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Characteristic classes of some higher order tangent bundles of complex projective spaces

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Introduction

Let M be a sufficiently smooth real compact differentiable manifold. In an earlier paper [**2**, Lemma (2.2)], we have computed elements of $KO(M)$ determined by higher order tangent bundles of M and applied them to find bounds for dimensions of odd order non-singular immersions of real projective spaces to real affine spaces. Purposes of the present paper are to express, in a similar manner, a pth order real tangent bundle of a complex projective space by means of symmetric ith power operations in KO-theory, and to compute characteristic classes of the bundle for $p=2, 3$. For some small p and complex projective spaces of small dimension, the pth order tangent bundle is completely determined in the KO-ring of the space. We compute it for $p=2$ and the space of complex dimension 4. As applications, we find bounds for dimensions of some higher order non-singular immersions of complex projective spaces to real affine spaces. One can see several properties of higher order non-singular immersion, in Feldman's work [**3**, II, Theorem 3.2].

Theorem (1.1) represents pth order real tangent bundles of the complex projective space by the operations in KO-theory. Theorems (1.3), (1.4) are computations of their Stiefel-Whitney classes for $p=2, 3$ and their Pontrjagin classes for $p=2$. These results are used in Theorem (1.5), Corollary (1.6) and we obtain necessary and sufficient conditions of second, third order non-singular immersions of complex projective spaces of certain dimensions to real affine spaces. Corollary (1.6) includes Feldman's example for the complex projective plane [**3**, I, Theorem 6.1, (b)][1].

We compute characteristic classes of powers and symmetric powers of certain real vector space bundles in Section 2, which are used, together with Theorem (1.1) concerning with KO-theory, to prove Theorems (1.3), (1.4) in Section 3. Theorem (1.5), Corollary (1.6) and other similar results are proved in the last section.

1) Details of [**3**, I] is stated in [**6**].

1. Statement of results

Let X be a finite CW-complex, and let $\mathcal{O}^i : KO(X) \to KO(X)$. $(i=0, 1, 2, \cdots)$ be the symmetric ith power operation in $KO(X)$. See [2] about it. We denote by CP^n the complex projective space of complex dimension n. Let $\tau_p(CP^n)$ be the bundle of pth order real tangent vectors on CP^n and also the element of $KO(CP^n)$ determined by the bundle. Let η denote the real plane bundle defined by the Hopf bundle which is the complex line bundle associated to the natural map $S^{2n+1} \to CP^n$. η denote also the element of $KO(CP^n)$ determined by the plane bundle.

THEOREM (1.1). *We have*

$$\tau_p(CP^n) = \mathcal{O}^p((n+1)\eta) - \mathcal{O}^{p-1}((n+1)\eta) - 1 \tag{1}$$

in $KO(X)$.

We can determine $\tau_p(CP^n)$ completely for small p and n by Theorem (1.1) and arguments of Pontrjagin classes. For instance, we have the following result. By Sanderson [4], $KO(CP^4)$ is the truncated polynomial ring over the integers with one generator $y = \eta - 2 \in KO(CP^4)$ and the relation $y^3 = 0$.

COROLLARY (1.2). *We have*

$$\tau_2(CP^4) = 15y^2 + 55y + 44 .$$

Again by Theorem (1.1), Stiefel-Whitney classes of $\tau_2(CP^n)$, $\tau_3(CP^n)$ and Pontrjagin classes of $\tau_2(CP^n)$ are computed as follows. Let $W(\tau_p(CP^n))$, $P(\tau_p(CP^n))$ be the total Stiefel-Whitney class and the total Pontrjagin class of the bundle $\tau_p(CP^n)$ respectively, and let g be the natural generator of $H^2(CP^n; Z)$. We set $\bar{g} = g \bmod 2$. $\bar{g}$ is the generator of $H^2(CP^n; Z_2)$.

THEOREM (1.3). *It follows that*

$$W(\tau_2(CP^n)) = (1+\bar{g})^{-(n+1)} \tag{2}$$

and

$$W(\tau_3(CP^n)) = (1+\bar{g}^4)^{\binom{n+1}{3}}(1+\bar{g}+\bar{g}^2+\bar{g}^3)^{n(n+1)}(1+\bar{g}^2)^{n+1} , \tag{3}$$

in $H^*(CP^n; Z_2)$.

THEOREM (1.4). *It follows that*

$$P(\tau_2(CP^n)) = (1+4g^2)^{\binom{n+2}{2}}(1+g^2)^{-(n+1)} \tag{4}$$

in $H^*(CP^n; Z)$.

