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Some examples of topological groups

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It might be interesting to ask to what extent the topological and algebraic structures of the group $H(M)$ of the homeomorphisms on a manifold M represent the topological structures of M. In spite of its importance, unfortunately, little has been known about it. Though it seems very difficult to determine the structures of $H(M)$, many conjectures or problems have been set up by several authors.

Among them the following two seem to be interesting and important.

i) Does $H(M)$ contain a p-adic group?

ii) Does a homomorphic image of a vector group into $H(M)$ have the locally compact closure?

Related to the problem i), the following have been known:

a) $H(M)$ has no small compact connected subgroup [4].

b) $H(M)$ has no small finite group [5].

c) If i) is negative, then any locally compact subgroup of $H(M)$ is necessarily a Lie group [3], [4].

d) If i) is affirmative, i.e., a p-adic group P can act effectively on M, then the orbit space M/P has the dimension $\dim M+2$ or ∞ [7].

As for the problem ii), A.M. Gleason and R.S. Palais proposed a following problem in [2]:

Is the closure of a homomorphic image of any connected Lie group into $H(M)$ necessarily a Lie group?

The topology for $H(M)$ is of course the compact open topology.

In the previous paper [8] the author showed that if ii) is affirmative, then any homomorphic image of any connected Lie group into $H(M)$ has the locally compact closure.

It follows that the problem of Gleason and Palais is equivalent to the problems i) and ii) above. In fact, if i) is negative and ii) is affirmative, then their problem is affirmative. Conversely, if their problem is affirmative, then clearly ii) is affirmative. Moreover, we see that if a p-adic group can act effectively on a connected n-dimensional manifold, then there is a connected $n+1$-dimensional manifold on which a p-adic solenoid can act effectively. p-adic solenoid can be considered as the closure of a monomorphic image of the additive group of the real numbers. Thus, if their problem is affirmative, then i) is negative.

As far as the problem ii) is concerned, we need to know, first of all, whether or not there exists such a topology $\mathfrak{T}$ for R^n (n-dimensional vector group) that 1) $\mathfrak{T}$ is weaker than the ordinary topology for R^n, 2) $(R^n, \mathfrak{T})$ is a topological additive group and 3) the completion $(R^n, \mathfrak{T})$ is not locally compact.

In this paper, it will be shown that such a topology exists (Example I). Moreover, it will also be shown that there is a topology $\mathfrak{T}$ for R^2 with the following properties (Example II):

1) $\mathfrak{T}$ is weaker than the ordinary topology for R^2,
2) $(R^2, \mathfrak{T})$ is a topological additive group,
3) any one-parameter subgroup of R^2 is locally compact in $(R^2, \mathfrak{T})$,
4) the completion of $(R^2, \mathfrak{T})$ is not locally compact.

This Example II means that the problem ii) can not be reduced to the case of one-parameter group by a mere group theoretical method.

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1. Notations.

Let A_p be the additive group of the formal power series

$$\sum_{i=0}^{\infty} a_i p^i, \qquad 0 \leqq a_i < p,$$

where p is an integer satisfying $p \geqq 2$ (p is not necessarily a prime number). In this group A_p, the element -1 is expressed by

$$\sum_{i=0}^{\infty} (p-1)p^i.$$

Corresponding to the p-adic expansion of an integer, there is a natural isomorphism φ_p from the additive group Z of the integers into A_p, which satisfies $\varphi_p(1) = 1$ and $\varphi_p(-1) = \sum_{i=0}^{\infty} (p-1)p^i$.

Let A'_p be the subgroup of A_p such that $A'_p = \{x = \sum a_i p^i \in A_p$; there exists $r = r(x)$ such that $a_{r+i} = 0$ or $p-1$ for all $i > 0\}$.
Then we see that $\varphi_p(Z) = A'_p$.

