J. Math. Soc. Japan
Vol. 18, No. 4, 1966

Note on cohomological dimension for non-compact spaces

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(Received Feb. 9, 1966)

§1. Introduction

The purpose of the present paper is to develop the theory of cohomological dimension for non-compact spaces. Let us denote by $D(X, G)$ the cohomological dimension of a space X with respect to an abelian group G. In the first part of this paper we shall give a characterization of $D(X, G)$ in terms of continuous mappings of X into an Eilenberg-MacLane complex in case X is a collectionwise normal space. As an application of this characterization, we have sum theorems. Some of our sum theorems were proved by Okuyama [**20**] in case X is paracompact normal. In the second part of this paper we shall concern the cohomological dimension of the product of a compact space X and a paracompact normal space Y. We shall prove that $D(X\times Y, G)$ is the largest integer n such that $H^n((X, A)\times(Y, B): G)\neq 0$ for some closed sets A and B of X and Y. By our previous paper [**15**] or Boltyanskii [**3**] we know which compact spaces are dimensionally full-valued for compact spaces. However, a space which is known to be dimensionally full-valued for paracompact normal spaces is only a locally finite polytope. This was proved by Morita [**19**]. We shall prove that a locally compact paracompact normal space is dimensionally full-valued for paracompact normal spaces if and only if it is dimensionally full-valued for compact spaces. As an immediate consequence of this theorem we can know that $\dim(X\times Y)\geqq \dim Y+1$ in case X is a locally compact paracompact normal space with covering dimension $\geqq 1$ and Y is paracompact normal. Moreover, we shall show that, if a compact space X is an ANR (metric) and R is a rational field, then $D(X, R)+D(Y, G)\leqq D(X\times Y, G)\leqq \dim X+D(Y, G)$ for a paracompact normal space Y and an abelian group G.

Throughout this paper we assume that *all spaces are normal and mappings are continuous transformations.*

§ 2. Cohomological dimension

Let X be a space and let $\mathfrak{U}$ be an open covering of X. We mean by the *nerve* of $\mathfrak{U}$ the nerve of $\mathfrak{U}$ with weak topology. If $\mathfrak{U}$ is locally finite, then there is a canonical mapping of X into the nerve of $\mathfrak{U}$. (See Dowker [5].) We denote by $\phi_{\mathfrak{U}}$ a canonical mapping of X into the nerve of $\mathfrak{U}$. If $\mathfrak{U} = \{U_\alpha | \alpha \in \Omega\}$ is a covering of X and A is a closed set of X, then we denote the covering $\{U_\alpha \cap A | \alpha \in \Omega\}$ of A by $\mathfrak{U}|A$. We mean by $H^*(X, A : G)$ the Čech cohomology group of (X, A) with coefficients in G *based on locally finite open coverings* of X. If X is paracompact normal, then $H^*(X, A : G)$ is equal to the unrestricted Čech cohomology group.

DEFINITION 1. The cohomological dimension $D(X, G)$ of a space X with respect to an abelian group G is the least integer n such that, for each $m \geqq n$ and each closed set A of X the homomorphism $i^* : H^m(X : G) \to H^m(A : G)$ induced by the inclusion mapping $i : A \subset X$ is onto.

Recently, Skljarenko [23] proved that, if X is paracompact normal, then $D(X, G)$ is the largest integer n such that $H^n(X, A : G) \neq 0$ for some closed set A of X.

DEFINITION 2. A space X is called *collectionwise normal* if, for every locally finite collection $\{A_\lambda\}$ of mutually disjoint closed subsets of X, there is a collection $\{U_\lambda\}$ of mutually disjoint open sets such that $A_\lambda \subset U_\lambda$ for each λ (Bing [1]).

The following was proved by Dowker [6, Lemma 1].

LEMMA 1. (*Dowker*) *Let A be a closed subset of a collectionwise normal space X and let $\{U_\lambda\}$ be a locally finite open covering of A. Then there exists a locally finite open covering $\{V_\lambda\}$ of X such that, for each λ, $V_\lambda \cap A \subset U_\lambda$.*

DEFINITION 3. Let Q be a class of spaces. A space X is called an ANR(Q) if, whenever X is a closed subset of Y in Q, X is a retract of a neighborhood of X in Y.

LEMMA 2. (i) (*Dowker*) *A simplicial complex with metric topology is an* ANR (*collectionwise normal and perfectly normal*).

(ii) (*Hanner*) *A finite dimensional simplicial complex with metric topology is an* ANR (*collectionwise normal*).

The proof is found in Dowker [6] and Hanner [10].

For an abelian group G, we denote by $K(G, m)$, $m \geqq 1$, an Eilenberg-MacLane space which is a simplicial complex with metric topology (cf. Hu [11]). For $m = 0$, $K(G, 0)$ is G itself with discrete topology. For an integer q, denote by $(K(G, m))^q$ the q-section of $K(G, m)$. According to Wojdyslawski [24, p. 186] $(K(G, m))^q$ can be imbedded as a closed set of a convex subset D of a normed vector space. Since $(K(G, m))^q$ is an ANR (*metric*) by Lemma 2 (i), there is a neighborhood T of $(K(G, m))^q$ in D and a retraction $r: T \to (K(G, m))^q$. For each point k of $(K(G, m))^q$, take an open spherical neighborhood $S(k)$ such that $S(k) \subset T$. Put $\mathfrak{S} = \{S(k) \mid k \in (K(G, m))^q\}$. There is a subdivision K' of $(K(G, m))^q$ such that the open covering of $(K(G, m))^q$ consisting of the open stars of K' is a star refinement of the open covering $\mathfrak{S} \mid (K(G, m))^q$. We denote K' by $(K(G, m))^q$ again.

We say that two mappings f_1 and f_2 of a space X into a simplicial complex K is *contiguous* if, for each point x of X, there is a closed simplex $s(x)$ of K such that $f_1(x) \cup f_2(x) \subset s(x)$.

LEMMA 3. *Let A be a closed set of a collectionwise normal space X, and let f_1 and f_2 be contiguous mappings of A into $(K(G, m))^q$. If f_1 is extendable over X, then f_2 is extendable over X.*

PROOF. We shall prove the lemma by the same argument as in Dowker [4, Th. 2.1]. Put $(K(G, m))^q = K$. Let $F_1: X \to K$ be an extension of f_1. Since K is an ANR (*collectionwise normal*) by Lemma 2 (ii), f_2 is extendable over some open neighborhood U_1 of A in X. Denote by f' this extension. Since f_1 and f_2 are contiguous, we can take an open neighborhood U_2 of A such that (1) $\bar{U}_2 \subset U_1$ and (2), for each point x of U_2, there is some spherical neighborhood $S(k)$ of $\mathfrak{S}$ which contains $F_1(x) \cup f'(x)$. Let h_1 be the mapping of $U_2 \times I$ into T which maps (x, t) in the point dividing the segment $(F_1(x), f'(x))$ in the ratio $t : 1-t$. Define the mapping $h_2: X \times 0 \cup U_2 \times I \to \cup \{S(k) \mid S(k) \in \mathfrak{S}\} \subset T$ by $h_2 \mid X \times 0 = F_1$ and $h_2 \mid U_2 \times I = h_1$. Take an open set U_3 of X such that $A \subset U_3 \subset \bar{U}_3 \subset U_2$ and let g be a continuous function of X into I such that $g(x) = 1$ for $x \in A$ and $g(x) = 0$ for $x \in X - U_3$. Let h_3 be the mapping of $X \times I$ into T defined by $h_3(x, t) = h_2(x, t \cdot g(x))$. Define the mapping $F_2: X \to K$ by $F_2(x) = rh_3(x, 1)$ for $x \in X$. Since $r: T \to K$ is a retraction, F_2 is an extension of f_2.

REMARK. If X is paracompact normal, then Lemma 1 is proved simply. Since $X \times I$ is paracompact normal, it follows from the homotopy extension theorem.