We define integers $s_W(n)$, $s_P(n)$, $d_W(n)$ and $d_P(n)$ by

$$s_W(n) = \begin{cases} \max\left\{i \mid 0 < i \leq n,\ \binom{n+i}{i} \not\equiv 0 \bmod 2\right\} \\ 0 \quad \text{if there is no such integer } i , \end{cases}$$

$$s_P(n) = \max \left\{ i \mid 0 < i \leqq n,\ \sum_{j=0}^{i} (-1)^j 4^{(i-j)} \binom{n+j}{j} \binom{\binom{n+2}{2}}{i-j} \neq 0 \right\},$$

$$d_W(n) = \begin{cases} \max \left\{ i \mid 0 < i \leqq n,\ \binom{n+1}{i} \not\equiv 0 \bmod 2 \right\} \\ 0 \qquad \text{if there is no such integer } i \end{cases}$$

and

$$d_P(n) = \max \left\{ i \mid 0 < i \leqq n,\ \sum_{j=0}^{i} (-1)^j 4^j \binom{\binom{n+2}{2}+j-1}{j} \binom{n+1}{i-j} \neq 0 \right\}.$$

Applying Theorem (1.3) (2) and Theorem (1.4) to second order non-singular immersions of complex projective spaces to real affine spaces, we obtain following results by arguments of Stiefel-Whitney classes (cf. [6], [2]) and by similar arguments of Pontrjagin classes.

THEOREM (1.5). *If k is an integer such that*

$$-2 \max \{s_P(n), s_W(n)\} < k < 2 \max \{d_P(n), d_W(n)\},$$

then CP^n can not be immersed in $R^{\binom{2n+2}{2}+k-1}$ without affine singularities of order 2.

Putting together Theorem (1.5) and pth order non-singular immersion theorem by Feldman [3, 1], we obtain:

COROLLARY (1.6). *Suppose n is a positive even integer and k is a non-negative integer. CP^n can be immersed in the real affine space $R^{\binom{2n+2}{2}+k-1}$ without affine singularities of order 2 if and only if $k \geqq 2n$.*

This is a more detailed form of Feldman's example [3, 1, Theorem 6, (b)] or [6, Theorem 10.3 (b)].

For third order non-singular immersions of CP^n, we prove, by Theorem (1.3), (3):

THEOREM (1.7). *Suppose $n = 2^r$ (r, integers $\geqq 1$) and k is a non-negative integer. CP^n can be immersed in the real affine space $R^{\binom{2n+3}{3}-k-1}$ without affine singularities of order 3 if and only if $k \geqq 2n$, $4n-1 \leqq \binom{2n+3}{3}-k-1$.*

2. Lemmas on characteristic classes

We compute in this section Stiefel-Whitney classes, Pontrjagin classes of (tensor) products and of second, third symmetric powers of the canonical plane bundle η over CP^n. Let us begin with Stiefel-Whitney classes. It is clear that

$$W(\eta) = 1 + \bar{g}$$

in $H^*(CP^n; Z_2)$.

LEMMA (2.1). *Let W denote the total Stiefel-Whitney class. We have*

(5) $$W(\eta^2)=1,$$

(6) $$W(\mathcal{O}^2\eta)=1,$$

(7) $$W(\eta^3)=1+\bar{g}^4,$$

(8) $$W(\mathcal{O}^3\eta)=1+\bar{g}^2$$

and

(9) $$W(\eta(\mathcal{O}^2\eta))=1+\bar{g}+\bar{g}^2+\bar{g}^3,$$

in $H^*(CP^n; Z_2)$.

PROOF. Let

$$W(\eta)=1+\bar{g}=(1+\gamma_1)(1+\gamma_2)$$

be a formal factorization[2]. From the expression of $W(\eta)$, it follows that[3]

$$W(\eta^2)=(1+2\gamma_1)(1+\gamma_1+\gamma_2)^2(1+2\gamma_2)=1,$$

$$W(\mathcal{O}^2\eta)=(1+2\gamma_1)(1+\gamma_1+\gamma_2)(1+2\gamma_2)=1.$$

In similar manners, the formulas (7), (8) follow from

$$W(\eta^3)=(1+3\gamma_1)(1+2\gamma_1+\gamma_2)^3(1+\gamma_1+2\gamma_2)^3(1+3\gamma_2),$$

$$W(\mathcal{O}^3\eta)=(1+3\gamma_1)(1+2\gamma_1+\gamma_2)(1+\gamma_1+2\gamma_2)(1+3\gamma_2)$$

respectively. By the formula (6), we have

$$W(\eta(\mathcal{O}^2\eta))=(1+\gamma_1)^3(1+\gamma_2)^3,$$

and obtain the formula (9). Thus results of our lemma are completely proved.

