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The structure of K_Λ-rings of the lens space and their applications

By Tsunekazu KAMBE

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Introduction.

In [2] M. F. Atiyah used the Grothendieck ring $KO(M)$, of real vector bundles over a differentiable manifold M, to the problems of immersion and imbedding of M, and applied his methods to the n-dimensional real projective space RP^n whose $KO(RP^n)$ had been determined by J. F. Adams [1].

In this paper we shall consider the lens space $L^n(p)$ which is defined as follows: Let p be an integer >1 and γ be the rotation of $(2n+1)$-sphere

$$S^{2n+1}=[(z_0, z_1, \cdots, z_n)/\sum_{i=0}^{n}|z_i|^2=1]$$

of the complex $(n+1)$-space C^{n+1} given by

$$\gamma(z_0, z_1, \cdots, z_n)=(e^{2\pi i/p}z_0, e^{2\pi i/p}z_1, \cdots, e^{2\pi i/p}z_n)\,.$$

Then γ generates the topological transformation group Γ of S^{2n+1} of order p, and the lens space is defined to be the orbit space:

$$L^n(p)=S^{2n+1}/\Gamma\,.$$

This is the compact differentiable $(2n+1)$-manifold without boundary and in particular $L^n(2)=RP^{2n+1}$.

The reduced Grothendieck rings $\tilde{K}(L^n(p))$ (for prime p) and $\widetilde{KO}(L^n(p))$ (for odd prime p), of complex and real vector bundles over $L^n(p)$ respectively, are determined by the following two theorems.

Let η be the canonical complex line bundle over the complex projective space CP^n. Consider the natural projection

$$\pi: L^n(p)=S^{2n+1}/\Gamma\to S^{2n+1}/S^1=CP^n$$

and the element

$$\sigma=\pi^!(\eta-1_c)^{1)}\in\tilde{K}(L^n(p))$$

1) Throughout this paper, the trivial real (complex) bundle of dimension n will be simply denoted by n (n_c).

where $\pi^!: \tilde{K}(CP^n) \to \tilde{K}(L^n(p))$ is the induced homomorphism of π.

THEOREM 1. *Let p be prime, let n be an integer and let $n=s(p-1)+r$ $(0 \leqq r < p-1)$. Then*

$$\tilde{K}(L^n(p)) \cong (Z_{p^{s+1}})^r + (Z_{p^s})^{p-r-1}\ [2]$$

and $\sigma^1, \cdots, \sigma^r$ generate additively the first r factors and $\sigma^{r+1}, \cdots, \sigma^{p-1}$ the last $p-r-1$ factors. Moreover, the ring structure of $\tilde{K}(L^n(p))$ is given by

$$\sigma^p = -\sum_{i=0}^{p-1} \binom{p}{i} \sigma^i \qquad \sigma^{n+1} = 0\,.$$

Also, consider the operator $r: \tilde{K}(L^n(p)) \to \widetilde{KO}(L^n(p))$ which sends complex vector bundles to the corresponding naturally defined real vector bundles, and the element

$$\bar{\sigma} = r\sigma \in \widetilde{KO}(L^n(p))\,.$$

THEOREM 2. *Let p be an odd prime, $q=(p-1)/2$, and $n=s(p-1)+r$ $(0 \leqq r < p-1)$. Then*

$$\widetilde{KO}(L^n(p)) \cong \begin{cases} (Z_{p^{s+1}})^{[r/2]} + (Z_{p^s})^{q-[r/2]} & (\text{if } n \not\equiv 0 \mod (4)) \\ Z_2 + (Z_{p^{s+1}})^{[r/2]} + (Z_{p^s})^{q-[r/2]} & (\text{if } n \equiv 0 \mod (4))\,, \end{cases}$$

and the direct summand $(Z_{p^{s+1}})^{[r/2]}$ and $(Z_{p^s})^{q-[r/2]}$ are generated additively by $\bar{\sigma}, \cdots, \bar{\sigma}^{[r/2]}$ and $\bar{\sigma}^{[r/2]+1}, \cdots, \bar{\sigma}^q$ respectively. Moreover its ring structure is given by

$$\bar{\sigma}^{q+1} = \sum_{i=1}^{q} \frac{-(2q+1)}{2i-1} \binom{q+i-1}{2i-2} \bar{\sigma}^i, \quad \bar{\sigma}^{[n/2]+1} = 0\,.$$

As an application of Theorem 2, we obtain following,

THEOREM 3[3]. *Let p be odd prime, then*

(1) *The lens space $L^n(p)$ cannot be immersed in $R^{2n+2L(n,p)-1}$,*

(2) *$L^n(p)$ cannot be imbedded in $R^{2n+2L(n,p)}$,*

where $L(n, p)$ is the integer defined by

$$L(n, p) = \max \{ i \leqq [n/2] \mid \binom{n+i}{i} \not\equiv 0 \mod (p^{1+[(n-2i)/(p-1)]}) \}\,.$$

In §1, we recall the basic properties of the rings $K(X)$ and $KO(X)$ of a finite CW-complex X, which are necessary in the latter sections. Theorem 1 is proved in §2 by the use of $K(CP^n)$ determined by J. F. Adams, and Theorem 2 is proved in §3. In §4, the Grothendieck operator γ^i in $KO(L^n(p))$ are determined, and Theorem 3 is proved by the methods of M. F. Atiyah.

