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Theory of Finsler spaces based on the contact structure

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Most of the studies on Finsler spaces seem to aim at the generalization of the fruitful results on Riemannian spaces as are resumed in the book of H. Rund [4]. Another approach to Finsler spaces is found in the book of E. Cartan [2] and C. Caratheodory [1]. We treated in the previous paper [3] some results given in [2] from a modern geometrical point of view, but a further step was not taken. Here we develop along this way the theory of Finsler spaces based upon a certain contact structure associated with the spaces.

1. Contact structure on Finsler spaces

Finsler spaces can be defined as follows. Let M be an n-dimensional differentiable manifold and the local coordinates of a point x on M be $x^1, \cdots, x^n$. In the tangent space $T(x)$ at x we take a natural frame and denote the components of a vector y in $T(x)$ by $y^1, \cdots, y^n$ and those of a vector p in the dual tangent space ${}^cT(x)$ by $p_1, \cdots, p_n$. We take a function $F=F(x, y)$ on the tangent bundle $T(M)$ of M which has the following properties:

(I) $F(x, y)$ is of class C^4 and is positively homogeneous of degree one with respect to $y^1, \cdots, y^n$,

(II) the rank of the matrix $(\partial^2 F/\partial y^i \partial y^j)$ $(i, j=1, \cdots, n)$ is $n-1$.

The manifold M with such a function $F(x, y)$ is called a Finsler space. The length of a curve $x=x(t)$ on the Finsler space is defined by $\int F(x, \dot{x})dt$, where $\dot{x}=dx/dt$.

A mapping $\varphi: T(M)\to {}^cT(M)$ is defined by $(x, y)\to(x, p)$ with

$$p_i=\frac{\partial F}{\partial y^i}. \qquad (i=1, \cdots, n). \tag{1.1}$$

Then $N=\varphi(T(M))$ is a submanifold of ${}^cT(M)$ by virtue of (I) and (II) (cf. [3]). We call N a p-manifold of M and denote its local equation by

$$G(x, p)=0, \tag{1.2}$$

where $(G_{p_1}, \cdots, G_{p_n}) \neq (0, \cdots, 0)$. We denote $\partial G/\partial p_i$ by G_{p_i} and $\partial^2 F/\partial y^i \partial y^j$ by $F_{y^i y^j}$ etc.. In a certain neighborhood of each point of $T(M)$ we can assume

$$\det (F_{y^a y^b}) \neq 0 \qquad (a, b = 1, \cdots, n-1) \tag{1.3}$$

without loss of generality. Then the local equation (1.2) of N can be written as

$$p_n = -H(x^1, \cdots, x^n, p_1, \cdots, p_{n-1}). \tag{1.4}$$

Throughout our discussion we use indices as

$$i, j, k, h = 1, \cdots, n \qquad a, b, c, e = 1, \cdots, n-1.$$

As (1.1) satisfies (1.2), we get by differentiation with respect to y^j $G_{p_i}\partial p_i/\partial y^j = 0$, namely $G_{p_i}F_{y^i y^j} = 0$. On the other hand we have by (I) $y^i F_{y^i y^j} = 0$. Hence we get

$$y^i = \lambda G_{p_i}, \tag{1.5}$$

and so

$$F = y^i F_{y^i} = p_i y^i = \lambda p_i G_{p_i}. \tag{1.6}$$

For a point $(x, p) \in {}^cT(M)$ corresponding to $(x, \dot{x}) \in T(M)$ along a curve $x = x(t)$ we have

$$\dot{x}^i = \lambda G_{p_i}. \tag{1.7}$$

We consider a 1-form

$$\omega = p_i dx^i \tag{1.8}$$

on N. We can show that $\omega \wedge (d\omega)^{n-1} \neq 0$ at the point (x, p) corresponding to (x, y) such that $F(x, y) \neq 0$, which means that ω defines a contact structure on N with the exception of the case $F(x, y) = 0$ (cf. [3]). We can assume $G_{p_n} \neq 0$ without loss of generality. Then we have

$$dp_n = -G_{p_n}^{-1}(G_{p_a}dp_a + G_{x^i}dx^i)$$

and get

$$\alpha = d\omega = dp_a \wedge dx^a - G_{p_n}^{-1}(G_{p_a}dp_a + G_{x^i}dx^i) \wedge dx^n.$$

Hence by taking interior products with $\partial/\partial p_a$, $\partial/\partial x^a$, $\partial/\partial x^n$ we get

$$\theta^a = i(\partial/\partial p_a)\alpha = dx^a - G_{p_n}^{-1}G_{p_a}dx^n \tag{1.9}$$

$$\rho_a = i(\partial/\partial x^a)\alpha = -dp_a - G_{p_n}^{-1}G_{x^a}dx^n \tag{1.10}$$

$$\rho_n = i(\partial/\partial x^n)\alpha = -dp_n = G^{-1}_{n}(G_{x^a}\theta^a - G_{p_a}\rho_a). \tag{1.11}$$

We get by (1.10) (1.6)

$$\begin{aligned} \omega = p_i dx^i &= p_a(\theta^a + G_{p_n}^{-1}G_{p_a}dx^n) + p_n dx^n \\ &= p_a\theta^a + G_{p_n}^{-1}G_{p_i}p_i dx^n \\ &= p_a\theta^a + G_{p_n}^{-1}\lambda^{-1}F dx^n \end{aligned} \tag{1.12}$$

Hence ω, θ^a, ρ_a are independent 1-forms on N. For (x, p) corresponding to $(x, \dot{x})$ along a curve $x=x(t)$ on M we have by (1.7) (1.9)

$$\theta^a = 0 . \tag{1.13}$$

The solution curves of differential equations

$$\theta^a = 0, \qquad \rho_a = 0 \qquad (a=1, \cdots, n-1) \tag{1.14}$$

are curves on N which we call e-curves. The projections of e-curves of M are extremals of the Finsler space M (cf. [3]).

