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On a model in the ordinal numbers

By Toshio UMEZAWA

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In his famous work [1] Gödel constructed the model Δ on the basis called Σ, consisting of the axiom groups A, B, C and D. The first axiom of the group C, i.e., C1, is the axiom of infinity and the fourth, C4, is the axiom of replacement. Now we take as our basis the axiom system, named S_1, obtained from Σ by the following replacements: C1 by $(\exists a)(0 \in a \cdot (x)(x \in a \cdot \supset \cdot x+1 \in a))$ and C4 by both the Aussonderungsaxiom and the axiom $(\alpha, A)(\mathfrak{Un}(A) \cdot \mathfrak{W}(A) \subseteqq \mathrm{On} : \supset \cdot \mathfrak{M}(A``\mathfrak{P}(\alpha)))$ where 0, $x+1$, and $\mathfrak{P}(\alpha)$ denote the empty class, the sum of x and $\{x\}$, and the power set of α respectively and α ranges over the ordinal numbers. S_1 is a subsystem of Σ.

In this paper we prove that if S_1 is consistent, then Σ is also consistent. For this purpose we follow Gödel [1] except that constructible sets are ordinal numbers. Such a method to construct the model Δ in the ordinal numbers appears in Takeuti [2], [3], [4] and [5] where the different bases of both axiom system and logic are taken.

The theory of Gödel [1] is assumed to be known and the symbols which are not defined in this paper have the same meaning as in [1].

§ 1. Sets and classes in S.

First we introduce the following axioms:

C1′. $$(\exists a)(0 \in a \cdot (x)(x \in a \cdot \supset \cdot x+1 \in a)).$$

C4′a. $$(x, A)(A \subseteqq x \cdot \supset \cdot \mathfrak{M}(A)).$$

C4′b. $$(x, A)(\mathfrak{Un}(A) \cdot \mathfrak{W}(A) \subseteqq \mathrm{On} \cdot x \in \mathrm{On} : \supset \cdot (\exists y)(u)(u \in y \cdot \equiv \cdot (\exists v)(v \in x \cdot \langle uv \rangle \in A))).$$

C4″. $$(x, A)(\mathfrak{Un}(A) \cdot \mathfrak{W}(A) \subseteqq \mathrm{On} \cdot x \in \mathrm{On} : \supset \cdot (\exists y)(u)(u \in y \cdot \equiv \cdot (\exists v)(v \subseteqq x \cdot \langle uv \rangle \in A))).$$

The system obtained from Σ by replacing C1 by C1′ and again C4 by both C4′a and C4′b is called S. Let S_1 be the system obtained from S by adding C4″. In S_1, C4′b is superfluous because C4″ implies C4′b.

In the course of arguments we take S as the basis except only for the relativised axiom of power set.

The theorems which may be proved by 5.12 instead of 5.1 in [1] are, of course, provable in S.

1.1. $\mathfrak{M}(\mathfrak{D}(x))$.

$\mathfrak{D}(x) \subseteq \mathfrak{S}(\mathfrak{S}(x))$ and $\mathfrak{M}(\mathfrak{S}(\mathfrak{S}(x)))$. Hence, by C4′a, $\mathfrak{M}(\mathfrak{D}(x))$.

1.2. $\mathfrak{M}(\mathfrak{W}(x))$.

1.3. $\mathfrak{Un}(A)\cdot\mathfrak{W}(A) \subseteq \mathrm{On}+\mathrm{On}^2+\mathrm{On}^3\cdot x \subseteq \mathrm{On} : \supset : \mathfrak{M}(A``x)\cdot\mathfrak{M}(A \restriction x)$.

We define A_1, A_2 and A_3 as follows: $(u)(u \in A_i \cdot \equiv : u \in A\cdot(\exists v, \beta)(u=\langle v\beta\rangle\cdot v \in \mathrm{On}^i))$ for $i=1, 2, 3$. Let y be an arbitrary element of $A``x$. There exists a $\beta \in x$ such that $\langle y\beta\rangle \in A$. By the premise, we obtain $y \in \mathrm{On}+\mathrm{On}^2+\mathrm{On}^3$. If $y \in \mathrm{On}$, then $\langle y\beta\rangle \in A_1$ and hence $y \in A_1``x$. If $y \in \mathrm{On}^2$, then $\langle y\beta\rangle \in A_2$ and hence $y \in (P_1 \mid A_2)``x \times (P_2 \mid A_2)``x$, using $y=\langle P_1`yP_2`y\rangle$. If $y \in \mathrm{On}^3$, then $\langle y\beta\rangle \in A_3$ and hence $y \in (P_1 \mid A_3)``x \times (((P_1 \mid P_2) \mid A_3)``x \times ((P_2 \mid P_2) \mid A_3)``x)$, using $y=\langle P_1`y(P_1 \mid P_2)`y (P_2 \mid P_2)`y\rangle$. Since $x \subseteq \mathrm{On}$, there exists an $\alpha$ such that $x \subseteq \alpha$. Hence $A``x \subseteq A_1``\alpha+(P_1 \mid A_2)``\alpha \times (P_2 \mid A_2)``\alpha+(P_1 \mid A_3)``\alpha \times (((P_1 \mid P_2) \mid A_3)``\alpha \times ((P_2 \mid P_2) \mid A_3)``\alpha)$. Let X denote one of $A_1, P_1 \mid A_2, P_2 \mid A_2, P_1 \mid A_3, (P_1 \mid P_2) \mid A_3, (P_2 \mid P_2) \mid A_3$. Then $\mathfrak{Un}(X)\cdot\mathfrak{W}(X) \subseteq \mathrm{On}$. Hence, in virtue of C4′b, $\mathfrak{M}(X``\alpha)$ and, by C4′a, $\mathfrak{M}(A``x)$. It is easy to prove that $\mathfrak{M}(A \restriction x)$.

1.4. $A\ \mathfrak{Fn}\ x\cdot\mathfrak{W}(A) \subseteq \mathfrak{P}(\mathrm{On}^2)\cdot x \subseteq \mathrm{On} : \supset : \mathfrak{M}(A``x)\cdot\mathfrak{M}(A \restriction x)$.

$\sup_R A$ is defined as follows: $(u)(u \in \sup_R A \cdot \equiv \cdot (\exists \alpha, \beta)(u \in \langle\alpha\beta\rangle \cdot A \subseteq R``\{\langle\alpha\beta\rangle\} \cdot(\gamma, \delta)(\langle\gamma\delta\rangle R \langle\alpha\beta\rangle \cdot \supset \cdot \sim(A \subseteq R``\{\langle\gamma\delta\rangle\}))))$. For any A, it holds that $A \subseteq \mathrm{On}^2 \cdot\mathfrak{M}(A) : \supset \cdot (\exists \alpha, \beta)(\langle\alpha\beta\rangle=\sup_R A)$. Let B be a class such that $(\alpha, \beta)(\langle\beta\alpha\rangle \in B \cdot \equiv \cdot \alpha \in x \cdot \beta=\sup_R A`\alpha)\cdot B \subseteq \mathrm{On}^2$. Then $B\ \mathfrak{Fn}\ x\cdot\mathfrak{W}(B) \subseteq \mathrm{On}^2$. Hence $\mathfrak{M}(B``x)$ and there are $\alpha$ and $\beta$ such that $\langle\alpha\beta\rangle=\sup_R B``x$. Take a y in $A``x$ arbitrarily. Then for any $\langle\xi\eta\rangle$ in $y$, $\langle\xi\eta\rangle R \langle\alpha\beta\rangle$ and hence $\xi, \eta < \mathfrak{Max}\{\alpha\beta\}+1$. Therefore $y \subseteq (\mathfrak{Max}\{\alpha\beta\}+1)^2$ and hence $A``x \subseteq \mathfrak{P}((\mathfrak{Max}\{\alpha\beta\}+1)^2)$. Then we obtain $\mathfrak{M}(A``x)$. It is easy to prove that $\mathfrak{M}(A \restriction x)$.

Next we treat the problem of existence in S of recursive functions.

1.5. $\mathfrak{W}(G_1) \subseteq \mathfrak{P}(\mathrm{On}^2)\cdot\mathfrak{W}(G_2) \subseteq \mathfrak{P}(\mathrm{On}^2) : \supset \cdot (\exists !\, F_1, F_2)(F_1 \mathfrak{Fn}\ \mathrm{On}\cdot F_2 \mathfrak{Fn}\ \mathrm{On}$
$(\alpha)(F_1`\alpha=G_1`\langle F_1 \restriction \alpha F_2 \restriction \alpha\rangle\cdot F_2`\alpha=G_2`\langle F_1 \restriction \alpha F_2 \restriction \alpha\rangle))$.

Let K be defined as follows: $(f_1, f_2)(\langle f_1 f_2\rangle \in K \cdot \equiv \cdot (\exists \beta)(f_1\ \mathfrak{Fn}\ \beta\cdot f_2\ \mathfrak{Fn}\ \beta\cdot(\alpha)(\alpha \in \beta \cdot \supset \cdot f_1`\alpha=G_1`\langle f_1 \restriction \alpha f_2 \restriction \alpha\rangle \cdot f_2`\alpha=G_2`\langle f_1 \restriction \alpha f_2 \restriction \alpha\rangle)))\cdot K \subseteq V^2$. We set $F_1=\mathfrak{S}(\mathfrak{W}(K))$ and $F_2=\mathfrak{S}(\mathfrak{D}(K))$. The proof is carried out successively as follows. We omit the proofs of (i)-(iv).

(i) $\langle f_1 f_2\rangle, \langle g_1 g_2\rangle \in K\cdot \alpha \in \mathfrak{D}(f_1)\cdot\mathfrak{D}(g_1) : \supset : f_1`\alpha=g_1`\alpha\cdot f_2`\alpha=g_2`\alpha$.

(ii) $\mathfrak{Fnc}(F_1)\cdot\mathfrak{Fnc}(F_2)$.

(iii) $\langle f_1 f_2\rangle \in K\cdot\gamma \subseteqq \mathfrak{D}(f_1) : \supset : f_1 \upharpoonright \gamma = F_1 \upharpoonright \gamma \cdot f_2 \upharpoonright \gamma = F_2 \upharpoonright \gamma$.

(iv) $\alpha \in \mathfrak{D}(F_1) \cdot \supset \cdot F_1{}^{\backprime}\alpha = G_1{}^{\backprime}\langle F_1 \upharpoonright \alpha F_2 \upharpoonright \alpha\rangle :$

$\alpha \in \mathfrak{D}(F_2) \cdot \supset \cdot F_2{}^{\backprime}\alpha = G_2{}^{\backprime}\langle F_1 \upharpoonright \alpha F_2 \upharpoonright \alpha\rangle$.

(v) $\mathfrak{D}(F_1) = \mathrm{On}\cdot\mathfrak{D}(F_2) = \mathrm{On}$.

It holds that $\mathfrak{D}(F_1) = \mathfrak{D}(\mathfrak{S}(\mathfrak{W}(K))) = \mathfrak{S}(\mathrm{Do}\,``\mathfrak{W}(K))$ and $\mathrm{Do}\,``\mathfrak{W}(K) \subseteqq \mathrm{On}$. Hence $\mathfrak{Ord}\,(\mathfrak{D}(F_1))$. Similarly for $\mathfrak{Ord}\,(\mathfrak{D}(F_2))$. Therefore $\mathfrak{D}(F_1) \leqq \mathrm{On}\cdot\mathfrak{D}(F_2) \leqq \mathrm{On}$. Assume that $\mathfrak{D}(F_1) < \mathrm{On}$ or $\mathfrak{D}(F_2) < \mathrm{On}$. Without loss of generality, we may assume $\mathfrak{D}(F_1) \leqq \mathfrak{D}(F_2)$. Set $\gamma = \mathfrak{D}(F_1)$. We define two classes H_1 and H_2 such that $H_1{}^{\backprime}\alpha = G_1{}^{\backprime}\langle F_1 \upharpoonright \alpha F_2 \upharpoonright \alpha\rangle$ and $H_2{}^{\backprime}\alpha = G_2{}^{\backprime}\langle F_1 \upharpoonright \alpha F_2 \upharpoonright \alpha\rangle$ for any α in $\gamma+1$. Then $H_1\,\mathfrak{Fn}\,\gamma+1\cdot H_2\,\mathfrak{Fn}\,\gamma+1$. By (iv) and the definition of γ, we obtain $F_1 \upharpoonright \alpha = H_1 \upharpoonright \alpha \cdot F_2 \upharpoonright \alpha = H_2 \upharpoonright \alpha$ for α in $\gamma+1$. Now it holds that $\mathfrak{W}(H_1) \subseteqq \mathfrak{W}(G_1) + \{0\} \subseteqq \mathfrak{P}(\mathrm{On}^2)$ and $\mathfrak{W}(H_2) \subseteqq \mathfrak{P}(\mathrm{On}^2)$. Hence $\mathfrak{M}(H_1)\cdot\mathfrak{M}(H_2)$. If we set $h_1 = H_1$ and $h_2 = H_2$, then $(\alpha)(\alpha \in \gamma+1 \cdot \supset : h_1{}^{\backprime}\alpha = G_1{}^{\backprime}\langle h_1 \upharpoonright \alpha h_2 \upharpoonright \alpha\rangle \cdot h_2{}^{\backprime} \upharpoonright \alpha = G_2{}^{\backprime}\langle h_1 \upharpoonright \alpha h_2 \upharpoonright \alpha\rangle)$. Hence $\langle h_1 h_2\rangle \in K$. Then it follows that $\gamma \in \mathfrak{D}(h_1)$, which is contradictory to $\gamma = \mathfrak{D}(h_1)$. Therefore $\mathfrak{D}(F_1) = \mathrm{On}\cdot\mathfrak{D}(F_2) = \mathrm{On}$.

1.6. $\mathfrak{W}(G) \subseteqq \mathrm{On}+\mathrm{On}^2+\mathrm{On}^3 \cdot \supset \cdot (\exists\,!\,F)(F\,\mathfrak{Fn}\,\mathrm{On}\cdot(\alpha)(F{}^{\backprime}\alpha = G{}^{\backprime}(F \upharpoonright \alpha)))$.

The proof is carried in the similar way as in 1.5.

1.7. $(\exists\,!\,F)(F\,\mathfrak{Isom}_{SE}(9\times\mathrm{On}^2, \mathrm{On}))$.