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2) $(Z_a)^b$ indicates the direct sum of b-copies of a cyclic group Z_a of order a.
3) Immersion and imbedding mean C^∞-differentiable ones.

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§1. The Grothendieck ring

Let X be a finite CW-complex with a base point x_0 and let $\mathcal{E}_\Lambda(X)$ denote the set of equivalence classes of Λ-vector bundles over X (Λ denotes either the real field R or the complex field C). The Whitney sum of bundles makes $\mathcal{E}_\Lambda(X)$ a semi-group. The Grothendieck group $K_\Lambda(X)$ is the associated abelian group. The tensor product of vector bundles defines a ring structure in $K_\Lambda(X)$.

For a continuous map $f: Y \to X$ we have the natural ring homomorphism $f^! : K_\Lambda(X) \to K_\Lambda(Y)$ induced by the lifting of bundles under f. The reduced ring $\tilde{K}_\Lambda(X)$ is defined to be the kernel of $i^! : K_\Lambda(X) \to K_\Lambda(x_0)$ where map $i: x_0 \to X$ is an imbedding of a point x_0.

Let Y be a subcomplex of X, define $K_\Lambda^{-n}(X, Y) = \tilde{K}_\Lambda(S^n(X/Y))$. Here X/Y is the complex obtained from X by collapsing Y to a point and $S^n(X/Y)$ is the n-times iterated suspension of X/Y. For negative n, $K_\Lambda^{-n}(X, Y)$ is defined by using isomorphisms $K_c^{-n-2}(X, Y) \cong K_c^{-n}(X, Y)$, $K_R^{-n-8}(X, Y) \cong K_R^{-n}(X, Y)$. In this way we have periodic cohomology theories $K_c^*(\ ,\ )$ and $K_R^*(\ ,\ )$, of periods 2 and 8 respectively [3].

Then we have the exact sequence

$$\cdots \to \tilde{K}_\Lambda^{-n-1}(Y) \to K_\Lambda^{-n}(X, Y) \to \tilde{K}_\Lambda^{-n}(X) \to \tilde{K}_\Lambda^{-n}(Y) \to \cdots \tag{1.1}$$

In what follows we use the notations K and KO in place of K_C and K_R respectively.

From [1] we have operators

$$r: K(X) \to KO(X), \qquad c: KO(X) \to K(X), \qquad t: K(X) \to K(X)$$

such that

$$rc = 2: KO(X) \to KO(X), \qquad cr = 1+t: K(X) \to K(X). \tag{1.2}$$

These operators are natural with respect to maps and c and t are ring homomorphisms.

The values of $\tilde{K}(S^n)$ and $\tilde{KO}(S^n)$ are as follows [1].

(1.3)

n	$\equiv 0$	1	2	3	4	5	6	7 mod 8
$\tilde{K}(S^n)$	$\cong Z$	0	Z	0	Z	0	Z	0
$\tilde{KO}(S^n)$	$\cong Z$	Z_2	Z_2	0	Z	0	0	0

The structure of $K(CP^n)$ is stated as follows, [1].

(1.4) Adams' theorem
Let η be the canonical complex line bundle over CP^n and put $\mu=\eta-1_c$ then $K(CP^n)$ is a truncated polynomial ring over the integers with one generator μ and one relation $\mu^{n+1}=0$.

§ 2. The ring $\tilde{K}(L^n(p))$

In this section we shall determine the ring $\tilde{K}(L^n(p))$ for prime p.

As is well-known, $L^n(p)$ has a cell structure given by

$$L^n(p)=S^1\cup e^2\cup e^3\cup\cdots\cup e^{2n}\cup e^{2n+1}$$

and

$$H^i(L^n(p),Z)=\begin{cases} Z_p & i=2,4,\cdots,2n \\ Z & i=0,\ 2n+1 \\ 0 & \text{for other } i. \end{cases} \tag{2.1}$$

In the followings, we use the subcomplex

$$L_0^n(p)=S^1\cup e^2\cup e^3\cup\cdots\cup e^{2n},$$

being the $2n$-skelton of the above CW-complex $L^n(p)$. Then we have obviously

$$L_0^n(p)/L_0^{n-1}(p)=S^{2n-1}\underset{p}{\cup} e^{2n} \tag{2.2}$$

where attaching map $p:S^{2n-1}\to S^{2n-1}$ means the map of degree p.