LEMMA 4. *Let X be a collectionwise normal space such that* $\dim X < q$, *where* $\dim X$ *means the covering dimension of X. In order that every mapping from a closed set A into $(K(G, m))^q$ be extendable over X it is necessary and sufficient that the homomorphism $i^*: H^m(X : G) \to H^m(A : G)$ induced by the inclusion mapping $i: A \subset X$ be onto.*

PROOF OF THE NECESSITY. Take an element e of $H^m(A : G)$. Let $\mathfrak{U}$ be a locally finite open covering of A with order $\leqq q$ such that, if $N_{\mathfrak{U}}$ is the nerve of $\mathfrak{U}$, there is a cocycle $z_{\mathfrak{U}}$ of $Z^m(N_{\mathfrak{U}} : G)$ which represents e. Denote $(K(G, m))^q$ by K and let k_0 be a fixed vertex of K. Let $f_{\mathfrak{U}}$ be a mapping from the m-section $(N_{\mathfrak{U}})^m$ of $N_{\mathfrak{U}}$ into K such that $f_{\mathfrak{U}}((N_{\mathfrak{U}})^{m-1}) = k_0$ and, for each m-simplex

$\sigma, f_{\mathfrak{U}}|\sigma$ represents the element $z_{\mathfrak{U}}(\sigma)$ of the homotopy group $\pi_m(K, k_0)=G$. Since $z_{\mathfrak{U}}$ is a cocycle over G, $f_{\mathfrak{U}}$ is extendable over $N_{\mathfrak{U}}$. (See Hu [**11**, Chap. VI].) Denote this extension by $f_{\mathfrak{U}}$ again. We say that $f_{\mathfrak{U}}$ is determined by the cocycle $z_{\mathfrak{U}}$. Let $\bar{f}_{\mathfrak{U}}$ be a simplicial approximation of $f_{\mathfrak{U}}$. Then $\bar{f}_{\mathfrak{U}}$ is a simplicial mapping from a subdivision $\bar{N}_{\mathfrak{U}}$ of $N_{\mathfrak{U}}$ into K such that $\bar{f}_{\mathfrak{U}} \sim f_{\mathfrak{U}} j: \bar{N}_{\mathfrak{U}} \rightarrow K$, where $j: \bar{N}_{\mathfrak{U}} \rightarrow N_{\mathfrak{U}}$ is the identity mapping. Let $\phi_{\mathfrak{U}}$ be a canonical mapping of A into $N_{\mathfrak{U}}$. Put $\bar{\phi}_{\mathfrak{U}}=j^{-1}\phi_{\mathfrak{U}}$. Let $\mathfrak{V}'$ be the open covering of $\bar{N}_{\mathfrak{U}}$ consisting of the open stars of $\bar{N}_{\mathfrak{U}}$. We may assume that $\bar{N}_{\mathfrak{U}}$ is the nerve of the covering $\mathfrak{V}=\bar{\phi}_{\mathfrak{U}}^{-1}(\mathfrak{V}')$. By the assumption the mapping $\bar{f}_{\mathfrak{U}}\bar{\phi}_{\mathfrak{U}}: A \rightarrow K$ has an extension $g: X \rightarrow K$. Denote by $\mathfrak{U}_0$ the open covering consisting of the open stars of K. Let $\mathfrak{W}$ be a locally finite open covering of X with order $\leqq q$ such that $\mathfrak{W}$ is a refinement of $g^{-1}(\mathfrak{U}_0)$ and $\mathfrak{W}|A$ is a refinement of $\mathfrak{V}$. The existence of such a covering follows from Lemma 1. Let $M_{\mathfrak{W}}$ be the nerve of $\mathfrak{W}$. We denote by w the vertex of $M_{\mathfrak{W}}$ corresponding to an element W of $\mathfrak{W}$. Define a simplicial mapping $f_{\mathfrak{W}}: M_{\mathfrak{W}} \rightarrow K$ by $f_{\mathfrak{W}}(w)=u$ for a vertex w of $M_{\mathfrak{W}}$, where $W \subset g^{-1}(U)$, $U \in \mathfrak{U}_0$, and u is the vertex of K corresponding to U. Let us denote by $N_{\mathfrak{W}}$ the nerve of $\mathfrak{W}|A$, and let $\bar{\pi}_{\mathfrak{W}\mathfrak{U}}: N_{\mathfrak{W}} \rightarrow \bar{N}_{\mathfrak{U}}$, $\pi_{\mathfrak{W}\mathfrak{U}}: N_{\mathfrak{W}} \rightarrow N_{\mathfrak{U}}$ and $\pi: \bar{N}_{\mathfrak{U}} \rightarrow N_{\mathfrak{U}}$ be projections. Since $\bar{f}_{\mathfrak{U}}\bar{\pi}_{\mathfrak{W}\mathfrak{U}}$ and $f_{\mathfrak{W}}|N_{\mathfrak{W}}$ are contiguous, they are homotopic. Also, we have $f_{\mathfrak{U}}\pi \sim f_{\mathfrak{U}} j \sim \bar{f}_{\mathfrak{U}}: \bar{N}_{\mathfrak{U}} \rightarrow K$. Thus, we know $f_{\mathfrak{U}}\pi_{\mathfrak{W}\mathfrak{U}} \sim f_{\mathfrak{W}}|N_{\mathfrak{W}}: N_{\mathfrak{W}} \rightarrow K$. Since $f_{\mathfrak{U}}\pi_{\mathfrak{W}\mathfrak{U}}((N_{\mathfrak{W}})^{m-1})=k_0$, K has the homotopy extension property in $M_{\mathfrak{W}}$ and K is $(m-1)$-connected, there is a mapping $g_{\mathfrak{W}}: M_{\mathfrak{W}} \rightarrow K$ such that $g_{\mathfrak{W}}((M_{\mathfrak{W}})^{m-1})=k_0$ and $g_{\mathfrak{W}}|N_{\mathfrak{W}}=f_{\mathfrak{U}}\pi_{\mathfrak{W}\mathfrak{U}}$. For each m-simplex σ of $M_{\mathfrak{W}}$, if we assign the element of $\pi_m(K)=G$ represented by $g_{\mathfrak{W}}|\sigma$ to σ, then we have a cocycle $z_{\mathfrak{W}}$ of $M_{\mathfrak{W}}$ (cf. Hu [**11**, Chap. VI]). We say that $z_{\mathfrak{W}}$ is determined by the mapping $g_{\mathfrak{W}}$. The restriction of $z_{\mathfrak{W}}$ to $N_{\mathfrak{W}}$ is the cocycle $(\pi_{\mathfrak{W}\mathfrak{U}})^* z_{\mathfrak{U}}$. This proves that $i^*: H^m(X:G) \rightarrow H^m(A:G)$ is onto.

PROOF OF THE SUFFICIENCY. Let f be a mapping of A into K. We shall use the same notation in the proof of the necessity. Take a locally finite open covering $\mathfrak{U}$ of A such that order of $\mathfrak{U} \leqq q$ and $\mathfrak{U}$ is a refinement of $f^{-1}(\mathfrak{U}_0)$. There is a mapping $f_{\mathfrak{U}}: N_{\mathfrak{U}} \rightarrow K$ such that $f_{\mathfrak{U}}\phi_{\mathfrak{U}}$ and f are contiguous. Since K is $(m-1)$-connected, we can take a mapping $f': N_{\mathfrak{U}} \rightarrow K$ such that $f'((N_{\mathfrak{U}})^{m-1}) = k_0$ and $f' \sim f_{\mathfrak{U}}$. The mapping f' determines a cocycle $z_{\mathfrak{U}}$ of $Z^m(N_{\mathfrak{U}}:G)$. Let $f'_{\mathfrak{U}}$ be a mapping from a subdivision $\bar{N}_{\mathfrak{U}}$ of $N_{\mathfrak{U}}$ into K which is a simplicial approximation of f'. Put $\bar{\phi}_{\mathfrak{U}}=j^{-1}\phi_{\mathfrak{U}}$. Let $\mathfrak{U}'$ be the open covering of A consisting of the inverse images of the open stars of $\bar{N}_{\mathfrak{U}}$ under $\bar{\phi}_{\mathfrak{U}}$. We may assume that $\bar{N}_{\mathfrak{U}}$ is the nerve of $\mathfrak{U}'$ and $\bar{\phi}_{\mathfrak{U}}$ is a canonical mapping of A into $\bar{N}_{\mathfrak{U}}$. Take a locally finite open covering $\mathfrak{V}$ of X with order $\leqq q$ such that (1) $\mathfrak{V}|A$ is a refinement of $\mathfrak{U}'$ and (2) there is a cocycle $z_{\mathfrak{V}}$ of $Z^m(M_{\mathfrak{V}}:G)$ whose restriction to $N_{\mathfrak{V}}$ is $(\pi_{\mathfrak{V}\mathfrak{U}})^* z_{\mathfrak{U}}$, where $(M_{\mathfrak{V}}, N_{\mathfrak{V}})$ is the pair of the nerves of $\mathfrak{V}$

