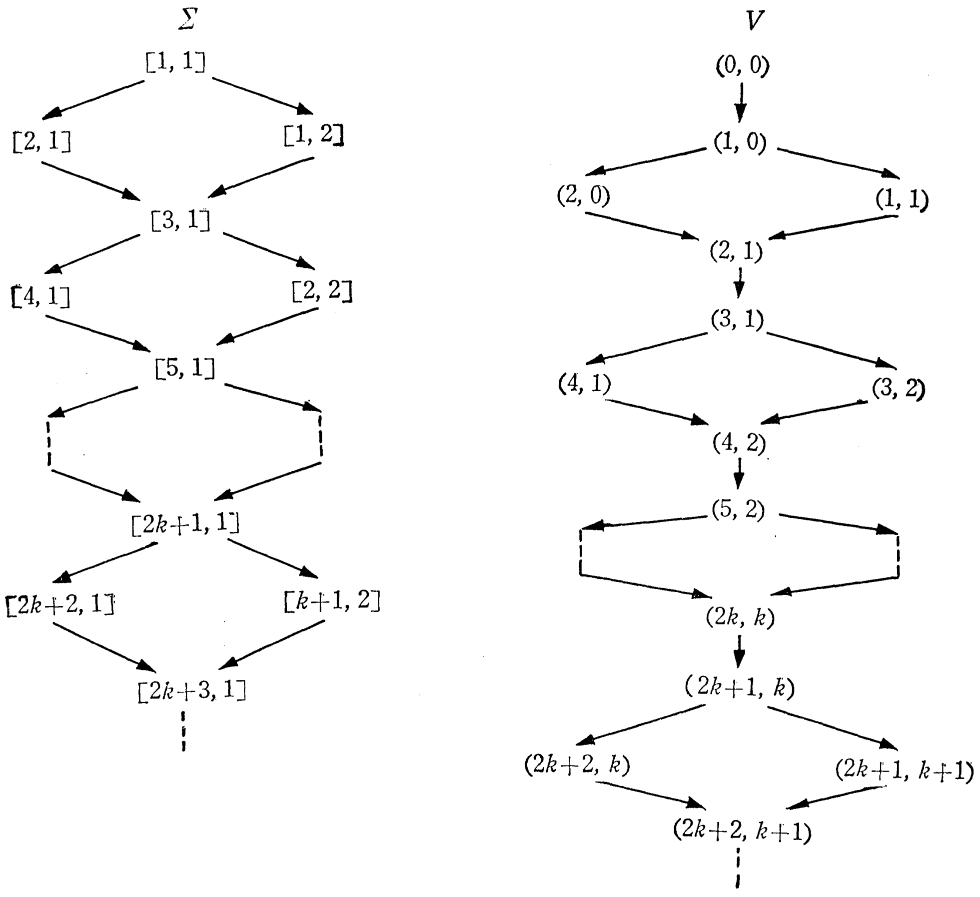
Suppose that $A^k+L^k \in V^k$, $[\eta, t] \leqq [\phi, j]$ in Σ^k and $(B^k+L^k)_t<\eta$, where $[(B^k+L^k)_j, j] \in \Omega^k$. Then $[\eta, t] \neq [1, 0]$ and $[\phi, j]=[1, 0]$, otherwise $B_0^k=A_0^k=0$, a contradiction. Hence $[\eta, t] \leqq [\phi, j]$ in Σ. Since $(B^k+L^k)|J \in V$, $(B^k+L^k)_j<\phi$ and then $B^k+L^k \in V^k$. Therefore $V_{A^k}^k=V_{B^k}^k$ and $A^k \sim B^k$, completing (4) of 1.2.