As ω, θ^a, ρ_a are linearly independent forms on N, they can be taken as local base of differential forms on N. We have by virtue of (1.9) (1.10) (1.11)

$$\begin{aligned} d\omega &= dp_a \wedge dx^a + dp_n \wedge dx^n \\ &= -(\rho_a + G_{pn}^{-1} G_{x^a} dx^n) \wedge (\theta^a + G_{pn}^{-1} G_{p_a} dx^n) - G_{pn}^{-1}(G_{x^a}\theta^a - G_{p_a}\rho_a) \wedge dx^n \\ &= \theta^a \wedge \rho_a . \end{aligned}$$

Thus we get

THEOREM 1. *Forms ω, θ^a, ρ_a defined by* (1.8) (1.9) (1.10) *are linearly independent and*

$$d\omega = \theta^a \wedge \rho_a . \tag{1.15}$$

(x, p) along a curve $x=x(t)$ on M is characterized by (1.13) *and (x, p) along an extremal by* (1.14).

In the case (1.4) we have $G(x, p) = p_n + H$ and so

$$\omega = p_a dx^a - H dx^n, \quad \theta^a = dx^a - H_{p_a} dx^n, \quad \rho_a = -dp_a - H_{x^a} dx^n . \tag{1.16}$$

We calculate $d\theta^a$, $d\rho_a$ and get

$$\begin{aligned} d\theta^a &= -(H_{p_a p_b} dp_b + H_{p_a x^i} dx^i) \wedge dx^n = (H_{p_a p_b}\rho_b - H_{p_a x^b}\theta^b) \wedge dx^n \\ d\rho_a &= -(H_{x^a x^b} dx^b + H_{x^a p_b} dp_b) \wedge dx^n = -H(_{x^a x^b}\theta^b - H_{x^a p_b}\rho_b) \wedge dx^n . \end{aligned}$$

We have by virtue of (1.12)

$$dx^n = -(H - p_c H_{p_c})^{-1}(\omega - p_c\theta^c)$$

Hence

$$d\theta^a = -(H - p_c H_{p_c})^{-1}(H_{p_a p_b}\rho_b - H_{p_a x^b}\theta^b) \wedge (\omega - p_c\theta^c) \tag{1.17}$$

$$d\rho_a = -(H - p_c H_{p_c})^{-1}(-H_{x^a x^b}\theta^b + H_{x^a p_b}\rho_b) \wedge (\omega - p_c\theta^c) \tag{1.18}$$

Here we calculate the relations between the derivatives of H and F. By (1.1) (1.4) we have $F_{y^n} = -H(x^1, \cdots, x^n, F_{y^1}, \cdots, F_{y^{n-1}})$. As F is of homogeneous degree one we get

$$F_{y^a} y^a - y^n H = F . \tag{1.19}$$

Differentiating with respect to y^b we get $F_{y^a y^b}(y^a - y^n H_{p_a}) = 0$. As in our case det $(F_{y^a y^b}) \neq 0$, we get

$$y^a = y^n H_{p_a}\,. \tag{1.20}$$

Differentiating with respect to y^c we get $\delta_{ac} = y^n H_{p_a p_b} F_{y^b y^c}$.
Hence

$$(H_{p_a p_b}) = 1/y^n (F_{y^a y^b})^{-1}\,. \tag{1.21}$$

By (1.19) (1.20) we get

$$-y^n(H - p_c H_{p_c}) = F \tag{1.22}$$

and so

$$-(H - p_c H_{p_c})^{-1}(H_{p_a p_b}) = F^{-1}(F_{y^a y^b})^{-1}\,. \tag{1.23}$$

2. Adapted orthogonal coframe

Here we consider a coframe ω, θ^a, ρ_a which is more general than that in Theorem 1. We assume that

(A) $\omega = p_i dx^i$ is the fundamental 1-form

(B) $\theta^a (a=1, \cdots, n-1)$ are forms which are linear combinations of $dx^1, \cdots, dx^n$

(C) ω, $\theta^1, \cdots, \theta^{n-1}$, $\rho_1, \cdots, \rho_{n-1}$ are independent

(D) $d\omega = \theta^a \wedge \rho_a$.

We call the coframe with these properties an adapted coframe of the p-manifold N. The transformation between two adapted coframes ω, θ^b, ρ_a and ω, $\bar{\theta}^a$, $\bar{\rho}_a$ is given by

$$\bar{\theta}^a = s^a_b \theta^b\,, \qquad \bar{\rho}_a = t^b_a \rho_b + r_{ab}\theta^b\,, \tag{2.1}$$

where

$$s^a_b t^b_c = \delta^a_c = \begin{cases} 1 & (a=c) \\ 0 & (a \neq c) \end{cases}, \qquad s^a_b r_{ac} = s^a_c r_{ab}\,. \tag{2.2}$$

This can be verified by the relations

$$\bar{\theta}^a = s^a_b \theta^b + s^a \omega, \quad \bar{\rho}_a = t^b_a \rho_b + r_{ab}\theta^b + r_a \omega, \quad d\omega = \theta^a \wedge \rho_a = \bar{\theta}_a \wedge \bar{\rho}_a\,.$$

By virtue of (2.1) we can conclude

THEOREM 2. *The assertion in Theorem* 1 *holds good for adapted coframes.*

We can put

$$d\theta^a = \frac{1}{2} k^a_{bc} \theta^b \wedge \theta^c + l^a_b \omega \wedge \theta^b + m^{ab}_c \rho_b \wedge \theta^c + h^{ab} \rho_b \wedge \omega\,, \tag{2.3}$$

$$d\rho_a = \frac{1}{2} u^{bc}_a \rho_b \wedge \rho_c + v^b_a \omega \wedge \rho_b + w_{ab} \omega \wedge \theta^b + y^b_{ac} \theta^c \wedge \rho_b + \frac{1}{2} z_{abc} \theta^b \wedge \theta^c\,, \tag{2.4}$$

where

$$k^a_{bc} = -k^a_{cb}, \quad u^{bc}_a = -u^{cb}_a, \quad z_{abc} = -z_{acb}\,.$$