for (X, A) and $\pi_{\mathfrak{B}\mathfrak{U}}$ is a projection: $N_{\mathfrak{B}}\to N_{\mathfrak{U}}$. Since $\mathfrak{B}|A$ is a refinement of $f^{-1}(\mathfrak{U}_0)$, there is a mapping $f_{\mathfrak{B}}:N_{\mathfrak{B}}\to K$ such that f and $f_{\mathfrak{B}}\phi_{\mathfrak{B}}|A$ are contiguous. where $\phi_{\mathfrak{B}}:X\to M_{\mathfrak{B}}$ is a canonical mapping. Then we have homotopies $f_{\mathfrak{B}}\sim f'_{\mathfrak{U}}\bar{\pi}_{\mathfrak{B}\mathfrak{U}}\sim f_{\mathfrak{U}}\pi_{\mathfrak{U}\mathfrak{B}}:N_{\mathfrak{B}}\to K$, where $\bar{\pi}_{\mathfrak{B}\mathfrak{U}}:N_{\mathfrak{B}}\to N_{\mathfrak{U}}$ is a projection. Since the cocycle $(\pi_{\mathfrak{B}\mathfrak{U}})^*z_{\mathfrak{U}}$ determined by the mapping $f_{\mathfrak{U}}\pi_{\mathfrak{B}\mathfrak{U}}$ is extended to the cocycle $z_{\mathfrak{B}}$ of $M_{\mathfrak{B}}, f_{\mathfrak{U}}\pi_{\mathfrak{B}\mathfrak{U}}$ is extendable over $M_{\mathfrak{B}}$. Since K has the homotopy extension property in $M_{\mathfrak{B}}, f_{\mathfrak{B}}$ is extendable over $M_{\mathfrak{B}}$. Denote this extension by $f_{\mathfrak{B}}$ again. Since $f_{\mathfrak{B}}\phi_{\mathfrak{B}}|A$ and f are contiguous, by Lemma 3, f is extendable over X. This completes the proof.

The following is a consequence of Lemma 4 and an analogous theorem in terms of homology is proved in [**15**, II, p. 103].

COROLLARY 1. *If X is a collectionwise normal space with covering dimension >0, then $D(X, G)\geqq 1$ for an abelian group G.*

PROOF. By Morita [**17**, I, Th. 3.1], there exist disjoint closed subsets A and B of X such that for any open set U, $A\subset U\subset \bar{U}\subset X-B$, we have $\bar{U}-U\neq\phi$. Put $K=K(G, 0)$. K is G itself with discrete topology. Take two distinct points a and b of K. Define a mapping f of $A\cup B$ into K by $f(A)=a$ and $f(B)=b$. If the homomorphism $i^*:H^0(X:G)\to H^0(A\cup B:G)$ is onto, then we can prove by the same argument as in the proof of the sufficiency of Lemma 4 for $m=0$ that f is extendable over X. Since K has discrete topology, we have a contradiction.

We need the following lemma in § 4.

LEMMA 5. *Let X be a collectionwise normal space with covering dimension $<q$, and let A and A' be closed sets of X such that $A\subset A'$. If there is a mapping f of A into $(K(G, m))^q$ such that* (1) *f is extendable over A' and* (2) *f is not extendable over X, then the homomorphism $i^*:H^{m+1}(X, A':G)\to H^{m+1}(X, A:G)$ induced by the inclusion mapping $i:(X, A)\subset(X, A')$ is not zero.*

PROOF. Let $f':A'\to K=(K(G, m))^q$ be an extension of f. There is a locally finite open covering $\mathfrak{U}$ of A' with order $\leqq q$ and a mapping $f'_{\mathfrak{U}}$ from the nerve $L_{\mathfrak{U}}$ of $\mathfrak{U}$ into K such that f' and $f'_{\mathfrak{U}}\phi_{\mathfrak{U}}$ are contiguous. Take a mapping $f_{\mathfrak{U}}:L_{\mathfrak{U}}\to K$ such that $f'_{\mathfrak{U}}\sim f_{\mathfrak{U}}$ and $f_{\mathfrak{U}}((L_{\mathfrak{U}})^{m-1})=k_0$. Let $N_{\mathfrak{U}}$ be the nerve of $\mathfrak{U}|A$. Denote by $z'_{\mathfrak{U}}$ and $z_{\mathfrak{U}}$ the cocycles of $L_{\mathfrak{U}}$ and $N_{\mathfrak{U}}$ determined by the mappings $f_{\mathfrak{U}}$ and $f_{\mathfrak{U}}\mid N_{\mathfrak{U}}$. Then the restriction of $z'_{\mathfrak{U}}$ to $N_{\mathfrak{U}}$ is $z_{\mathfrak{U}}$. Let e' and e be the elements of $H^m(A':G)$ and $H^m(A:G)$ represented by $z'_{\mathfrak{U}}$ and $z_{\mathfrak{U}}$. We have $e=j^*e'$, where $j:A\subset A'$. Take a locally finite open covering $\mathfrak{B}$ of X with order $\leqq q$ such that $\mathfrak{B}|A'$ is a refinement of $\mathfrak{U}$. By Lemma 1, any locally finite open covering of X has such a covering $\mathfrak{B}$ as a refinement. Let $M_{\mathfrak{B}}$ and $N_{\mathfrak{B}}$ be the nerves of $\mathfrak{B}$ and $\mathfrak{B}|A$. Assume that there is a cocycle z of $M_{\mathfrak{B}}$ whose restriction to $N_{\mathfrak{B}}$ is cohomologous to $(\pi_{\mathfrak{B}\mathfrak{U}})^*z_{\mathfrak{U}}$ in $N_{\mathfrak{B}}$, where $\pi_{\mathfrak{B}\mathfrak{U}}:N_{\mathfrak{B}}\to N_{\mathfrak{U}}$ is a projection. By the same argument as in the proof of the sufficiency of Lemma 4, we can know that the mapping $f: A\to K$ is extendable over X. Thus we proved that $e\notin j_1^*H^m(X:G)$, where $j_1: A\subset X$. Consider the following diagram:

$$\begin{array}{ccccccc} \longrightarrow H^m(X:G) & \xrightarrow{j_2^*} & H^m(A':G) & \xrightarrow{\delta_1^*} & H^{m+1}(X, A':G) & \longrightarrow \\ \| & & \downarrow j^* & & \downarrow i^* & \\ \longrightarrow H^m(X:G) & \xrightarrow{j_1^*} & H^m(A:G) & \xrightarrow{\delta^*} & H^{m+1}(X, A:G) & \longrightarrow. \end{array}$$

It is known by Lemma 1 that the Čech cohomology theory based on locally finite open coverings in collectionwise normal spaces satisfies axioms 3 and 4 of Eilenberg-Steenrod [**9**]. Thus we have $i^*\delta_1^*e'=\delta^*j^*e'=\delta^*e\neq 0$. This completes the proof.

The following theorem is an immediate consequence of Lemma 4 and Definition 1.

THEOREM 1. *Let X be a collectionwise normal space with covering dimension $<q$. The cohomological dimension $D(X:G)$ is the least integer n such that, for each $m\geqq n$ and each closed set A of X, every mapping from A into $(K(G, m))^q$ is extendable over X.*

Let X be collectionwise normal and perfectly normal. If we make use of Lemma 2 (i) in place of Lemma 2 (ii) in the proofs of Lemmas 3 and 4, then we know that Lemmas 3 and 4 are true without restriction of finite dimension. Thus we have:

THEOREM 2. *If X is collectionwise normal and perfectly normal, then the cohomological dimension $D(X:G)$ is the least integer n such that, for each $m\geqq n$ and each closed set A of X, every mapping from A into $K(G, m)$ is extendable over X.*

§ 3. Sum theorems

DEFINITION 4. Let $\{A_\lambda\}$ be a closed covering of a space X. We say that *X has the weak topology with respect to $\{A_\lambda\}$*, if the union of any subcollection $\{A_\mu\}$ of $\{A_\lambda\}$ is closed in X and any subset of $\bigcup_\mu A_\mu$ whose intersection with each A_μ is closed relative to the subspace topology of A_μ is necessarily closed in the subspace $\bigcup_\mu A_\mu$ (Morita [18]).