Let G be a class such that $(y, x)(\langle yx\rangle \in G \cdot \equiv : y \in (9\times\mathrm{On}^2 - \mathfrak{W}(x))\cdot(9\times\mathrm{On}^2\mathfrak{W}(x)) \cdot S``\{y\} = 0)\cdot G \subseteqq V^2$. Since $\mathfrak{W}(G) \subseteqq \mathrm{On}+\mathrm{On}^2+\mathrm{On}^3$, there is, by 1.6, an F such that $F\,\mathfrak{Fn}\,\mathrm{On}\cdot(\alpha)(F{}^{\backprime}\alpha = G{}^{\backprime}(F \upharpoonright \alpha))$. Then $\mathfrak{W}(F) \subseteqq \mathfrak{W}(G) \subseteqq \mathrm{On}+\mathrm{On}^2+\mathrm{On}^3$ and $F \upharpoonright \alpha\,\mathfrak{Fn}\,\alpha$. Hence $\mathfrak{M}(F \upharpoonright \alpha)$. Then, in virtue of $\langle G{}^{\backprime}xx\rangle \in G$, we obtain $\langle F{}^{\backprime}\alpha F \upharpoonright \alpha\rangle \in G$. Hence $F{}^{\backprime}\alpha \in 9\times\mathrm{On}^2 - F``\alpha\cdot(9\times\mathrm{On}^2 - F``\alpha)\cdot S``\{F{}^{\backprime}\alpha\} = 0$. Therefore $F$ is a one-to-one correspondence and $F``\mathrm{On} \subseteqq 9\times\mathrm{On}^2$. Now assume that $9\times\mathrm{On}^2 - F``\mathrm{On} \neq 0$. Then there is the least element, $\langle i\alpha\beta\rangle$, of $9\times\mathrm{On}^2 - F``\mathrm{On}$ with respect to S. Let x be an arbitrary predecessor of $\langle i\alpha\beta\rangle$ with respect to S. Then $F^{-1}{}^{\backprime}x \in \mathrm{On}$. Set $y = F{}^{\backprime}((F^{-1}{}^{\backprime}x)+1)$ and then xSy and $yS\langle i\alpha\beta\rangle$. Hence there is no immediate predecessor of $\langle i\alpha\beta\rangle$ with respect to S. Therefore $i=0$ and there are the following four cases: (i) $\alpha \in K_{II}\cdot\beta = 0$ or (ii) $\alpha > \beta\cdot\beta \in K_{II}$ or (iii) $\alpha = 0\cdot\beta \in K_{II}$ or (iv) $\alpha \leqq \beta\cdot\alpha \in K_{II}$. Let (i) be the case. We define A by that $(\xi, \eta)(\langle\eta\xi\rangle \in A \cdot \equiv : \xi \in \alpha\cdot\eta = F^{-1}{}^{\backprime}\langle 0\xi 0\rangle)\cdot A \subseteqq \mathrm{On}^2$. Clearly $A\,\mathfrak{Fn}\,\alpha\cdot\mathfrak{W}(A) \subseteqq \mathrm{On}$. Hence $\mathfrak{M}(A``\alpha)$ and so there is a $\mu$ such that $A``\alpha \subseteqq \mu$. It holds that $(\xi)(\xi < \alpha \cdot \supset \cdot \langle 0\xi 0\rangle S(F{}^{\backprime}\mu))$. Let $F{}^{\backprime}\mu = \langle j\theta\zeta\rangle$. Then $\langle j\theta\zeta\rangle \in F``\mathrm{On}$ and hence $\langle j\theta\zeta\rangle S\langle 0\alpha 0\rangle$. So we obtain $\theta, \zeta < \alpha$. Since $\alpha \in K_{II}$, there is a ρ such that $\theta, \zeta < \rho < \alpha$. Hence $(F{}^{\backprime}\mu)S\langle 0\rho 0\rangle$, which is a contradiction. Let (ii) be the case. B is defined by that $(\xi, \eta)(\langle\eta\xi\rangle \in B \cdot \equiv : \xi \in \beta\cdot\eta = F^{-1}{}^{\backprime}\langle 0\alpha\xi\rangle)\cdot B \subseteqq \mathrm{On}^2$. In the same way as before, there is a μ such that $B``\beta \subseteqq \mu$ and hence $(\xi)(\xi < \beta \cdot \supset \cdot \langle 0\alpha\xi\rangle S(F{}^{\backprime}\mu))$. Let $F{}^{\backprime}\mu = \langle k\sigma\tau\rangle$. Since $\beta \in K_{II}, 0 < \beta$ and hence $\langle 0\alpha 0\rangle S\langle k\sigma\tau\rangle$. Therefore

$\alpha \leqq \mathfrak{Max}\{\sigma\tau\}$. On the other hand, $\langle k\sigma\tau\rangle S\langle 0\alpha\beta\rangle$ and $\alpha > \beta$. Hence we obtain $\alpha = \mathfrak{Max}\{\sigma\tau\}$ and furthermore $\sigma = \alpha\cdot\tau < \beta$, since $\langle\sigma\tau\rangle$ Le $\langle\alpha\beta\rangle$. Since $\beta \in K_{II}$, there is a κ such that $\tau < \kappa < \beta$. Hence $(F`\mu)S\langle 0\alpha\kappa\rangle$, which is a contradiction. Similarly for cases (iii) and (iv). Therefore we obtain $9\times\mathrm{On}^2 - F``\mathrm{On} = 0$ and $F\,\mathfrak{Isom}_{ES}(\mathrm{On}, 9\times\mathrm{On}^2)$. Uniqueness is easily proved and hence $(\exists ! F)(F\,\mathfrak{Isom}_{ES}(\mathrm{On}, 9\times\mathrm{On}^2))$. Hence we may obtain the theorem.

By 1.7, we may give the following definition.

Dfn $$J\,\mathfrak{Isom}_{SE}(9\times\mathrm{On}^2, \mathrm{On})\,.$$

The part relating to the axiom of infinity is slightly changed.

1.8. $$\mathfrak{M}(\omega)\,.$$

By C1′ there exists a set a such that $0\in a\cdot(x)(x\in a\cdot\supset\cdot x+1\in a)$. Hence, by the principle of induction, $\omega\subseteqq a$. Then we obtain $\mathfrak{M}(\omega)$.

§ 2. Preparation of model construction.

Dfn $$\langle\mu\gamma\rangle\in K_0\cdot\equiv\cdot(\exists\alpha,\beta)(\gamma = J`\langle\mu\alpha\beta\rangle) : K_0\subseteqq\mathrm{On}^2\,.$$

Dfn $$\xi\in\{\beta\gamma\}^*\cdot\equiv\cdot(\exists\eta)(\xi\in\eta\cdot(\eta=\mathfrak{Min}\{J_1`\langle\beta\gamma\rangle J_1`\langle\gamma\beta\rangle\}\cdot\langle\beta\gamma\rangle\neq\langle 10\rangle\cdot\langle\beta\gamma\rangle\neq\langle 01\rangle)$$
$$\vee\,(\eta = J_0`\langle 10\rangle\cdot(\langle\beta\gamma\rangle = \langle 01\rangle \vee \langle\beta\gamma\rangle=\langle 10\rangle)))) : \{\beta\gamma\}^*\subseteqq\mathrm{On}\,.$$

2.1. $$\sim(\langle\beta\gamma\rangle\in\{\langle 01\rangle\langle 10\rangle\})\cdot\supset\cdot\{\beta\gamma\}^* = \mathfrak{Min}\{J_1`\langle\beta\gamma\rangle J_1`\langle\gamma\beta\rangle\}\cdot$$
$$\langle\beta\gamma\rangle\in\{\langle 01\rangle\langle 10\rangle\}\cdot\supset\cdot\{\beta\gamma\}^* = J_0`\langle 10\rangle\,.$$

Dfn $$\langle\beta\gamma\rangle^* = \{\{\beta\}^*\{\beta\gamma\}^*\}^*\,.$$

Dfn $$\langle\beta\gamma\delta\rangle^* = \langle\beta\langle\gamma\delta\rangle^*\rangle^*\,.$$

Dfn $$\langle yx\rangle\in G_1\cdot\equiv\cdot(K_0`\mathfrak{D}(P_1`x) = 0\cdot(\gamma,\eta)(\langle\gamma\eta\rangle\in y\cdot\equiv : \eta = \mathfrak{D}(P_1`x)\cdot\gamma<\mathfrak{D}(P_1`x)))$$
$$\vee\,(K_0`\mathfrak{D}(P_1`x) = 1\cdot(\gamma,\eta)(\langle\gamma\eta\rangle\in y\cdot\equiv : \eta=\mathfrak{D}(P_1`x)\cdot\gamma = K_1`\mathfrak{D}(P_1`x)\vee\gamma = K_2`\mathfrak{D}(P_1`x)))$$
$$\vee\,(K_0`\mathfrak{D}(P_1`x) = 2\cdot(\gamma,\eta)(\langle\gamma\eta\rangle\in y\cdot\equiv : \eta = \mathfrak{D}(P_1`x)\cdot\gamma\in\mathfrak{W}((P_1`x)`K_1`\mathfrak{D}(P_1`x))\,.$$
$$(\exists\mu,\delta)(\langle\mu\langle\mu\delta\rangle^*\rangle\in\mathfrak{S}(\mathfrak{W}(P_2`x))\cdot(\exists\xi)(\langle\mu\xi\rangle\in\mathfrak{S}(\mathfrak{W}(P_2`x))\cdot\gamma\in\mathfrak{W}((P_1`x)`K_1`\mathfrak{D}(P_1`x)))))$$
$$\vee\,(K_0`\mathfrak{D}(P_1`x) = 3\cdot(\gamma,\eta)(\langle\gamma\eta\rangle\in y\cdot\equiv\cdot\eta = \mathfrak{D}(P_1`x)\cdot\gamma\in\mathfrak{W}((P_1`x)`K_1`\mathfrak{D}(P_1`x))\cdot$$
$$\sim(\exists\xi)(\langle\gamma\xi\rangle\in\mathfrak{S}(\mathfrak{W}(P_2`x))\cdot\xi\in\mathfrak{W}(P_1`x)`K_2`\mathfrak{D}(P_1`x)))))$$
$$\vee\,(K_0`\mathfrak{D}(P_1`x) = 4\cdot(\gamma,\eta)(\langle\gamma\eta\rangle\in y\cdot\equiv\cdot\eta = \mathfrak{D}(P_1`x)\cdot\gamma\in\mathfrak{W}((P_1`x)`K_1`\mathfrak{D}(P_1`x))\,.$$
$$(\exists\mu,\nu,\xi)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(\mathfrak{W}(P_2`x))\cdot\langle\nu\xi\rangle\in\mathfrak{S}(\mathfrak{W}(P_2`x))\cdot\xi\in\mathfrak{W}((P_1`x)`K_2`\mathfrak{D}(P_1`x)))))$$
$$\vee\,(K_0`\mathfrak{D}(P_1`x) = 5\cdot(\gamma,\eta)(\langle\gamma\eta\rangle\in y\cdot\equiv : \eta = \mathfrak{D}(P_1`x)\cdot\gamma\in\mathfrak{W}((P_1`x)`K_1`\mathfrak{D}(P_1`x))\,.$$

$(\exists\mu,\nu,\xi)(\langle\gamma\xi\rangle\in\mathfrak{S}(\mathfrak{W}(P_2\text{‘}x))\cdot\langle\mu\langle\nu\xi\rangle^*\rangle\in\mathfrak{S}(\mathfrak{W}(P_2\text{‘}x))\cdot\mu\in\mathfrak{W}((P_1\text{‘}x)\text{‘}K_2\text{‘}\mathfrak{D}(P_1\text{‘}x)))))$

$\vee(K_0\text{‘}\mathfrak{D}(P_1\text{‘}x)=6\cdot(\gamma,\eta)(\langle\gamma\eta\rangle\in y\cdot\equiv:\eta=\mathfrak{D}(P_1\text{‘}x)\cdot\gamma\in\mathfrak{W}((P_1\text{‘}x)\text{‘}K_1\text{‘}\mathfrak{D}(P_1\text{‘}x)).$

$(\exists\mu,\nu,\xi)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(\mathfrak{W}(P_2\text{‘}x))\cdot\langle\langle\nu\mu\rangle^*\xi\rangle\in\mathfrak{S}(\mathfrak{W}(P_2\text{‘}x))\cdot\xi\in\mathfrak{W}((P_1\text{‘}x)\text{‘}K_2\text{‘}\mathfrak{D}(P_1\text{‘}x)))))$

$\vee(K_0\text{‘}\mathfrak{D}(P_1\text{‘}x)=7\cdot(\gamma,\eta)(\langle\gamma\eta\rangle\in y\cdot\equiv:\eta=\mathfrak{D}(P_1\text{‘}x)\cdot\gamma\in\mathfrak{W}((P_1\text{‘}x)\text{‘}K_1\text{‘}\mathfrak{D}(P_1\text{‘}x)).$

$(\exists\mu,\nu,\kappa,\xi)(\langle\gamma\langle\alpha\mu\nu\rangle^*\rangle\in\mathfrak{S}(\mathfrak{W}(P_2\text{‘}x))\cdot\langle\langle\mu\nu\kappa\rangle^*\xi\rangle\in\mathfrak{S}(\mathfrak{W}(P_2\text{‘}x))\cdot\xi\in\mathfrak{W}((P_1\text{‘}x)\text{‘}K_2\text{‘}\mathfrak{D}(P_1\text{‘}x)))))$

$\vee(K_0\text{‘}\mathfrak{D}(P_1\text{‘}x)=8\cdot(\gamma,\eta)(\langle\gamma\eta\rangle\in y\cdot\equiv:\eta=\mathfrak{D}(P_1\text{‘}x)\cdot\gamma\in\mathfrak{W}((P_1\text{‘}x)\text{‘}K_1\text{‘}\mathfrak{D}(P_1\text{‘}x)).$

$(\exists\mu,\nu,\kappa,\xi)(\langle\gamma\langle\mu\kappa\nu\rangle^*\rangle\in\mathfrak{S}(\mathfrak{W}(P_2\text{‘}x))\cdot\langle\langle\mu\nu\kappa\rangle^*\xi\rangle$

$\in\mathfrak{S}(\mathfrak{W}(P_2\text{‘}x))\cdot\xi\in\mathfrak{W}(P_1\text{‘}x)\text{‘}K_2\text{‘}\mathfrak{D}(P_1\text{‘}x))))):G_1\subseteqq\mathfrak{P}(\mathrm{On}^2)\times V.$

Dfn $\langle yx\rangle\in G_2\cdot\equiv\cdot(u)(u\in y\cdot\equiv\cdot(\exists\beta,\gamma)(u=\langle\beta\gamma\rangle\cdot\mathfrak{Max}\{\beta\gamma\}=\mathfrak{D}(P_2\text{‘}x).$

$(\xi)(\xi\in\mathfrak{W}(G_1\text{‘}\langle(P_1\text{‘}x)\restriction\beta(P_2\text{‘}x)\restriction\beta\rangle)\cdot\supset\cdot(\exists\eta)(\eta$

$\in\mathfrak{W}(G_1\text{‘}\langle(P_1\text{‘}x)\restriction\gamma(P_2\text{‘}x)\restriction\gamma\rangle)\cdot\langle\xi\eta\rangle\in\mathfrak{S}(\mathfrak{W}(P_2\text{‘}x)))).$

$(\xi)(\xi\in\mathfrak{W}(G_1\text{‘}\langle(P_1\text{‘}x)\restriction\gamma(P_2\text{‘}x)\restriction\gamma\rangle)\cdot\supset\cdot(\exists\eta)(\eta\in\mathfrak{W}(G_1,\langle(P_1\text{‘}x)\restriction\beta(P_2\text{‘}x)\restriction\beta\rangle)\cdot$

$\langle\xi\eta\rangle\in\mathfrak{S}(\mathfrak{W}(P_2\text{‘}x)))))):G_2\subseteqq\mathfrak{P}(\mathrm{On}^2)\times V.$

By 1.5, we may give the next definition.

