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Closed hypersurfaces with constant mean curvature in a Riemannian manifold

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It has been proved by H. Liebmann [3] and W. Süss [4] that the only convex closed hypersurface with constant mean curvature is a sphere. To prove this theorem we need integral formulas of Minkowski.

Prof. Y. Katsurada [1], [2] derived integral formulas of Minkowski type which are valid in an Einstein space and proved the following generalisation of the theorem of Liebmann-Süss.

THEOREM. *Let M be an $(m+1)$-dimensional orientable Einstein space and S a closed orientable hypersurface in M whose first mean curvature is constant. We suppose that M admits a one-parameter group of conformal transformations such that the inner product α of the generating vector v^h and the normal C^h to the hypersurface does not change the sign (and $\neq 0$) on S. Then every point of S is umbilical.*

The main purpose of the present paper is to derive three integral formulas which are valid in a general Riemannian manifold and to generalise Katsurada's theorem to the case of general Riemannian manifolds admitting a one-parameter group of homothetic transformations.

§ 0. Preliminaries.

We consider an orientable $(m+1)$-dimensional Riemannian manifold M with positive definite metric and denote by g_{ji}, ∇_j, $K_{kji}{}^h$, $K_{ji}=K_{kji}{}^k$, the fundamental metric tensor, the covariant differentiation with respect to the Riemannian connection, the curvature tensor, and the Ricci tensor of M respectively, where and in the sequel the indices $h, i, j, k, \cdots$ run over the range $1, 2, \cdots, m, m+1$.

We assume that there is given an orientable hypersurface S whose local expression is

$$\xi^h=\xi^h(\eta^a)\,, \tag{0.1}$$

where ξ^h are local coordinates in M and η^a are local parameters on the hypersurface S, where and in the sequel the indices $a, b, c, d, \cdots$ run over the range $\dot{1}, \dot{2}, \cdots, \dot{m}$.

If we put

$$B_b{}^h = \partial_b \xi^h, \qquad \partial_b = \partial/\eta^b, \tag{0.2}$$

then the first fundamental tensor of S is given by

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i. \tag{0.3}$$

We assume that $B_b{}^h$ ($b = \dot{1}, \dot{2}, \cdots, \dot{m}$) give the positive direction in S and choose the unit normal C^h to S in such a way that $B_b{}^h$, C^h give the positive direction in M.

Denoting by ∇_c the van der Waerden-Bortolotti covariant differentiation along the S, we can write equations of Gauss and Weingarten in the form

$$\nabla_c B_b{}^h = h_{cb} C^h, \tag{0.4}$$

$$\nabla_c C^h = -h_c{}^a B_a{}^h \tag{0.5}$$

respectively, where h_{cb} is the second fundamental tensor of S and $h_c{}^a = h_{cb} g^{ba}$.

If we denote by $k_1, k_2, \cdots, k_m$ the principal curvatures of S, that is, the roots of the characteristic equation

$$|h_{cb} - k g_{cb}| = 0, \tag{0.6}$$

then the first mean curvature H_1 and the second mean curvature H_2 of S are respectively given by

$$m H_1 = \sum_a k_a = h_c{}^c \tag{0.7}$$

and

$$\binom{m}{2} H_2 = \sum_{c<b} k_c k_b = \frac{1}{2}(h_c{}^b h_b{}^c - h_c{}^b h_b{}^c). \tag{0.8}$$

Now, the equations of Gauss and those of Codazzi are respectively written as

$$K_{kjih} B_d{}^k B_c{}^j B_b{}^i B_a{}^h = K_{dcba} - (h_{da} h_{cb} - h_{ca} h_{db}) \tag{0.9}$$

and

$$K_{kjih} B_d{}^k B_c{}^j B_b{}^i C^h = \nabla_d h_{cb} - \nabla_c h_{db}. \tag{0.10}$$

Transvecting g^{cb} to the equations of Codazzi and remembering $g^{cb} B_c{}^j B_b{}^i = g^{ji} - C^j C^i$, we find

$$K_{kh} B_d{}^k C^h = \nabla_d h_c{}^c - \nabla_c h_d{}^c. \tag{0.11}$$

We now assume that there is given a global vector field $v^h(\xi)$ in M and denote by $\mathcal{L}$ the Lie differentiation with respect to v^h. (See [**5**].) The vector field v^h is said to be conformal, homothetic or Killing when it satisfies

$$\mathcal{L} g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji},$$

$$\mathcal{L} g_{ji} = 2c g_{ji},$$

or

$$\mathcal{L} g_{ji} = 0$$

respectively, where ρ is a function and c is a constant. When v^h is conformal, it satisfies