By $\|x\| \leqq k$ we mean that $a_{k+j} = 0$ or $p-1$ for all $j > 0$, where $x = \sum a_i p^i$. By $D_{p,n}(x)$ is meant the sequence of the integers

$$(a_\lambda, a_{\lambda+1}, \cdots, a_\mu), \quad \lambda = n(n+1), \quad \mu = (n+1)(n+2)-1, \; n \geqq 0,$$

where $x=\sum_{i=0}^{\infty} a_i p^i$. If $D_{p,n}(x)=(0, 0, \cdots, 0)$ (or $(p-1, p-1, \cdots, p-1)$), then we denote $D_{p,n}(x)=(0)$ (or $(p-1)$) briefly. By $\xi_{p,n}$ is meant the element of A'_p such that $D_{p,i}=(0)$ for $i \neq n$, and $D_{p,n}(x)=(1, 0, \cdots, 0)$. By $\| D_{p,n}(x) \| \leqq k$ is meant that

$$D_{p,n}(x)=(\underbrace{*, *, \cdots, *}_{k}, 0, \cdots, 0) \quad \text{or} \quad (\underbrace{*, *, \cdots, *}_{k}, p-1, \cdots, p-1).$$

Since $\|\cdot\|$ is defined with respect to x and $D_{p,n}(x)$, we can define $\|\cdot\|=k$, if $\|\cdot\| \leqq k$ is true and $\| \ \| \leqq k-1$ is false, and define $\|\cdot\| \geqq k$, if $\|\cdot\| \leqq k-1$ is false.

These notations are fixed throughout this paper.

By an elementary calculation we see:

A) if $D_{p,i}(x)=(0)$ and $D_{p,i}(y)=(0)$ for $0 \leqq i \leqq n$, then $D_{p,i}(-x)=(0)$ and $D_{p,i}(x+y)=(0)$ for $0 \leqq i \leqq n$.

B) $\| D_{p,m}(x+y) \| \leqq \max \{\| D_{p,m}(x) \|, \| D_{p,m}(y) \|\}+1$.

C) $\| D_{p,m}(-x) \| \leqq \| D_{p,m}(x) \|+1$.

2. Example I.

First, we begin with the definition of a topology $\mathfrak{T}$ for Z satisfying the following properties:

1) $(Z, \mathfrak{T})$ is a topological additive group.
2) $(Z, \mathfrak{T})$ satisfies the second countability axiom.
3) The completion of $(Z, \mathfrak{T})$ is not locally compact.

Putting

$$V_n=\{x \in A'_p\,;\, D_{p,i}(x)=(0),\ 1 \leqq i \leqq n,\ \| D_{p,n+j}(x) \| \leqq j,\ j>0\}$$

we see that $\{V_n\}$ determines a topology $\mathfrak{T}$ for A'_p such that $(A'_p, \mathfrak{T})$ is a topological additive group. In fact, since A'_p is abelian, we have only to prove that

(a) $\cap V_n=\{0\}$, $V_n \supset V_{n+1}$,

(b) $-V_n \subset V_{n-1}$, $V_n+V_n \subset V_{n-1}$,

(c) for any $x \in V_n$, there is V_m such that $x+V_m \subset V_n$.

By A), B) and C) above, we see that (a), (b) are satisfied. Thus, we have only to prove (c). Since $x \in A'_p$, there is an integer m such that $m \geqq n+1$ and $D_{p,k}(x)=(0)$ or $(p-1)$ for all $k \geqq m$. It follows that $x+V_m \subset V_n$.

Since $\{V_n\}$ and A'_p consist of countably many elements, we see that $(A'_p, \mathfrak{T})$ satisfies the second countability axiom.

Obviously, $(A'_p, \mathfrak{T})$ is not discrete.

It will be shown below that the completion $\mathrm{Cl}(A'_p, \mathfrak{T})$ of $(A'_p, \mathfrak{T})$ is not compact.

Assume that $\mathrm{Cl}(A'_p, \mathfrak{T})$ is compact. Then $(A'_p, \mathfrak{T})$ is totally bounded. Therefore, for any V_n there is a finite set of points

$$\{a_1, a_2, \cdots, a_k\}$$

such that $\cup\{a_i+V_n\}=A'_p$.