(SDβ) If $M, N \geqq X$ where $(\pi(M), \varphi(N)) \in K$, then the extension $[\Sigma^k, H^k]$ of type β described in 3.1 is the required $(\pi(M), \varphi(N))$-extension. Let $A^k|J \in \pi$ and $B^k|J \in \varphi$. Then $A^k|J=M+mZ$ and $B^k|J=N+nZ$ by the definition of $\simeq$, where $m, n \geqq 0$ are integers. Then we may write $A^k=A(m)^k$ and $B^k=B(n)^k$. By (i) of 3.1 $[2m+1, 0] \ll [N_r+mZ, r]$. Then $B(m)_0^k \geqq 2m+1$ by (2)$'$ of 9.2 of [**6**]. By (ii) of 3.1 $[N_r+mZ_r+1, r] \ll [2m+2, 0]$. Then $B(m)_0^k<2m+2$ by (2) of 9.2 of [**6**]. Hence $B(m)_0^k=2m+1$. Since $M_r+1<N_r$, $[M_r+mZ_r+1, r]<[2m+1, 0] \ll [N_r+mZ_r, r]$ by (i) of 3.1 and $A(m)_0^k<2m+1$. In particular if $m=0$ then $A(0)_0^k=0$. Since $M_r+Z_r>N_r+1$, $[N_r+mZ_r+1, r]<[M_r+(m+1)Z_r, r]$. Then by (ii) of 3.1 $[N_r+mZ_r+1, r] \ll [2m+2, 0]$, and $[2m+2, 0]<[M_r+(m+1)Z_r, r]$. Therefore $2m+2 \leqq A(m+1)_0^k$ by (2)$'$ of 9.3 of [**6**], and then $A(m)_0^k=2m$. Hence $h^k(A(p)^k) \neq h^k(B(q)^k)$ for any integers $p, q \geqq 0$, completing (3) of 1.2. Suppose that $A^k|J=A \simeq B=B^k|J$ and $h^k(A^k)=h^k(B^k)$. Then it is assumed that $A \neq B$, otherwise (4) of 1.2 is trivially satisfied. If $A^k_r < N^k_r$ then A^k_0 is either 0 or 1 because $[N_r+1, r] \ll [2, 0]$ by (ii) of 3.1. If $N_r+mZ^k_r \leqq A^k_r \leqq N_r+mZ^k_r+1$, then A^k_0 is either $2m+1$ or $2m+2$, because of $[N_r+mZ_r+1, r] < [M_r+(m+1)Z_r, r] < [2m+3, 0]$ and (i) of 3.1. And if $N_r+mZ^k_r+1 < A^k_r < N_r+(m+1)Z^k_r+1$ then A^k_0 is either $2m+2$ or $2m+3$ by $[N_r+(m+1)Z^k_r+1, r] \ll [2m+4, 0]$ and (i), (ii) of 3.1. Since we may assume $B^k|J=(A^k|J)+sZ$ for some integer $s>0$, the same is true if m is replaced by $m+s$. Hence if $A<B$ then $B^k = A^k+sZ^k$, because $Z^k_0=2$.

Now let us prove that $V^k_{A^k} = V^k_{A^k+sZ^k}$ using (iii) of 3.2. Suppose that $A^k+L^k \in V^k$, $[\theta, p] \leqq [\phi, q]$ in Σ^k and $(A^k+sZ^k+L^k)_p < \theta$, where $[A^k_j+sZ^k_j+L^k_j, j] \in \Omega^k$. Then by (iii) $[\theta - sZ^k_p, p] \leqq [\phi - sZ^k_q, q]$ and $(A^k+L^k)_q < \theta - sZ^k_q$, because $X \leqq A^k|J$, $B^k|J$. Then $(A^k+L^k)_q < \phi - sZ^k_q$ and $(A^k+sZ^k+L^k)_q < \phi$, that is, $A^k+L^k+sZ^k \in V^k$. By the similar argument, if $A^k+L^k+sZ^k \in V^k$ then $A^k+L^k \in V^k$. Hence $V^k_{A^k} = V^k_{A^k+sZ^k}$. Hence $A^k \sim A^k+sZ^k = B^k$, completing the proof of (4) of 1.2.

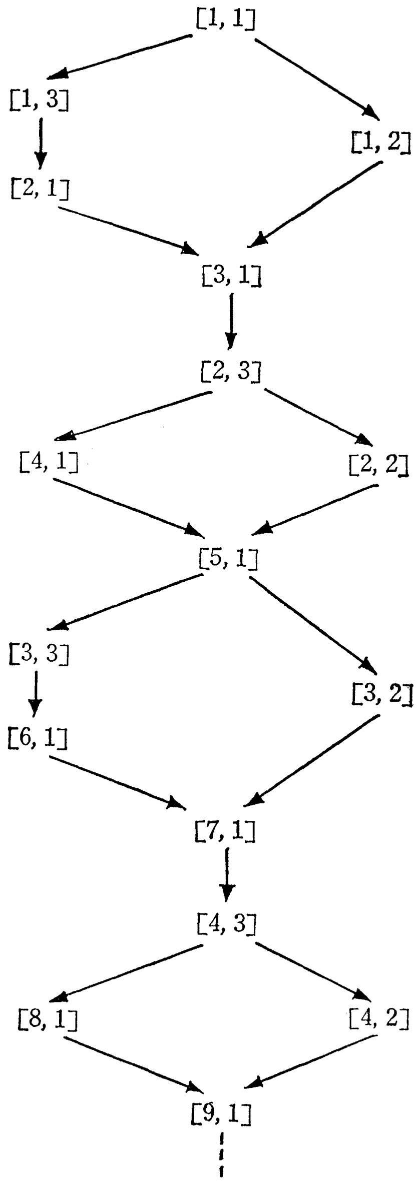
§ 4. An example

Let us synthesize a simple chart. Since our chart is binary, it is enough to give the change diagram Σ or distributive subset V which are shown below where $X \to Y$ means that Y covers X.