Now, consider the pair $(S^{2n-1}\underset{p}{\cup} e^{2n}, S^{2n-1})$. We have the following exact sequence of (1.1).

$$\cdots\to\tilde{K}^{-1}(S^{2n})\to\tilde{K}^{-1}(S^{2n-1}\underset{p}{\cup} e^{2n})\to\tilde{K}^{-1}(S^{2n-1})\xrightarrow{\delta}\tilde{K}(S^{2n})$$
$$\to\tilde{K}(S^{2n-1}\underset{p}{\cup} e^{2n})\to\tilde{K}(S^{2n-1})\to\cdots$$

In this sequence, the coboundary homomorphism δ is the composition $S_0p^!$: $\tilde{K}^{-1}(S^{2n-1})\to\tilde{K}^{-1}(S^{2n-1})\underset{S_0}{\cong}\tilde{K}(S^{2n})$ where the homomorphism $p^!$ induced by p is obviously given by $p^!(x)=px$, $x\in\tilde{K}^{-1}(S^{2n-1})$. Hence, using (1.3) we have $\tilde{K}^{-1}(S^{2n-1}\underset{p}{\cup} e^{2n})=0$, $\tilde{K}(S^{2n-1}\underset{p}{\cup} e^{2n})=Z_p$ and so

$$K^{\pm1}(L_0^n(p), L_0^{n-1}(p))=0\,,\qquad K(L_0^n(p), L_0^{n-1}(p))=Z_p\,. \tag{2.3}$$

LEMMA (2.4). *$\tilde{K}(L_0^n(p))$ consists of p^n elements and $\tilde{K}^{\pm1}(L_0^n(p))=0$.*

PROOF. We prove by induction on n. For the case $n=0$, our assertions are trivial, since $L_0^0(p)$ is one point. Suppose that (2.4) is true for $n-1$ and consider the following exact sequence (1.1) of the pair $(L_0^n(p), L_0^{n-1}(p))$:

$$\cdots\to K^{-1}(L_0^n(p), L_0^{n-1}(p))\to\tilde{K}^{-1}(L_0^n(p))\to\tilde{K}^{-1}(L_0^{n-1}(p))\to K(L_0^n(p), L_0^{n-1}(p))$$
$$\to\tilde{K}(L_0^n(p))\to\tilde{K}(L_0^{n-1}(p))\to K^1(L_0^n(p), L_0^{n-1}(p))\to\cdots.$$

From (2.3) and the inductive assumption, $K^1(L_0^n(p), L_0^{n-1}(p))=\tilde{K}^{-1}(L_0^{n-1}(p)) =0$, $K(L_0^n(p), L_0^{n-1}(p))=Z_p$ and $\tilde{K}(L_0^{n-1}(p))$ consists of p^{n-1} elements. Therefore the above exact sequence implies that $\tilde{K}(L_0^n(p))$ contains exactly p^n elements and $\tilde{K}^{-1}(L_0^n(p))=0$, and (2.4) is true for n. q. e. d.

Comparing $\tilde{K}(L^n(p))$ with $\tilde{K}(L_0^n(p))$, we have

LEMMA (2.5). *The inclusion map $i: L_0^n(p)\subset L^n(p)$ induces the isomorphism $i^!: \tilde{K}(L^n(p))\cong\tilde{K}(L_0^n(p))$, and $\tilde{K}(L^n(p))$ consists of p^n elements.*

PROOF. Since $L^n(p)/L_0^n(p)=S^{2n+1}$, we have the following exact sequence:

$$\cdots\rightarrow\tilde{K}(S^{2n+1})\rightarrow\tilde{K}(L^n(p))\xrightarrow{i^!}\tilde{K}(L_0^n(p))\xrightarrow{\delta}\tilde{K}^1(S^{2n+1})\rightarrow\cdots$$

Here, $\tilde{K}(L_0^n(p))$ is a finite group by (2.4) and $\tilde{K}^1(S^{2n+1})$ is an infinite cyclic group, then the above homomorphism δ is trivial. This and $\tilde{K}(S^{2n+1})=0$ and (2.4) imply the lemma. q. e. d.

Let $\pi': S^{2n+1}\rightarrow L^n(p)$ be a natural projection and define the map $\pi: L^n(p)\rightarrow CP^n$ by $\pi(\pi'(z_0, z_1\cdots, z_n))=[z_0, z_1, \cdots, z_n]$. Then $(L^n(p), \pi, CP^n)$ is the locally trivial fibre space with fibre S^1. Consider its Gysin's sequence:

$$\cdots\rightarrow H^{i-2}(CP^n)\rightarrow H^i(CP^n)\rightarrow H^i(L^n(p))\rightarrow H^{i-1}(CP^n)\rightarrow\cdots.$$

Since $H^{i-1}(CP^n)=0$ for each even i $i\neq 1$ and $H^i(L^n(p))=0$ for each odd $i<2n$, thus we have

(2.6) The homomorphism π^*: $H^i(CP^n)\rightarrow H^i(L^n(p))$ $i<2n+1$ is an epimorphism.