On account of the relation $d\theta^a \wedge \rho_a - \theta^a \wedge d\rho_a = d(d\omega) = 0$ we get

$$k^a_{bc}=y^a_{bc}-y^a_{cb},\quad l^a_b=-v^a_b,\quad m^{ab}_c-m^{ba}_c=u^{ab}_c\,,$$

$$h^{ab}=h^{ba},\quad w_{ab}=w_{ba},\quad z_{abc}+z_{bca}+z_{cab}=0\,.$$

Now we consider the effect of the transformation (2.1). We denote the coefficients in the structure equations for $\bar{\theta}^a$, $\bar{\rho}_a$ corresponding to (2.3) (2.4) by $\bar{k}^a_{bc}$, $\bar{l}^a_b$, $\cdots$, etc.. We substitute (2.1) into

$$d\bar{\theta}^a=\frac{1}{2}\bar{k}^a_{bc}\bar{\theta}^b\wedge\bar{\theta}^c+\bar{l}^a_b\omega\wedge\bar{\theta}^b+\bar{m}^{ab}_c\bar{\rho}_b\wedge\bar{\theta}^c+\bar{h}^{ab}\bar{\rho}_b\wedge\omega \tag{2.5}$$

and get

$$ds^a_b\wedge\theta^b+s^a_b d\theta^b=\cdots+s^a_b h^{bc}\rho_c\wedge\omega=\cdots+\bar{h}^{ab}(t^e_b\rho_e+r_{bc}\theta^c)\wedge\omega\,.$$

We compare the coefficients of $\rho_c\wedge\omega$ and we get

$$\bar{h}^{ab}=s^a_c s^b_e h^{ce}\,. \tag{2.6}$$

We denote the matrices inversive to $(\bar{h}^{ab})$, (h^{ab}) by $(\bar{h}_{ab})$, (h_{ab}) respectively and get $\bar{h}_{ab}=t^c_a t^e_b h_{ce}$. Hence

$$A=h_{ab}\theta^a\theta^b\,. \tag{2.7}$$

$$B=\omega^2+h_{ab}\theta^a\theta^b \tag{2.8}$$

are quadratic differential forms on N which are independent with the choice of adapted coframes. We can show that these are forms which are fundamental in the theory of Finsler spaces hitherto studied, namely

$$A=F\frac{\partial^2 F}{\partial y^i\partial y^j}dx^i dx^j\,, \tag{2.9}$$

$$B=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i\partial y^j}dx^i dx^j\,. \tag{2.10}$$

These can be verified by using (1.16) as follows. In this case

$$\theta^a=dx^a-H_{p_a}dx^n,\quad \omega=p_a dx^a-Hdx^n=F_{y^i}dx^i \tag{2.11}$$

and by comparing (1.17) (2.3) we get

$$h^{ab}=-(H-p_cH_{p_c})^{-1}H_{p_ap_b}\,.$$

Hence we get by (1.22)

$$h_{ab}=F\,F_{y^ay^b} \tag{2.12}$$

and so

$$\begin{aligned}F^{-1}h_{ab}\theta^a\theta^b&=F_{y^ay^b}(dx^a-H_{p_a}dx^n)(dx^b-H_{p_b}dx^n)\\&=F_{y^ay^b}dx^a dx^b-2F_{y^ay^b}H_{p_b}dx^a dx^n+F_{y^ay^b}H_{p_a}H_{p_b}(dx^n)^2\,.\end{aligned}$$

As $F_{y^n}=-H$ we get

$$F_{y^ny^a}=-H_{p_b}F_{y^by^a},\quad F_{y^ny^n}=-H_{p_a}F_{y^ay^n}=H_{p_a}H_{p_b}F_{y^ay^b}\,,$$

and so

$$F^{-1}h_{ab}\theta^a\theta^b = F_{y^iy^j}dx^idx^j\,.$$

(2.10) can also be verified by remarking

$$\frac{\partial^2 F^2}{\partial y^i \partial y^j} = 2\frac{\partial}{\partial y^i}\left(F\frac{\partial F}{\partial y^j}\right) = 2(F F_{y^iy^j} + F_{y^i}F_{y^j})\,.$$

Here we make the third assumption on our Finsler space.

(III) The matrix $(F\partial^2 F/\partial y^i\partial y^j)$ is positive semidefinite.

This means that (h_{ab}) is positive definite in the case $\det(F_{y_a y_b}) \neq 0$, which we can assume without loss of generality. In fact $(FF_{y_a y_b})$ is then positive definite and so (h_{ab}) with respect to special frame (2.11) is positive definite. This means the same holds good for the general frame.

Now we can take suitable $\bar{\theta}^a$, $\bar{\rho}_a$ in such a way that

$$h_{ab} = \delta_{ab} \tag{2.13}$$

hold good. The transformation between the coframes satisfying (2.13) is given by (2.1) with orthogonal (s^a_b). We take a special one among these transformations, namely

$$\bar{\theta}^a = \theta^a, \quad \bar{\rho}_a = \rho_a + r_{ab}\theta^b\,. \qquad (r_{ab} = r_{ba})\,. \tag{2.14}$$

By (2.3) and (2.14) we get

$$d\bar{\theta}^a = d\theta^a = \frac{1}{2}k^a_{bc}\theta^b \wedge \theta^c + l^a_b\omega \wedge \theta^b$$
$$+ m^{ab}_c(\bar{\rho}_b - r_{be}\theta^e) \wedge \theta^c + (\bar{\rho}_a - r_{ab}\theta^b) \wedge \omega\,.$$

By comparing the terms $\omega \wedge \bar{\theta}^b = \omega \wedge \theta^b$ we get by (2.5)

$$\bar{l}^a_b = l^a_b + r_{ab}\,.$$

So if we take r_{ab} such that $r_{ab} = -\frac{1}{2}(l^a_b + l^b_a)$, we get

$$\bar{l}^a_b = \frac{1}{2}(l^a_b - l^b_a) = \bar{l}^b_a\,.$$

Thus corresponding to any frame θ^a, ρ_a we can take $\bar{\theta}^a = \theta^a$, $\bar{\rho}_a$ uniquely so as to (2.13) and

$$l^a_b = -l^b_a \tag{2.15}$$

hold good. We call an adapted coframe satisfying these conditions an adapted orthogonal coframe. Then we can prove