Let $r=\max\{\|a_1\|, \cdots, \|a_k\|\}$. We choose a sufficiently large m such that $m(m+1)>r$ and $m \geqq n+2$. From B) in section 1, we see that $\|D_{p,m}(a_i+V_n)\| \leqq m-n+1$ for all i. This contradicts the assumption that $\cup\{a_i+V_n\}=A^i_p$, because there is an element $x \in A'_p$ such that $\|D_{p,m}(x)\| \geqq m-n+2$. It follows that $\mathrm{Cl}(A'_p, \mathfrak{T})$ is not compact.

Since $A'_p \cong Z$, we can prove that $\mathrm{Cl}(A'_p, \mathfrak{T})$ is not locally compact from the lemmas 2.1-2.3 below.

Let $(G, \mathfrak{T}_0)$ be a fixed Lie group, where G is the underlying group and $\mathfrak{T}_0$ is the underlying topology for G. Then we denote by $T(G, \mathfrak{T}_0)$ the set of pairs of the abstract group G and the topology $\mathfrak{T}$ for G such that (1) $\mathfrak{T}$ is weaker than $\mathfrak{T}_0$ (2) $(G, \mathfrak{T})$ is a topological group with Hausdorff's separation axiom and the first countability axiom.

LEMMA 2.1. *Let $(Z, \mathfrak{T}_0)$ and $(R, \mathfrak{T}_0)$ be the group of integers with discrete topology and the group of real numbers with the ordinary topology respectively. Then, there exists a mapping ι from $T(Z, \mathfrak{T}_0)$ into $T(R, \mathfrak{T}_0)$ satisfying the following properties:*

1) *ι is injective and denoting by r the restriction of the topology for R to the subgroup Z, $\iota \circ r=$ identity.*
2) $\mathrm{Cl}(Z, \mathfrak{T})$ *is (locally) compact if and only if* $\mathrm{Cl}(\iota(Z, \mathfrak{T}))$ *is (locally) compact.*

PROOF. Since $(Z, \mathfrak{T})$ satisfies the first countability axiom, there is a system $\{V_n\}$ of countable many neighborhoods of the identity 0 satisfying the following properties:

$$a)\ V_n=-V_n,\ V_n \subset V_{n-1} \qquad b)\ V_{n-1} \supset V_n+V_n\,.$$

Let U_n be an open interval $(-1/2^n, 1/2^n)$. Put $W_n=V_n+U_n$. Then we see easily that $\{W_n\}$ determines a topology for R satisfying 1). Since the closure of $(Z, \mathfrak{T})$ in $\mathrm{Cl}(\iota(Z, \mathfrak{T}))$ is identical with $\mathrm{Cl}(Z, \mathfrak{T})$ and $\mathrm{Cl}(\iota(Z, \mathfrak{T}))/\mathrm{Cl}(Z, \mathfrak{T})$ is a homomorphic image of the circle group R/Z, we see that $\mathrm{Cl}(\iota(Z, \mathfrak{T}))/\mathrm{Cl}(Z, \mathfrak{T})$ is compact. Thus, we obtain 2).

LEMMA 2.2. *Let $(R, \mathfrak{T}) \in T(R, \mathfrak{T}_0)$. If $\mathrm{Cl}(R, \mathfrak{T})$ is locally compact, then either $\mathrm{Cl}(R, \mathfrak{T})$ is compact or $(R, \mathfrak{T})=(R, \mathfrak{T}_0)$.*

PROOF. Let K be the maximal compact subgroup of the locally compact group $\mathrm{Cl}(R, \mathfrak{T})$. It is well-known that $\mathrm{Cl}(R, \mathfrak{T})/K$ is a vector group. Obviously, there is a natural homomorphism ϕ from $(R, \mathfrak{T})$ into $\mathrm{Cl}(R, \mathfrak{T})/K$ and $\phi(R)$ is dense in $\mathrm{Cl}(R, \mathfrak{T})/K$. If $\mathrm{Cl}(R, \mathfrak{T})/K$ is non-trivial, then we see that ϕ is monomorphic and $\mathrm{Cl}(R, \mathfrak{T})/K$ is one dimensional vector group, that is $\mathrm{Cl}(R, \mathfrak{T})/K \cong (R, \mathfrak{T}_0)$. It follows $(R, \mathfrak{T}) = (R, \mathfrak{T}_0)$. If $\mathrm{Cl}(R, \mathfrak{T})/K$ is trivial, then $\mathrm{Cl}(R, \mathfrak{T})$ is compact.