The following proposition is basic in our computation of $\tilde{K}(L^n(p))$.

PROPOSITION (2.7). *The ring homomorphism $\pi^!: \tilde{K}(CP^n)\rightarrow\tilde{K}(L^n(p))$ is an epimorphism.*

PROOF. Let $\pi_0=\pi\mid L_0^n(p): L_0^n(p)\rightarrow CP^n$ be the restriction of π. Then, by Lemma (2.5), it is sufficient to prove that the homomorphism $\pi_0^!: \tilde{K}(CP^n)\rightarrow\tilde{K}(L_0^n(p))$ is an epimorphism. This is trivial for $n=0$ and we suppose inductively that $\pi_0^!: \tilde{K}(CP^{n-1})\rightarrow\tilde{K}(L_0^{n-1}(p))$ is an epimorphism. Consider the following commutative diagram where the horizontal sequences are exact:

$$\begin{array}{ccccccccc}
\cdots\rightarrow & \tilde{K}(CP^n/CP^{n-1}) & \rightarrow & \tilde{K}(CP^n) & \rightarrow & \tilde{K}(CP^{n-1}) & \rightarrow & \tilde{K}^1(CP^n/CP^{n-1}) & \rightarrow\cdots \\
 & \downarrow\pi_0^! & & \downarrow\pi_0^! & & \downarrow\pi_0^! & & \downarrow\pi_0^! & \\
\cdots\rightarrow & \tilde{K}(L_0^n(p)/L_0^{n-1}(p)) & \rightarrow & \tilde{K}(L_0^n(p)) & \rightarrow & \tilde{K}(L_0^{n-1}(p)) & \rightarrow & \tilde{K}^1(L_0^n(p)/L_0^{n-1}(p)) & \rightarrow\cdots
\end{array}$$

Here, $\pi_0^!: \tilde{K}^1(CP^n/CP^{n-1})\rightarrow\tilde{K}^1(L_0^n(p)/L_0^{n-1}(p))$ is a monomorphism, since $\tilde{K}^1(CP^n/CP^{n-1})=\tilde{K}^1(S^{2n})=0$, and $\pi_0^!: \tilde{K}(CP^{n-1})\rightarrow\tilde{K}(L_0^{n-1}(p))$ is an epimorphism by the hypothesis of induction. Also, since $\pi_0: L_0^n(p)/L_0^{n-1}(p)=S^{2n-1}\cup_p e^{2n}\rightarrow CP^n/CP^{n-1} =S^{2n}$ is nothing but the map collapsing S^{2n-1} to a point, $\pi_0^!: \tilde{K}(CP^n/CP^{n-1})\rightarrow\tilde{K}(L_0^n(p)/L_0^{n-1}(p))$ is an epimorphism. Therefore $\pi_0^!: \tilde{K}(CP^n)\rightarrow\tilde{K}(L_0^n(p))$ is an epimorphism by the Five Lemma, and this completes the proof by induction. q. e. d.

Now, concerning the element

$$\sigma = \pi^! \mu \in \tilde{K}(L^n(p)),$$

we have two lemmas.

LEMMA (2.8). $(1_c+\sigma)^p = 1_c, \quad \sigma^{n+1} = 0.$

PROOF. The total Chern class $C(\eta)$ of the canonical complex line bundle η is geven by $C(\eta) = 1+x$, where $x \in H^2(CP^n) = Z$ is a generater [4]. By the naturality of Chern class and (2.1) and (2.6), we have

$$C((\pi^! \eta)^p) = \pi^* C(\eta \otimes \cdots \otimes \eta) = \pi^*(1+px) = 1+p\pi^* x = 1.$$

and so $(\pi^! \eta)^p = 1_c$, for the complex line bundles are classified by their first Chern class. Thus

$$(\sigma+1_c)^p = (\pi^! \mu+1_c)^p = (\pi^! \eta)^p = 1_c.$$

$\sigma^{n+1} = 0$ is an immediate consequence of $\mu^{n+1} = 0$ of (1.4). q. e. d.

LEMMA (2.9). *Lep p be prime, then*

$$p^{+1\left[\frac{i}{p-1}\right]} \sigma^{n-i} = 0.$$

PROOF. Multiplying σ^{n-i-1} to the first equation of (2.8), we have

$$\binom{p}{1} \sigma^{n-i} + \binom{p}{2} \sigma^{n-i+1} + \cdots + \sigma^{n-i+p-1} = 0.$$

For $i=0$, this and $\sigma^{n+1} = 0$ imply $p\sigma^n = 0$. Suppose inductively that the lemma is true for $j \leqq i-1$. Multiply $p^{\left[\frac{i}{p-1}\right]}$ to the above equation, we have

$$P^{\left[\frac{i}{p-1}\right]} \binom{p}{1} \sigma^{n-i} + p^{\left[\frac{i}{p-1}\right]} \binom{p}{2} \sigma^{n-(i-1)} + \cdots + p^{\left[\frac{i}{p-1}\right]} \sigma^{n-i+(p-1)} = 0.$$