THEOREM 3. *Let X be a finite dimensional collectionwise normal space or a collectionwise normal and perfectly normal space.*

(1) *If $\{A_i; i=1, 2, \cdots\}$ is a closed covering of X, then* $D(X, G)=\mathrm{Max}\{D(A_i, G); i=1, 2, \cdots\}$.

(2) *If X has the weak topology with respect to $\{A_\lambda | \lambda\in\Gamma\}$, then* $D(X, G)=\mathrm{Max}\{D(A_\lambda, G); \lambda\in\Gamma\}$.

(3) *If A is a closed subset of X such that the complement $X-A$ and X*

are both collectionwise normal or collectionwise normal and perfectly normal, then $D(X, G) \leqq \text{Max}\{D(X-A, G), D(A, G)\}$. *Moreover, A is* G_δ, *then the equality holds.*

REMARK 1. If $\{A_\lambda\}$ is a locally finite closed covering of X, then X has the weak topology with respect to $\{A_\lambda\}$ by Morita [**18**].

REMARK 2. In case X is paracompact normal, (1), (3) and (2) in which $\{A_\lambda\}$ is replaced by a locally finite closed covering are proved by Okuyama [**20**].

By an analogous argument as in Morita [**18**, I, Th. 2], Theorem 3 can be deduced from Theorems 1 and 2 and the following Lemma.

LEMMA 6. *Let K be a space having the neighborhood extension property in X. Under the assumptions of Theorem 4, if K has the extension property in subsets* A_i, A_λ, A *and* $X-A$, *then K has the extension property in X.*

PROOF. Let $\{A_i; i=1, 2, \cdots\}$ be a closed covering of X, and let f_0 be a mapping from a closed set F_0 of X into K. Since K has the extension property in A_1, f_0 is extendable over $F_0 \cup A_1$. Since K has the neighborhood extension property in X, there is a closed neighborhood F_1 of $F_0 \cup A_1$ over which f is extendable. Continuing such procedure, we know that there exist sequences of closed sets $\{F_k; k=1, 2, \cdots\}$ and mappings $\{f_k; k=1, 2, \cdots\}$ such that (1) F_k is a closed neighborhood of $A_k \cup F_{k-1}$ and (2) $f_k: F_k \to K$ is an extension of $f_{k-1}: F_{k-1} \to K$, $k=1, 2, \cdots$. Define a mapping $f: X \to K$ by $f(x)=f_k(x)$ for $x \in F_k$. Since each point x of X is contained in the interior of some F_k, f is continuous. Thus f_0 has a continuous extension. Others are proved similarly.

DEFINITION 5. A compact space X is called to be a *Cantor manifold for an abelian group G* if, whenever X is a union of non empty closed subsets A and B, then $D(A \cap B, G) \geqq D(X, G)-1$.

It is obvious that X is a Cantor manifold if and only if it is a Cantor manifold for Z, where Z is the additive group of integers.

THEOREM 4. *Every finite dimensional compact space X contains a Cantor manifold C for G such that* $D(X, G)=D(C, G)$.

By Hurewicz-Wallman [**12**, Th. VI, 8] the theorem is a consequence of Theorem 1 and the following lemma.

LEMMA 7. *Let X be a finite dimensional paracompact normal space such that* $D(X, G) < m-1$. *Then every mapping* $f: X \to (K(G, m))^q$ *is homotopic to a constant mapping, where* $q > \dim X$.

PROOF. In the next section, it is proved that $D(X \times I, G)=D(X, G)+1$ for a finite dimensional paracompact normal space X. Thus, the lemma is a consequence of Theorem 1.

§4. The cohomological dimension of product spaces

THEOREM 5. *If X is a finite dimensional locally compact metric space and Y is a finite dimensional paracompact normal space, then $D(X\times Y, G)$ is the largest integer n such that, for some closed sets $A_2\subset A_1\subset X$ and $B_2\subset B_1\subset Y$, $H^n((A_1, A_2)\times(B_1, B_2): G)\neq 0$.*

REMARK 1. In case X and Y are locally compact paracompact normal, the theorem is proved by Dyer [8]. The local compactness of $X\times Y$ is essential in his proof.

PROOF. Put $d_1(X\times Y, G)=\operatorname{Max}\{n: H^n((A_1, A_2)\times(B_1, B_2): G)\neq 0$ for some closed sets A_i and B_i of X and $Y\}$ and $d_2(X\times Y, G)=\operatorname{Max}\{n: H^n(X\times Y, F: G)\neq 0$ for some closed set F of $X\times Y\}$. Since $X\times Y$ is paracompact normal by Morita [19], we have the equality $D(X\times Y, G)=d_2(X\times Y, G)$ by Skljarenko [23]. Using the exact sequence of triples, we know that $d_2(X\times Y, G)=\operatorname{Max}\{n: H^n(F_1, F_2: G)\neq 0$ for some closed sets $F_2\subset F_1\subset X\times Y\}$. Thus, we have $D(X\times Y, G)\geqq d_1(X\times Y, G)$. It is sufficient to prove that $D(X\times Y, G)\leqq d_1(X\times Y, G)$. By Theorem 3 or Okuyama [20] we may assume that X is a compact metric space. Let us set the following assumption.

$$\text{Assumption } (*): \begin{cases} D(X\times Y, G)=n, \textit{ and } H^m((A_1, A_2)\times(B_1, B_2): G)=0 \\ \textit{for } m\geqq n, \textit{ any closed sets } A_i \textit{ and } B_i,\ i=1, 2. \end{cases}$$

We shall prove that Assumption (*) gives us a contradiction. Since the inequality $D(X\times Y, G)>d_1(X\times Y, G)$ means (*), we have the theorem. The proof is devided in five steps.

1st step. Since X is a compact metric space, it is the inverse limit of a countable sequence $\{M_i: i=1, 2, \cdots\}$ of finite simplicial complexes such that (i) $\dim M_i\leqq\dim X$, and (ii) the projection $\pi_i^{i+1}: M_{i+1}\to M_i$ is linear in each simplex of M_{i+1}, $i=1, 2, \cdots$. (See Isbell [13].) Denote by $\pi_i^j: M_j\to M_i, j>i$, the composition of π_k^{k+1}, $k=i, \cdots, j-1$, and by μ_i the projection: $X\to M_i$. We have $\mu_i=\pi_i^j\mu_j$ for $j>i$. Let $\mathfrak{U}_i$, $i=1, 2, \cdots$, be the open covering of X consisting of the inverse images of the open stars of M_i under μ_i. We can assume without loss of generality that $\{\mathfrak{U}_i; i=1, 2, \cdots\}$ forms a cofinal system of open coverings of X.

2nd step. By Theorem 1 and Assumption (*), there is a closed set F of $X\times Y$ and a mapping f of F into $(K(G, n-1))^q$ such that f is not extendable over $X\times Y$, where $q>\dim X+\dim Y$. Put $K=(K(G, n-1))^q$. Since K has the neighborhood extension property in $X\times Y$ by Lemma 2, f is extendable over some open neighborhood S of F. We denote an extension by f again. Let $\mathfrak{U}$ be a locally finite open covering of K which is a refinement of the open covering of K consisting of the open stars of K. Since the covering $f^{-1}\mathfrak{U}|F$ of F

is a locally finite collection in $X\times Y$, there exists a locally finite open covering $\mathfrak{W}=\{W_\alpha|\alpha\in\Omega\}$ of Y with order $\leqq \dim Y+1$ satisfying the following conditions:

(i) For each $\alpha\in\Omega$ there is an open covering $\mathfrak{U}_{i(\alpha)}$ of X such that the collection $\mathfrak{V}=\{\mathfrak{U}_{i(\alpha)}\times W_\alpha|\alpha\in\Omega\}$ is a locally finite open covering of $X\times Y$. (See 1st step for $\mathfrak{U}_{i(\alpha)}$.)