Dfn $H\,\mathfrak{Fn}\,\mathrm{On}\cdot(\alpha)(H\text{‘}\alpha=G_1\text{‘}\langle H\restriction\alpha I\restriction\alpha\rangle)\cdot I\,\mathfrak{Fn}\,\mathrm{On}\cdot(\alpha)(I\text{‘}\alpha=G_2\text{‘}\langle H\restriction\alpha I\restriction\alpha\rangle).$

Dfn $x\in^* A\cdot\equiv\cdot\begin{cases}x\in A & \text{if } A\subseteqq\mathrm{On}\text{ and }\sim(A\in\mathrm{On}),\\ x\in\mathfrak{W}(H\text{‘}A) & \text{if } A\in\mathrm{On},\\ \text{a false statement} & \text{otherwise}.\end{cases}$

2.2. $(\gamma)(\gamma\in^*\alpha\cdot\supset\cdot\gamma<\alpha).$

For $K_0\text{‘}\alpha=0$, $(\gamma)(\gamma\in^*\alpha\cdot\equiv\cdot\gamma<\alpha).$

For $K_0\text{‘}\alpha=1$, $(\gamma)(\gamma\in^*\alpha\cdot\equiv\cdot\gamma=K_1\text{‘}\alpha\vee\gamma=K_2\text{‘}\alpha).$

For $K_0\text{‘}\alpha=2$, $(\gamma)(\gamma\in^*\alpha\cdot\equiv:\gamma\in^*K_1\text{‘}\alpha\cdot(\exists\mu,\nu)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(I\text{“}\alpha)\cdot$

$(\xi\mu)(\langle\mu\xi\rangle\in\mathfrak{S}(I\text{“}\alpha)\cdot\xi\in^*\nu))).$

For $K_0\text{‘}\alpha=3$, $(\gamma)(\gamma\in^*\alpha\cdot\equiv:\gamma\in^*K_1\text{‘}\alpha\cdot\sim(\exists\xi)(\langle\gamma\xi\rangle\in\mathfrak{S}(I\text{“}\alpha)\cdot\xi\in^*K_2\text{‘}\alpha)).$

For $K_0\text{‘}\alpha=4$, $(\gamma)(\gamma\in^*\alpha\cdot\equiv:\gamma\in^*K_1\text{‘}\alpha\cdot(\exists\mu,\nu,\xi)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(I\text{“}\alpha)\cdot$

$\langle\nu\xi\rangle\in\mathfrak{S}(I\text{“}\alpha)\cdot\xi\in^*K_2\text{‘}\alpha)).$

For $K_0\text{‘}\alpha=5$, $(\gamma)(\gamma\in^*\alpha\cdot\equiv:\gamma\in^*K_1\text{‘}\alpha\cdot(\exists\mu,\nu,\xi)(\langle\gamma\xi\rangle\in\mathfrak{S}(I\text{“}\alpha)\cdot$

$\langle\mu\langle\nu\xi\rangle^*\rangle\in\mathfrak{S}(I\text{“}\alpha)\cdot\mu\in^*K_2\text{‘}\alpha)).$

For $K_0‘\alpha=6,\quad (\gamma)(\gamma\in^*\alpha\cdot\equiv:\gamma\in^* K_1‘\alpha\cdot(\exists\mu,\nu,\xi)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot$
$\langle\langle\nu\mu\rangle^*\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in^* K_2‘\alpha))$.

For $K_0‘\alpha=7,\quad (\gamma)(\gamma\in^*\alpha\cdot\equiv:\gamma\in^* K_1‘\alpha\cdot(\exists\mu,\nu,\kappa,\xi)(\langle\gamma\langle\kappa\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot$
$\langle\langle\mu\nu\kappa\rangle^*\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in^* K_2‘\alpha))$.

For $K_0‘\alpha=8,\quad (\gamma)(\gamma\in^*\alpha\cdot\equiv:\gamma\in^* K_1‘\alpha\cdot(\exists\mu,\nu,\kappa,\xi)(\langle\gamma\langle\mu\kappa\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot$
$\langle\langle\mu\nu\kappa\rangle^*\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in^* K_2‘\alpha))$.

$$(\beta,\gamma)(\langle\beta\gamma\rangle\in I‘\alpha\cdot\equiv:\mathfrak{Max}\,\{\beta\gamma\}=\alpha\cdot(\xi)(\xi\ni^*\beta\cdot\supset\cdot$$
$$(\exists\eta)(\eta\in^*\gamma\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I“\alpha)))\cdot(\xi)(\xi\in^*\gamma\cdot\supset\cdot$$
$$(\exists\eta)(\eta\in^*\beta\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I“\alpha))))\cdot I‘\alpha\subseteqq \mathrm{On}^2\,.$$

We assume that the theorem holds for any ordinal less than α. By the definition of G_1, $(x)(\exists!\,y)(\langle xy\rangle\in G_1)$ and hence $\langle G_1‘xx\rangle\in G_1$. Since $\langle H\restriction\alpha I\restriction\alpha\rangle$ is a set, we obtain $\langle H‘\alpha\langle H\restriction\alpha I\restriction\alpha\rangle\rangle\in G_1$ and similarly $\langle I‘\alpha\langle H\restriction\alpha I\restriction\alpha\rangle\rangle\in G_2$. Since $\mathfrak{W}(G_1)\subseteqq\mathfrak{P}(\mathrm{On}^2)$, it holds that $H‘\alpha\in\mathfrak{P}(\mathrm{On}^2)$ for any α and hence $\mathfrak{W}(H)\subseteqq\mathfrak{P}(\mathrm{On}^2)$, using $H\,\mathfrak{Fn}\,\mathrm{On}$. Therefore, by 1.4, $\mathfrak{M}(H\restriction\alpha)$ and similarly $\mathfrak{M}(I\restriction\alpha)$.

$\gamma\in^*\alpha\cdot\supset\cdot\gamma<\alpha$ holds by inductive hypothesis, provided that each formula for $K_0‘\alpha=i$ $(i=0,\cdots,8)$ is proved. Let $K_0‘\alpha=0$. Then we obtain that $(\eta)(\langle\gamma\eta\rangle\in H‘\alpha\cdot\equiv:\eta=\alpha\cdot\gamma<\alpha)$. Hence $\gamma\in^*\alpha\cdot\equiv\cdot\gamma<\alpha$. Let $K_0‘\alpha=1$. It holds that $(\eta)(\langle\gamma\eta\rangle\in H‘\alpha\cdot\equiv:\eta=\alpha\cdot\gamma=K_1‘\alpha\vee\gamma=K_2‘\alpha)$ and hence $\gamma\in^*\alpha\cdot\equiv:\gamma=K_1‘\alpha\vee\gamma=K_2‘\alpha$. Let $K_0‘\alpha=2$. It holds that $(\eta)(\langle\gamma\eta\rangle\in H‘\alpha\cdot\equiv:\eta=\alpha\cdot\gamma\in\mathfrak{W}(H‘K_1‘\alpha)\cdot(\exists\mu,\nu)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot(\exists\xi)(\langle\mu\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in\mathfrak{W}((H\restriction\alpha)‘\nu)))$. If $\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)$, then, by inductive hypothesis, $\mathfrak{Max}\{\gamma\langle\mu\nu\rangle^*\}<\nu$ and hence $\nu<\alpha$. Hence $\mathfrak{W}((H\restriction\alpha)‘\nu)=\mathfrak{W}(H‘\nu)$. Therefore $\gamma\in^*\alpha\cdot\equiv:\gamma\in^* K_1‘\alpha\cdot(\exists\mu,\nu)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot(\exists\xi)(\langle\mu\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in^*\nu))$. Let $K_0‘\alpha=3$. It holds that $(\eta)(\langle\gamma\eta\rangle\in H‘\alpha\cdot\equiv:\eta=\alpha\cdot\gamma\in\mathfrak{W}(H‘K_1‘\alpha)\cdot\sim(\exists\xi)(\langle\gamma\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in\mathfrak{W}(H‘K_2‘\alpha)))$. Hence we obtain that $\gamma\in^*\alpha\cdot\equiv:\gamma\in^* K_1‘\alpha\cdot\sim(\exists\xi)(\langle\gamma\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in^* K_2‘\alpha)$. Let $K_0‘\alpha=4$. It holds that $(\eta)(\langle\gamma\eta\rangle\in H‘\alpha\cdot\equiv:\eta=\alpha\cdot\gamma\in\mathfrak{W}(H‘K_1‘\alpha)\cdot(\exists\mu,\nu,\xi)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot\langle\nu\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in\mathfrak{W}(H‘K_2‘\alpha)))$. Hence $\gamma\in^*\alpha\cdot\equiv:\gamma\in^* K_1‘\alpha\cdot(\exists\mu,\nu,\xi)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot\langle\nu\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in^* K_2‘\alpha)$. Let $K_0‘\alpha=5$. It holds that $(\eta)(\langle\gamma\eta\rangle\in H‘\alpha\cdot\equiv:\eta=\alpha\cdot\gamma\in\mathfrak{W}(H‘K_1‘\alpha)\cdot(\exists\mu,\nu,\xi)(\langle\gamma\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\langle\mu\langle\nu\xi\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot\mu\in\mathfrak{W}(H‘K_2‘\alpha)))$. Hence $\gamma\in^*\alpha\cdot\equiv:\gamma\in^* K_1‘\alpha(\exists\mu,\nu,\xi)(\langle\gamma\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\langle\mu\langle\nu\xi\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot\mu\in^* K_2‘\alpha)$. Let $K_0‘\alpha=6$. It holds that $(\eta)(\langle\gamma\eta\rangle\in H‘\alpha\cdot\equiv:\eta=\alpha\cdot\gamma\in\mathfrak{W}(H‘K_1‘\alpha)\cdot(\exists\mu,\nu,\xi)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot\langle\langle\nu\mu\rangle^*\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in\mathfrak{W}(H‘K_2‘\alpha)))$. Hence $\gamma\in^*\alpha\cdot\equiv:\gamma\in^* K_1‘\alpha\cdot(\exists\mu,\nu,\xi)(\langle\gamma\langle\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot\langle\langle\nu\mu\rangle^*\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in^* K_2‘\alpha)$. Let $K_0‘\alpha=7$. It holds that $(\eta)(\langle\gamma\eta\rangle\in H‘\alpha\cdot\equiv:\eta=\alpha\cdot\gamma\in\mathfrak{W}(H‘K_1‘\alpha)\cdot(\exists\mu,\nu,\kappa,\xi)(\langle\gamma\langle\kappa\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot\langle\langle\mu\nu\kappa\rangle^*\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in\mathfrak{W}(H‘K_2‘\alpha)))$. Hence $\gamma\in^*\alpha\cdot\equiv:\gamma\in^* K_1‘\alpha\cdot(\exists\mu,\nu,\kappa,\xi)(\langle\gamma\langle\kappa\mu\nu\rangle^*\rangle\in\mathfrak{S}(I“\alpha)\cdot\langle\langle\mu\nu\kappa\rangle^*\xi\rangle\in\mathfrak{S}(I“\alpha)\cdot\xi\in^* K_2‘\alpha))$. Similarly for $K_0‘\alpha=8$.

By the definition of G_2, it holds that $(u)(u\in I‘\alpha\cdot\equiv:(\exists\beta,\gamma)(u\in\langle\beta\gamma\rangle\cdot\mathfrak{Max}\,\{\beta\gamma\}$

$=\alpha\cdot(\xi)(\xi\in\mathfrak{W}(G_1`\langle H\restriction\beta I\restriction\beta\rangle)\cdot\supset\cdot(\exists\eta)(\eta\in\mathfrak{W}(G_1`\langle H\restriction\gamma I\restriction\gamma\rangle)\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I``\alpha)))\cdot(\xi)(\xi\in\mathfrak{W}(G_1`\langle H\restriction\gamma I\restriction\gamma\rangle)\cdot\supset\cdot(\exists\eta)(\eta\in\mathfrak{W}(G_1`\langle H\restriction\beta I\restriction\beta\rangle)\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I``\alpha)))))$. Hence $(\beta,\gamma)(\langle\beta\gamma\rangle\in I`\alpha\cdot\equiv:\mathfrak{Max}\{\beta\gamma\}=\alpha\cdot(\xi)(\xi\in^*\beta\cdot\supset\cdot(\exists\eta)(\eta\in^*\gamma\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I``\alpha)))\cdot(\xi)(\xi\in^*\gamma\cdot\supset\cdot(\exists\eta)(\eta\in^*\beta\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I``\alpha))))\cdot I`\alpha\subseteqq\mathrm{On}^2$. Therefore the theorem holds for α. Hence we obtain the theorem, since the kernel of induction is normal.

Dfn $\quad\beta\in[A]\cdot\equiv\cdot(\exists\gamma)(\langle\beta\gamma\rangle\in\mathfrak{S}(I``\mathrm{On})\cdot\gamma\in^*A):[A]\subseteqq\mathrm{On}$.

2.3. $$\alpha\in^*A\cdot\supset\cdot\alpha\in[A].$$

2.4. $[\beta]=[\gamma]\cdot\equiv\cdot\langle\beta\gamma\rangle\in\mathfrak{S}(I``\mathrm{On}):[\beta]=[\gamma]\cdot\equiv\cdot\langle\beta\gamma\rangle\in I`\mathfrak{Max}\{\beta\gamma\}$.

First we prove a lemma: $(\alpha,\beta,\gamma)(\langle\alpha\beta\rangle,\langle\beta\gamma\rangle\in\mathfrak{S}(I``\mathrm{On})\cdot\supset\cdot\langle\alpha\gamma\rangle\in\mathfrak{S}(I``\mathrm{On}))$. Let A be a class such that $\lambda\in A\cdot\equiv\cdot(\alpha,\beta,\gamma)(\lambda=\mathfrak{Max}\{\alpha\beta\gamma\}\cdot\langle\alpha\beta\rangle,\langle\beta\gamma\rangle\in\mathfrak{S}(I``\mathrm{On}):\supset\cdot\langle\alpha\gamma\rangle\in\mathfrak{S}(I``\mathrm{On})):A\subseteqq\mathrm{On}$. Assume that $\lambda\subseteqq A$. Let $\lambda=\mathfrak{Max}\{\alpha\beta\gamma\}$ and $\langle\alpha\beta\rangle,\langle\beta\gamma\rangle\in\mathfrak{S}(I``\mathrm{On})$. Take an arbitrary $\xi$ such that $\xi\in^*\alpha$. Since $\langle\alpha\beta\rangle\in\mathfrak{S}(I``\mathrm{On})$, there is an η such that $\eta\in^*\beta\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I``\mathrm{On})$ and also there is a $\zeta$ such that $\zeta\in^*\gamma\cdot\langle\eta\zeta\rangle\in\mathfrak{S}(I``\mathrm{On})$, since $\langle\beta\gamma\rangle\in\mathfrak{S}(I``\mathrm{On})$. It holds that $\mathfrak{Max}\{\xi\eta\zeta\}<\lambda$. Hence, by inductive hypothesis, $\langle\xi\zeta\rangle\in\mathfrak{S}(I``\mathrm{On})$. Hence $\langle\xi\zeta\rangle\in I`\mathfrak{Max}\{\xi\zeta\}$ and so $\langle\xi\zeta\rangle\in\mathfrak{S}(I``\mathfrak{Max}\{\alpha\gamma\})$. Hence it holds that $(\xi)(\xi\in^*\alpha\cdot\supset\cdot(\exists\zeta)(\zeta\in^*\gamma\cdot\langle\xi\zeta\rangle\in\mathfrak{S}(I``\mathfrak{Max}\{\alpha\gamma\})))$. Similarly we obtain that $(\xi)(\xi\in^*\gamma\cdot\supset\cdot(\exists\zeta)(\zeta\in^*\alpha\cdot\langle\xi\zeta\rangle\in\mathfrak{S}(I``\mathfrak{Max}\{\alpha\gamma\})))$. Therefore $\langle\alpha\gamma\rangle\in I`\mathfrak{Max}\{\alpha\gamma\}$ and so $\langle\alpha\gamma\rangle\in\mathfrak{S}(I``\mathrm{On})$. Hence $\lambda\in A$ and so $\mathrm{On}\subseteqq A$.