Now we compute Pontrjagin classes. Let ξ_1, ξ_2 be real vector space bundles over a finite CW-complex X, which come from complex vector space bundles of complex dimensions n_1, n_2 respectively. Let P denote the total Pontrjagin

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2) Let ξ be a real vector space bundle with group $O(n)$ over a finite CW-complex X and $E(\xi)$ be the total space of the principal bundle associated to ξ. Let $S=\underbrace{Z_2\times\cdots\times Z_2}_{n}\to O(n)$ $(Z_2=\{\pm 1\})$ be the natural inclusion as diagonal elements. We denote by ρ the natural projection $E(\xi)/S\to X$ and denote by γ_i the first Stiefel-Whitney class of a line bundle over $E(\xi)/S$ associated to the ith factor of S. It follows that $\rho^*{}_2W(\xi)=\prod_{i=1}^{n}(1+\gamma_i)$ and $\rho^*{}_2: H^*(X;Z_2)\to H^*(E(\xi)/S;Z_2)$ induced by ρ is a monomorphism.

3) Cf. [7] and [8].

class. $P(\xi_i)$ $i=1, 2$ are elements of $H^*(X; Z)$. We have formal factorizations[4)]

$$P(\xi_1) \equiv \prod_{i=1}^{n} (1+x_i^2),$$

$$P(\xi_2) \equiv \prod_{j=1}^{n} (1+y_j^2)$$

modulo any odd prime number. Then the total Pontrjagin class of $\xi_1 \cdot \xi_2$ is

$$P(\xi_1 \cdot \xi_2) \equiv \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (1+(x_i+y_j)^2)(1+(x_i-y_j)^2) \tag{10}$$

modulo any odd prime number. See, for instance, [**5**, 10.6 (f)].

LEMMA (2.2). *We have*

$$P(\eta^2) = 1+4g^2 \tag{11}$$

and

$$P(\mathcal{O}^2\eta) = 1+4g^2 \tag{12}$$

in $H^*(CP^n; Z)$.

PROOF. It is known that

$$P(\eta) = 1+g^2 .$$

See, e.g., [**1**]. We apply the relation (10) to $\xi_1 = \xi_2 = \eta$. Since $H^*(CP^n; Z)$ has no torsions, the formula (11) is obtained. Let η_c, $(\mathcal{O}^2\eta)_c$ be complexifications of η, $\mathcal{O}^2\eta$. We have the total Chern class of η_c,

$$C(\eta_c) = 1-g^2 .$$

See also [**1**]. Let

$$C(\eta_c) = (1+w_1)(1+w_2)$$

be a formal factorization[5)]. From the expression of $C(\eta_c)$, it follows that

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4) Let ξ be a real vector space bundle with group $SO(2n)$ over a finite CW-complex X, which comes from a complex vector space bundle. We denote by $E(\xi)$ the total space of the principal bundle associated to ξ. Let T be the standard maximal torus of $SO(2n)$ and $\rho: E(\xi)/T \to X$ be the natural projection. We denote by x_i the first Chern class of a complex line bundle associated to the ith factor of T. It follows that $\rho^*P(\xi) = \prod_{i=1}^{n} (1+x_i^2)$ in $H^*(E(\xi)/T; Z)$ and $\rho_p^*: H^*(X; Z_p) \to H^*(E(\xi)/T; Z_p)$ induced by ρ is a monomorphism for any odd prime p.

5) Let ζ be a complex vector space bundle with group $U(n)$ over a finite CW-complex X and $E(\zeta)$ be the total space of the principal bundle associated to ζ. Let T be the standard maximal torus of $U(n)$ and $\rho: E(\zeta)/T \to X$ be the natural projection. We denote by w_i the first Chern class of a complex line bundle over $E(\zeta)/T$ associated to the ith factor of T. It follows that $\rho^*C(\zeta) = \prod_{i=1}^{n} (1+w_i)$ in $H^*(E(\zeta)/T; Z)$ and $\rho^*: H^*(X; Z) \to H^*(E(\zeta)/T; Z)$ is a monomorphism.

$$C((\mathcal{O}^2\eta)_c) = C(\mathcal{O}^2(\eta_c))^{6)}$$
$$= (1+2w_1)(1+w_1+w_2)(1+2w_2)$$
$$= 1-4g^2 .$$

By the definition of the Pontrjagin class, one obtain immediately the formula (12). Thus the proof of our lemma is completed.

3. Characteristic classes of $\tau_2(CP^n)$ and $\tau_3(CP^n)$

Let M be a compact connected real differentiable (C^r, $r \geqq p$) manifold. We denote by $\tau_p(M)$ the bundle of pth order tangent vectors on M and also the element of $KO(M)$ defined by this bundle. Sometimes we use a notation $\tau(M)$ for $\tau_1(M)$ which is the tangent bundle of M.