Here, $\binom{p}{j} = 0 \bmod (p)$ for $1 \leqq j \leqq p-1$ since p is prime. Then this and the assumption of induction imply

$$p^{\left[\frac{i}{p-1}\right]} \binom{p}{2} \sigma^{n-(i-1)} = \cdots = p^{\left[\frac{i}{p-1}\right]} \binom{p}{p-1} \sigma^{n-(i-(p-1))} = 0$$

and

$$p^{\left[\frac{i}{p-1}\right]} \sigma^{n-i+p-1} = p^{1+\left[\frac{i-(p-1)}{p-1}\right]} \sigma^{n-(i-(p-1))} = 0.$$

Hence we have $p^{1+\left[\frac{i}{p-1}\right]} \sigma^{n-i} = 0$ from the above equation. q. e. d.

Now, we are ready to prove Theorem 1.

PROOF OF THEOREM 1. $\tilde{K}(L^n(p))$ is generated by σ by (1.4) and Proposition (2.7). This and the relation of (2.8) imply that $\tilde{K}(L^n(p))$ is additively generated by $\sigma, \sigma^2, \cdots, \sigma^{p-1}$. On the other hand (2.9) implies

$$p^{s+1+\left[\frac{r-1}{p-1}\right]}\sigma^i = p^{1+\left[\frac{n-1}{p-1}\right]}\sigma^i = 0 \qquad \text{for} \quad i=1, 2, \cdots, p-1$$

and also we have $(p^{s+1})^r \times (p^s)^{p-1-r} = p^n$. Therefore we have Theorem 1 using Lemma (2.5). q. e. d.

LEMMA (2.10). *Element σ^i is of order $p^{1+\left[\frac{n-i}{p-1}\right]}$.*

PROOF. Lemma (2.9) shows that σ^n is 0 or of order p. If $\sigma^n = 0$, it follows $p^{\left[\frac{n-i}{p-1}\right]}\sigma^i = 0$ by the similar way as (2.9). But this does not happen for elements $\sigma, \sigma^2, \cdots, \sigma^{p-1}$ by Theorem 1. Therefore σ^n must be of order p. Clearly

$$\text{order } \sigma^n \leqq \text{order } \sigma^{n-1} \leqq \cdots \leqq \text{order } \sigma^{n-(p-2)}$$

and $p\sigma^{n-(p-2)} = 0$ by (2.9). Therefore elements $\sigma^n, \sigma^{n-1}, \cdots, \sigma^{n-(p-2)}$ are of order p. Assume that the elements $\sigma^{n-j(p-1)}, \sigma^{n-j(p-1)-1}, \cdots, \sigma^{n-j(p-1)-(p-2)}$ are of order p^{j+1}. Then multipling $\sigma^{n-j(p-1)-p}$ to the equation

$$\binom{p}{1}\sigma + \binom{p}{2}\sigma^2 + \cdots + \sigma^p = 0,$$

we have

$$\binom{p}{1}\sigma^{n-(j+1)(p-1)} = -\sigma^{n-j(p-1)}.$$

Therefore $\sigma^{n-(j+1)(p-1)}$ is of order p^{j+2}. Since

$$\text{order } \sigma^{n-(j+1)(p-1)-1} \leqq \cdots \leqq \text{order } \sigma^{n-(j+1)(p-1)-(p-2)}$$

and $p^{j+2}\sigma^{n-(j+1)(p-1)-(p-2)} = 0$ by (2.9), the elements $\sigma^{n-(j+1)(p-1)}, \sigma^{n-(j+1)(p-1)-1}, \cdots, \sigma^{n-(j+1)(p-1)-(p-2)}$ are of order p^{j+2}. q. e. d.

§ 3. The ring $\widetilde{KO}(L^n(p))$.

Throughout this section, we assume that p is an odd prime. As our first step, we can take the similar arguments with § 2. Consider the exact sequence:

$$\cdots \to \widetilde{KO}(S^{2n}) \xrightarrow{\times p} \widetilde{KO}(S^{2n}) \to KO(L_0^n(p),\ L_0^n(p)) \to \widetilde{KO}^1(S^{2n}) \xrightarrow{\times p} \widetilde{KO}^1(S^{2n}) \to \cdots$$

where $\widetilde{KO}^i(S^{2n})$ is isomorphic to Z, Z_2, or 0 by (1.3). For $n \equiv 0 \mod (2)$, the above sequence is

$$Z \xrightarrow{\times p} Z \to KO(L_0^n(p),\ L_0^{n-1}(p)) \to 0$$

and for $n \not\equiv 0 \mod (2)$, we have the following two cases:

$$Z_2 \xrightarrow{\times p} Z_2 \to KO(L_0^n(p),\ L_0^{n-1}(p)) \to Z_2 \xrightarrow{\times p} Z_2$$

$$0 \to KO(L_0^n(p),\ L_0^{n-1}(p)) \to 0.$$

Thus we have

$$(3.1) \qquad KO(L_0^n(p),\ L_0^{n-1}(p)) = \begin{cases} Z_p & n \equiv 0 \mod (2) \\ 0 & n \not\equiv 0 \mod (2) \end{cases}$$