Now let $[\beta]=[\gamma]$. Take a $\xi\in^*\beta$ arbitrarily. $\xi\in[\beta]$ and so $\xi\in[\gamma]$. Hence $(\exists\eta)(\eta\in^*\gamma\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I``\mathrm{On}))$ and so $(\exists\eta)(\eta\in^*\gamma\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I``\mathfrak{Max}\{\beta\gamma\}))$. It is proved similarly that if $\xi\in^*\gamma$, then $(\exists\eta)(\eta\in^*\beta\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I``\mathfrak{Max}\{\beta\gamma\}))$. Hence $\langle\beta\gamma\rangle\in\mathfrak{S}(I``\mathrm{On})$. Conversely assume that $\langle\beta\gamma\rangle\in\mathfrak{S}(I``\mathrm{On})$. Take an arbitrary $\xi$ such that $\xi\in[\beta]$. There is an $\eta$ such that $\eta\in^*\beta\cdot\langle\xi\eta\rangle\in\mathfrak{S}(I``\mathrm{On})$. Since $\langle\beta\gamma\rangle\in\mathfrak{S}(I``\mathrm{On})$, there is a $\zeta$ such that $\zeta\in^*\gamma\cdot\langle\eta\zeta\rangle\in\mathfrak{S}(I``\mathfrak{Max}\{\beta\gamma\})$. Then we obtain $\langle\xi\zeta\rangle\in\mathfrak{S}(I``\mathrm{On})$ by the above lemma, since $\langle\xi\eta\rangle,\langle\eta\zeta\rangle\in\mathfrak{S}(I``\mathrm{On})$. Hence $\xi\in[\gamma]$. Consequently $[\beta]\subseteqq[\gamma]$ holds. Similarly we obtain that $[\gamma]\subseteqq[\beta]$ and hence $[\beta]=[\gamma]$. The second part is clear from the first.

2.5. $$\beta\in[A]\cdot\equiv\cdot(\exists\gamma)([\beta]=[\gamma]\cdot\gamma\in^*A).$$

It is clear from the definition of $[A]$ and 2.4.

2.6. $$[A]=[B]\cdot\equiv:(\xi)(\xi\in^*A\cdot\supset\cdot(\exists\eta)([\xi]=[\eta]\cdot\eta\in^*B))\cdot(\xi)(\xi\in^*B\cdot\supset\cdot(\exists\eta)([\xi]=[\eta]\cdot\eta\in^*A)).$$

Assume that $[A]=[B]$. Take an arbitrary ξ such that $\xi\in^*A$. Then $\xi\in[A]$ and hence $\xi\in[B]$. By 2.5, $(\exists\eta)([\xi]=[\eta]\cdot\eta\in^*B)$. Similar for $(\xi)(\xi\in^*B\cdot\supset\cdot(\exists\eta)([\xi]=[\eta]\cdot\xi\in^*A))$. Conversely assume that the right-hand side of 2.6 holds. Take a $\xi\in[A]$ arbitrarily. By 2.5, there is an η such that $[\xi]=[\eta]\cdot$

$\eta \in^* A$. Then, by the assumption, there is a ζ such that $[\eta]=[\zeta]\cdot\zeta\in^* B$. Hence it holds that $[\xi]=[\zeta]\cdot\zeta\in^* B$ and so $\xi\in[B]$. Hence $[A]\subseteqq[B]$ holds. Similar for $[B]\subseteqq[A]$ and so $[A]=[B]$.

2.7. $$(\alpha, \beta, A)([\alpha]=[\beta]\cdot\supset:\alpha\in[A]\cdot\equiv\cdot\beta\in[A]).$$

Assume that $[\alpha]=[\beta]$. Let $\alpha\in[A]$. Then there is a γ such that $[\alpha]=[\gamma]\cdot\gamma\in^* A$. Hence $[\beta]=[\gamma]\cdot\gamma\in^* A$ and so $\beta\in[A]$. Similar for the converse.

2.8. $$\xi\in[\{\alpha\beta\}^*]\cdot\equiv\cdot[\xi]=[\alpha]\vee[\xi]=[\beta].$$

It holds that $\{\alpha\beta\}^*=\mathfrak{Min}\,\{J_1`\langle\alpha\beta\rangle J_1`\langle\beta\alpha\rangle\}\vee\{\alpha\beta\}^*=J_0`\langle 10\rangle$. Let $\{\alpha\beta\}^*=\mathfrak{Min}\,\{J_1`\langle\alpha\beta\rangle J_1`\langle\beta\alpha\rangle\}$. Take an arbitrary ξ in $[\{\alpha\beta\}^*]$. Then there is an η such that $[\xi]=[\eta]\cdot\eta\in^*\{\alpha\beta\}^*$. By 2.2, $\eta=\alpha\vee\eta=\beta$ and hence $[\xi]=[\alpha]\vee[\xi]=[\beta]$. Conversely assume that $[\xi]=[\alpha]\vee[\xi]=[\beta]$. Since $\alpha\in^*\{\alpha\beta\}^*$ and $\beta\in^*\{\alpha\beta\}^*$, we obtain $\xi\in[\{\alpha\beta\}^*]$ by 2.5. Let $\{\alpha\beta\}^*=J_0`\langle 10\rangle$. Assume that $\xi\in[\{\alpha\beta\}^*]$ and $[\xi]=[\eta]\cdot\eta\in^* J_0`\langle 10\rangle$. Then $\eta=0\vee\eta=1\vee[\eta]=0$. Hence $[\eta]=[0]\vee[\eta]=[1]$ and so $[\xi]=[0]\vee[\xi]=[1]$. Since $\langle\alpha\beta\rangle=\langle 01\rangle$ or $\langle\alpha\beta\rangle=\langle 10\rangle$, we obtain that $[\xi]=[\alpha]\vee[\xi]=[\beta]$. Conversely assume that $[\xi]=[\alpha]\vee[\xi]=[\beta]$. Then $[\xi]=[0]\vee[\xi]=[1]$. Since $0\in^* J_0`\langle 10\rangle$ and $1\in^* J_0`\langle 10\rangle$, it holds that $\xi\in[J_0`\langle 10\rangle]$, i.e., $\xi\in[\{\alpha\beta\}^*]$.

2.9. $$[\langle\alpha\beta\rangle^*]=[\langle\gamma\delta\rangle^*]\cdot\equiv:[\alpha]=[\gamma]\cdot[\beta]=[\delta].$$

We omit the proof, since it may be carried out as usual.

2.10. $$\alpha\leqq\gamma\cdot\beta\leqq\delta:\supset\cdot\{\alpha\beta\}^*\leqq\{\gamma\delta\}^*\cdot$$
$$\alpha\leqq\gamma\cdot\beta\leqq\delta\cdot\langle\alpha\beta\rangle\neq\langle\gamma\delta\rangle:\supset\cdot\{\alpha\beta\}^*<\{\gamma\delta\}^*.$$

It is sufficient to prove the latter formula. Let $\{\gamma\delta\}^*=\mathfrak{Min}\,\{J_1`\langle\gamma\delta\rangle J_1`\langle\delta\gamma\rangle\}$. By the premise, $\langle\alpha\beta\rangle R\langle\gamma\delta\rangle$ and $\langle\beta\alpha\rangle R\langle\delta\gamma\rangle$. If $\{\gamma\delta\}^*=J_1`\langle\gamma\delta\rangle$, then $J_1`\langle\alpha\beta\rangle<J_1`\langle\gamma\delta\rangle$. Since $\{\alpha\beta\}^*\leqq J_1`\langle\alpha\beta\rangle$, it holds that $\{\alpha\beta\}^*<\{\gamma\delta\}^*$. If $\{\gamma\delta\}^*=J_1`\langle\delta\gamma\rangle$, then $J_1`\langle\beta\alpha\rangle<J_1`\langle\delta\gamma\rangle$ and so $\{\alpha\beta\}^*<\{\gamma\delta\}^*$. Let $\{\gamma\delta\}^*=J_0`\langle 10\rangle$. Then $\langle\gamma\delta\rangle=\langle 10\rangle$ or $\langle\gamma\delta\rangle=\langle 01\rangle$. Hence, by the premise, $\langle\alpha\beta\rangle=\langle 00\rangle$ and so $\{\alpha\beta\}^*=J_0`\langle 00\rangle$. Consequently $\{\alpha\beta\}^*<J_0`\langle 10\rangle=\{\gamma\delta\}^*$.

2.11. $$\alpha\leqq\gamma\cdot\beta\leqq\delta:\supset\cdot\langle\alpha\beta\rangle^*\leqq\langle\gamma\delta\rangle^*\cdot$$
$$\alpha\leqq\gamma\cdot\beta\leqq\delta\cdot\langle\alpha\beta\rangle\neq\langle\gamma\delta\rangle:\supset\cdot\langle\alpha\beta\rangle^*<\langle\gamma\delta\rangle^*.$$

It is sufficient to prove the latter formula. By the premise and 2.10, $\{\alpha\}^*\leqq\{\gamma\}^*$ and $\{\alpha\beta\}^*<\{\gamma\delta\}^*$. Hence $\langle\{\alpha\}^*\{\alpha\beta\}^*\rangle\neq\langle\{\gamma\}^*\{\gamma\delta\}^*\rangle$ and again, by 2.10, $\langle\alpha\beta\rangle^*=\{\{\alpha\}^*\{\alpha\beta\}^*\}<\{\{\gamma\}^*\{\gamma\delta\}^*\}^*=\langle\gamma\delta\rangle^*$.

2.12. $$(\exists\mu, \nu)([\alpha]=[\langle\mu\nu\rangle^*])\cdot\supset\cdot(\exists\sigma, \tau)([\alpha]=[\langle\sigma\tau\rangle^*]\cdot\langle\sigma\tau\rangle^*\leqq\alpha)$$

By the premise, there are a μ and a ν such that $[\alpha]=[\langle\mu\nu\rangle^*]$. It holds that $\{\mu\}^*\in[\langle\mu\nu\rangle^*]$. Then there is an x such that $[\{\mu\}^*]=[x]\cdot x\in^*\alpha$. The least among such x's is named π. In the same way, there is the least, ρ, among x's such that $[\{\mu\nu\}^*]=[x]\cdot x\in^*\alpha$, using that $\{\mu\nu\}^*\in[\langle\mu\nu\rangle^*]$. For any x such that $x\in^*\alpha$, it holds that $x\in[\langle\mu\nu\rangle^*]$ and so $[x]=[\{\mu\}^*]\vee[x]=[\{\mu\nu\}^*]$. There also exists an x such that $[\mu]=[x]\cdot x\in^*\rho$, since $[\{\mu\nu\}^*]=[\rho]$. Hence there is an x such that $[\mu]=[x]$ and $x\in^*\pi\vee x\in^*\rho$. The least among such x's is named σ. There also exists the least, τ, among x's such that $[\nu]=[x]\cdot x\in^*\rho$.

Let $K_0'\alpha=0$. Assume that $\pi\leqq\rho$. Since $[\{\mu\}^*]=[\pi]$, there is an x such that $x\in^*\pi$ and so $x<\pi$. It holds that $\pi<\alpha$, since $\pi\in^*\alpha$. Hence $x<\alpha$ and so $x\in^*\alpha$, since $K_0`\alpha=0$. Then we obtain an $x$ such that $x<\pi$, $x<\rho$, and $x\in^*\alpha$, which is contrary to the definitions of $\pi$ and $\rho$. Similarly we obtain a contradiction, assuming that $\rho\leqq\pi$. Hence $K_0`\alpha\neq 0$ holds. Let $K_0`\alpha=1$. Since $\pi,\rho\in^*\alpha$, it holds that $\pi,\rho\in\{K_1`\alpha K_2`\alpha\}$ and so $\mathfrak{Max}\{\pi\rho\}\leqq\mathfrak{Max}\{K_1`\alpha K_2`\alpha\}$. If $\mathfrak{Max}\{\pi\rho\}<\mathfrak{Max}\{K_1`\alpha K_2`\alpha\}$, then $\langle\pi\rho\rangle R\langle K_1`\alpha K_2`\alpha\rangle$. Let $\mathfrak{Max}\{\pi\rho\}=\mathfrak{Max}\{K_1`\alpha K_2`\alpha\}$. Assume that $\pi<\rho$. If $K_1`\alpha\leqq K_2`\alpha$, then $\rho=K_2`\alpha$ and so $\pi<K_2`\alpha$. Hence $\langle\rho\pi\rangle R\langle K_1`\alpha K_2`\alpha\rangle$. If $K_2`\alpha<K_1`\alpha$, then $\rho=K_1`\alpha$ and $\pi=K_2`\alpha$, since $\pi=K_1`\alpha\vee\pi=K_2`\alpha$. So $\langle\rho\pi\rangle=\langle K_1`\alpha K_2`\alpha\rangle$. Similarly we obtain that $\langle\pi\rho\rangle R\langle K_1`\alpha K_2`\alpha\rangle$ or $\langle\pi\rho\rangle=\langle K_1`\alpha K_2`\alpha\rangle$, assuming that $\rho<\pi$. Assume that $\pi=\rho$. Then $\pi=\mathfrak{Max}\{K_1`\alpha K_2`\alpha\}$. On the other hand, by the definitions of $\pi$ and $\rho$, it holds that $\pi\leqq K_1`\alpha$ and $\pi\leqq K_2`\alpha$. Hence $\pi=\rho=K_1`\alpha=K_2`\alpha$ and so $\langle\pi\rho\rangle=\langle K_1`\alpha K_2`\alpha\rangle$. In each case, it holds that $J_1`\langle\pi\rho\rangle\leqq\alpha$ or $J_1`\langle\rho\pi\rangle\leqq\alpha$. Hence $\mathfrak{Min}\{J_1`\langle\pi\rho\rangle J_1`\langle\rho\pi\rangle\}\leqq\alpha$. Since $\pi\neq 0$ and $\rho\neq 0$, $\{\pi\rho\}^*=\mathfrak{Min}\{J_1`\langle\pi\rho\rangle J_1`\langle\rho\pi\rangle\}$ and so $\{\pi\rho\}^*\leqq\alpha$. Let $K_0`\alpha=i$ $(i=2,\cdots,8)$. By 2.2, $(x)(x\in^*\alpha\cdot\supset\cdot x\in^*K_1`\alpha)$ and hence $\pi,\rho\in^*K_1`\alpha$. Hence $\mathfrak{Max}\{\pi\rho\}<\mathfrak{Max}\{K_1`\alpha K_2`\alpha\}$ and so $\langle\pi\rho\rangle R\langle K_1`\alpha K_2`\alpha\rangle$. Hence $J_1`\langle\pi\rho\rangle<\alpha$ and so $\{\pi\rho\}^*<\alpha$. Consequently it holds that $\{\pi\rho\}^*\leqq\alpha$.