THEOREM 3. *The frame transformation between adapted orthogonal coframes ω, θ^a, ρ_a and ω, $\bar{\theta}^a$, $\bar{\rho}_a$ is given by*

$$\bar{\theta}^a = s^a_b\theta^b, \qquad \bar{\rho}_a = s^a_b\rho_b\,, \tag{2.16}$$

where (s^a_b) *is an orthogonal matrix.*

PROOF. First we verify that $\bar{\theta}^a$, $\bar{\rho}_a$ given by (2.16) is an adapted orthogonal coframe with ω when θ^a, ρ_a is so. We have

$$d\bar{\theta}^a = ds^a_b \wedge \theta^b + s^a_b d\theta^b = ds^a_b s^c_b \wedge \bar{\theta}^c + s^a_b(\cdots + l^b_c\omega \wedge s^e_c\bar{\theta}^e + \cdots).$$

The coefficient of $\omega \wedge \bar{\theta}^e$ on the right side is skew symmetric with respect to a, e because $ds^a_b s^e_b$, $s^a_b l^b_c s^e_c$ are so. Thus $\bar{\theta}^a$, $\bar{\rho}_a$ is an adapted orthogonal coframe. As ρ_a is uniquely determined by $\theta^1, \cdots, \theta^{n-1}$ in an adapted orthogonal coframe, our theorem is proved.

With respect to adapted orthogonal coframes

$$C = \rho_a \rho_a \tag{2.17}$$

is a quadratic differential form on N in addition to $A = \theta^a\theta^a$ and $B = \omega^2 + \theta^a\theta^a$.

Next we prove the following fundamental theorem.

THEOREM 4. *With respect to adapted orthogonal coframes the structure equations can be uniquely represented in the following way.*

$$d\omega = \theta^a \wedge \rho_a \tag{2.18}$$

$$d\theta^a = \theta^b \wedge \lambda^a_b + \rho_b \wedge \mu^{ba} + \rho_a \wedge \omega \tag{2.19}$$

$$d\rho_a = \theta^b \wedge \nu_{ba} + \rho_b \wedge \lambda^a_b - w_{ab}\theta^b \wedge \omega + \Phi_a, \tag{2.20}$$

where w_{ab} *are functions,* λ^a_b, μ^{ab}, ν_{ab} *are* 1*-forms and* Φ_a *are* 2*-forms in* $\theta^1, \cdots, \theta^{n-1}$, *each satisfying the following conditions*

$$w_{ab} = w_{ba}, \qquad \lambda^a_b = -\lambda^b_a$$

$$\mu^{ab} = \mu^{ba} \equiv 0 \pmod{\theta^1, \cdots, \theta^{n-1}}$$

$$\nu_{ab} = \nu_{ba} \equiv 0 \pmod{\rho_1, \cdots, \rho_{n-1}}.$$

PROOF. We consider (2.3) (2.4) with $h_{ab} = \delta_{ab}$ and $l^a_b = -l^b_a$. We put

$$\begin{aligned}
\lambda^a_b &= \frac{1}{2}(k^a_{bc} + k^c_{ba} + k^b_{ca})\theta^c - l^a_b\omega + \frac{1}{2}(u^{bc}_a + u^{ba}_c + u^{ca}_b)\rho_c \\
\mu^{ab} &= \frac{1}{2}(m^{ab}_c + m^{ba}_c + u^{cb}_a + u^{ca}_b)\theta^c \\
\nu_{ab} &= \frac{1}{2}(k^a_{bc} + k^b_{ca} + y^c_{ab} + y^c_{ba})\rho_c \\
\Phi_a &= \frac{1}{2} z_{abc}\theta^b \wedge \theta^c
\end{aligned} \tag{2.21}$$

Then all the relations in Theorem 4 are satisfied. Uniqueness follows as is shown in the next. When $\frac{1}{2}k^a_{bc}\theta^b \wedge \theta^c$ is given, Γ^a_{bc} satisfying the relations

$$\frac{1}{2}k^a_{bc}\theta^b\wedge\theta^c=\theta^b\wedge\Gamma^a_{bc}\theta^c\,, \qquad \Gamma^a_{bc}=-\Gamma^b_{ac}$$

are uniquely given by $\Gamma^a_{bc}=\frac{1}{2}(k^a_{bc}+k^c_{ba}+k^b_{ca})$ and similarly for $\frac{1}{2}u^{bc}_a\rho_b\wedge\rho_c$. Thus λ^a_b and hence w_{ab}, μ^{ab}, ν_{ab}, Φ_a are uniquely determined.

Next we examine how w_{ab}, λ^a_b, μ^{ab}, ν_{ab}, Φ_a are transformed by the frame transformation (2.16). We have

$$d\bar{\theta}^a=ds^a_b\wedge\theta^b+s^a_bd\theta^b=ds^a_b\wedge\theta^b+s^a_b(\theta^c\wedge\lambda^b_c+\rho_c\wedge\mu^{cb}+\rho_b\wedge\omega)$$

$$d\bar{\rho}_a=ds^a_b\wedge\rho_b+s^a_bd\rho_b=ds^a_b\wedge\rho_b+s^a_b(\theta^c\wedge\nu_{cb}+\rho_c\wedge\lambda^b_c-w_{bc}\theta^c\wedge\omega+\Phi_b)\,.$$

We put $\theta^b=s^c_b\bar{\theta}^c$ and $\rho_b=s^c_b\bar{\rho}_c$ into these and by the comparison with (2.5) we get

$$\bar{\lambda}^a_b=s^a_cs^b_e\lambda^c_e-ds^a_cs^b_c\,,\qquad \bar{\mu}^{ab}=s^a_cs^b_e\mu^{ce}\,,\qquad \bar{\nu}_{ab}=s^a_cs^b_e\nu_{ce}\,,$$
$$\bar{w}_{ab}=s^a_cs^b_ew_{ce}\,,\qquad \bar{\Phi}_a=s^a_b\Phi_b\,. \tag{2.22}$$

Hence w_{ab}, μ^{ab}, ν_{ab}, Φ_a are tensorial forms with respect to adapted orthogonal coframes, while λ^a_b are connection-like.

Here we resume the fundamental properties of adapted orthogonal coframes ω, θ^a, ρ_a on the p-manifold N of M.

(i) For a curve (x, p) in N induced by a curve $x=x(t)$ in M we have $\theta^a=0$. When $x=x(t)$ is an extremal, we have $\theta^a=0$ and $\rho_a=0$.