(ii) The covering $\mathfrak{V}|F$ is a star refinement of $f^{-1}\mathfrak{U}|F$.

(iii) Every element of $\mathfrak{V}$ does not intersect both F and $X\times Y-S$.

(iv) If $\Omega_\alpha=\{\beta|W_\alpha\cap W_\beta\neq\phi\}$, then $\text{Max}\,\{i(\beta)|\beta\in\Omega_\alpha\}<\infty$ for each $\alpha\in\Omega$. The existence of $\mathfrak{V}$ satisfying (iv) is proved by taking locally finite refinements and star refinements.

3rd step. Let N be the nerve of $\mathfrak{W}$. Denote by w_α the vertex of N corresponding to an element W_α of $\mathfrak{W}$. Let T^0 be a topological sum of the sets $M_{i(\alpha)}\times w_\alpha$, $\alpha\in\Omega$. Suppose that T^l is constructed for $0\leqq l<j$. For a j-simplex σ of N, put $i(\sigma)=\text{Max}\,\{i(\mu):\mu$ is a $(j-1)$-face of $\sigma\}$. Let T^j be a topological sum of the sets $M_{i(\sigma)}\times\sigma$, where σ ranges over all j-simplexes of N. For 1-simplex $s=(w_\alpha, w_\beta)$ of N, since $i(s)=\text{Max}\,\{i(\alpha), i(\beta)\}$, the projections $\pi^{i(s)}_{i(\alpha)}$ and $\pi^{i(s)}_{i(\beta)}$ induce a mapping g_s of the subcomplex $M_{i(s)}\times(w_\alpha\cup w_\beta)$ of T^1 into T^0. If we identify the corresponding points of T^1 and T^0 under these mappings g_s, we obtain a set P_1. Let f_1 be the identification mapping: $T^0\cup T^1\to P_1$. Since the projection π^j_i, $i<j$, is linear in each simplex of M_j, we see that P_1 is a CW complex whose closed cells are topological cells. The closure finiteness of P_1 is guaranteed by the condition (iv) satisfied by the covering $\mathfrak{W}$. (See 2nd step.) Assume that the CW complex P_{j-1} is constructed for $j-1>0$ and $f_{j-1}:\bigcup_{i=0}^{j-1}T_i\to P_{j-1}$ is the identification mapping. Consider the cell complex $T^j=\cup\{M_{i(\sigma)}\times\sigma|\sigma$ is a j-simplex of $N\}$. If μ is a $(j-1)$-face of σ, then we have $i(\mu)\leqq i(\sigma)$. Put $S^j=\cup\{M_{i(\sigma)}\times\dot{\sigma}\}$, where $\dot{\sigma}$ is the boundary of σ. Then S^j is a subcomplex of T^j. Define the mapping $g_j:S^j\to P_{j-1}$ by $g_j(x, y)=f_{j-1}(\pi^{i(\sigma)}_{i(\mu)}(x), y)$ for $x\in M_{i(\sigma)}$ and $y\in\mu$, where μ is a $(j-1)$-face of σ. If s is a k-face of σ, $k\leqq j-2$, and $y\in s$, then we have $f_{j-1}(\pi^{i(\sigma)}_{i(\mu)}(x), y)=f_k(\pi^{i(\sigma)}_{i(s)}(x), y)$, where μ is a $(j-1)$-face of σ containing the simplex s. Thus we see that g_j is a continuous mapping. By identifying the corresponding points of T^j and P_{j-1} under the mapping g_j, we obtain a CW complex P_j. Denote by P the CW complex P_j for $j=\dim Y$. Each closed cell τ of P is obtained from a product cell $\nu\times\sigma$ by contracting some simplexes of $\nu\times\dot{\sigma}$, where ν and σ are simplexes of $M_{i(\sigma)}$ and N. Thus each closed cell of P is a topological cell. We say that P is the CW complex associated with the product covering $\mathfrak{V}$ of $X\times Y$.

4th step. Consider the cell complex $T^j=\cup\{M_{i(\sigma)}\times\sigma\}$. (See 3rd step.) Let ϕ be a canonical mapping of Y into N. Put $B_\sigma=\phi^{-1}(\sigma)$ for a j-simplex σ of

N. Define the mapping $\bar{g}_\sigma: X\times B_\sigma\to T^j$ by $\bar{g}_\sigma(x, y)=(\mu_{i(\sigma)}(x), \phi(y))$ for $x\in X$ and $y\in B_\sigma$. Since the mapping $\bar{g}_\sigma, j=1, 2, \cdots$, dim Y and $\sigma\in N$, is compatible with the identification mapping: $\cup\{T^j, j=1, 2, \cdots$. dim $Y\}\to P$ (cf. 3rd step), $\bar{g}_\sigma$ induces the mapping $\psi: X\times Y\to P$. It is easy to see that ψ is continuous. Moreover, the mapping ψ has the following property: For each closed cell τ of P there are simplexes σ of N and $\nu_{i(\sigma)}$ of $M_{i(\sigma)}$ such that $\psi^{-1}(\tau, \dot{\tau})=(\mu^{-1}_{i(\sigma)}(\nu_{i(\sigma)})\times\phi^{-1}(\sigma), \mu^{-1}_{i(\sigma)}(\dot{\nu}_{i(\sigma)})\times\phi^{-1}(\sigma)\cup\mu^{-1}_{i(\sigma)}(\nu_{i(\sigma)})\times\phi^{-1}(\dot{\sigma}))$, where $\dot{\tau}$ means the boundary of τ. Put $A_\tau=\mu^{-1}_{i(\sigma)}(\nu_{i(\sigma)})$, $A_{\dot{\tau}}=\mu^{-1}_{i(\sigma)}(\dot{\nu}_{i(\sigma)})$, $B_\tau=\phi^{-1}(\sigma)$ and $B_{\dot{\tau}}=\phi^{-1}(\dot{\tau})$ for a closed cell τ of P. Then we have $\psi^{-1}(\tau, \dot{\tau})=((A_\tau, A_{\dot{\tau}})\times(B_\tau, B_{\dot{\tau}}))$.

5th step. Let Q be the minimal closed subcomplex of P such that $\psi(F)\subset Q$. By the condition (iii) satisfied by the covering $\mathfrak{W}$ (2nd step), we have $\psi(X\times Y-S)\cap Q=\phi$. By an analogous argument as in the proof of Lemma 4 we see that there is a mapping g of Q into K such that $g\psi|F\sim f: F\to K$. Denote Q^j the j-section of Q. Since K is $(n-2)$-connected, we may assume that $g(Q^{n-2})=k_0$ (=a base point of K). Let L be the closed subcomplex of P consisting of closed cells which do not intersect Q. Let us extend g over $Q\cup L\cup P^{n-1}$ such that $g(L)=k_0$ and, if μ is an $(n-1)$-cell of P^{n-1} whose interior is in $P-Q$, $g(\mu)=k_0$. Take an n-cell τ such that $\tau\notin Q\cup L$. Then we have $\psi^{-1}(\tau, \dot{\tau})=((A_\tau, A_{\dot{\tau}})\times(B_\tau, B_{\dot{\tau}}))$ by 4th step. Denote by h_τ the mapping $g\psi|\psi^{-1}(\dot{\tau}): \psi^{-1}(\dot{\tau})\to K$. Since $H^n((A_\tau, A_{\dot{\tau}})\times(B_\tau, B_{\dot{\tau}}): G)=0$ by Assumption (*), the homomorphism: $H^{n-1}(A_\tau\times B_\tau: G)\to H^{n-1}(A_\tau\times B_{\dot{\tau}}\cup A_{\dot{\tau}}\times B_\tau: G)$ is onto. By Lemma 4 h_τ is extendable over $\psi^{-1}(\tau)=A_\tau\times B_\tau$. Continuing this procedure, we see that the mapping $g\psi|F: F\to K$ is extendable over $X\times Y$. Since $f\sim g\psi|F: F\to K$, the mapping f is extendable over $X\times Y$. We obtain a contradiction. This completes the proof.