Similarly we have

$$(3.2) \qquad KO^{\pm 1}(L_0^n(p),\ L_0^{n-1}(p))=0$$

Using (3.1), (3.2) and the exact sequences of the pair $(L_0^n(p),\ L_0^{n-1}(p))$ and the pair $(L^n(p),\ L_0^n(p))$, we can prove the following two lemmas by the similar way to the proof of Lemmas (2.4), (2.5).

LEMMA (3.3). *$\widetilde{KO}(L_0^n(p))$ contains exactly $p^{\left[\frac{n}{2}\right]}$ elements.*

LEMMA (3.4).

$$\widetilde{KO}(L^n(p)) \cong \begin{cases} \widetilde{KO}(L_0^n(p)) & \text{if } n \not\equiv 0 \bmod (4) \\ Z_2+\widetilde{KO}(L_0^n(p)) & \text{if } n \equiv 0 \bmod (4) \end{cases}$$

Now, consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{K}(CP^n) & \xrightarrow{\pi^!} & \widetilde{K}(L^n(p)) \\ r\downarrow\uparrow c & & r\downarrow\uparrow c \\ \widetilde{KO}(CP^n) & \xrightarrow{\pi^!} & \widetilde{KO}(L^n(p)) \end{array}$$

where r, c are operators recalled in §1.

For the elements $\mu \in \widetilde{K}(CP^n)$ and $\sigma \in \widetilde{K}(L^n(p))$, we put

$$\bar{\mu}=r(\mu) \in \widetilde{KO}(CP^n), \qquad \bar{\sigma}=r(\sigma) \in \widetilde{KO}(L^n(p)).$$

Since the inclusion map $i: L_0^n(p) \subset L^n(p)$ induces the isomorphism $i^!: \widetilde{K}(L^n(p)) \cong \widetilde{K}(L_0^n(p))$ by (2.5), we identify $\widetilde{K}(L_0^n(p))$ and $\operatorname{Im} i^!$ and regard σ as an elements of $\widetilde{K}(L_0^n(p))$ and also regard $\bar{\sigma}=r(\sigma) \in \widetilde{KO}(L_0^n(p))$.

LEMMA (3.5).

i) *The homomorphism $r: \widetilde{K}(L_0^n(p)) \to \widetilde{KO}(L_0^n(p))$ is an epimorphism.*

ii) *The homomorphism $c: \widetilde{KO}(L_0^n(p)) \to \widetilde{K}(L_0^n(p))$ is a monomorphism and*

$$c(\bar{\sigma})=\sigma^2-\sigma^3+\sigma^4-\cdots=\frac{\sigma^2}{1+\sigma}.$$

PROOF. $\widetilde{KO}(L_0^n(p))$ has not an element of order 2 by (3.3). Therefore $rc=2: \widetilde{KO}(L_0^n(p)) \to \widetilde{KO}(L_0^n(p))$ is an isomorphism and so r is epimorphic and c is monomorphic. For the complex line bundle η, the conjugation operator t satisfies clearly $\eta\cdot(t\eta)=1$, and we have $(1+\mu)(1+t\mu)=1$ and so $t(\mu) =-\mu+\mu^2-\mu^3+\cdots$. Therefore by (1.2) $c(\bar{\mu})=c(r(\mu))=(1+t)\mu=\mu^2-\mu^3+\mu^4-\cdots$, and the equation of ii) follows from the naturality of the operator c. q.e.d.

(2.10) and (3.5) ii) give

(3.6) $\bar{\sigma}^i$ is of order $p^{1+\left[\frac{n-2i}{p-1}\right]}$.

The following lemma is necessary to determine the ring structure of $\widetilde{KO}(L^n(p))$.

LEMMA (3.7).

$$\binom{2q}{j}=\sum_{i=1}^{j+1}\binom{q+i-1}{j}\binom{j+1}{2i-1}$$

$$\binom{2q+1}{j}=\sum_{i=1}^{j+1}\binom{q+i-1}{j}\binom{j+1}{2i-2}.$$

PROOF. For $q=1$, two formulas are true for any j. Assuming the first equation, we have

$$\begin{aligned}\binom{2q+1}{j}&=\binom{2q}{j}+\binom{2q}{j-1}\\&=\sum_{i=1}^{j+1}\binom{q+i-1}{j}\binom{j+1}{2i-1}+\sum_{i=1}^{j}\binom{q+i-1}{j-1}\binom{j}{2i-1}\\&=\sum_{i=1}^{j}\binom{q+i-1}{j}\binom{j}{2i-1}+\sum_{j=1}^{j+1}\binom{q+i-1}{j}\binom{j}{2i-1}\\&\quad+\sum_{i=1}^{j}\binom{q+i-1}{j-1}\binom{j}{2i-1}\\&=\sum_{i=1}^{j}\binom{q+i}{j}\binom{j}{2i-1}+\sum_{i=1}^{j+1}\binom{q+i-1}{j}\binom{j}{2i-2}\\&=\sum_{i=1}^{j+1}\binom{q+i-1}{j}\binom{j}{2i-3}+\sum_{i=1}^{j+1}\binom{q+i-1}{j}\binom{j}{2i-2}\\&=\sum_{i=1}^{j+1}\binom{q+i-1}{j}\binom{j+1}{2i-2}.\end{aligned}$$