(0.12) $$\mathcal{L}\{{}^{h}_{ji}\}=\nabla_j\nabla_i v^h+K_{kji}{}^{h}v^k=\delta^h_j\rho_i+\delta^h_i\rho_j-\rho^h g_{ji}\,,$$

where $\{{}^{h}_{ji}\}$ are Christoffel symbols and $\rho_i=\nabla_i\rho$, $\rho^h=\rho_i g^{ih}$. When v^h is homothetic, it satisfies

(0.13) $$\mathcal{L}\{{}^{h}_{ji}\}=\nabla_j\nabla_i v^h+K_{kji}{}^{h}v^k=0$$

and thus it defines an infinitesimal affine collineation.

On the hypersurface S we can put

(0.14) $$v^h=B_a{}^h v^a+\alpha C^h\,.$$

Since we have

$$B_c{}^jB_b{}^i\mathcal{L}g_{ji}=B_c{}^jB_b{}^i(\nabla_j v_i+\nabla_i v_j)$$
$$=\nabla_c v_b+\nabla_b v_c-2\alpha h_{cb}\,,$$

denoting also by $\mathcal{L}$ the Lie differentiation with respect to v^a in S, we have

(0.15) $$B_c{}^jB_b{}^i(\mathcal{L}g_{ji})=\mathcal{L}g_{cb}-2\alpha h_{cb}\,.$$

Transvecting v^d to (0.11), we find

$$K_{kh}B_d{}^k v^d C^h=v^d\nabla_d h_c{}^c-v^d(\nabla_c h_d{}^c)\,,$$
$$K_{ji}(v^j-\alpha C^j)C^i=v^d\nabla_d h_c{}^c-\nabla_c(h_d{}^c v^d)+h^{cb}\nabla_c v_b$$

and consequently

(0.16) $$K_{ji}v^jC^i-\alpha K_{ji}C^jC^i=v^d\nabla_d h_c{}^c-\nabla_c(h_d{}^c v^d)+\frac{1}{2}h^{cb}(\mathcal{L}g_{cb})$$

or

(0.17) $$K_{ji}v^jC^i-\alpha K_{ji}C^jC^i=v^d\nabla_d h_c{}^c-\nabla_c(h_d{}^c v^d)+\alpha h_c{}^b h_b{}^c$$
$$+\frac{1}{2}h^{cb}B_c{}^jB_b{}^i(\mathcal{L}g_{ji})$$

by virtue of (0.15).

§1. The first integral formula.

We have

$$v_b=B_b{}^i v_i$$

from which, by covariant differentiation along S,

$$\nabla_c v_b=\alpha h_{cb}+B_c{}^jB_b{}^i(\nabla_j v_i)\,.$$

Transvecting g^{cb} to this, we get

$$g^{cb}\nabla_c v_b=\alpha h_c{}^c+\frac{1}{2}g^{cb}B_c{}^jB_b{}^i(\mathcal{L}g_{ji})$$

or

$$g^{cb}\nabla_c v_b = m\alpha H_1 + \frac{1}{2} g^{cb} B_c{}^j B_b{}^i (\mathcal{L} g_{ji}) .$$

Thus, assuming S to be compact, we get the integral formula

$$(1.1) \qquad \int_S m\alpha H_1 \, dS + \frac{1}{2} \int_S g^{cb} B_c{}^j B_b{}^i (\mathcal{L} g_{ji}) dS = 0 ,$$

where dS denotes the surface element of S. (See [6].)

If the vector field v^h is conformal, that is, if $\mathcal{L} g_{ji} = 2\rho g_{ji}$, we have, from the formula above,

$$(1.2) \qquad \int_S \alpha H_1 \, dS + \int_S \rho \, dS = 0 .$$

§ 2. The second integral formula.