From the proof of Theorem 5 (3rd step), we can see the following fact. Let $D(X\times Y, G)=n$. Then there exist; (1) closed sets $A_2\subset A_1\subset X$ and $B_2\subset B_1\subset Y$, (2) closed simplexes ν and σ, (3) mappings $f:(A_1, A_2)\to(\nu, \dot{\nu})$, $g:(B_1, B_2)\to(\sigma, \dot{\sigma})$ and $h:(\nu\times\sigma)^{\cdot}=\nu\times\dot{\sigma}\cup\dot{\nu}\times\sigma\to(K(G, n-1))^q$, and (4) the mapping $h(f\times g)|A_1\times B_2\cup A_2\times B_1: A_1\times B_2\cup A_2\times B_1\to(K(G, n-1))^q$ is not extendable over $A_1\times B_1$. Extend the mappings f and g over X and Y, respectively. We denote by f and g such extensions, again. Put $f^{-1}(\dot{\nu})=A$ and $g^{-1}(\dot{\sigma})=B$. Then the mapping $h(f\times g)|X\times B\cup A\times Y: X\times B\cup A\times Y\to(K(G, n-1))^q$ is not extendable over $X\times Y$. By Theorem 1, the homomorphism: $H^{n-1}(X\times Y: G)\to H^{n-1}(X\times B\cup A\times Y: G)$ is not onto.

Consider the mapping $h: (\nu\times\sigma)^{\cdot}=\nu\times\dot{\sigma}\cup\dot{\nu}\times\sigma\to(K(G, n-1))^q$. Since $(K(G, n-1))^q$ has the neighborhood extension property in $\nu\times\sigma$, h is extendable over some neighborhood U of $(\nu\times\sigma)^{\cdot}$ in $\nu\times\sigma$. Denote this extension by h again. By the compactness of $\nu\times\sigma$, there are closed neighborhoods s_1 and s_2 of $\dot{\nu}$ and $\dot{\sigma}$ such that $\nu\times s_2\cup s_1\times\sigma\subset U$. Put $A'=f^{-1}(s_1)$ and $B'=g^{-1}(s_2)$.

Then A' and B' are closed neighborhoods of A and B. Moreover, the mapping $h(f\times g)|X\times B'\cup A'\times Y: X\times B'\cup A'\times Y\to(K(G, n-1))^q$ is not extendable over $X\times Y$. By Lemma 5, the homomorphism $i^*: H^n((X, A')\times(Y, B'): G)\to H^n((X, A)\times(Y, B): G)$ is not zero, where $i:(X, A)\times(Y, B)\subset(X, A')\times(Y, B')$. Thus, we have the following corollaries.

COROLLARY 2. *$D(X\times Y, G)=\text{Max}\,\{n: H^n((X, A)\times(Y, B): G)\neq 0$ for some closed sets A and B of X and Y respectively$\}$.*

COROLLARY 3. *Let $D(X\times Y, G)=n$. Then there exist closed sets $A_2\subset A_1\subset X$ and $B_2\subset B_1\subset Y$ such that (1) A_1 and B_1 are closed neighborhoods of A_2 and B_2 respectively, and (2) the homomorphism: $H^n((X, A_1)\times(Y, B_1): G)\to H^n((X, A_2)\times(Y, B_2): G)$ is not zero.*

Let X be a finite simplicial complex and let Y be a finite dimensional paracompact normal space. For an open covering $\mathfrak{V}=\{\mathfrak{U}_{i(\sigma)}\times W_\sigma|\sigma\in\Omega\}$ of $X\times Y$, where $\mathfrak{W}=\{W_\alpha|\alpha\in\Omega\}$ is a locally finite open covering of Y and $\mathfrak{U}_{i(\sigma)}$ is the open covering of X consisting of the open stars of the $i(\sigma)$-th barycentric subdivision of X, construct a CW complex P associated with $\mathfrak{V}$ (cf. the proof of Theorem 5). Then P is a subdivision of the cell complex $X\times N_{\mathfrak{W}}$, where $N_{\mathfrak{W}}$ is the nerve of $\mathfrak{W}$. If $\mathfrak{W}'$ is a locally finite refinement of $\mathfrak{W}$ and $\pi_{\mathfrak{W}'\mathfrak{W}}: N_{\mathfrak{W}'}\to N_{\mathfrak{W}}$ is a projection, let us define a mapping $\bar{\pi}_{\mathfrak{W}'\mathfrak{W}}: X\times N_{\mathfrak{W}'}\to X\times N_{\mathfrak{W}}$ by $\bar{\pi}_{\mathfrak{W}'\mathfrak{W}}(x, y)=(x, \pi_{\mathfrak{W}'\mathfrak{W}}(y))$ for $x\in X$ and $y\in N_{\mathfrak{W}'}$. Then we have:

COROLLARY 4. *Let (X, A) be a pair of finite simplicial complexes and let (Y, B) be a pair of finite dimensional paracompact normal spaces. Then $H^n((X, A)\times(Y, B): G)$ is the direct limit of the system $\{H^n((X, A)\times(M_{\mathfrak{W}}, N_{\mathfrak{W}}): G)|(\pi_{\mathfrak{W}'\mathfrak{W}})^*\}$, where $\mathfrak{W}$ ranges over all locally finite open coverings of Y and $(M_{\mathfrak{W}}, N_{\mathfrak{W}})$ is the pair of the nerves of $\mathfrak{W}$ for (X, A).*

COROLLARY 5. *If X is a locally finite polytope and Y is a finite dimensional paracompact normal space, then $D(X\times Y, G)=\dim X+D(Y, G)$.*

PROOF. It is sufficient to prove the corollary in case $X=I$. Let $D(I\times Y, G)=n$. By Corollary 2, there are closed subsets A and B of I and Y such that $H^n((I, A)\times(Y, B): G)\neq 0$. We may assume that $A=\dot{I}$ (= the boundary of I). By Corollary 4, $H^n((I, \dot{I})\times(Y, B): G)=\lim\{H^n((I, \dot{I})\times(M_{\mathfrak{W}}, N_{\mathfrak{W}}): G)|(\pi_{\mathfrak{W}'\mathfrak{W}})^*\}$. It is well known that $H^n((I, \dot{I})\times(M_{\mathfrak{W}}, N_{\mathfrak{W}}): G)\approx H^{n-1}(M_{\mathfrak{W}}, N_{\mathfrak{W}}: G)$. Thus, we have $H^{n-1}(Y, B: G)\neq 0$. This proves that $D(I\times Y, G)\leqq D(Y, G)+1$. The converse relation $D(I\times Y, G)\geqq D(Y, G)+1$ is proved similarly.

Recently O'Neil [21] proved the following Künneth theorem.

THEOREM. (O'Neil) *If X is compact and Y is paracompact normal, then the sequence*

$$0\to\sum_{q=0}^{n}H^q(X:Z)\otimes H^{n-q}(Y:Z)\to H^n(X\times Y:Z)\to\sum_{q=0}^{n}H^{q+1}(X:Z)*H^{n-q}(Y:Z)\to 0$$

is exact.

From his proof we have the following exact sequence:

$$0\to\sum_{q=0}^{n}H^q(X:Z)\otimes H^{n-q}(Y:G)\to H^n(X\times Y:G)\to\sum_{q=0}^{n}H^{q+1}(X:Z)*H^{n-q}(Y:G)\to 0\,.$$

Here G is any abelian group.

REMARK 2. For compact spaces, the Künneth sequence in relative forms is exact (Dyer [8, Appendix]). But, it is not known whether or not it is true for non compact spaces.

REMARK 3. The following theorem was proved by Peterson [**22**, Appendix].

THE UNIVERSAL COEFFICIENT THEOREM. *If X is compact and G is an abelian group or X is paracompact normal and G is finitely generated, the sequence*

$$0\to H^n(X:Z)\otimes G\to H^n(X:G)\to H^{n+1}(X:Z)*G\to 0$$

is exact.

But, as the following simple example shows, if G is not finitely generated, the universal coefficient theorem does not hold even for a finite dimensional countable simplicial complex. Let Y be a one point union of a countable infinite number of the segments $s_i=(x_0, x_i)$, $i=1, 2, \cdots$, such that $s_i\cup s_j=x_0$ for $i\neq j$. Denote by X' the product of Y and an $(n-1)$-sphere S^{n-1}. Let $q=(p_1, p_2, \cdots)$ be a sequence of all prime integers. Let f_i be a simplicial mapping from the subspace $x_i\times S^{n-1}$ of X' into an $(n-1)$-sphere S_i^{n-1} with degree p_i. The simplicial complex X is obtained by identifying points of $x_i\times S^{n-1}$ mapped to the same point under the mapping f_i, $i=1, 2, \cdots$. Then we have:

(1) $H^n(X:Z)$ contains an element with infinite order.