(ii) $A=\theta^a\theta^a$, $B=\omega^2+\theta^a\theta^a$, $C=\rho_a\rho_a$ are quadratic forms on N.

(iii) The structure equations are given by (2.18) (2.19) (2.20).

(iv) Transformations of w_{ab}, λ^a_b, μ^{ab}, ν_{ab}, Φ_a induced by the transformation of adapted orthogonal coframes are given by (2.22).

3. Connections

We consider connections on the p-manifold N of the Finsler spaces. Firstly we summerise fundamental concepts of an affine connection on an n-dimensional differentiable manifold. An affine connection is given by a matrix $\Gamma=(\omega^j_i)$ of 1-forms ω^j_i (on the i-th row and on the j-th column) with respect to fundamental coframe $\pi=(\omega^1, \cdots, \omega^n)$. Torsion forms $\tau=(\tau^1, \cdots, \tau^n)$ and curvature forms $\Omega=(\Omega^j_i)$ are given respectively by

$$\tau=d\pi-\pi\wedge\Gamma, \quad \text{namely} \quad \tau^i=d\omega^i-\omega^j\wedge\omega^i_j\,,$$

$$\Omega=d\Gamma-\Gamma\wedge\Gamma, \quad \text{namely} \quad \Omega^j_i=d\omega^j_i-\omega^k_i\wedge\omega^j_k$$

and the relations

$$d\tau+\tau\wedge\Gamma=\pi\wedge\Omega\,,\qquad d\Omega+\Omega\wedge\Gamma-\Gamma\wedge\Omega=0 \tag{3.1}$$

hold good. A covariant differential of a contravariant vector $v=(v^1, \cdots, v^n)$

is given by $Dv=dv+v\Gamma$. A covariant tensor field $A=(a_{ij})$ is parallel with respect to our connection when and only when

$$dA-A^t\Gamma-\Gamma A=0. \tag{3.2}$$

We take on the p-manifold N of the n-dimensional Finsler space M adapted orthogonal coframes ω, θ^a, ρ_a. We put

$$\pi=(\omega, \theta^1, \cdots, \theta^{n-1}, \rho_1, \cdots, \rho_{n-1}) \tag{3.3}$$

$$\Lambda=(\lambda_b^a) \qquad (a, b=1, \cdots, n-1). \tag{3.4}$$

By virtue of (2.22) we can define a connection on N by

$$\Gamma=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix} \tag{3.5}$$

with respect to adapted orthogonal coframes. We denote torsion forms by

$$\tau=(\tau^p)=d\pi-\pi\wedge\Gamma. \qquad (p=0, 1, \cdots, 2n-2)$$

Then we get by virtue of (2.18) (2.19) (2.20)

$$\tau^0=d\omega=\theta^a\wedge\rho_a$$

$$\tau^a=d\theta^a-\theta^b\wedge\lambda_b^a=\rho_b\wedge\mu^{ba}+\rho_a\wedge\omega$$

$$\tau^{a+n-1}=d\rho_a-\rho_b\wedge\lambda_b^a=\theta^b\wedge\nu_{ba}-w_{ab}\theta^b\wedge\omega+\Phi_a.$$

As for curvature forms $d\Gamma-\Gamma\wedge\Gamma$ we have essentially

$$\Psi_0=d\Lambda-\Lambda\wedge\Lambda. \tag{3.6}$$

We take up the tensors on N corresponding to the forms

$$\omega, \qquad A=\theta^a\theta^a, \qquad C=\rho_a\rho_a, \qquad d\omega=\theta^a\wedge\rho_a.$$

These tensors have components

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{n-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_{n-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & E_{n-1} \\ 0 & -E_{n-1} & 0 \end{pmatrix}, \tag{3.7}$$

where E_{n-1} is a unit matrix of degree $n-1$.

We can easily verify the following result by (3.2).

THEOREM 5. *For an affine connection* (3.5) *the tensors given by* (3.7) *are parallel.*

Next we define a second connection by

$$\Gamma_1=\begin{pmatrix} 0 & -{}^t\rho & 0 \\ \rho & \Lambda & 0 \\ 0 & 0 & \Lambda' \end{pmatrix}, \tag{3.8}$$

where ${}^t\rho=(\rho_1, \cdots, \rho_{n-1})$. This is well defined by virtue of (2.22) and (2.16). The torsion components of the connection are

$$\begin{aligned} \tau^0 &= d\omega-\theta^a \wedge \rho_a=0 \\ \tau^a &= d\theta^a-\theta^b \wedge \lambda_b^a-\omega \wedge(-\rho_a)=\rho_b \wedge \mu^{ba} \\ \tau^{a+n-1} &= d\rho_a-\rho_b \wedge \lambda_b^a=\theta^b \wedge \nu_{ba}-w_{ab}\theta^b \wedge \omega+\Phi_a . \end{aligned} \tag{3.9}$$

When we denote curvature forms by

$$\Omega_1=d\Gamma_1-\Gamma_1 \wedge \Gamma_1,$$

we get

$$\Omega_1=\begin{pmatrix} 0 & -d{}^t\rho+{}^t\rho \wedge \Lambda & 0 \\ d\rho-\Lambda \wedge \rho & \Psi & 0 \\ 0 & 0 & \Psi_0 \end{pmatrix},$$

where

$$\Psi=\Psi_0+\rho \wedge {}^t\rho=d\Lambda-\Lambda \wedge \Lambda+\rho \wedge {}^t\rho . \tag{3.10}$$

We can easily verify

THEOREM 6. *For an affine connection* (3.8) *tensors given by*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_{n-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & E_{n-1} \\ 0 & -E_{n-1} & 0 \end{pmatrix}, \quad \begin{pmatrix} E_n & 0 \\ 0 & 0 \end{pmatrix}$$

are all parallel.