Next we show that $\{\sigma\}^*\leqq\pi$ and $\{\sigma\tau\}^*\leqq\rho$. Let $K_0`\pi=0$. Since $\pi\neq 0$, it holds that $J_0`\langle 10\rangle\leqq\pi$ and hence $0\in^*\pi\cdot 1\in^*\pi$ where $\sim([0]=[1])$. On the other hand, $(x)(x\in^*\pi\cdot\supset\cdot[x]=[\sigma])$, which is a contradiction. Hence it holds that $K_0`\pi\neq 0$. Let $K_0`\pi=1$. By the definition, $\sigma\leqq\mathfrak{Min}\{K_1`\alpha K_2`\alpha\}$ and so $J_1`\langle\sigma\sigma\rangle\leqq J_1`\langle K_1`\pi K_2`\pi\rangle$. Hence $\{\sigma\}^*\leqq\pi$. Let $K_0`\pi=i$ $(i=2,\cdots,8)$. Since $\sigma<K_1`\pi$, $\sigma<\mathfrak{Max}\{K_1`\pi K_2`\pi\}$ holds and hence $\{\sigma\}^*<\pi$. Hence we obtain that $\{\sigma\}^*\leqq\pi$. Let $K_0`\rho=0$. Assume that $\rho\neq J_0`\langle 10\rangle$. Since $\rho\neq 0$, $J_0`\langle 10\rangle<\rho$ holds and so $0\in^*\rho\cdot 1\in^*\rho\cdot J_0`\langle 10\rangle\in^*\rho$ where $\sim([0]=[1])$, $\sim([0]=[J_0`\langle 10\rangle])$, and $\sim([1]=J_0`\langle 10\rangle])$. On the other hand, $(x)(x\in^*\rho\cdot\supset\cdot[x]=[\sigma]\vee[x]=[\tau])$, which is a contradiction. Hence it holds that $\rho=J_0`\langle 10\rangle$. So, for any x such that $x\in^*\rho$, it holds that $x=0$ or $x=1$ or $[x]=0$. Since $\tau\in^*\rho$, we obtain, by the definition of ρ, that $\tau\leqq 1$. Let σ_1 be the least x such that $[x]=[\sigma]\cdot x\in^*\rho$. σ_1 exists and $\sigma\leqq\sigma_1\cdot\sigma_1\in^*\rho$. Hence $\sigma_1\leqq 1$ and so $\sigma\leqq 1$. By the definition, $\sigma=0\vee\tau=0$ and hence $\langle\sigma\tau\rangle\neq\langle 11\rangle$. Hence $\{\sigma\tau\}^*\leqq J_0`\langle 10\rangle$ and so $\{\sigma\tau\}^*\leqq\rho$. Let $K_0`\rho=1$. It holds that $\mathfrak{Max}\{\sigma\tau\}\leqq\mathfrak{Max}\{K_1`\rho K_2`\rho\}$. In the same way as before, we obtain that $\{\sigma\tau\}^*\leqq\rho$. Let $K_0`\rho=i$ $(i=2,\cdots,8)$. It holds that

$\mathfrak{Max}\,\{\sigma\tau\} < \mathfrak{Max}\,\{K_1{}^{\backprime}\rho K_2{}^{\backprime}\rho\}$ and similarly $\{\sigma\tau\}^* < \rho$. Consequently we obtain that $\{\sigma\}^* \leqq \pi \cdot \{\sigma\tau\}^* \leqq \rho$.

Hence, by 2.10, $\langle\sigma\tau\rangle^* = \{\{\sigma\}^*\{\sigma\tau\}^*\}^* \leqq \{\pi\rho\}$. On the other hand, $\{\pi\rho\}^* \leqq \alpha$ and so we obtain that $\langle\sigma\tau\rangle^* \leqq \alpha$. By the definition, $[\mu] = [\sigma] \cdot [\nu] = [\tau]$ and and hence, by 2.9, $[\alpha] = [\langle\sigma\tau\rangle^*]$. Hence the theorem holds.

2.13. $(\exists\lambda, \mu, \nu)([\alpha] = [\langle\lambda\mu\nu\rangle^*]) \cdot \supset \cdot (\exists\rho, \sigma, \tau)([\alpha] = [\langle\rho\sigma\tau\rangle^*] \cdot \langle\rho\sigma\tau\rangle^* \leqq \alpha)$.

Let $[\alpha] = [\langle\lambda\mu\nu\rangle^*]$. By 2.12, there are a ρ and a κ such that $[\langle\lambda\mu\nu\rangle^*] = [\langle\rho\kappa\rangle^*]$ and $\langle\rho\kappa\rangle^* \leqq \alpha$. Hence, by 2.9, $[\kappa] = [\langle\mu\nu\rangle^*]$ and so again there are a σ and a τ such that $[\kappa] = [\langle\sigma\tau\rangle^*]$ and $\langle\sigma\tau\rangle^* \leqq \kappa$. So, by 2.11, $\langle\rho\sigma\tau\rangle^* \leqq \langle\rho\kappa\rangle^*$ and hence $\langle\rho\sigma\tau\rangle^* \leqq \alpha$. It is clear that $[\alpha] = [\langle\rho\sigma\tau\rangle^*]$. So the theorem holds.

Dfn $x \in E \cdot \equiv \cdot (\exists\mu, \nu)([x] = [\langle\mu\nu\rangle^*] \cdot \mu \in [\nu])$.

Dfn $\langle\gamma\beta\rangle \in Q_4 \cdot \equiv \cdot (\exists\mu)([\gamma] = [\langle\mu\beta\rangle^*]) : Q_4 \subseteqq \mathrm{On}^2$.

Dfn $\langle\gamma\beta\rangle \in Q_5 \cdot \equiv \cdot (\exists\nu)([\beta] = [\langle\nu\gamma\rangle^*]) : Q_5 \subseteqq \mathrm{On}^2$.

Dfn $\langle\gamma\beta\rangle \in Q_6 \cdot \equiv \cdot (\exists\mu, \nu)([\gamma] = [\langle\nu\mu\rangle^*] \cdot [\beta] = [\langle\mu\nu\rangle^*]) : Q_6 \subseteqq \mathrm{On}^2$.

Dfn $\langle\gamma\beta\rangle \in Q_7 \cdot \equiv \cdot (\exists\mu, \nu, \kappa)([\gamma] = [\langle\kappa\mu\nu\rangle^*] \cdot [\beta] = [\langle\mu\nu\kappa\rangle^*]) : Q_7 \subseteqq \mathrm{On}^2$.

Dfn $\langle\gamma\beta\rangle \in Q_8 \cdot \equiv \cdot (\exists\mu, \nu, \kappa)([\gamma] = [\langle\mu\kappa\nu\rangle^*] \cdot [\beta] = [\langle\mu\nu\kappa\rangle^*]) : Q_8 \subseteqq \mathrm{On}^2$.

2.14.
$$(\gamma)(\gamma \in^* J_0{}^{\backprime}\langle\alpha\beta\rangle \cdot \equiv \cdot \gamma < J_0{}^{\backprime}\langle\alpha\beta\rangle).$$
$$(\gamma)(\gamma \in^* J_1{}^{\backprime}\langle\alpha\beta\rangle \cdot \equiv \cdot \gamma = \alpha \vee \gamma = \beta).$$
$$(\gamma)(\gamma \in^* J_2{}^{\backprime}\langle\alpha\beta\rangle \cdot \equiv \cdot \gamma \in^* \alpha \cdot \gamma \in E).$$
$$(\gamma)(\gamma \in^* J_3{}^{\backprime}\langle\alpha\beta\rangle \cdot \equiv \cdot \gamma \in^* \alpha \cdot \sim(\gamma \in [\beta])).$$
$$(\gamma)(\gamma \in^* J_i{}^{\backprime}\langle\alpha\beta\rangle \cdot \equiv \cdot \gamma \in^* \alpha \cdot \gamma \in Q_i{}^{\backprime\backprime}[\beta]) \quad \text{for } i = 4, \cdots, 8.$$

The formula regarding $J_0{}^{\backprime}\langle\alpha\beta\rangle$ and $J_1{}^{\backprime}\langle\alpha\beta\rangle$ are clear from 2.2. Let $\gamma \in^* J_2{}^{\backprime}\langle\alpha\beta\rangle$. By 2.2 and 2.4, we obtain that $\gamma \in^* \alpha \cdot (\exists\mu, \nu)([\gamma] = [\langle\mu\nu\rangle^*] \cdot (\exists\xi)([\mu] = [\xi] \cdot \xi \in^* \nu))$. Hence $\gamma \in^* \alpha \cdot (\exists\mu, \nu)([\gamma] = [\langle\mu\nu\rangle^*] \cdot \mu \in [\nu])$ and so $\gamma \in^* \alpha \cdot \gamma \in E$. Conversely assume that $\gamma \in^* \alpha \cdot \gamma \in E$. For suitable μ and ν, it holds that $[\gamma] = [\langle\mu\nu\rangle^*]$ $\mu \in [\nu]$ where we may assume by 2.12 that $\langle\mu\nu\rangle^* \leqq \gamma$. Clearly $\gamma < \alpha$ and so $\mathfrak{Max}\,\{\gamma\langle\mu\nu\rangle^*\} < J_2{}^{\backprime}\langle\alpha\beta\rangle$. Hence, by 2.4, $\langle\gamma\langle\mu\nu\rangle^*\rangle \in \mathfrak{S}(I^{\backprime\backprime}J_2{}^{\backprime}\langle\alpha\beta\rangle)$. Since $\mu \in [\nu]$, there is a ξ such that $[\mu] = [\xi] \cdot \xi \in^* \nu$. It holds that $\mu, \xi < \alpha$ and so $\langle\mu\xi\rangle \in \mathfrak{S}(I^{\backprime\backprime}J_2{}^{\backprime}\langle\alpha\beta\rangle)$. Hence $\gamma \in^* J_2{}^{\backprime}\langle\alpha\beta\rangle$. So it holds that $\gamma \in^* J_2{}^{\backprime}\langle\alpha\beta\rangle \cdot \equiv \cdot \gamma \in^* \alpha \cdot \gamma \in E$. Let $\gamma \in^* J_3{}^{\backprime}\langle\alpha\beta\rangle$. Then $\gamma \in^* \alpha \cdot \sim(\exists\xi)(\langle\gamma\xi\rangle \in \mathfrak{S}(I^{\backprime\backprime}J_3{}^{\backprime}\langle\alpha\beta\rangle) \cdot \xi \in^* \beta)$. Assume that there is a ξ such that $\langle\gamma\xi\rangle \in \mathfrak{S}(I^{\backprime\backprime}\mathrm{On}) \cdot \xi \in^* \beta$. Then $\gamma < \alpha \cdot \xi < \beta$ and so $\langle\gamma\xi\rangle \in \mathfrak{S}(I^{\backprime\backprime}\,\mathfrak{Max}\,\{\alpha\beta\})$. Since $\mathfrak{Max}\,\{\alpha\beta\} < J_3{}^{\backprime}\langle\alpha\beta\rangle$, it holds that $\langle\gamma\xi\rangle \in \mathfrak{S}(I^{\backprime\backprime}J_3{}^{\backprime}\langle\alpha\beta\rangle)$, which is a contradiction. Hence $\sim(\exists\xi)(\langle\gamma\xi\rangle \in \mathfrak{S}(I^{\backprime\backprime}\mathrm{On}) \cdot \xi \in^* \beta)$, i.e., $\sim(\exists\xi)([\gamma] = [\xi] \cdot \xi \in^* \beta)$. Hence $\sim(\gamma \in [\beta])$. Conversely assume that $\gamma \in^* \alpha \cdot \sim(\gamma \in [\beta])$. Then $\sim(\exists\xi)(\langle\gamma\xi\rangle \in \mathfrak{S}(I^{\backprime\backprime}\mathrm{On}) \cdot \xi \in^* \beta)$ and a fortiori $\sim(\exists\xi)$

$(\langle\gamma\xi\rangle \in \mathfrak{S}(I``J_3`\langle\alpha\beta\rangle) \cdot \xi \in^* \beta)$. So $\gamma \in^* J_3`\langle\alpha\beta\rangle$. Let $\gamma \in^* J_4`\langle\alpha\beta\rangle$. It is easy to see that $\gamma \in^* \alpha \cdot (\exists\mu, \nu)([\gamma] = [\langle\mu\nu\rangle^*] \cdot (\exists\xi)([\nu] = [\xi] \cdot \xi \in^* \beta))$ and hence $\gamma \in^* \alpha \cdot \gamma \in Q_4``[\beta]$. Conversely assume that $\gamma \in^* \alpha \cdot \gamma \in Q_4``[\beta]$. There is a ν such that $\langle\gamma\nu\rangle \in Q_4 \cdot \nu \in [\beta]$. Hence there also exists a μ such that $[\gamma] = [\langle\mu\nu\rangle^*]$. In virtue of 2.12 we assume that $\langle\mu\nu\rangle^* \leqq \gamma$. $\gamma < \alpha$ and so $\mathfrak{Max}\,\{\gamma\langle\mu\nu\rangle^*\} < J_4`\langle\alpha\beta\rangle$. Hence $\langle\gamma\langle\mu\nu\rangle^*\rangle \in \mathfrak{S}(I``J_4`\langle\alpha\beta\rangle)$. Since $\nu \in [\beta]$, there is a $\xi$ such that $[\nu] = [\xi] \cdot \xi \in^* \beta$. $\mathfrak{Max}\,\{\nu\xi\} < J_4`\langle\alpha\beta\rangle$ and so $\langle\nu\xi\rangle \in \mathfrak{S}(I``J_4`\langle\alpha\beta\rangle)$. Hence $\gamma \in^* J_4`\langle\alpha\beta\rangle$. Let $\gamma \in^* J_5`\langle\alpha\beta\rangle$. It is easy to see that $\gamma \in^* \alpha \cdot \gamma \in Q_5``[\beta]$. Assume that $\gamma \in^* \alpha \cdot \gamma \in Q_5``[\beta]$. There are a μ and a ν such that $[\mu] = [\langle\nu\gamma\rangle^*] \cdot \mu \in^* \beta$. In virtue of 2.12, it holds that for suitable σ and τ, $[\mu] = [\langle\sigma\tau\rangle^*] \cdot \langle\sigma\tau\rangle^* \leqq \mu$. Clearly $[\gamma] = [\tau]$ and $\tau < \beta$. So $\mathfrak{Max}\,\{\gamma\tau\} < J_5`\langle\alpha\beta\rangle$ and hence $\langle\gamma\tau\rangle \in \mathfrak{S}(I``J_5`\langle\alpha\beta\rangle)$. Similarly $\langle\mu\langle\sigma\tau\rangle^*\rangle \in \mathfrak{S}(I``J_5`\langle\alpha\beta\rangle)$ holds. Hence $\gamma \in^* J_5`\langle\alpha\beta\rangle$. Let $\gamma \in^* J_6`\langle\alpha\beta\rangle$. Easily we obtain that $\gamma \in^* \alpha \cdot \gamma \in Q_6``[\beta]$. Assume that $\gamma \in^* \alpha \cdot \gamma \in Q_6``[\beta]$. There is a δ such that $\langle\gamma\delta\rangle \in Q_6 \cdot \delta \in [\beta]$. Hence there is a ξ such that $[\delta] = [\xi] \cdot \xi \in^* \beta$. There also exist a μ and a ν such that $[\gamma] = [\langle\mu\nu\rangle^*] \cdot [\xi] = [\langle\nu\mu\rangle^*]$. By 2.12, it holds that for suitable σ_1, τ_1, σ_2, τ_2, $[\gamma] = \langle\sigma_1\tau_1\rangle^*] \cdot \langle\sigma_1\tau_1\rangle^* \leqq \gamma \cdot [\xi] = [\langle\tau_2\sigma_2\rangle^*] \cdot \langle\tau_2\tau_2\rangle^* \leqq \xi$. Set $\sigma = \mathfrak{Min}\,\{\sigma_1\sigma_2\}$ and $\tau = \mathfrak{Min}\,\{\tau_1\tau_2\}$. Then $[\sigma_1] = [\mu] = [\sigma_2]$ and $[\tau_1] = [\nu] = [\tau_2]$. So $[\gamma] = [\langle\sigma\tau\rangle^*]$ and $[\xi] = [\langle\tau\sigma\rangle^*]$. By 2.11, $\langle\sigma\tau\rangle^* \leqq \langle\sigma_1\tau_1\rangle^* \cdot \langle\tau\sigma\rangle^* \leqq \langle\tau_2\sigma_2\rangle^*$ and hence $\langle\sigma\tau\rangle^* < \alpha$ and $\langle\tau\sigma\rangle^* < \beta$. So $\langle\gamma\langle\sigma\tau\rangle^*\rangle \in \mathfrak{S}(I``J_6`\langle\alpha\beta\rangle)$ and $\langle\langle\tau\sigma\rangle^*\xi\rangle \in \mathfrak{S}(I``J_6`\langle\alpha\beta\rangle)$. Consequently $\gamma \in^* J_6`\langle\alpha\beta\rangle$. Similar for the cases of $J_7`\langle\alpha\beta\rangle$ and $J_8`\langle\alpha\beta\rangle$, using 2.13 instead of 2.12.