(2) For every prime p, $H^n(X:Z)$ contains an element with order p.

(3) Let $R=$ the additive group of rationals, $R_p=$ the additive group of rationals whose denominators are coprime with p, $Q_p=$ the additive group of p-adic rationals reduced mod 1 and $Z_p=$ the cyclic group of order p. If G is one of the groups R, R_p, Q_p and Z_p, p a prime, then $H^n(X:G)=0$.

The properties (1) and (3) imply that the universal coefficient theorem does not hold for the group R or R_p.

THEOREM 6. *Let X be a compact* ANR (*metric*) *and let Y be a finite dimensional paracompact normal space. Then we have the relation:*

$$D(X, R)+D(Y, G)\leqq D(X\times Y, G)\leqq \dim X+D(Y, G)\,.$$

REMARK 4. As the following example shows, we can not replace a compact ANR(*metric*) X by a metric Cantor manifold. Consider the 2-dimensional Cantor manifold M_0 constructed in [**16**, p. 44]. By [**16**, Lemma 9], we have $D(M_0, R)=2$ and $D(M_0, Q_p)=D(M_0, Z_p)=1$ for a prime p. In case G is Q_p or a finite group, we have $D(M_0, G)=1$. If Y is a compact space such that dim $Y=D(Y, G)$, then $D(M_0\times Y, G)\leqq D(M_0, G)+D(Y, G)=1+D(Y, G)$ by Bockstein [2]. Thus we have $D(M_0, R)+D(Y, G)=2+D(Y, G)>D(M_0\times Y, G)$.

We need the following lemmas.

LEMMA 8. *Let X be an LC^∞ compact space and let A_2 be a closed subset of X. For a closed neighborhood A_1 of A_2 there are a pair (K, L) of finite simplicial complexes, mappings $f:(X, A_2)\rightarrow(K, L)$ and $g:(K, L)\rightarrow(X, A_1)$ such that $g\cdot f\sim i:(X, A_2)\rightarrow(X, A_1)$, where $i:(X, A_2)\subset(X, A_1)$.*

The proof is given by a similar way to [**14**].

Following Dyer [8, p. 144], a group H is said to have property $P(p)$, p a prime, if there is some element of H/H_p which is not divisible by p, where H_p is the p-primary part of H.

LEMMA 9. *If X is an LC^∞ compact space such that $D(X, R)=m$, then there is a closed set A of X such that* (1) *$H^m(X, A:Z)$ contains an element with infinite order which is not divisible by any integer >1 and* (2) *$H^m(X, A:Z)$ has property $P(p)$ for every prime p.*

PROOF. There is a closed set A_2 of X such that $H^m(X, A_2:R)\neq 0$. By the universal coefficient theorem [**22**], $H^m(X, A_2:Z)$ contains an element e with infinite order. Take a closed neighborhood A_1 of A_2 such that, if $i:(X, A_2)\subset(X, A_1)$, then $e\in i^*H^m(X, A_1:Z)$. Let (K, L), f and g be complexes and mappings in Lemma 8. Put $H=g^*H^m(X, A_1:Z)\subset H^m(K, L:Z)$. Then H is finitely generated. Take an element e' of H such that (1) e' is of infinite order and (2) e' is not divisible by any integer >1 in H. Let e'' be an element of $H^m(X, A_1:Z)$ such that $g^*e''=e'$. Then e'' is of infinite order and it is not divisible by any integer >1. Since H is finitely generated and contains an element with infinite order, H has property $P(p)$. Thus, $H^m(X, A_1:Z)$ has property $P(p)$ for any prime p.

PROOF OF THE RELATION $D(X, R)+D(Y, G)\leqq D(X\times Y, G)$. We shall give the proof by an analogous argument as in Morita [**19**, p. 220]. Let $s\leqq D(X, R)$ and $t\leqq D(Y, G)$. For some $m\geqq s$, there is a closed set A of X satisfying the conclusion of Lemma 9. Put $X_0=X/A$ and denote by x_0 the point corresponding to A. Take a closed set B of Y such that $H^n(Y, B:G)\neq 0$, $n\geqq t$. Put $Y_0=Y/B$ and denote by y_0 the point corresponding to B. Then, $H^m(X_0:Z)$ contains an element with infinite order which is not divisible by any integer >1 and it has property $P(p)$ for every prime p. Also, we have $H^n(Y_0:G)\neq 0$. Thus, by Dyer [8, Lemmas 1.6 and 1.7], $H^m(X_0:Z)\otimes H^n(Y_0:G)\neq 0$. By O'Neil [**21**] we can conclude that $H^{m+n}(X_0\times Y_0:G)\neq 0$ and $D(X_0\times Y_0:G)\geqq m+n$. We may assume that A and B are G_δ. Let $X-A=\bigcup_{i=1}^{\infty}A_i$ and $Y-B=\bigcup_{i=1}^{\infty}B_i$. Then we have $X_0\times Y_0=x_0\times y_0\cup(\bigcup_{i=1}^{\infty}A_i\times y_0)\cup(\bigcup_{i=1}^{\infty}x_0\times B_i)\cup(\bigcup_{i=1}^{\infty}A_i\times B_i)$. By Theorem 3 or Okuyama [**20**], we have $D(A_i\times B_i, G)\geqq m+n$ for some i. Since

$A_i\times B_i$ is closed in $X\times Y$, this proves that $D(X\times Y, G)\geqq m+n$.

PROOF OF THE RELATION $D(X\times Y, G)\leqq \dim X+D(Y, G)$. If $D(Y, G)=0$, then, since $\dim Y=0$ by Corollary 1, we have $D(X\times Y, G)\leqq \dim(X\times Y)=\dim X=\dim X+D(Y, G)$. If $\dim X=0$, then X consists of a finite number of points. If $\dim X=\infty$, then the relation is obvious. Therefore, it is sufficient to prove the relation in case $0<\dim X<\infty$ and $0<D(Y, G)<\infty$. Let $D(X\times Y, G)=m$ and $\dim X=n$. Let us assume that $m>n+D(Y, G)$. We shall prove that this assumption gives us a contradiction. Since $D(Y, G)\geqq 1$, we have $m>n+1$. By Corollary 3, there are closed sets $A_2\subset A_1\subset X$ and $B\subset Y$ such that (1) A_1 is a closed neighborhood of A_2 and (2) the homomorphism $i_1^*: H^m((X, A_1)\times(Y, B): G)\rightarrow H^m((X, A_2)\times(Y, B): G)$ is not zero, where $i_1:(X, A_2)\times(Y, B)\subset(X, A_1)\times(Y, B)$. Applying Lemma 8 to the inclusion $i:(X, A_2)\subset(X, A_1)$, we find a pair (K, L) of n-dimensional finite simplicial complexes, mappings $f:(X, A_2)\rightarrow(K, L)$ and $g:(K, L)\rightarrow(X, A_1)$ such that $gf\sim i:(X, A_2)\rightarrow(X, A_1)$. Define mappings $\bar{f}:(X, A_2)\times(Y, B)\rightarrow(K, L)\times(Y, B)$ and $\bar{g}:(K, L)\times(Y, B)\rightarrow(X, A_1)\times(Y, B)$ by $\bar{f}(x, y)=(f(x), y)$, $x\in X$ and $y\in Y$, and $\bar{g}(k, y)=(g(k), y)$, $k\in K$ and $y\in Y$. Then we have $\bar{g}\bar{f}\sim i:(X, A_2)\times(Y, B)\rightarrow(X, A_1)\times(Y, B)$. Since the homomorphism $i_1^*=(\bar{g}\bar{f})^*$ is not zero, we can conclude that $H^m((K, L)\times(Y, B): G)\neq 0$. By Corollary 4, $H^m((K, L)\times(Y, B): G)=\varinjlim\{H^m((K, L)\times(M_{\mathfrak{W}}, N_{\mathfrak{W}}): G)\,|\,(\pi_{\mathfrak{W}'\mathfrak{W}})^*\}$, where $\mathfrak{W}$ ranges over all locally finite open coverings of Y and $(M_{\mathfrak{W}}, N_{\mathfrak{W}})$ is the pair of the nerves of $\mathfrak{W}$ for (Y, B). Take a locally finite open covering $\mathfrak{W}$ such that some element e of $H^m((K, L)\times(M_{\mathfrak{W}}, N_{\mathfrak{W}}): G)$ represents a non-zero element of $H^m((K, L)\times(Y, B): G)$. Put $K/L=K_0$ and $M_{\mathfrak{W}}/N_{\mathfrak{W}}=M_{\mathfrak{W}}^0$, and let k_0 and m_0 be the points corresponding to L and $N_{\mathfrak{W}}$. Consider the following exact sequence:

$$\rightarrow H^{m-1}(K_0\times m_0\cup k_0\times M_{\mathfrak{W}}^0: G)\xrightarrow{\delta^*}H^m((K_0, k_0)\times(M_{\mathfrak{W}}^0, m_0): G)\xrightarrow{j^*}H^m(K_0\times M_{\mathfrak{W}}^0: G)$$