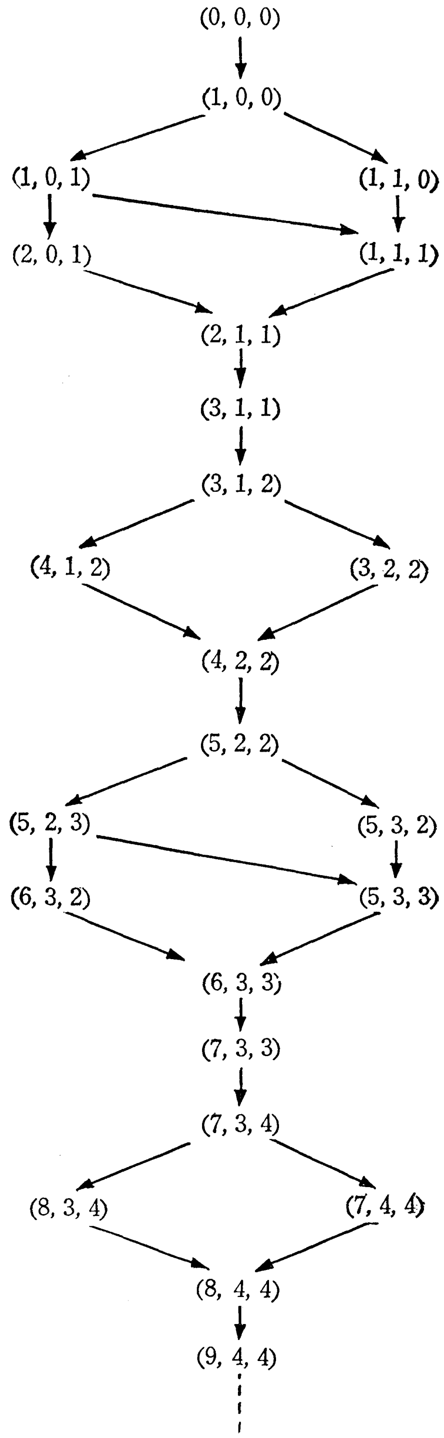


The change chart could be a change chart for a binary counter for every change on node 2 implies two changes occuring at node 1. The chart is not digital, because V has a cycle $z=(4, 2)$ spanning nodes 1 and 2, contradicting 10.9 of [7]. The chart has eight similarity classes which are represented by (0, 0), (1, 0), (2, 0), (1, 1), (2, 1), (3, 1), (4, 1) and (3, 2) respectively, where each representative is the minimum state of the similarity class. The chart will be treated by 2.4 (PD) and 3.7 (SD). In view of 2.4 $[\theta, i]=[1, 1]$, $X=(1, 1)$ and $Z=z$. Since there is no harm in this case, we will apply 3.7 (SD) by taking $X=(0, 0)$, that is, the synthetic relation is the similarity relation. Here $K=\{((0, 0), (2, 0)), ((1, 0), (3, 2)), ((1, 1), (3, 1)), ((2, 1), (4, 1))\}$. The following Σ^1 is the ((0, 0), (2, 0))-extension of Σ of type β where $r=1$ and the new node is 3. In fact, Σ^1 is also a ((1, 0), (3, 2))-extension of Σ. Similary Σ^2 is the ((1, 1), (3, 1))-extension of Σ of type β where $r=1$ and the new node is 4. It turns out that Σ^2 is also a ((2, 1), (4, 1))-extension of Σ. $\Sigma^1 \cup \Sigma^2 = \Sigma^e$ is the digital extension required of Σ.