PROOF OF THEOREM (1.1). By Lemma (2.2) of [2], we have

$$\tau_p(M) = \mathcal{O}^p(\tau(M)+1)-1$$

in $KO(M)$. We apply this formula to $M = CP^n$ and obtain the required relation,

$$\tau_p(CP^n) = \mathcal{O}^p((n+1)\eta-1)-1$$
$$= \mathcal{O}^p((n+1)\eta)-\mathcal{O}^{p-1}((n+1)\eta)-1 .$$

To prove Corollary (1.2), we need following lemma on the element defined by $\mathcal{O}^2\eta$ in $KO(CP^4)$.

LEMMA (3.1). *We have*

(13) $$\mathcal{O}^2\eta = \eta^2-1$$

in $KO(CP^n)$.

PROOF. We have noted that $KO(CP^4)$ is the truncated polynomial ring over the integers with one generator $y = \eta-2$ and the relation $y^3 = 0$. It follows that any element of $KO(CP^4)$ has a unique form

$$a\eta^2+b\eta+c$$

where a, b and c are integers. We set

(14) $$\mathcal{O}^2\eta = a\eta^2+b\eta+c$$

and determine the coefficients. Computing Pontrjagin classes of both side of (14) by Lemma (2.2), one obtains

$$1+4g^2 = (1+4g^2)^a(1+g^2)^b ,$$

6) The symmetric power operation $\mathcal{O}^2$ in the right hand side is that for complex vector space bundles, i.e. for K-theory. See [8].

since $H^*(CP^n; Z)$ has no torsions. It follows immediately that

(15) $$4a+b=4,$$

(16) $$16\binom{a}{2}+\binom{b}{2}+4ab=0.$$

Thus one obtains

$$a=1, \quad b=0.$$

Comparing dimensions of vector space bundles, we have

$$c=-1,$$

which completes the proof of our lemma.

PROOF OF COROLLARY (1.2). By Theorem (1.1) and the above lemma we have

$$\begin{aligned}\tau_2(CP^4) &= \mathcal{O}^2(5\eta)-5\eta-1\\ &= 15\eta^2-5\eta-6\\ &= 15y^2+55y+44.\end{aligned}$$

In the remainder of this section, one proves results on characteristic classes of $\tau_2(CP^n)$ and $\tau_3(CP^n)$.

PROOF OF THEOREM (1.3). The formula (2) follows immediately from Theorem (1.1) and Lemma (2.1) (5), (6). The formula (3) follows also from Theorem (1.1) and from Lemma (2.1) (7), (8), (9).

PROOF OF THEOREM (1.4). Applying the Whitney formulas for Pontrjagin classes modulo any odd prime number, (cf. [1]) to the relation of Theorem (1.1) for $p=2$, we obtain

$$P(\tau_2(CP^n))=P\left(\binom{n+1}{2}\eta^2\right)P((n+1)\mathcal{O}^2\eta)P((n+1)\eta)^{-1},$$

since $H^*(CP^n; Z)$ has no torsions. The result (4) of our theorem directly follows from Lemma (2.2).

4. Proofs of Theorems (1.5), (1.7)

One proves theorems on bounds for dimension of second and third order non-singular immersions of CP^n to the real affine spaces.

PROOF OF THEOREM (1.5). From Theorem (1.1) of [2] on Stiefel-Whitney classes and from Theorem (1.3) (2), it follows that if k is an integer such that $2s_W(n)<k<2d_W(n)$, then CP^n can not be immersed in the affine space $R^{\binom{2n+2}{2}+k-1}$ without affine singularities of order 2. From a similar argument on Pontrjagin classes and Theorem (1.4), it also follows that if $2s_P(n)<k<2d_P(n)$

then CP^n can not be immersed in $R^{\binom{2n+2}{2}+k-1}$ without affine singularities of order 2. Putting together these results, we obtain the proof of our theorem.

Since we have $d_W(n)=n$, Corollary (1.6) is an easy consequence of Theorem (1.5) and Feldman's theorem [**3**, I, Theorem 3.1] or [**6**, Theorem 6.2].

PROOF OF THEOREM (1.7). From Theorem (1.3) (3) and arguments on Stiefel-Whitney classes, similar to Theorem (1.5), it follows that if we have $n=2^r$ $(r\geqq 1)$ and $0\leqq k<2n$, then CP^n can not be immersed in $R^{\binom{2n+3}{3}-k-1}$ without affine singularities of order 3. It is known that CP^n is differentiably embedded in R^{4n-1} and this is the best possible immersion for $n=2^r$, (cf. [**9**]). By Feldman's theorem mentioned in the proof of Corollary (1.6), the proof of our theorem is completed.

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