If we put

$$(2.1) \qquad w_b = h_b{}^a v_a ,$$

we have, by covariant differentiation along S,

$$\nabla_c w_b = \nabla_c (h_{db} v^d) .$$

Transvecting g^{cb} to this, we get

$$g^{cb}\nabla_c w_b = \nabla_c (h_d{}^c v^d) ,$$

from which, taking account of (0.17)

$$(2.2) \qquad g^{cb}\nabla_c w_b = v^d \nabla_d h_c{}^c + \alpha h_c{}^b h_b{}^c - K_{ji} v^j C^i + \alpha K_{ji} C^j C^i + \frac{1}{2} h^{cb} B_c{}^j B_c{}^i (\mathcal{L} g_{ji}) .$$

On the other hand, we have, from (0.7) and (0.8),

$$h_c{}^c = mH_1 , \qquad h_c{}^b h_b{}^c = m^2 H_1{}^2 - m(m-1) H_2 ,$$

and consequently, we have, from (2.2),

$$g^{cb}\nabla_c w_b = m v^d \nabla_d H_1 + m\alpha \{ m H_1^2 - (m-1) H_2 \}$$
$$- K_{ji} v^j C^i + \alpha K_{ji} C^j C^i + \frac{1}{2} h^{cb} B_c{}^j B_b{}^i (\mathcal{L} g_{ji}) .$$

Thus, assuming S to be compact, we get the second integral formula

$$(2.3) \qquad \int_S [m v^d \nabla_d H_1 + m\alpha \{ m H_1^2 - (m-1) H_2 \}$$
$$- K_{ji} v^j C^i + \alpha K_{ji} C^j C^i + \frac{1}{2} h^{cb} B_c{}^j B_b{}^i (\mathcal{L} g_{ji})] \, dS = 0 .$$

If the vector field v^h is conformal, then we get from (2.3)

(2.4) $$\int_S [mv^d\nabla_d H_1+m\rho H_1+m\alpha\{mH_1^2-(m-1)H_2\} -K_{ji}v^jC^i+\alpha K_{ji}C^jC^i]\,dS=0\,.$$

§ 3. The third integral formula.

We have

(3.1) $$\alpha=v^hC_h\,,$$

from which, by covariant differentiation along S,

$$\nabla_b\alpha=(B_b{}^i\nabla_i v^h)C_h-h_b{}^a v_a$$

and

$$\nabla_c\nabla_b\,\alpha=h_{cb}(\nabla_j v_i)C^jC^i+B_c{}^jB_b{}^i(\nabla_j\nabla_i v^h)C_h-h_c{}^aB_b{}^i(\nabla_i v^h)B_{ah}-\nabla_c(h_b{}^a v_a)\,.$$

Transvecting g^{cb} to this, we get

$$g^{cb}\nabla_c\nabla_b\,\alpha=\frac{1}{2}h_c{}^c(\mathcal{L}g_{ji})C^jC^i+g^{cb}B_c{}^jB_b{}^i(\mathcal{L}\{{}^h_{ji}\})C_h -K_{ji}v^jC^i-\frac{1}{2}h^{cb}B_c{}^jB_b{}^i(\mathcal{L}g_{ji})-g^{cb}\nabla_c(h_b{}^a v_a)$$

by virtue of

$$\nabla_j\nabla_i\,v^h=\mathcal{L}\{{}^h_{ji}\}-K_{kji}{}^h v^k\,.$$

Thus, assuming S to be compact, we get the third integral formula

(3.3) $$\int_S\Big[\frac{1}{2}h_c{}^c(\mathcal{L}g_{ji})C^jC^i +g^{cb}B_c{}^jB_b{}^i(\mathcal{L}\{{}^h_{ji}\})C_h-K_{ji}v^jC^i-\frac{1}{2}h^{cb}B_c{}^jB_b{}^i(\mathcal{L}g_{ji})\Big]dS=0\,.$$

We now assume that the vector field v^h is conformal, then we have

$$\mathcal{L}g_{ji}=2\rho g_{ji}\,,\qquad \mathcal{L}\{{}^h_{ji}\}=\delta^h_j\rho_i+\delta^h_i\rho_j-g_{ji}\rho^h\,.$$

Thus we find from (3.3)

(3.4) $$\int_S[m\rho_iC^i+K_{ji}v^jC^i]\,dS=0\,.$$

Moreover if v^h is homothetic,we get

(3.5) $$\int_S K_{ji}v^jC^i\,dS=0\,.$$

§ 4. Integral formulas for the case H_1 = constant.