4. Riemannian space

In the case of Riemannian spaces

$$B=\omega^2+\theta^a\theta^a=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}dx^i dx^j$$

is represented by $x^1, \cdots, x^n$ exclusively and does not contain auxiliary variables, which we denote here as $u_1, \cdots, u_{n-1}$. When we take vector fields $X=\xi^i(x)\,\partial/\partial x^i$, $Y=\eta_a(u)\,\partial/\partial u_a$, we have $[X, Y]=0$ and hence by virtue of (2.18) we have

$$X(\omega(Y))-Y(\omega(X))=\theta^a(X)\rho_a(Y)-\theta^a(Y)\rho_a(X).$$

As $\omega(Y)=0$, $\theta^a(Y)=0$ in this case, we get

$$Y(\omega(X))=-\rho_a(Y)\theta^a(X).$$

Similarly we get by (2.19)

$$Y(\theta^a(X))=-\lambda^a_b(Y)\theta^b(X)+\rho_b(Y)\mu^{ba}(X)+\rho_a(Y)\omega(X).$$

Hence

$$Y(\omega(X)^2+\theta^a(X)\theta^a(X))$$
$$=2(\omega(X)Y(\omega(X))+\theta^a(X)Y(\theta^a(X)))=2\rho_b(Y)\theta^a(X)\mu^{ba}(X). \tag{4.1}$$

A condition for our Finsler space to be Riemannian is that (4.1) vanishes for every X and Y. The condition reduces to $M^{ba}_c+M^{bc}_a=0$ when we put $\mu^{ba}=M^{ba}_c\theta^c$. As we have $M^{ab}_c=M^{ba}_c$ moreover, we get $M^{ab}_c=0$. Thus we get

THEOREM 7. *A necessary and sufficient condition for a Finsler space to be Riemannian is*

$$\mu^{ab}=0 \tag{4.2}$$

for its p-manifold.

In this case the connection on N given by (3.8) induces the Riemannian connection

$$\Gamma_0=\begin{pmatrix}0 & -{}^t\rho\\ \rho & \Lambda\end{pmatrix} \tag{4.3}$$

on M, which can be verified by (3.9) and $\lambda^a_b=-\lambda^b_a$. The curvature forms of our Riemannian space M are given by

$$\Omega_0=d\Gamma_0-\Gamma_0\wedge\Gamma_0=\begin{pmatrix}0 & -{}^t(d\rho-\Lambda\wedge\rho)\\ d\rho-\Lambda\wedge\rho & \Psi\end{pmatrix}$$

with respect to orthogonal coframes $\omega, \theta^1, \cdots, \theta^{n-1}$. When we put

$$d\rho-\Lambda\wedge\rho=(\Omega^0_a), \qquad \Psi=(\Omega^b_a),$$

Ω^0_a and Ω^b_a are differential forms which are linear combinations of $\omega\wedge\theta^a$ and $\theta^a\wedge\theta^b$ by the property of curvature forms. On the other hand we have by (2.20)

$$\Omega^0_a=d\rho_a-\lambda^b_a\wedge\rho_b=d\rho_a-\rho_b\wedge\lambda^a_b=\theta^b\wedge\nu_{ba}-w_{ab}\theta^b\wedge\omega+\Phi_a. \tag{4.4}$$

Hence we have $\nu_{ba}=0$ on account of (2.21). Thus we get

THEOREM 8. *For Riemannian spaces we have necessarily*

$$\nu_{ab}=0$$

By (4.4) we have then

$$\Omega^0_a=-w_{ab}\theta^b\wedge\omega+\Phi_a \qquad \left(\Phi_a=\frac{1}{2}z_{abc}\theta^b\wedge\theta^c\right).$$

Hence for the Riemannian space with constant curvature K we have

$$w_{ab}=K\delta_{ab}\ (K \text{ const.}), \qquad \Phi_a=0. \tag{4.5}$$

We assume that these are satisfied. Then we have

$$\Omega_a^0=-K\theta^a\wedge\omega.$$

On the other hand we have by Bianchi's identity (3.1) $d\Omega_a^0-\lambda_a^b\wedge\Omega_b^0+\Omega_a^b\wedge\rho_b=0$. Here

$$\begin{aligned} d\Omega_a^0-\lambda_a^b\wedge\Omega_b^0&=-Kd(\theta^a\wedge\omega)+K\lambda_a^b\wedge\theta^b\wedge\omega \\ &=-K(d\theta^a-\theta^b\wedge\lambda_b^a)\wedge\omega+K\theta^a\wedge d\omega=-K\rho_a\wedge\omega\wedge\omega+K\theta^a\wedge\theta^b\wedge\rho_b. \end{aligned}$$

Hence

$$\rho_b\wedge(\Omega_a^b+K\theta^a\wedge\theta^b)=0.$$

As Ω_a^b does not contain ρ_c we get

$$\Omega_a^b=-K\theta^a\wedge\theta^b.$$

Thus we have got

THEOREM 9. *A necessary and sufficient condition for a Riemannian space to be of constant curvature K is* (4.5).

Any unit tangent vector on a Riemannian manifold has components $v=(v^0, v^1, \cdots, v^{n-1})=(1, 0, \cdots, 0)$ with respect to a suitably chosen coframe $\omega, \theta^1, \cdots, \theta^{n-1}$, and on account of (4.3) the covariant derivative $Dv=dv+v\Gamma_0$ is given by

$$Dv^0=0, \qquad Dv^a=-\rho_a.$$

Hence the third form C is in this case

$$C=\rho_a\rho_a=Dv^kDv^k. \qquad (k=0, 1, \cdots, n-1) \tag{4.6}$$

This metric is of constant curvature 1, if we restrict to a fixed fiber of ${}^cT(M)$ for a fixed point x on M.

As an analogy to the Riemannian space we take up a Finsler space on which a volume element can be given by

$$\sigma=\omega\wedge\theta^1\wedge\cdots\wedge\theta^{n-1}. \tag{4.7}$$

This form contains auxiliary variables $u_1, \cdots, u_{n-1}$ in general. If they do not appear, a volume element can be defined. We put

$$\mu^{ab}=M_c^{ab}\theta^c. \qquad \text{and} \qquad M_b^{ab}=M^a.$$

Then we have $d\sigma=-M^a\rho_a\wedge\omega\wedge\theta^1\wedge\cdots\wedge\theta^{n-1}$ by (2.18) (2.19). Hence

THEOREM 10. *A necessary and sufficient condition for a Finsler space to have a volume element defined by* (4.7) *is the vanishing of* (M^a).