By Lemmas 2.1 and 2.2 we have immediately the following:

LEMMA 2.3. *Let* $(Z, \mathfrak{T}) \in T(Z, \mathfrak{T}_0)$. *If* $\mathrm{Cl}(Z, \mathfrak{T})$ *is locally compact, then either* $\mathrm{Cl}(Z, \mathfrak{T})$ *is compact or* $(Z, \mathfrak{T})$ *is discrete.*

The desired example is obtained in the following way. Since $\mathrm{Cl}(A'_p, \mathfrak{T})$ which is constructed above is not locally compact, there is a topology $\mathfrak{T}'$ for Z such that $(Z, \mathfrak{T}')$ has non-locally compact completion. Thus, $\iota(Z, \mathfrak{T}')$ is a desired one.

3. Example II.

As in the section 2, we begin with the construction of a topology $\mathfrak{T}$ for Z^2 satisfying the following a), b) and c): a) $(Z^2, \mathfrak{T}) \in T(Z^2, \mathfrak{T}_0)$ where $\mathfrak{T}_0$ is the discrete topology, b) any one generated subgroup of $(Z^2, \mathfrak{T})$ is discrete under the relative topology in $(Z^2, \mathfrak{T})$ and c) $(Z^2, \mathfrak{T})$ is not discrete.

Let p, q be integers satisfying $q \geqq p^3$, $p \geqq 2$. Considering the direct product $A'_p \times A'_q$, we denote $\eta'_n = (\xi_{p,n}, \xi_{q,n})$. The topology $\mathfrak{T}$ for $A'_p \times A'_q$ is defined by giving a system of neighborhoods $\{V_n\}$ of the identity,

$$V_n = \{x \in A'_p \times A'_q\, ;\, x = \sum a_i \eta'_i \text{ (finite summation)},\ a_i = 0 \text{ for } 0 \leqq i \leqq n \text{ and } |a_{n+j}| \leqq 2^j \text{ for } j \geqq 1,\ a_i \in Z\}\,.$$

In fact, we see immediately by this definition that $\cap V_n = \{0\}$, $V_n = -V_n$, $V_n + V_n \subset V_{n-1}$. Thus, to prove that $\{V_n\}$ is a system of neighborhoods of the identity of a topological additive group, we have only to show that for any $x \in V_n$ there exists a neighborhood V_m such that $x + V_m \subset V_n$.

Let $x \in V_n$ and $x = \sum_{i=n+1}^{m} a_i \eta'_i$, then we see by an elementary calculation that $x + V_m \subset V_n$. It follows that $(A'_p \times A'_q, \mathfrak{T})$ is a topological group satisfying the first countability axiom.

Since there is a natural isomorphism $\varphi = \varphi_p \times \varphi_q$ from Z^2 onto $A'_p \times A'_q$, the topological group $(A'_p \times A'_q, \mathfrak{T})$ determines the topology $\mathfrak{T}$ (denoted by the same notation) for Z^2 such that $(Z^2, \mathfrak{T}) \in T(Z^2, \mathfrak{T}_0)$.

It will be shown below that this topology $\mathfrak{T}$ is a desired one.

Let $\eta_n = (p^{n(n+1)}, q^{n(n+1)}) \in Z^2$. Then $\varphi(\eta_n) = \eta'_n$. It follows that

$$\varphi^{-1}(V_n) = \{x \in Z^2\, ;\, x = \sum a_i \eta_i,\ a_i = 0 \text{ for } 0 \leqq i \leqq n,\ |a_{n+j}| \leqq 2^j \text{ for all } j \geqq 1\}\,.$$