We assume in this section that the Riemannian manifold admits an infinitesimal conformal transformation v^h and the first mean curvature H_1 of the hypersurface S is constant. Then, the first integral formula (1.2) and the second integral formula (2.4) become respectively

$$H_1 \int_S \alpha \, dS + \int_S \rho \, dS = 0 \tag{4.1}$$

and

$$H_1 \int_S \rho \, dS + \int_S \alpha \{mH_1^2 - (m-1)H_2\} \, dS - \frac{1}{m} \int_S (K_{ji} v^j C^i - \alpha K_{ji} C^j C^i) \, dS = 0 \,. \tag{4.2}$$

Eliminating $\int_S \rho \, dS$ from these equations, we find

$$\int_S (m-1)\alpha(H_1^2 - H_2) \, dS - \frac{1}{m} \int_S (K_{ji} v^j C^i - \alpha K_{ji} C^j C^i) \, dS = 0 \,. \tag{4.3}$$

If the Riemannian manifold M under consideration is an Einstein space, then

$$K_{ji} = \lambda g_{ji}$$

and consequently we have from (4.3)

$$\int_S \alpha(H_1^2 - H_2) \, dS = 0 \,, \tag{4.4}$$

where

$$H_1^2 - H_2 = \frac{1}{m^2(m-1)} \sum_{a \neq b} (k_a - k_b)^2 \,. \tag{4.5}$$

Using (4.4) and (4.5), Prof. Katsurada proved the theorem mentionned in the introduction of the present paper.

§ 5. Hypersurfaces with constant first mean curvature in a Riemannian manifold admitting an infinitesimal homothetic transformation.

We assume in this section that the Riemannian manifold admits an infinitesimal homothetic transformation v^h and the first mean curvature H_1 of the hypersurface is constant. Then the first, the second and the third integral formulas become respectively

$$H_1 \int_S \alpha \, dS + c \int_S dS = 0 \tag{5.1}$$

$$cH_1 \int_S dS + \int_S \alpha \{mH_1^2 - (m-1)H_2\} \, dS + \frac{1}{m} \int_S \alpha K_{ji} C^j C^i \, dS = 0 \tag{5.2}$$

$$\int_S K_{ji} v^j C^i \, dS = 0 \,. \tag{5.3}$$

Eliminating $\int_S dS$ from (5.1) and (5.2), we find

$$\int_S \alpha\Big[(m-1)(H_1^2-H_2)+\frac{1}{m}K_{ji}C^jC^i\Big]\,dS=0\,. \tag{5.4}$$

From this we have

THEOREM 5.1. *Let M be an $(m+1)$-dimensional orientable Riemannian manifold and S a closed orientable hypersurface in M whose first mean curvature is constant. We suppose that M admits a one-parameter group of homothetic transformations such that the inner product of the generating vector v^h and the normal C^h to the hypersurface does not change the sign (and $\neq 0$) on S and that the Ricci curvature K_{ji} with respect to the normal C^h is non-negative on S. Then every point of S is umbilical and $K_{ji}C^jC^i=0$ on S.*

We assume next that the Riemannian manifold under consideration is an Einstein space: $K_{ji}=\lambda g_{ji}$. Then from (5.3) we have

$$\lambda\int_S \alpha\, dS=0\,, \tag{5.7}$$

λ being a constant.

Thus if α does not change the sign and is not identically zero on S, we must have $\lambda=0$ and consequently $K_{ji}=0$. Thus we have

THEOREM 5.2. *Let M be an $(m+1)$-dimensional orientable Einstein space and S a closed orientable hypersurface in M whose first mean curvature is constant. We suppose that M admits a one-parameter group of homothetic transformations such that the inner product of the generating vector v^h and the normal C^h to S does not change the sign and is not identically zero on S. Then the curvature scalar of the space vanishes and every point of the hypersurface is umbilical.*

If $\alpha=0$, then (1.2) becomes

$$c\int_S dS=0\,,$$

from which

$$c=0\,.$$

Thus we have

THEOREM 5.3. *Let M be an $(m+1)$-dimensional orientable Riemannian manifold and S a closed orientable hypersurface in M. If we suppose that M admits a one-parameter group of homothetic transformations such that the generating vector v^h is always tangent to M. Then the group is that of motions.*

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