2.15.
$$(\alpha, \beta)([J_2`\langle\alpha\beta\rangle] = [\alpha] \cdot E)\,.$$
$$(\alpha, \beta)([J_3`\langle\alpha\beta\rangle] = [\alpha] - [\beta])\,.$$
$$(\alpha, \beta)([J_i`\langle\alpha\beta\rangle] = [\alpha] \cdot Q_i``[\beta]) \quad \text{for } i = 4, \cdots, 8\,.$$

Let $\xi \in [J_2`\langle\alpha\beta\rangle]$. There is a $\gamma$ such that $[\xi] = [\gamma] \cdot \gamma \in^* J_2`\langle\alpha\beta\rangle$. So, by 2.14, $\gamma \in^* \alpha \cdot \gamma \in E$. Hence $\xi \in [\alpha] \cdot \xi \in E$. Conversely assume that $\xi \in [\alpha] \cdot E$. Then there is a γ such that $[\xi] = [\gamma] \cdot \gamma \in^* \alpha$. Since $\xi \in E$, it holds that $\gamma \in E$. So $\gamma \in^* J_2`\langle\alpha\beta\rangle$ and hence $\xi \in [J_2`\langle\alpha\beta\rangle]$. Similarly $[J_3`\langle\alpha\beta\rangle] = [\alpha] - [\beta]$ is proved, using that $(\delta, \gamma)([\delta] = [\gamma] \cdot \gamma \in [\beta] : \supset \cdot \delta \in [\beta])$. Also $[J_i`\langle\alpha\beta\rangle] = [\alpha] \cdot Q_i``[\beta]$ is proved, using that $(\delta, \gamma)([\delta] = [\gamma] \cdot \gamma \in Q_i``[\beta] : \supset \cdot \delta \in Q_i``[\beta])$.

2.16.
$$(\alpha, \beta)(\exists\gamma)([\gamma] = [\alpha] \cdot [\beta])\,.$$

By 2.15, it is clear, since $[\alpha] \cdot [\beta] = [\alpha] - ([\alpha] - [\beta])$.

Dfn $\overline{\mathfrak{Cls}}(A) \cdot \equiv \cdot (\exists B)(A = [B] \cdot B \subseteq \mathrm{On} \cdot (\alpha)(\exists\beta)([\beta] = [\alpha] \cdot B))\,.$

Dfn $$\overline{\mathfrak{M}}(A) \cdot \equiv \cdot (\exists\alpha)(A = [\alpha])\,.$$

Dfn $$A \mathrel{\overline{\in}} B \cdot \equiv \cdot (\exists\alpha)(A = [\alpha] \cdot \alpha \in^* B)\,.$$

Dfn $\bar{x}$, $\bar{y}$, $\bar{z}$, $\cdots$ will be used as variables for X's such that $\overline{\mathfrak{M}}(X)$ and

$\bar{A}, \bar{B}, \bar{C}, \cdots$ as variables for A's such that $\overline{\mathfrak{Cls}}(A)$.

2.17. $$(\alpha, \beta)(\exists \gamma)([\alpha]=[\gamma] \cdot \beta<\gamma).$$

For arbitrary α and β, it holds that $[\alpha]=[\alpha]-[J_3`\langle\beta\beta\rangle]=[J_3`\langle\alpha J_3`,\langle\beta\beta\rangle\rangle]$ where $\beta<J_3`\langle\alpha J_3`\langle\beta\beta\rangle\rangle$. Hence $(\exists\gamma)([\alpha]=[\gamma]\cdot\beta<\gamma)$.

2.18. $$A \subseteqq \mathrm{On} \cdot [A] \neq 0 : \supset : \mathfrak{Pr}([A]) \cdot \sim([A] \in \mathrm{On}).$$

Assume that $\mathfrak{M}([A])$. Then there is a β such that $[A] \subseteqq \beta$. Since $[A] \neq 0$, there is a γ such that $\gamma \in^* A$. By 2.17, there is a δ such that $[\gamma]=[\delta]\cdot\beta<\delta$. Hence $\delta \in [A]$ and so $\delta<\beta$ which is a contradiction. Hence $\mathfrak{Pr}([A])$ and consequently $\sim([A] \in \mathrm{On})$.

2.19. $$A \subseteqq \mathrm{On} \cdot \equiv \cdot [[A]]=[A].$$

If $[A]=0$, then it is obvious, since $[0]=0$. Let $[A] \neq 0$. By 2.18, $\sim([A]\in\mathrm{On})$ and so $(\gamma)(\gamma \in^* [A] \cdot \equiv \cdot \gamma \in [A])$ since $[A] \subseteqq \mathrm{On}$. Hence, by 2.3, $[A] \subseteqq [[A]]$. Conversely assume that $\gamma \in [[A]]$. There is a ξ such that $[\gamma]=[\xi]\cdot\xi \in^* [A]$. Hence $\xi \in [A]$ and there is an η such that $[\xi]=[\eta]\cdot\eta\in^* A$. Then $[\gamma]=[\eta]\cdot \eta \in^* A$ and so $\gamma \in [A]$. Hence $[[A]] \subseteqq [A]$.

2.20. $$(\alpha)([\alpha] \mathrel{\overline{\in}} \bar{A} \cdot \equiv \cdot \alpha \in \bar{A}).$$

For any $\bar{A}$, there is a B such that $\bar{A}=[B]\cdot B \subseteqq \mathrm{On}$. By the definition of $\overline{\in}$, it holds that $[\alpha] \mathrel{\overline{\in}} \bar{A} \cdot \equiv \cdot (\exists\beta)([\alpha]=[\beta]\cdot\beta\in^*[B])$. If $[B]\neq 0$, then we obtain by 2.18 that $\beta \in^* [B] \cdot \equiv \cdot \beta \in [B]$. Hence it holds that $[\alpha] \mathrel{\overline{\in}} \bar{A} \cdot \equiv \cdot (\exists\beta)([\alpha] =[\beta]\cdot\beta\in[B]) \cdot\equiv\cdot (\exists\gamma)([\alpha]=[\gamma]\cdot\gamma\in^* B) \cdot\equiv\cdot \alpha \in [B] \cdot\equiv\cdot \alpha\in\bar{A}$.

2.21. $$(\alpha)(\exists\beta)([\beta]=[\alpha]\cdot\bar{A}).$$

By $\overline{\mathfrak{Cls}}(\bar{A})$, there is a B such that $\bar{A}=[B]\cdot B\subseteqq \mathrm{On}\cdot(\alpha)(\exists\beta)([\beta]=[\alpha]\cdot B)$. If there is no γ such that $\gamma\in^* B$, then $[B]=0$ and hence the theorem is clear. So we may assume that there is a γ such that $\gamma\in^* B$. It holds that $\gamma \in B$. Assume that $B \in \mathrm{On}$. Then there is a δ such that $[\delta]=[B]\cdot B$. $\gamma\in[B]\cdot B$ and so $[\beta]\neq 0$. Hence, by 2.18, $\mathfrak{Pr}([\beta])$. On the other hand, $\mathfrak{M}([B]\cdot B)$ and so $\mathfrak{M}([\beta])$ which is a contradiction. Hence $\sim(B\in\mathrm{On})$ and so $(\eta)(\eta\in^* B \cdot \equiv\cdot \eta\in B)$. Take an α arbitrarily. There is a β such that $[\beta]=[\alpha]\cdot B$. Let $\xi\in[\beta]$. Then $\xi\in[\alpha]\cdot\xi\in B$. So $\xi\in^* B$ and hence $\xi\in[B]$. So we obtain $[\beta]\leqq[\alpha]\cdot[B]$. Conversely assume that $\xi\in[\alpha]\cdot[B]$. There is an η such that $[\xi]=[\eta]\cdot\eta\in^* B$. It holds that $\eta\in[\alpha]\cdot\eta\in B$ and hence $\eta\in[\beta]$. So $[\alpha]\cdot[B]\subseteqq[\beta]$. Hence $[\beta]=[\alpha]\cdot[B]$ and we obtain that $(\alpha)(\exists\beta)([\beta]=[\alpha]\cdot\bar{A})$.

2.22. $$\overline{\mathfrak{Cls}}(E).$$

We prove that $E=[E]$. Assume that $E\in\mathrm{On}$. Set $\gamma=E$ and $\delta=\langle\gamma J_1`\langle\gamma\gamma\rangle\rangle^*$. $\gamma\in^* J_1`\langle\gamma\gamma\rangle$ and hence $\gamma\in[J_1`\langle\gamma\gamma\rangle]$. Hence $\delta\in E$ and so $\delta<\gamma$. On the other hand, $\gamma<\delta$ since $\delta=\langle\gamma J_1`\langle\gamma\gamma\rangle\rangle^*$, which is a contradiction. Hence $\sim(E\in\mathrm{On})$.

Since $E \subseteqq \mathrm{On}$, it holds that $(\xi)(\xi \in^* E \cdot \equiv \cdot \xi \in E)$. Hence we obtain that $\beta \in E \cdot \equiv \cdot (\exists \gamma)([\beta] = [\gamma] \cdot \gamma \in E) \cdot \equiv \cdot (\exists \gamma)([\beta] = [\gamma] \cdot \gamma \in^* E) \cdot \equiv \cdot \beta \in [E]$. Hence $E = [E]$. By 2.15, $(\alpha)(\exists \beta)([\beta] = [\alpha] \cdot E)$ and so we obtain that $\mathfrak{Cls}(E)$.

2.23. $$\overline{\mathfrak{Cls}}(\bar{A} - \bar{B}).$$

Let $\bar{A} = [A]$ and $\bar{B} = [B]$. If $\bar{A} - \bar{B} = 0$, then the theorem holds since $\overline{\mathfrak{Cls}}(0)$. Let $\bar{A} - \bar{B} \neq 0$. There is a γ such that $\gamma \in [A] - [B]$. Assume that $\bar{A} - \bar{B} \in \mathrm{On}$. Then, by 2.16, there is a ξ such that $[\gamma] = [\xi] \cdot \bar{A} - \bar{B} < \xi$. It holds that $\xi \in [A] - [B]$ and hence $(\exists \xi)(\bar{A} - \bar{B} \in \xi \cdot \xi \in \bar{A} - \bar{B})$ which is a contradiction. Hence $\sim(\bar{A} - \bar{B} \in \mathrm{On})$. So it is obtained that $\alpha \in [\bar{A} - \bar{B}] \cdot \equiv \cdot (\exists \beta)([\alpha] = [\beta] \cdot \beta \in^* \bar{A} - \bar{B}) \cdot \equiv \cdot (\exists \beta)([\alpha] = [\beta] \cdot \beta \in \bar{A} - \bar{B}) \cdot \equiv \cdot \alpha \in \bar{A} - \bar{B}$ for any α and hence $\bar{A} - \bar{B} = [\bar{A} - \bar{B}]$. Take an α arbitrarily. For suitable β, γ and δ it holds that $[\alpha] \cdot (\bar{A} - \bar{B}) = [\alpha] \cdot \bar{A} - [\alpha] \cdot \bar{B} = [\beta] - [\gamma] = [\delta]$. So we obtain that $\overline{\mathfrak{Cls}}(\bar{A} - \bar{B})$.

2.24. $$\overline{\mathfrak{Cls}}(\bar{A} \cdot \bar{B})$$

By 2.23, it is clear.

2.25. $$\overline{\mathfrak{Cls}}(Q_i``\bar{A}) \quad \text{for } i = 4, \cdots, 8.$$

If $Q_i``\bar{A} = 0$, then it is clear. Let $Q_i``\bar{A} \neq 0$. Then there is an α such that $\alpha \in Q_i``\bar{A}$. Assume that $Q_i``\bar{A} \in \mathrm{On}$. There is a β such that $[\alpha] = [\beta] \cdot Q_i``\bar{A} < \beta$. Since $[\alpha] = [\beta]$ and $\alpha \in Q_i``\bar{A}$, it holds that $\beta \in Q_i``\bar{A}$, which is a contradiction. Hence $\sim(Q_i``\bar{A} \in \mathrm{On})$. So we may obtain that $Q_i``\bar{A} = [Q_i``\bar{A}]$. Hence the proof is completed, provided that $(\alpha)(\exists \beta)[\beta] = [\alpha] \cdot Q_i``\bar{A})$ is proved. Take an α arbitrarily. Let B be a class defined as follows: $\langle \eta\gamma \rangle \in B \cdot \equiv : \gamma \in^* \alpha \cdot \langle \gamma\eta \rangle \in Q_i \cdot \eta \in \bar{A} \cdot \sim(\exists \xi)(\xi < \eta \langle \gamma\xi \rangle \in Q_i \cdot \xi \in \bar{A}) : B \subseteqq \mathrm{On}^2$. It is easily seen that $\mathfrak{Un}(B) \cdot \mathfrak{W}(B) \subseteqq \mathrm{On} \cdot \mathfrak{W}(\mathfrak{D}(B)) \cdot \mathfrak{D}(B) \subseteqq \mathrm{On}$. Hence, by 1.3, $\mathfrak{M}(B``\mathfrak{D}(B))$ and so $\mathfrak{M}(\mathfrak{W}(B))$. Set $u = \mathfrak{W}(B)$. There is a δ such that $u \subseteqq \delta \cdot K_0`\delta = 0$ and hence $u \subseteqq [\delta]$. Since $\overline{\mathfrak{Cls}}(\bar{A})$, there is a μ such that $[\mu] = [\delta] \cdot \bar{A}$. $u \subseteqq \bar{A}$ and so $u \subseteqq [\mu]$. Let $\xi \in [\alpha] \cdot Q_i``\bar{A}$. There are a $\gamma$ and an $\eta$ such that $[\xi] = [\gamma] \cdot \langle \eta\gamma \rangle \in B$. So $\gamma \in Q_i``[\mu]$ and hence $\xi \in Q_i``[\mu]$. So we obtain that $[\alpha] \cdot Q_i``\bar{A} \subseteqq [\alpha] \cdot Q_i``[\mu]$. On the other hand, $[\alpha] \cdot Q_i``[\mu] \subseteqq [\alpha] \cdot Q_i``[A]$ since $[\mu] \subseteqq \bar{A}$. Hence it holds that $[\alpha] \cdot Q_i``\bar{A} = [\alpha] \cdot Q_i``[\mu]$ and so, by 2.15, $[J_i`\langle \alpha\mu \rangle] = [\alpha] \cdot Q_i``\bar{A}$. Consequently we obtain that $(\alpha)(\exists \beta)([\beta] = [\alpha] \cdot Q_i``\bar{A})$ and hence $\overline{\mathfrak{Cls}}(Q_i``\bar{A})$.