Z^2 can be naturally imbedded in R^2. Clearly, $\varphi^{-1}(V_n)$ is contained in $\{(x_1, x_2) \in R^2\, ;\, x_1 \cdot x_2 \geqq 0\}$. It will be shown below that φ^{-1}, $V_n (n \geqq 5)$ is contained in

$$\{(x_1, x_2)\,;\, x_1 \geqq 0,\, x_1^2 \leqq x_2\} \cup \{(x_1, x_2)\,;\, x_1 \leqq 0,\, -x_1^2 \geqq x_2\}\,.$$

Assume that $z = \sum_{i=n+1}^{m} a_i \eta_i$ is contained in V_n and $a_m \neq 0$. If $a_m > 0$, then by denoting $z = (z_1, z_2)$ we see that $z_1 \leqq p^{(m+1)(m+2)}$ and $z_2 \geqq q^{m(m+1)-1}$. Since $q \geqq p^3$, if $n \geqq 5$, then $z_2 \geqq z_1^2$. If $a_m < 0$, then $-z_2 \geqq z_1^2$.

By the definition of $\{V_n\}$, the subgroup generated by $(0, 1)$ is discrete under the relative topology in $(Z^2, \mathfrak{T})$.

Let S be a subgroup of Z^2 generated by an element which is not $(0, 1)$. We see easily that $S \cap \varphi^{-1}(V_5)$ consists of a finite number of points. Thus, there is a neighborhood V_m such that $S \cap \varphi^{-1}(V_m) = \{0\}$. This means that S is discrete under the relative topology in $(Z^2, \mathfrak{T})$.

On the other hand, $(Z^2, \mathfrak{T})$ is not discrete. So we see that $(Z^2, \mathfrak{T})$ is a desired one.

In the same way as in Lemma 2.1, we can define a topology $\mathfrak{T}'$ for R^2 satisfying the following properties:

1) $(R^2, \mathfrak{T}') \in T(R^2, \mathfrak{T}_0)$ where $\mathfrak{T}_0$ is the ordinary topology for R^2.
2) Any one parameter subgroup of R^2 with the relative topology in $(R^2, \mathfrak{T}')$ is isomorphic to $(R, \mathfrak{T}_0)$.
3) $(R^2, \mathfrak{T}')$ is not equal to $(R^2, \mathfrak{T}_0)$.

In fact, putting $W_n = V_n + U_n$ where $U_n = \{(x_1, x_2)\,;\, x_1^2 + x_2^2 < 1/2^n\}$, there is no difficulty to show that $\{W_n\}$ determines a topology for R^2 satisfying 1)-3) above.

For this group $(R^2, \mathfrak{T}')$, which is defined above, it will be shown below that the completion $\mathrm{Cl}\,(R^2, \mathfrak{T}')$ of $(R^2, \mathfrak{T}')$ is not locally compact.

Assume that $\mathrm{Cl}\,(R^2, \mathfrak{T}')$ is locally compact. Let K be a maximal compact subgroup of $\mathrm{Cl}(R^2, \mathfrak{T}')$. Since the identity mapping from $(R^2, \mathfrak{T}')$ into $\mathrm{Cl}(R^2, \mathfrak{T}')$ is continuous, so is the identity mapping i from $(R^2, \mathfrak{T}_0)$ into $\mathrm{Cl}\,(R^2, \mathfrak{T}')$. If $K \neq \{0\}$, then by Lemma 2.3 in [8], we see that there is a one-parameter subgroup R in R^2 such that the closure of $i(R)$ in $\mathrm{Cl}\,(R^2, \mathfrak{T}')$ is contained in K. This contradicts the property 2) of $(R^2, \mathfrak{T}')$. It follows that $\mathrm{Cl}\,(R^2, \mathfrak{T}')$ is a vector group. Since iR^2 is dense in $\mathrm{Cl}\,(R^2, \mathfrak{T}')$, we see that

$$\mathrm{Cl}\,(R^2, \mathfrak{T}') = (R^2, \mathfrak{T}') = (R^2, \mathfrak{T}_0)\,.$$

This also contradicts the property 3) of $(R^2, \mathfrak{T}')$. Consequently, we see that $\mathrm{Cl}\,(R^2, \mathfrak{T}')$ is not locally compact.

This example II is a desired one announced in the introduction.

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