Σ^1

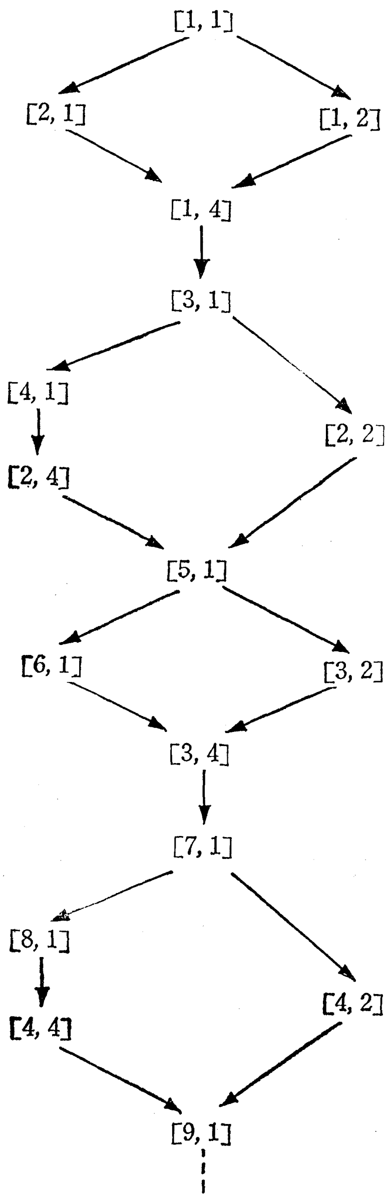


$\mu(\Sigma^1)$

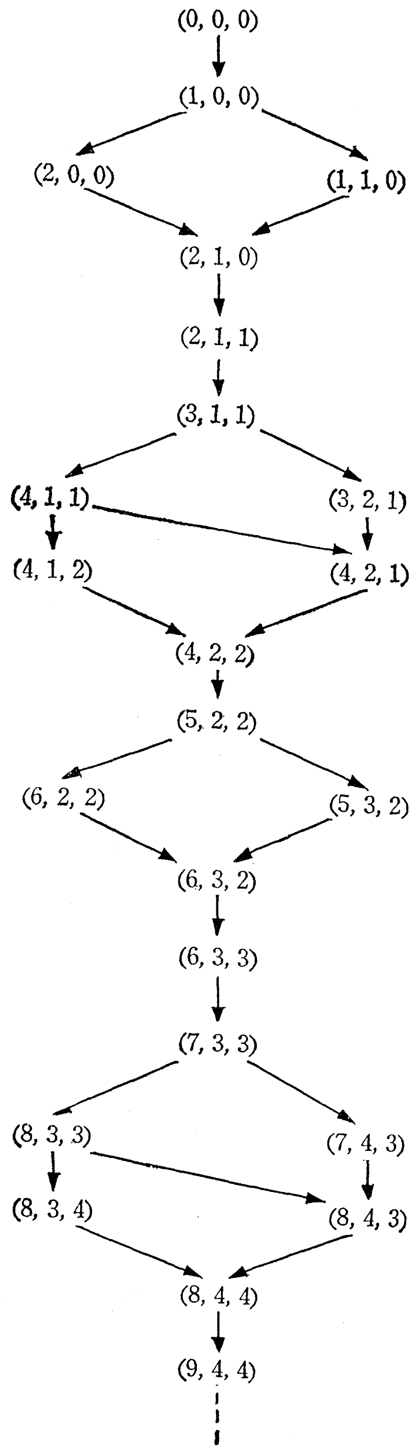


The third node in $\mu(\Sigma^1)$ is the new node 3.