5. Dilatation

A dilatation in the Euclidean space is to translate each plane element to the direction orthogonal to it by a constant length, and it is proved that the dilatation is a contact homogeneous transformation. The geodesic flow in the Riemannian space is a generalization of the dilatation in the Euclidean space. We can define the dilatation in the Finsler space as the translation along an e-curve of a point (x, p) in the p-manifold N of M, which we proved to be a contact homogeneous transformation in [3]. This means that the fundamental form $\omega = p_i dx^i$ is preserved by the dilatation. As a result we showed in [3] that a volume element in N is invariant under a dilatation, and in the special case of Riemannian space of constant curvature even a certain Riemannian metric is invariant. Here we will prove a generalization of the latter theorem.

THEOREM 11. *When the tensor* (w_{ab}) *satisfies the relation*

$$w_{ab} = K\delta_{ab} \qquad (K \text{ const}),$$

the Riemannian metric $KB+C$ *on* N *is invariant under any dilatation.*

PROOF. We take a dilatation φ_s, where s means an arc-length along an extremal which is a projection of an e-curve. We consider the product space $N\times l$, where l is a segment $0 \leqq s \leqq s_1$. We define a mapping $\varphi: N\times l \to N$ by $(x, p, s) \to \varphi_s(x, p)$. We denote the forms $\varphi^*\omega$, $\varphi^*\theta^a$, $\varphi^*\rho_a$ conventionally by ω, θ^a, ρ_a. We take a vector field $S=\partial/\partial s$ and a vector field X which is independent with s. As $\varphi_s(x, p)$ describes an e-curve for fixed x, p, we have

$$\omega(S) = 1, \qquad \theta^a(S) = 0, \qquad \rho_a(S) = 0.$$

As $[S, X] = 0$, we get by (2.18)

$$S(\omega(X)) - X(\omega(S)) = \theta^a(S)\rho_a(X) - \theta^a(X)\rho_a(S)$$

and hence

$$S(\omega(X)) = 0.$$

Similarly by (2.19) and (2.20)

$$S(\theta^a(X)) = -\lambda^a_b(S)\theta^b(X) - \rho_a(X)$$

$$S(\rho_a(X)) = -\rho_b(X)\lambda^a_b(S) + w_{ab}\theta^b(X).$$

By these relations we get

$$\begin{aligned} &S(K(\omega(X)^2 + \theta^a(X)\theta^a(X)) + \rho_a(X)\rho_a(X)) \\ &\qquad = 2\{K(\omega(X)S(\omega(X)) + \theta^a(X)S(\theta^a(X))) + \rho_a(X)S(\rho_a(X))\} \\ &\qquad = 2\{-K\theta^a(X)\rho_a(X) + w_{ab}\rho_a(X)\theta^b(X)\}. \end{aligned}$$

This vanishes by the assumption $w_{ab}=K\delta_{ab}$. Hence $K(\omega(X)^2+\theta^a(X)\theta^a(X))+\rho_a(X)\rho_a(X)$ does not contain s.

In the case of Riemannian space of constant curvature K the metric $KB+C$ is $Kds^2+Dv^kDv^k$ by virtue of (4.6), where ds^2 is the Riemannian metric of the space.

6. Minkowskian space

A Minkowskian space is a one for which $F(x, y)$ does not contain x when the coordinates $x=(x^1, \cdots, x^n)$ are suitably chosen. In these coordinates the p-manifold N can be represented as $G(p)=0$, and hence

$$p_n=-H(p_1, \cdots, p_{n-1}) \tag{6.1}$$

when $G_{p_n}\neq 0$. Conversely, when (6.1) holds good, we can represent p_a by y^i by solving (1.20) with respect to p_a and so $F(x, y)$ does not contain $x^1, \cdots, x^n$ by virtue of (1.22). Thus the space in Minkowskian.

Here we will seek for the tensorial condition for a Finsler space M to be Minkowskian. We assume (6.1) and take a local coframe (1.15). Then we get

$$\omega=p_a dx^a-Hdx^n, \qquad \theta^a=dx^a-H_{p_a}dx^n, \qquad \rho_a=-dp_a. \tag{6.2}$$

Hence by (1.17) (1.18)

$$d\theta^a=h^{ab}\rho_b\wedge(\omega-p_c\theta^c), \qquad d\rho_a=0, \tag{6.3}$$

where

$$h^{ab}=-(H-p_cH_{p_c})^{-1}H_{p_ap_b}. \tag{6.4}$$

ω, θ^a, ρ_a is not an adapted orthogonal coframe in general and we can transform it by

$$\bar\theta^a=s^a_b\theta^b, \qquad \bar\rho_a=t^b_a\rho_b \qquad (s^a_bt^b_c=\delta^a_c) \tag{6.5}$$

into orthogonal $\bar\theta^a$, $\bar\rho_a$ as follows. We have

$$d\bar\theta^a=ds^a_b\wedge\theta^b+s^a_bd\theta^b=ds^a_bt^b_c\wedge\bar\theta^b+s^a_bh^{bc}s^e_c\bar\rho_e\wedge(\omega-p_ct^c_b\bar\theta^b).$$

As h^{bc} do not contain $x^1, \cdots, x^n$, we get $s^a_bh^{bc}s^e_c=\delta^{ae}$ for suitably chosen s^a_b which are functions of $p_1, \cdots, p_{n-1}$. Thus we get

$$d\bar\theta^a=ds^a_bt^b_c\wedge\bar\theta^c+\bar\rho_a\wedge(\omega-p_ct^c_b\bar\theta^b), \qquad d\bar\rho_a=dt^b_as^c_b\wedge\bar\rho_c. \tag{6.6}$$

Here $dt^b_as^c_b$ does not contain $x^1, \cdots, x^n$ and is a linear combination of $\bar\rho_1, \cdots, \bar\rho_{n-1}$. We denote $\bar\theta^a$, $\bar\rho_a$ by θ^a, ρ_a anew. Hence we get by the comparison of (6.6) with (2.3) (2.4)

$$k^a_{bc}=0, \qquad l^b_a=0, \qquad w_{ab}=0, \qquad y^b_{ac}=0, \qquad z_{abc}=0.$$