§ 3. Model construction and proof of the relativised axioms.

Now we consider the model defined by the following:

1. *The classes in the model are X's such that* $\overline{\mathfrak{Cls}}(X)$.
2. *The sets in the model are X's such that* $\overline{\mathfrak{M}}(X)$.
3. *The membership relation in the model is* $\bar{\in}$.

The relativised formula of axiom A1 is designated by $\overline{\mathrm{A1}}$ and similar for any other formulas.

Group A.

$\overline{\mathbf{A1}}$. $$(\bar{x})\,\overline{\mathfrak{Cls}}\,(\bar{x})\,.$$

Let $\bar{x}=[\gamma]$. By 2.19, $\bar{x}=[[\gamma]]$. Also, by 2.16, $(\alpha)(\exists\beta)([\beta]=[\alpha]\cdot[\gamma])$. Hence $\overline{\mathfrak{Cls}}\,(\bar{x})$.

$\overline{\mathbf{A2}}$. $$(\bar{X},\,\bar{Y})(\bar{X}\,\overline{\in}\,\bar{Y}\cdot\supset\cdot\overline{\mathfrak{M}}(\bar{X}))\,.$$

By the definitions of $\overline{\in}$ and $\overline{\mathfrak{M}}(A)$, it is clear.

$\overline{\mathbf{A3}}$. $$(\bar{X},\,\bar{Y})((\bar{u})(\bar{u}\,\overline{\in}\,\bar{X}\cdot\equiv\cdot\bar{u}\,\overline{\in}\,\bar{Y})\cdot\supset\cdot\bar{X}=\bar{Y})\,.$$

Take an $x\in\bar{X}$ arbitrarily. Since $\bar{X}\subseteq\mathrm{On}$, $x\in\mathrm{On}$ holds. Hence, by 2.20, $[x]\,\overline{\in}\,\bar{X}$. $\overline{\mathfrak{M}}([x])$ and so, by the premise, $[x]\,\overline{\in}\,\bar{Y}$. Hence $x\in\bar{Y}$. Similar for the converse.

$\overline{\mathbf{A4}}$. $$(\bar{x},\,\bar{y})(\exists\bar{z})(\bar{u})(\bar{u}\,\overline{\in}\,\bar{z}\cdot\equiv\cdot\bar{u}=\bar{x}\vee\bar{u}=\bar{y})\,.$$

Take $\bar{x}$ and $\bar{y}$ arbitrarily. Let $\bar{x}=[\alpha]$ and $\bar{y}=[\beta]$. We prove that $(\bar{u})(\bar{u}\,\overline{\in}\,[\{\alpha\beta\}^*]\cdot\equiv\cdot\bar{u}=\bar{x}\vee\bar{u}=\bar{y})$. Take a $\bar{u}$ arbitrarily. Let $\bar{u}=[\gamma]$. By $\overline{\mathrm{A1}}$, $\overline{\mathfrak{Cls}}([\{\alpha\beta\}^*])$ and so, by 2.20, $\bar{u}\,\overline{\in}\,[\{\alpha\beta\}^*]\cdot\equiv\cdot\gamma\in[\{\alpha\beta\}^*]$. By 2.8, it holds that $\gamma\in[\{\alpha\beta\}^*]\cdot\equiv\cdot\bar{u}=\bar{x}\vee\bar{u}=\bar{y}$. Hence $\bar{u}\,\overline{\in}\,[\{\alpha\beta\}^*]\cdot\equiv\cdot\bar{u}=\bar{x}\vee\bar{u}=\bar{y}$. Since $\overline{\mathfrak{M}}([\{\alpha\beta\}^*])$, we obtain that $(\exists\bar{z})(\bar{u})(\bar{u}\,\overline{\in}\,\bar{z}\cdot\equiv\cdot\bar{u}=\bar{x}\vee\bar{u}=\bar{y})$.

By $\overline{\mathrm{A3}}$ and $\overline{\mathrm{A4}}$, the relativised of $\{xy\}$ exists uniquely. Let $\bar{x}=[\alpha]$ and $\bar{y}=[\beta]$. Then $[\{\alpha\beta\}^*]$ satisfies $\overline{\mathrm{A4}}$, as shown in the proof of $\overline{\mathrm{A4}}$, and hence $\overline{\{[\alpha][\beta]\}}=[\{\alpha\beta\}^*]$. Also $\overline{\langle[\alpha][\beta]\rangle}=\overline{\{\{[\alpha]\},\,\{[\alpha][\beta]\}\}}=\overline{\{[\{\alpha\}^*][\{\alpha\beta\}^*]\}}=[\{\{\alpha\}^*\{\alpha\beta\}^*\}^*]=[\langle\alpha\beta\rangle^*]$ and similar for $\overline{\langle[\alpha][\beta][\gamma]}=[\langle\alpha\beta\gamma\rangle^*]$.

Group B.

$\overline{\mathbf{B1}}$. $$(\exists,\,\bar{A})(\bar{x},\,\bar{y})(\overline{\langle\bar{x}\bar{y}\rangle}\,\overline{\in}\,\bar{A}\cdot\equiv\cdot\bar{x}\,\overline{\in}\,\bar{y})\,.$$

In virtue of 2.22, it is sufficient to prove that $(\bar{x},\,\bar{y})(\overline{\langle\bar{x}\bar{y}\rangle}\,\overline{\in}\,E\cdot\equiv\cdot\bar{x}\,\overline{\in}\,\bar{y})$. Let $\bar{x}=[\alpha]$ and $\bar{y}=[\beta]$. Then it holds that $\overline{\langle\bar{x}\bar{y}\rangle}\,\overline{\in}\,E\cdot\equiv\cdot[\langle\alpha\beta\rangle^*]\,\overline{\in}\,E\cdot\equiv\cdot\langle\alpha\beta\rangle^*\in E\cdot\equiv\cdot\alpha\in[\beta]\cdot\equiv\cdot\alpha\in\bar{y}\cdot\equiv\cdot\bar{x}\,\overline{\in}\,\bar{y}$.

$\overline{\mathbf{B2}}$. $$(\bar{A},\,\bar{B})(\exists\bar{C})(\bar{x})(\bar{x}\,\overline{\in}\,\bar{C}\cdot\equiv:\bar{x}\,\overline{\in}\,\bar{A}\cdot\bar{x}\,\overline{\in}\,\bar{B})\,.$$

Let $\bar{x}=[\alpha]$. By 2.24, $\overline{\mathfrak{Cls}}\,(\bar{A}\cdot\bar{B})$. Then it holds that $\bar{x}\,\overline{\in}\,\bar{A}\cdot\bar{B}\cdot\equiv\cdot\alpha\in\bar{A}\cdot\bar{B}\cdot\equiv:\alpha\in\bar{A}\cdot\alpha\in\bar{B}:\equiv:\bar{x}\,\overline{\in}\,\bar{A}\cdot\bar{x}\,\overline{\in}\,\bar{B}$. Hence $\overline{\mathrm{B2}}$ holds.

$\overline{\mathbf{B3}}$. $$(\bar{A})(\exists\bar{B})(\bar{x})(\bar{x}\in\bar{B}\cdot\equiv\cdot\sim(\bar{x}\in\bar{A}))\,.$$

It is easy to see that $\overline{\mathfrak{Cls}}\,(\mathrm{On})$. Hence, by 2.23, $\overline{\mathfrak{Cls}}\,(\mathrm{On}-\bar{A})$. Take an $\bar{x}$ arbitrarily. Let $\bar{x}=[\alpha]$. It holds that $\bar{x}\,\overline{\in}\,\mathrm{On}-\bar{A}\cdot\equiv\cdot\alpha\in\mathrm{On}-\bar{A}\cdot\equiv\cdot\sim(\alpha\in\bar{A})\cdot$

$\equiv \cdot \sim(\bar{x} \mathrel{\bar{\in}} \bar{A})$. So $\overline{\mathrm{B}3}$ holds.

$\overline{\mathbf{B4}}$. $$(\bar{A})(\exists \bar{B})(\bar{x})(\bar{x} \in \bar{B} \cdot \equiv \cdot (\exists \bar{y})(\overline{\langle \bar{y}\bar{x} \rangle} \mathrel{\bar{\in}} \bar{A})).$$

Let $\bar{x} = [\alpha]$. By 2.25, $\overline{\mathfrak{Cls}}(Q_5``\bar{A})$. It holds that $\bar{x} \mathrel{\bar{\in}} Q_5``\bar{A} \cdot \equiv \cdot \alpha \in Q_5``\bar{A} \cdot \equiv \cdot (\exists \beta)(\langle \alpha\beta \rangle \in Q_5 \cdot \beta \in \bar{A}) \cdot \equiv \cdot (\exists \beta, \nu)([\beta] = [\langle \nu\alpha \rangle^*] \cdot \beta \in \bar{A}) \cdot \equiv \cdot (\exists \nu)(\langle \nu\alpha \rangle^* \in \bar{A}) \cdot \equiv \cdot (\exists \nu)([\langle \nu\alpha \rangle^*] \mathrel{\bar{\in}} \bar{A}) \cdot \equiv \cdot (\exists \bar{y})(\overline{\langle \bar{y}\bar{x} \rangle} \mathrel{\bar{\in}} \bar{A})$. So $\overline{\mathrm{B}4}$ holds.

$\overline{\mathbf{B5}}$. $$(\bar{A})(\exists \bar{B})(\bar{x}, \bar{y})(\overline{\langle \bar{y}\bar{x} \rangle} \mathrel{\bar{\in}} \bar{B} \cdot \equiv \cdot \bar{x} \mathrel{\bar{\in}} \bar{A}).$$

By 2.25, $\overline{\mathfrak{Cls}}(Q_4``\bar{A})$. Let $\bar{x} = [\alpha]$ and $\bar{y} = [\beta]$. Then it holds that $\overline{\langle \bar{y}\bar{x} \rangle} \mathrel{\bar{\in}} Q_4``\bar{A} \cdot \equiv \cdot [\langle \beta\alpha \rangle^*] \mathrel{\bar{\in}} Q_4``\bar{A} \cdot \equiv \cdot \langle \beta\alpha \rangle^* \in Q_4``\bar{A} \cdot \equiv \cdot (\exists \gamma)(\langle \langle \beta\alpha \rangle^* \gamma \rangle \in Q_4 \cdot \gamma \in \bar{A}) \cdot \equiv \cdot (\exists \gamma)((\exists \mu)([\langle \beta\alpha \rangle^*] = [\langle \mu\gamma \rangle^*]) \cdot \gamma \in \bar{A}) \cdot \equiv \cdot (\exists \gamma)([\alpha] = [\gamma] \cdot \gamma \in \bar{A}) \cdot \equiv \cdot \alpha \in \bar{A} \cdot \equiv \cdot \bar{x} \mathrel{\bar{\in}} \bar{A}$. So $\overline{\mathrm{B}5}$ holds.

$\overline{\mathbf{B6}}$. $$(\bar{A})(\exists \bar{B})(\bar{x}, \bar{y})(\overline{\langle \bar{x}\bar{y} \rangle} \mathrel{\bar{\in}} \bar{B} \cdot \equiv \cdot \overline{\langle \bar{y}\bar{x} \rangle} \mathrel{\bar{\in}} \bar{A}).$$

By 2.25, $\overline{\mathfrak{Cls}}(Q_6``\bar{A})$. Let $\bar{x} = [\alpha]$ and $\bar{y} = [\beta]$. Then it holds that $\overline{\langle \bar{x}\bar{y} \rangle} \mathrel{\bar{\in}} Q_6``\bar{A} \cdot \equiv \cdot \langle \alpha\beta \rangle^* \in Q_6``\bar{A} \cdot \equiv \cdot (\exists \gamma)(\langle \langle \alpha\beta \rangle^* \gamma \rangle \in Q_6 \cdot \gamma \in \bar{A}) \cdot \equiv \cdot (\exists \gamma)([\gamma] = [\langle \beta\alpha \rangle^*] \cdot \gamma \in \bar{A}) \cdot \equiv \cdot \langle \beta\alpha \rangle^* \in \bar{A} \cdot \equiv \cdot \overline{\langle \bar{y}\bar{x} \rangle} \mathrel{\bar{\in}} \bar{A}$. So $\overline{\mathrm{B}6}$ holds.

$\overline{\mathbf{B7}}$. $$(\bar{A})(\exists \bar{B})(\bar{x}, \bar{y}, \bar{z})(\overline{\langle \bar{x}\bar{y}\bar{z} \rangle} \mathrel{\bar{\in}} \bar{B} \cdot \equiv \cdot \overline{\langle \bar{y}\bar{z}\bar{x} \rangle} \mathrel{\bar{\in}} \bar{A}).$$

By 2.25, $\overline{\mathfrak{Cls}}(Q_7``\bar{A})$. Let $\bar{x} = [\alpha]$, $\bar{y} = [\beta]$, and $\bar{z} = [\gamma]$. Then it holds that $\overline{\langle \bar{x}\bar{y}\bar{z} \rangle} \mathrel{\bar{\in}} Q_7``\bar{A} \cdot \equiv \cdot \langle \alpha\beta\gamma \rangle^* \in Q_7``\bar{A} \cdot \equiv \cdot (\exists \delta)(\langle \langle \alpha\beta\gamma \rangle^* \delta \rangle \in Q_7 \cdot \delta \in \bar{A}) \cdot \equiv \cdot \langle \beta\gamma\alpha \rangle^* \in \bar{A} \cdot \equiv \cdot \overline{\langle \bar{y}\bar{z}\bar{x} \rangle} \in \bar{A}$. So $\overline{\mathrm{B}7}$ holds

$\overline{\mathbf{B8}}$. $$(\bar{A})(\exists \bar{B})(\bar{x}, \bar{y}, \bar{z})(\overline{\langle \bar{x}\bar{y}\bar{z} \rangle} \mathrel{\bar{\in}} \bar{B} \cdot \equiv \cdot \langle \bar{x}\bar{y}\bar{z} \rangle \mathrel{\bar{\in}} \bar{A}).$$

It is similar for $\overline{\mathrm{B}7}$, taking $Q_8``\bar{A}$ instead of $Q_7``\bar{A}$.

Group C.