Σ^2

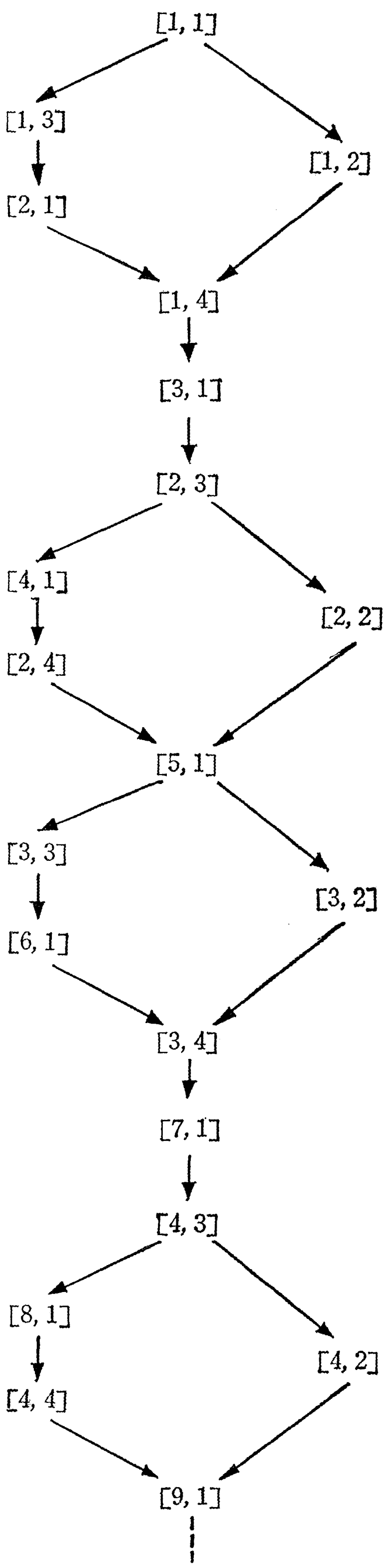


$\mu(\Sigma^2)$

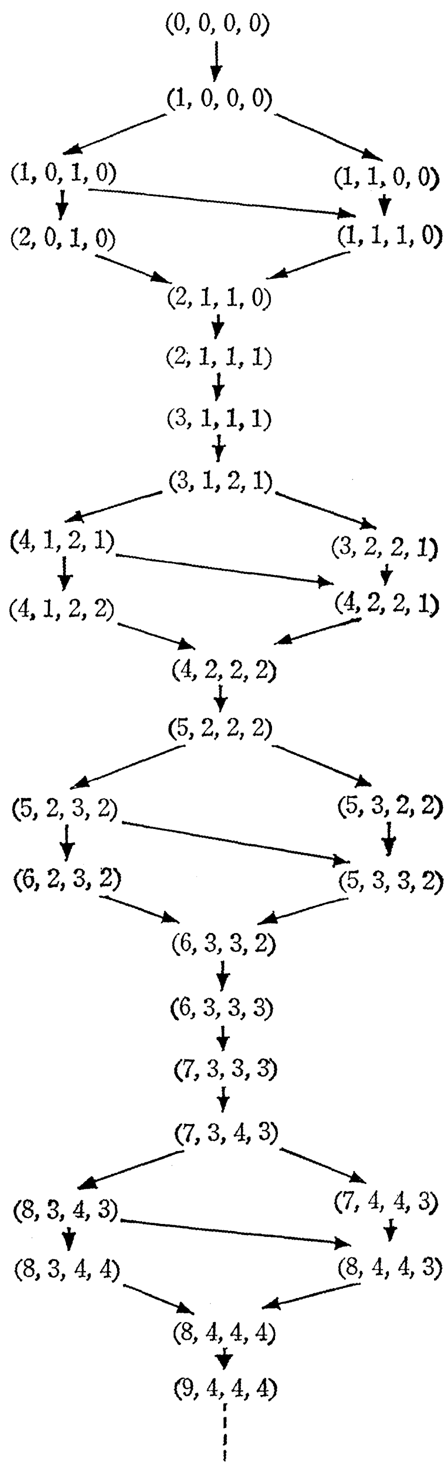


The third node in $\mu(\Sigma^2)$ is the new node 4.

$\Sigma^e = \Sigma^1 \cup \Sigma^2$



$V^e = \mu(\Sigma^e)$



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References

[1] D. E. Muller and W. S. Bartky, A Theory of Asynchronous Circuits I, Report 75, Univ. of Illinois Digital Computer Laboratory, 1956.
[2] D. E. Muller and W. S. Bartky, A Theory of Asynchronous Circuits II, Report 78, ibid, 1957.
[3] D. E. Muller and W. S. Bartky, A Theory of Asynchronous Circuits III, Report 96, ibid, 1960.
[4] R. E. Miller, R. E. Swartwout, D. E. Gillies and J. E. Robertson, Introduction speed Independent Circuits, Proc. A. I. E. E, Second Annual Symposium on Switching Circuit Theory and Logical Design, 1961.
[5] H. Noguchi, Mathematical Theory of Asynchronous Circuits. I, The Generalized Circuits, Mem. School Sci. Engrg. Waseda Univ. Tokyo, 27 (1963) 79–91.
[6] D. E. Muller and H. Noguchi, Mathematical Theory of Asynchronous Circuit. II, The Digital Circuits, Bull. Sci. Engrg. Research Laboratory, Waseda Univ., 27(1964) 85–101.
[7] M. Hattori, D. E. Muller and H. Noguchi, Mathematical Theory of Asynchronous Circuits. III, The Charts, Bull. Sci. Engrg. Research Laboratory, Waseda Univ., 28 (1964) 54–75.