Thus we get by (2.20)

$$w_{ab}=0\,,\qquad \nu_{ab}=0\,,\qquad \Phi_a=0\,, \tag{6.7}$$

and (6.6) can be written as

$$d\theta^a=\theta^b\wedge\lambda^a_b+\rho_b\wedge\mu^{ba}+\rho_a\wedge\omega\,,\qquad d\rho_a=\rho_b\wedge\lambda^a_b\,. \tag{6.8}$$

As ω and θ^a are linear combinations of $dx^1, \cdots, dx^n$ with coefficients which are functions of $p_1, \cdots, p_{n-1}$, there exists a matrix $P=(p^i_j)$ such that

$$(\omega, \theta^1, \cdots, \theta^{n-1})\;P=(dx^1, \cdots, dx^n)\,, \tag{6.9}$$

where p^i_j are functions of $p_1, \cdots, p_{n-1}$. We take an exterior differential of (6.9) and get

$$(d\omega, d\theta^1, \cdots, d\theta^{n-1})P-(\omega, \theta^1, \cdots, \theta^{n-1})\wedge dP=0\,. \tag{6.10}$$

Hence

$$(d\omega, d\theta^1, \cdots, d\theta^{n-1})=(\omega, \theta^1, \cdots, \theta^{n-1})\wedge(dPP^{-1})\,.$$

We write this briefly as

$$(d\omega, d\theta)=(\omega, \theta)\wedge\Gamma\,,\quad\text{where}\quad \Gamma=dPP^{-1}\,. \tag{6.11}$$

When we put

$$\mu^{ab}=M^{ab}_c\theta^c$$

and

$$\sigma^a_b=M^{ca}_b\rho_c\,,$$

we have

$$d\theta^a=\theta^b\wedge\lambda^a_b+\rho_b\wedge M^{ba}_c\theta^c+\rho_a\wedge\omega=-\omega\wedge\rho_a+\theta^b\wedge(\lambda^a_b-M^{ca}_b\rho_c)$$
$$=-\omega\wedge\rho_a+\theta^b\wedge(\lambda^a_b-\sigma^a_b)\,.$$

By comparing this with (6.11) we get

$$\Gamma=\begin{pmatrix}0 & -{}^t\rho\\ \rho & \Lambda-\Sigma\end{pmatrix}, \tag{6.12}$$

where we have put

$$\Lambda=(\lambda^b_a)\,,\qquad \Sigma=(\sigma^b_a)\,,\qquad {}^t\rho=(\rho_1, \cdots, \rho_{n-1})\,.$$

Since $dPP^{-1}=\Gamma$, we get

$$d\Gamma-\Gamma\wedge\Gamma=0\,. \tag{6.13}$$

Namely

$$\begin{pmatrix}0 & -d^t\rho+\rho^t\wedge(\Lambda-\Sigma)\\ d\rho-(\Lambda-\Sigma)\wedge\rho & d(\Lambda-\Sigma)-(\Lambda-\Sigma)\wedge(\Lambda-\Sigma)+\rho\wedge{}^t\rho\end{pmatrix}=0\,.$$

The components of $d\rho-(\Lambda-\Sigma)\wedge\rho$ vanish by (6.8) and $M^{bc}_a=M^{cb}_a$. As to the components of $-d^t\rho+{}^t\rho\wedge(\Lambda-\Sigma)$

$$-d\rho_a+\rho_b\wedge(\lambda^a_b-M^{ca}_b\rho_c)=-M^{ca}_b\rho_b\wedge\rho_c$$

and their vanishing means

$$M_b^{ca} = M_c^{ba}. \tag{6.14}$$

Next we put

$$d\Sigma - \Lambda \wedge \Sigma - \Sigma \wedge \Lambda + \Sigma \wedge \Sigma = T.$$

Then we get by the vanishing of $d(\Lambda-\Sigma)-(\Lambda-\Sigma)\wedge(\Lambda-\Sigma)+\rho\wedge{}^t\rho$

$$\Psi = T, \tag{6.15}$$

where Ψ is given by (3.10).

As (6.7) (6.14) (6.15) are tensorial relations with respect to adapted orthogonal coframes we get the following.

THEOREM 12. *In the Minkowskian space we have the following relations with respect to adapted orthogonal coframes.*

$$w_{ab} = 0, \qquad \nu_{ab} = 0, \qquad \Phi_a = 0, \qquad M_b^{ca} = M_c^{ba}, \qquad \Psi = T.$$

The converse is also true.

We will give the proof of the converse. From the first three conditions we get

$$d\rho_a = \rho_b \wedge \lambda_b^a,$$

and as $\lambda_b^a = -\lambda_a^b$ we see that $\rho_a\rho_a$ can be represented by suitably chosen variables $u_1, \cdots, u_{n-1}$. We can also find $\bar{\rho}_a = s_b^a \rho_b$ such that (s_b^a) is an orthogonal matrix and $\bar{\rho}_a$ are 1-forms in $u_1, \cdots, u_{n-1}$. Then there exist uniquely such $\bar{\lambda}_b^a$ that

$$d\bar{\rho}_a = \bar{\rho}_b \wedge \bar{\lambda}_b^a, \qquad \bar{\lambda}_b^a = -\bar{\lambda}_a^b.$$

We take $\bar{\theta}^a = s_b^a \theta^b$. Then $\bar{\theta}^a$, $\bar{\rho}_a$ is a new adapted orthogonal coframe and (6.8) holds good for this coframe. Then (6.13) holds good by the last two conditions in the Theorem 12. Hence there exists P such that $dPP^{-1} = \Gamma$, where Γ is given by (6.12) corresponding to the new frame. As $\bar{\lambda}_b^a$, $\bar{\sigma}_b^a$ are 1-forms which are linear combination of $\bar{\rho}_1, \cdots, \bar{\rho}_{n-1}$, each component of the matrix P can be taken as a function of $u_1, \cdots, u_{n-1}$. Hence we have (6.10) and there exist such $x^1, \cdots, x^n$ that (6.9) holds good. Hence $\omega = p_i dx^i$, where p_i are functions of $u_1, \cdots, u_{n-1}$, and our space is Minkowskian.

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