We shall assert that the element e does not belong to the image of δ^*. Let us assume that $e\in$ Image of δ^*. Since $H^{m-1}(K_0\times m_0\cup k_0\times M_{\mathfrak{W}}^0: G)=H^{m-1}(K_0: G)+H^{m-1}(M_{\mathfrak{W}}^0: G)$ and $\dim K_0=\dim K=n<m-1$, we have $H^{m-1}(M_{\mathfrak{W}}^0: G)\neq 0$. If $\mathfrak{W}'$ is a locally finite refinement of $\mathfrak{W}$, then $h^*: H^{m-1}(M_{\mathfrak{W}}^0: G)\rightarrow H^{m-1}(M_{\mathfrak{W}'}^0: G)$ is not zero, where h is the mapping induced by a projection $\pi_{\mathfrak{W}'\mathfrak{W}}:(M_{\mathfrak{W}'}, N_{\mathfrak{W}'})\rightarrow(M_{\mathfrak{W}}, N_{\mathfrak{W}})$. This shows that $D(Y, G)\geqq m-1$. Then we have $D(X\times Y, G)=m>\dim X+D(Y, G)=n+m-1\geqq m$. This contradiction proves that $e\notin$ Image of δ^*. Thus we have $0\neq j^*e\in H^m(K_0\times M_{\mathfrak{W}}^0: G)$. By O'Neil [**21**], there exist integers p and q such that (1) $p+q=m$ and $H^p(K_0: Z)\otimes H^q(M_{\mathfrak{W}}^0: G)\neq 0$ or (2) $p+q=m+1$ and $H^p(K_0: Z)*H^q(M_{\mathfrak{W}}^0: G)\neq 0$. In any case (1) or (2) we can conclude that $D(Y,G)\geqq q$. Since $\dim X=n\geqq p$, we have $m>n+q\geqq p+q=m$. This completes the proof.

As an immediate consequence of Theorem 6, we have:

COROLLARY 6. *If X is a compact* ANR(*metric*) *such that* $\dim X = D(X, R)$, *then* $D(X\times Y, G) = \dim X + D(Y, G)$ *for a finite dimensional paracompact normal space* Y.

REMARK 5. Let Y be paracompact normal and perfectly normal. If we make use of Theorem 2 in place of Theorem 1, then we can see that Theorems 5 and 6, and Corollaries 2, 3, 4, 5 and 6 are true without restriction of finite dimension.

THEOREM 7. *Let X be a locally compact paracompact normal space. If* $D(X, Q_p) \geqq k$ *for every prime p and* $D(X, R) \geqq k$, *then* $\dim X\times Y \geqq \dim Y + k$ *for a paracompact normal space* Y.

PROOF. If $\dim X=\infty$ or $\dim Y=\infty$, then the theorem is obvious. Moreover, by Theorem 3 and Morita [**17**], we may assume that X is compact. Let $\dim Y=n$. There exists a closed G_δ set B of Y such that $H^n(Y, B:Z)\neq 0$. Put $Y/B=Y_0$ and let y_0 be the point corresponding to B. We have the following two cases: (1) the p-primary part of $H^n(Y_0:Z)\neq 0$ for some prime p, or (2) $H^n(Y_0:Z)$ contains an element with infinite order. If (1) holds, take a closed set A of X such that $H^m(X, A:Q_p)\neq 0$, $m\geqq k$. Let $X/A=X_0$ and let x_0 be the point corresponding to A. Then we have $H^m(X_0:Q_p)\neq 0$. By Dyer [**8**, Theorem 1], we can conclude that (i) $H^m(X:Z)$ has property $P(p)$ or (ii) $H^{m+1}(X_0:Z)$ contains an element with order p. If (i) holds, then $H^m(X_0:Z)\otimes H(Y_0:Z)\neq 0$. If (ii) holds, then $H^{m+1}(X_0:Z)*H^n(Y_0:Z)\neq 0$. (See Dyer [**8**, Lemma 1.6].) In any case (i) or (ii), we can show that $H^{m+n}(X_0\times Y_0:Z)\neq 0$ by O'Neil [**21**]. Thus, we have $\dim X_0\times Y_0\geqq m+n$. By an analogous argument as in the proof of Theorem 6, we can prove that $\dim X\times Y\geqq m+n \geqq k+\dim Y$. The proof for the case (2) is given similarly.

DEFINITION 6. Let Q be a class of spaces. A space X is called *dimensionally full-valued for* Q if $\dim X\times Y=\dim X+\dim Y$ for every space Y of Q.

Let Q be the class of paracompact normal spaces.

THEOREM 8. *A locally compact paracompact normal space X is dimensionally full-valued for Q if and only if* $D(X, Q_p)=\dim X$ *for every prime p.*

PROOF. The proof of 'only if' part follows from [**15**] or Boltyanski [**3**]. Let $D(X, Q_p)=\dim X$ for every prime p. By Bockstein [**2**] or Dyer [**8**, Corollary 2.1 (c)], we have $D(X, Q_p)\leqq \mathrm{Max}\,\{D(X, R), D(X, R_p)-1\}\leqq \dim X$. This shows that $D(X, Q_p)=D(X, R)=\dim X$. The theorem follows from Theorem 7.

THEOREM 9. *If X is locally compact paracompact normal space such that* $\dim X>0$, *then* $\dim X\times Y\geqq \dim Y+1$ *for every paracompact normal space Y.*

The theorem follows from Corollary 1 and Theorem 7.

DEFINITION 7. A compact space C is called *a pseudo n-cell* if there exists a mapping f of an n-cell E onto C such that $f|$ the boundary of E is a homeomorph.

Theorem 10. *If a locally compact paracompact normal space X contains a pseudo n-cell, then* $D(X\times Y, G)\geqq D(Y, G)+n$ *for every paracompact normal space Y.*

Proof. There exists a mapping f of an n-cell E into X such that $f|$ the boundary of E is a homeomorph. Denote by S the boundary of E, and put $C=f(E)$ and $D=f(S)$. The mapping $f^{-1}: D\rightarrow S$ is extendable over C. Denote this extension by g. Then $gf\sim 1:(E, S)\rightarrow(E, S)$, where 1 means the identity mapping. Let $D(Y, G)=m$. Take a closed set B of Y such that $H^{m}(Y, B:G)\neq 0$. By an analogous argument as in the proof of Corollary 5, we can prove that $H^{m+n}((E, S)\times(Y, B):G)\neq 0$. This shows that $H^{m+n}((C, D)\times(Y, B):G)\neq 0$. Thus, we have $D(X\times Y, G)\geqq D(Y, G)+n$.

Corollary 7. *If a compact n-dimensional metric space X is* lc^{n} *(over Z), then it is dimensionally full-valued for Q if and only if* $D(X, R)=n$.

It follows from Dyer [**7**, Corollary 2], [**15**] and Theorem 9.

Corollary 8. *The following spaces are dimensionally full-valued for Q.*

(1) *A locally compact* 2*-dimensional* ANR *(metric).*

(2) *A* 1*-dimensional locally compact paracompact normal space.*

(3) *An n-dimensional locally compact paracompact normal space which contains a pseudo n-cell.*

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