$\overline{\mathbf{C1}}$. $$(\exists \bar{a})(\sim \overline{\mathfrak{Em}}(\bar{a}) \cdot (\bar{x})(\bar{x} \mathrel{\bar{\in}} \bar{a} \cdot \supset \cdot (\exists \bar{y})(\bar{y} \mathrel{\bar{\in}} \bar{a} \cdot \bar{x} \mathrel{\bar{\sqsubset}} \bar{y}))).$$

It holds that $\omega = J_0`\langle \omega 0 \rangle$ and so $K_0`\omega = 0$. It is easily seen that $[0] \mathrel{\bar{\in}} [\omega]$ and hence $(\exists \bar{u})(\bar{u} \mathrel{\bar{\in}} [\omega])$. Consequently $\sim \overline{\mathfrak{Em}}([\omega])$. Take an $\bar{x} \mathrel{\bar{\in}} [\omega]$ arbitrarily. Let $\bar{x} = [\alpha]$. Then $\alpha \in [\omega]$ and so there is the least k such that $[\alpha] = [k] \cdot k \in \omega$. There is an n such that $k < n < \omega \cdot K_0`n = 0$. Clearly $[n] \mathrel{\bar{\in}} [\omega]$. We prove that $\bar{x} \mathrel{\bar{\sqsubset}} [n]$. Take an arbitrary $\bar{u}$ and set $\bar{u} = [\xi]$. Let $\bar{u} \mathrel{\bar{\in}} \bar{x}$. Then $\xi \in \bar{x}$ and so $\xi \in [k]$. Hence there is an m such that $[\xi] = [m] \cdot m \in^* k \cdot m < k$ and so $m < n$. Since $K_0`n = 0$, $m \in^* n$ and so $\xi \in [n]$. Hence $\bar{u} \mathrel{\bar{\in}} [n]$. So we obtain that $(\bar{u})(\bar{u} \mathrel{\bar{\in}} \bar{x} \cdot \supset \cdot \bar{u} \mathrel{\bar{\in}} [n])$. Assume that $\bar{x} = [n]$. Then $[k] = [n]$. Since $k \in [n]$, $k \in [k]$ holds and so there is an l such that $[k] = [l] \cdot l \in^* k$. Hence there is an $l < k$ such that $[\alpha] = [l] \cdot l \in \omega$, which contradicts to the definition of k. So $\bar{x} \neq [n]$ and consequently $\bar{x} \mathrel{\bar{\sqsubset}} [n]$. Then we obtain that $(\bar{x})(\bar{x} \mathrel{\bar{\in}} [\omega] \cdot \supset \cdot (\exists \bar{y})(\bar{y} \mathrel{\bar{\in}} [\omega] \cdot \bar{x} \mathrel{\bar{\sqsubset}} \bar{y}))$. It is clear that $\overline{\mathfrak{M}}([\omega])$. So we obtain $\overline{\mathrm{C}1}$.

$\overline{\text{C2}}$. $$(\bar{x})(\exists \bar{y})(\bar{u}, \bar{v})(\bar{u} \,\bar{\in}\, \bar{v} \cdot \bar{v} \,\bar{\in}\, \bar{x} : \supset \cdot \bar{u} \,\bar{\in}\, \bar{y}).$$

Let $\bar{x}=[\alpha]$. There is a μ such $\alpha<\mu \cdot K_0{}^{\prime}\mu=0$. Take $\bar{u}$ and $\bar{v}$ arbitrarily. Let $\bar{u}=[\gamma]$ and $\bar{v}=[\beta]$. Assume that $\bar{u} \,\bar{\in}\, \bar{v} \cdot \bar{v} \,\bar{\in}\, \bar{x}$. Then $\gamma \subseteqq [\beta]$ and $\beta \in [\alpha]$. Hence, for suitable ζ and ξ, it holds that $[\gamma]=[\zeta] \cdot \zeta \in^* \beta \cdot [\beta]=[\xi] \cdot \xi \in^* \alpha$. Since $\zeta \in^* \beta \cdot [\beta]=[\xi]$, there is an η such that $[\zeta]=[\eta] \cdot \eta \in^* \xi$. So $[\gamma]=[\eta] \cdot \eta \in^* \xi \cdot \xi \in^* \alpha$. Hence $\eta<\alpha$ and so $\eta<\mu$. Since $K_0{}^{\prime}\mu=0$, $\eta \in^* \mu$ holds. Hence $\gamma \in [\mu]$ and so $\bar{u} \,\bar{\in}\, [\mu]$. So we obtain that $(\bar{u}, \bar{v})(\bar{u} \,\bar{\in}\, \bar{v} \cdot \bar{v} \,\bar{\in}\, \bar{x} : \supset \cdot \bar{u} \,\bar{\in}\, [\mu])$. Clearly $\overline{\mathfrak{M}}([\mu])$ and so $\overline{\text{C2}}$ holds.

$\overline{\text{C3}}$. $$(\bar{x})(\exists \bar{y})(\bar{u})(\bar{u} \,\bar{\subseteqq}\, \bar{x} \cdot \supset \cdot \bar{u} \,\bar{\in}\, \bar{y}).$$

Let $\bar{x}=[\alpha]$. We define $\mathfrak{H}(\gamma)$ as follows: $\mathfrak{H}(\gamma) \cdot \equiv \cdot \sim(\exists \xi)([\gamma]=[\xi] \cdot \xi<\gamma)$. Let A and $s(\gamma)$ be two classes such that $(\gamma)(\gamma \in A \cdot \equiv : \mathfrak{H}(\gamma) \cdot [\gamma] \,\bar{\subseteqq}\, [\alpha]) \cdot A \subseteqq \mathrm{On}$ and $(\xi)(\xi \in s(\gamma) \cdot \equiv : \xi \in [\gamma] \cdot \mathfrak{H}(\xi)) \cdot s(\gamma) \subseteqq \mathrm{On}$. First we show that $(\gamma)(\gamma \in A \cdot \supset \cdot s(\gamma) \subseteqq \alpha)$. Let $\gamma \in A$. Take a $\delta \in s(\gamma)$ arbitrarily. By the definition of $s(\gamma)$, $\delta \in [\gamma]$ and so $\delta \in [\alpha]$ since $[\gamma] \,\bar{\subseteqq}\, [\alpha]$. Hence there is an η such that $[\delta]=[\eta] \cdot \eta \in^* \alpha$. By $\mathfrak{H}(\delta)$, $\delta \leqq \eta$ and so $\delta<\alpha$. Hence $s(\gamma) \subseteqq \alpha$ and we obtain that $(\gamma)(\gamma \in A \cdot \supset \cdot s(\gamma) \subseteqq \alpha)$. Hence $(\gamma)(\gamma \in A \cdot \supset \cdot \mathfrak{M}(s(\gamma)))$. Let F be a class defined as follows: $(\gamma, u)(\langle \gamma u\rangle \in F \cdot \equiv : \gamma \in A \cdot u=s(\gamma)) \cdot F \subseteqq \mathrm{On} \times V$. From the above, it holds that $(\gamma)(\gamma \in A \cdot \supset \cdot (\exists x)(\langle \gamma x\rangle \in F))$ and hence $A \subseteqq \mathfrak{W}(F)$. From the definition of F, it is clear that $\mathfrak{W}(F) \subseteqq A$ and hence $A=\mathfrak{W}(F)$. Now assume that $s(\gamma)=s(\delta)$ for any $\gamma, \delta \in A$. Let $\xi \in [\gamma]$ and η be the least such that $[\xi]=[\eta]$. Then $\mathfrak{H}(\eta)$ and $\eta \in [\gamma]$. So $\eta \in s(\gamma)$ and, by the assumption, $\eta \in s(\delta)$ and so $\eta \in [\delta]$. Hence $\xi \in [\delta]$. Then it holds that $[\gamma] \subseteqq [\delta]$. Similar for the converse and so it holds that $[\gamma]=[\delta]$. Hence by $\mathfrak{H}(\eta)$ and $\mathfrak{H}(\delta)$ we obtain that $\gamma=\delta$. Therefore $\mathfrak{Un}(F)$. It is obvious that $\mathfrak{W}(F) \subseteqq \mathrm{On}$ and $\mathfrak{D}(F) \subseteqq \mathfrak{P}(\alpha)$. Hence, by the axiom C4″, $\mathfrak{M}(F``\mathfrak{P}(\alpha))$. Since $A=F``\mathfrak{D}(F) \subseteqq F``\mathfrak{P}(\alpha)$, $\mathfrak{M}(A)$ and so there is a β such that $A \subseteqq \beta$. Let β_0 be the least β such that $A \subseteqq \beta \cdot K_0{}^{\prime}\beta=0$. Set $\bar{y}=[\beta_0]$. Take a $\bar{u}$ such that $\bar{u} \,\bar{\subseteqq}\, \bar{x}$ arbitrarily. Let $\bar{u}=[\gamma]$. Let δ be the least ξ such that $[\gamma]=[\xi]$. Then $\mathfrak{H}(\delta)$ and $[\delta] \,\bar{\subseteqq}\, [\alpha]$. So $\delta \in A$ and hence $\delta \in \beta_0$. Since $K_0{}^{\prime}\beta_0=0$, $\delta \in^* \beta_0$ and so $\gamma \in [\beta_0]$. Hence $\bar{u} \,\bar{\in}\, \bar{y}$. So we obtain that $(\bar{u})(\bar{u} \,\bar{\subseteqq}\, \bar{x} \cdot \supset \cdot \bar{u} \,\bar{\in}\, \bar{y})$.

$\overline{\text{C4}}$. $$(\bar{x}, \bar{A})(\overline{\mathfrak{Un}}(\bar{A}) \cdot \supset \cdot (\exists \bar{y})(\bar{u})(\bar{u} \,\bar{\in}\, \bar{y} \cdot \equiv \cdot (\exists \bar{v})(\bar{v} \,\bar{\in}\, \bar{x} \cdot \overline{\langle \bar{u}\bar{v}\rangle} \,\bar{\in}\, \bar{A}))).$$

Set $B=Q_5``Q_6``(\bar{A} \cdot Q_4``\bar{x})$. It holds that $\mathfrak{Cls}(B)$. Let $\bar{u}=[\mu]$. Then it holds that $\bar{u} \,\bar{\in}\, B \cdot \equiv \cdot \mu \in B \cdot \equiv \cdot (\exists \beta)(\langle \mu\beta\rangle \in Q_5 \cdot \beta \in Q_6``(\bar{A} \cdot Q_4``\bar{x})) \cdot \equiv \cdot (\exists \beta)((\exists \nu)([\beta]=[\langle \nu\mu\rangle^*] \cdot (\exists \delta)(\langle \beta\delta\rangle \in Q_6 \cdot \delta \in \bar{A} \cdot Q_4``\bar{x})) \cdot \equiv \cdot (\exists \nu, \delta)(\langle\langle \nu\mu\rangle^*\delta\rangle \in Q_6 \cdot \delta \in \bar{A} \cdot Q_4``\bar{x}) \cdot \equiv \cdot (\exists \nu)(\langle \mu\nu\rangle^* \in \bar{A} \cdot Q_4``\bar{x}) \cdot \equiv \cdot (\exists \nu)(\langle \mu\nu\rangle^* \in \bar{A} \cdot (\exists \xi)(\langle\langle \mu\nu\rangle^*\xi\rangle \in Q_4 \cdot \xi \in \bar{x})) \cdot \equiv \cdot (\exists \nu)(\langle \mu\nu\rangle^* \in \bar{A} \cdot (\exists \xi, \eta)([\langle \mu\nu\rangle^*]=[\langle \eta\xi\rangle^*] \cdot \xi \in \bar{x})) \cdot \equiv \cdot (\exists \nu)(\nu \in \bar{x} \cdot \langle \mu\nu\rangle^* \in \bar{A})$. Hence we obtain that $(\bar{u})(\bar{u} \,\bar{\in}\, B \cdot \equiv \cdot (\exists \bar{v})(\bar{v} \,\bar{\in}\, \bar{x} \cdot \overline{\langle \bar{u}\bar{v}\rangle} \,\bar{\in}\, \bar{A}))$. Therefore $\overline{\text{C4}}$ holds, provided $\overline{\mathfrak{Un}}(\bar{A}) \cdot \supset \cdot \overline{\mathfrak{M}}(B)$ is proved. Let $\overline{\mathfrak{Un}}(\bar{A})$ and $\bar{x}=[\alpha]$. We define F as follows:

$(\mu, \nu)(\langle\mu\nu\rangle \in F \cdot \equiv : \langle\mu\nu\rangle^* \in \bar{A} \cdot \nu \in^* \alpha \cdot (\xi)(\xi < \mu \cdot \supset \cdot [\xi] \neq [\mu] \cdot F \subseteqq \mathrm{On}^2))$. In virtue of $\overline{\mathfrak{Un}}(\bar{A})$, it holds that $(\alpha, \beta, \gamma)(\langle\beta\alpha\rangle^* \in \bar{A} \cdot \langle\gamma\alpha\rangle^* \in \bar{A} : \supset \cdot [\beta] = [\gamma])$. Hence $\mathfrak{Un}(F)$ holds. Since $\mathfrak{W}(F) \subseteqq \mathrm{On}$ and $\mathfrak{D}(F) \subseteqq \alpha$, $\mathfrak{M}(\mathfrak{W}(F))$ holds and so there is a β such that $\mathfrak{W}(F) \subseteqq \beta$. Take a $\mu \in B$ arbitrarily. By the above shown equivalence, there is a ν such that $\nu \in [\alpha] \cdot \langle\mu\nu\rangle^* \in \bar{A}$. Hence there is a ρ such that $[\nu] = [\rho] \cdot \rho \in^* \alpha$. Let σ be the least ξ such that $[\mu] = [\xi]$. Then $\langle\sigma\rho\rangle^* \in \bar{A}$ and so $\langle\sigma\rho\rangle \in F$. Hence $\sigma \in [\beta]$ and so $\mu \in [\beta]$. Therefore $B \subseteqq [\beta]$. Since $\overline{\mathfrak{Cls}}(B)$, we obtain, by 2.21, that $(\exists\gamma)([\gamma] = [\beta] \cdot B)$ and hence $(\exists\gamma)([\gamma] = B)$. Consequently $\overline{\mathfrak{M}}(B)$.

Lastly we prove

$\overline{\mathbf{D}}$. $$\sim \overline{\mathfrak{Em}}(\bar{A}) \cdot \supset \cdot (\exists\bar{x})(\bar{x} \mathrel{\bar{\in}} \bar{A} \cdot \overline{\mathfrak{Ex}}(\bar{x}, \bar{A})) .$$

Let $\sim \overline{\mathfrak{Em}}(\bar{A})$. Then there is a $\bar{u}$ such that $\bar{u} \mathrel{\bar{\in}} \bar{A}$. Hence there is a ξ such that $\xi \in \bar{A}$. We define α to be the least ξ such that $\xi \in \bar{A}$. Let $\bar{x} = [\alpha]$. Then $\bar{x} \mathrel{\bar{\in}} \bar{A}$. Now assume that there is a $\bar{u}$ such that $\bar{u} \mathrel{\bar{\in}} \bar{x} \cdot \bar{u} \mathrel{\bar{\in}} \bar{A}$. Let $\bar{u} = [\beta]$. Then $\beta \in [\alpha]$. Hence there is a γ such that $[\beta] = [\gamma] \cdot \gamma \in^* \alpha$. $\bar{u} = [\gamma]$ and hence $\gamma \in \bar{A} \cdot \gamma < \alpha$ which contradicts to the definition of α. Hence $(\bar{u}) \sim (\bar{u} \mathrel{\bar{\in}} \bar{x} \cdot \bar{u} \mathrel{\bar{\in}} \bar{A})$, i.e., $\overline{\mathfrak{Ex}}(\bar{x}, \bar{A})$. Therefore we obtain that $(\exists\bar{x})(\bar{x} \mathrel{\bar{\in}} \bar{A} \cdot \overline{\mathfrak{Ex}}(\bar{x}, \bar{A}))$.

Shizuoka University

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