Therefore the first one implies the second one. Similarly the second one implies the first one for $q+1$, and the lemma is obtained by induction. q. e. d.

Now, we are ready to prove Theorem 2.

PROOF OF THEOREM 2. First, we prove the equation

$$\bar{\sigma}^{q+1}=\sum_{i=1}^{q}a_i\bar{\sigma}^i,\qquad a_i=\frac{-(2q+1)}{2i-1}\binom{q+i-1}{2i-2}.$$

Because c is monomorphic, this is equivalent to

$$(c(\bar{\sigma}))^{q+1}=\sum_{i=1}^{q}a_i(c(\bar{\sigma}))^i,$$

and to

$$-\Big(\sum_{j=1}^{2q}\binom{2q+1}{j}\sigma^{j+1}\Big)=\sum_{i=1}^{q}a_i\sigma^{2i}\Big(\sum_{k=0}^{q+1-i}\binom{q+1-i}{k}\sigma^k\Big)$$

by (2.8) and ii) of (3.5). Therefore it is sufficient to show

$$-\binom{2q+1}{j}=\sum_{i=1}^{j}a_i\binom{q-i+1}{j-2i+1},\quad a_i=\frac{-(2q+1)}{2i-1}\binom{q+i-1}{2i-2}.$$

This is clearly equivalent to

$$\binom{2q}{j-1}=\sum_{i=1}^{j}\frac{j}{2i-1}\binom{q+i-1}{2i-2}\binom{q-i+1}{j-2i+1}=\sum_{i=1}^{j}\binom{q+i-1}{j-1}\binom{j}{2i-1}$$

and this follows from Lemma (3.7).

Now, Lemma (3.5) i) and the above equation $\bar{\sigma}^{q+1}=\sum_{i=1}^{q}a_i\bar{\sigma}^i$ shows that $\widetilde{KO}(L_0^n(p))$ is generated additively by $\bar{\sigma},\cdots,\bar{\sigma}^q$. By (3.6) the order of $\bar{\sigma}^i$ is equal to p^{s+1} for $i=1,\cdots,\left[\frac{r}{2}\right]$ and to p^s for $i=\left[\frac{r}{2}\right]+1,\cdots,q$. This and (3.3) and

$$(p^{s+1})^{\left[\frac{r}{2}\right]}\times(p^s)^{q-\left[\frac{r}{2}\right]}=p^{\left[\frac{n}{2}\right]}$$

imply $\widetilde{KO}(L_0^n(p))\cong(Z_{p^{s+1}})^{\left[\frac{r}{2}\right]}+(Z_{p^s})^{q-\left[\frac{r}{2}\right]}$.

This and (3.4) complete the proof of Theorem 2. q. e. d.

§4. Immersions and imbeddings of lens spaces.

First we recall the Theorem of Atiyah [2]. Consider the exterior power operators λ^i which have the following properties in $\mathcal{E}_R(X)$.

(4.1) $$\lambda^0(x)=1,$$

(4.2) $$\lambda^1(x)=x,$$

(4.3) $$\lambda^i(x+y)=\sum_{j=0}^{i}\lambda^j(x)\otimes\lambda^{i-j}(y),$$

(4.4) $$\lambda^i(x)=0 \quad \text{for} \quad i>\dim x, \qquad \text{for any } x,y\in\mathcal{E}_R(X).$$

Let $A(X)$ denote the multiplicative group of formal power series in t with coefficient in $KO(X)$ and constant term 1. Then

$$\lambda_t(x)=\sum_{i=1}^{\infty}\lambda^i(x)t^i$$

defines a homomorphism $\mathcal{E}_R(X)\to A(X)$ by (4.3). Hence we have a homomorphism $\lambda_t: KO(X)\to A(X)$. Taking the coefficients of λ_t we have operator

$$\lambda^i: KO(X)\to KO(X).$$

Again we introduce the homomorphism

$$\gamma_t=\lambda_{t/1-t}: KO(X)\to A(X)$$

and the Grothendieck operator

$$\gamma^i: KO(X)\to KO(X)$$

is also defined as the coefficients of γ_t:

$$\gamma_t(x)=\sum_{i=1}^{\infty}\gamma^i(x)t^i .$$

Then Theorem of Atiyah is stated as follows.

(4.5) Let M be a compact n-dimensional manifold and let $\tau(M)$ denote its tangent bundle and put $\tau_0(M)=\tau(M)-n\in\widetilde{KO}(M)$. Then

i) Let M be immersible in R^{n+k}, then $\gamma^i(-\tau_0(M))=0$ for $i>k$,

ii) Let M be imbeddable in R^{n+k}, then $\gamma^i(-\tau_0(M))=0$ for $i\geqq k$.

Now, consider the differential bundle space $(L^n(p), \pi, CP^n, S^1)$ and let α be its bundle along the fibre. Then as is well-known,

$$\tau(L^n(p))=\pi^!(\tau(CP^n))+\alpha .$$

Here, since the manifolds $L^n(p)$ and CP^n are orientable and $\dim\alpha=1$, we have $\alpha=1$ and so

$$\tau(L^n(p))=\pi^!\tau(CP^n)+1 . \tag{4.6}$$

LEMMA (4.7).

$$\tau_0(L^n(p))=(n+1)\bar{\sigma}\in\widetilde{KO}(L^n(p)) .$$

PROOF. It is well-known that the complex tangent bundle $\tau_c(CP^n)$ is given by $\tau_c(CP^n)+1=(n+1)\eta$ [4]. Hence

$$\tau(L^n(p))+1=\pi^!(\tau(CP^n)+2)=\pi^!(r(\tau_c(CP^n)+1_c))=(n+1)r\pi^!\eta$$

by (4.6), and so

$$\tau_0(L^n(p))=\tau(L^n(p))-(2n+1)=(n+1)r\pi^!\eta-(n+1)_c=(n+1)\bar{\sigma} .$$

q. e. d.

LEMMA (4.8).

$$\gamma_t(\bar{\sigma})=1+\bar{\sigma}t-\bar{\sigma}t^2\in A(L^n(p)) .$$

PROOF. If x is an oriented real bundle of dimension n, then $\lambda^n(x)=1$ by the definition of the exterior power operations. Since $r(\eta)$ is an oriented bundle of dimension 2, $\lambda_t(r(\eta))=1+r(\eta)t+t^2$ by the properties (4.1), (4.2) and (4.4) and so $\lambda_t(\bar{\mu}+2)=(1+t)^2+\bar{\mu}t$. On the other hand, since λ_t is the homomorphism, we have

$$\lambda_t(\bar{\mu}+2)=\lambda_t(\bar{\mu})\cdot\lambda_t(1)^2=\lambda_t(\bar{\mu})(1+t)^2$$

and so $\lambda_t(\bar{\mu})=1+\dfrac{t}{(1+t)^2}\bar{\mu}$. Hence

$$\gamma_t(\bar{\mu})=\lambda_{t/1-t}(\bar{\mu})=1+\bar{\mu}t-\bar{\mu}t^2 .$$

Therefore $\gamma_t(\bar{\sigma})=1+\bar{\sigma}t-\bar{\sigma}t^2$ by the naturality of the operator γ_t. q. e. d.

PROOF OF THEOREM 3. Lemmas (4.7) and (4.8) imply

$$\gamma_t(-\tau_0(L^n(p))) = (\gamma_t(\bar{\sigma}))^{-(n+1)} = (1+\bar{\sigma}(t-t^2))^{-(n+1)}$$
$$= \sum_{i=0}^{\infty} \binom{-(n+1)}{i} \bar{\sigma}^i (t-t^2)^i$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{n+i}{i} \bar{\sigma}^i (t-t^2)^i .$$

Therefore we have

$L^n(p)$ cannot be immersed in $R^{2n+1+2L'(n,p)-1}$,
$L^n(p)$ cannot be imbedded in $R^{2n+1+2L'(n,p)}$.

by Theorem of Atiyah, where $L'(n, p) = \max \left\{ i \mid \binom{n+i}{i} \bar{\sigma}^i \neq 0 \right\}$. On the other hand, (3.6) and $\bar{\sigma}^{\left[\frac{n}{2}\right]+1} = 0$ imply $L'(n, p) = L(n, p)$, and Theorem 3 is obtained. q. e. d.

COROLLARY 1. *For odd prime* $p > n + \left[\frac{n}{2}\right]$, $L^n(p)$ *cannot be immersed in* $R^{2n+2\left[\frac{n}{2}\right]}$ *and cannot be imbedded in* $R^{2n+2\left[\frac{n}{2}\right]+1}$.

Here, we notice that the following immersibility theorem is obtained, using (4.6) and the notion of the geometric dimension [2].

THEOREM 4. *If* CP^n *is immersible in* R^{2n+s}, *then* $L^n(p)$ *is immersible in* R^{2n+s+1}.

This and Corollary 1 give the known result:

COROLLARY 2. CP^n *cannot be immersed in* $R^{2n+2\left[\frac{n}{2}\right]-1}$.

Kyoto University

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