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A duality theorem for the real unimodular group of second order

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Introduction.

Let G be the real special linear group of second order. G consists of all real matrices, such that,

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad-bc=1 .$$

The purpose of the present paper is to characterize G as a "dual group" of the space of its irreducible unitary representations. This space is given a law according to which the Kronecker product of any two representations is decomposed into irreducible components. This duality may be considered as an analogue of the Tannaka duality theorem for compact groups and of Pontrjagin duality theorem for locally compact abelian groups.

If A is a locally compact abelian group, there is the duality of Pontrjagin [1]. This duality is as follows. Any irreducible unitary representation of A is one-dimensional, i.e., a homomorphism of A into one-dimensional torus. This homomorphism is called a unitary character of A. In this case the Kronecker product of two irreducible representations is nothing but the ordinary product of characters as functions. Denote by $\hat{A}$ the totality of unitary characters χ, then $\hat{A}$ is an abelian group with the mentioned product. Moreover introducing the topology defined by uniform convergence on any compact set, $\hat{A}$ becomes a locally compact group.

We consider $\hat{\hat{A}}$ the group of the unitary characters on $\hat{A}$ in the same way, that is, an element of $\hat{\hat{A}}$ is a function $\tilde{\chi}(\chi)$ on $\hat{A}$, whose absolute value is unity. And we have

$$\tilde{\chi}(\chi_1)\tilde{\chi}(\chi_2)=\tilde{\chi}(\chi_1\chi_2) , \qquad \chi_1, \chi_2 \in \hat{A} .$$

This implies that $\tilde{\chi}$ commutes with the operation of product in $\hat{A}$, that is, the operation of Kronecker product of representations of A. When an element a of A is fixed, clearly the equality

$$\tilde{a}(\chi)=\chi(a)$$

defines an element of $\hat{\hat{A}}$. Hence A is imbedded in $\hat{\hat{A}}$. The Pontrjagin duality theorem affirms that A coincides with $\hat{\hat{A}}$ as a locally compact abelian group. This fact means that A is characterized as a " dual group " of the space of its irreducible unitary representations.

When a compact group K is given, Tannaka [2] gave a similar duality theorem. Consider the set X of all irreducible unitary representations of K. Let ρ, σ be two elements in X. From the general representation theory of compact groups, the Kronecker product $\rho\otimes\sigma$ is decomposed into a finite direct sum $\tau_1\oplus\tau_2\oplus\cdots\oplus\tau_m$ of irreducible representations. While an element k of K determines an operator field over X, which consists of unitary matrices $\rho(k)$ in each representation space, corresponding to the representation ρ. An important property of this operator field is that it commutes with the operation of the Kronecker product. More precisely, in above decomposition of Kronecker product, if $u\otimes v$ is decomposed into the sum $w_1\oplus w_2\oplus\cdots\oplus w_m$, then $\rho(k)u\otimes\sigma(k)v$ is decomposed into the sum $\tau_1(k)w_1\oplus\tau_2(k)w_2\oplus\cdots\oplus\tau_m(k)w_m$, where u, v, w_j are vectors in the spaces of representations ρ, σ, τ_j respectively.

Conversely, we consider an operator field $\{T(\rho)\}$ over X, such that each $T(\rho)$ is a unitary matrix in the representation space of ρ, and $(T(\rho)u)\otimes(T(\sigma)v)$ corresponds to $T(\tau_1)w_1\oplus T(\tau_2)w_2\oplus\cdots\oplus T(\tau_m)w_m$ for any u and v. Because this operator field consists of unitary operators and commutes with the operation of the Kronecker product, we may consider it as an analogue of a unitary character over X. Finally, we consider the totality $\hat{X}$ of these operator fields over X, and induce the weakest topology in $\hat{X}$ which makes all the matrix elements $\langle T(\rho)u, v\rangle$ continuous. The product TS of two elements T and S in $\hat{X}$ is defined as $(TS)(\rho)=T(\rho)S(\rho)$. With this topology and product operation, $\hat{X}$ becomes a compact group.

The duality theorem says that $\hat{X}$ is isomorphic to K under the mapping which maps k to the operator field $\{\rho(k)\}$, i.e. K is characterized as a " dual group " of the space X of its irreducible unitary representations.

For a non-compact non-abelian case, there is a study of Harish-Chandra [3]. This work treats a connected semi-simple Lie group L and finite-dimensional representations of L. His result is described in the same way as in the case of compact group, with only the difference of changing the unitarity in ρ by finite-dimensionality. In this result, the totality of operator fields is not isomorphic to L, but the factor group $L/N(L)$, in which $N(L)$ is the intersection of the kernels of all finite-dimensional representations L.

In the present paper, we consider the group $G=SL(2, \boldsymbol{R})$, which is non-compact and non-abelian, and its irreducible unitary (hence in general infinite-dimensional) representations.

Let Ω be the set of all equivalence classes of irreducible unitary representations of G. To each element ω of Ω, a unitary representation $\omega=\{U_g(\omega), \mathfrak{H}(\omega)\}$ of G is fixed as a representative of ω. Denote by $\boldsymbol{T}=\{T(\omega)\}$ an operator field over Ω, $T(\omega)$ being a bounded operator in $\mathfrak{H}(\omega)$. We consider the Kronecker product $\{U_g(\rho)\otimes U_g(\sigma), \mathfrak{H}(\rho)\otimes\mathfrak{H}(\sigma)\}$ in the usual way, and its decomposition, into irreducible components. In this decomposition, let

$$u\otimes v=\int_\Omega w(\omega)d\mu_{\rho,\sigma}(\omega),$$

where u, v, and $w(\omega)$ are vectors in $\mathfrak{H}(\rho)$, $\mathfrak{H}(\sigma)$, and $\mathfrak{H}(\omega)$ respectively, and $\mu_{\rho,\sigma}$ is a measure on Ω depending only on ρ, σ.

Now, we call a given operator field $\boldsymbol{T}$ *admissible* if for arbitrary u in $\mathfrak{H}(\rho)$ and v in $\mathfrak{H}(\sigma)$,

$$T(\rho)u\otimes T(\sigma)v=\int_\Omega (T(\omega)w(\omega))d\mu_{\rho,\sigma}(\omega),$$

i.e., shortly $\boldsymbol{T}$ commutes with the operation of the Kronecker product. $\boldsymbol{T}$ is called *unitary* if $T(\omega)$ are unitary operators for all ω. A unitary admissible operator field is an analogue of a unitary character over Ω.

For any element g in G, an operator field $\boldsymbol{U}_g$ is defined by the equality

$$\boldsymbol{U}_g=\{U_g(\omega)\},$$

then $\boldsymbol{U}_g$ is a unitary admissible operator field.

The totality $\mathfrak{R}$ of admissible unitary operator fields becomes a group with the multiplication defined by

$$(TS)(\omega)=T(\omega)S(\omega)$$

for two operator fields $\boldsymbol{T}=\{T(\omega)\}$ and $\boldsymbol{S}=\{S(\omega)\}$. And the mapping $g\to\boldsymbol{U}_g$ gives an homomorphism from G into $\mathfrak{R}$.

Our result is stated as follows.

THEOREM. *The mapping $g\to\boldsymbol{U}_g$ is an isomorphism from G* onto $\mathfrak{R}$. *And in this correspondence the natural topology in G coincides with the weakest topology, which makes all the matrix elements $\langle U_g(\omega)u, v\rangle$ (ω, u, v fixed) continuous.*

This theorem admits us to identify G with the totality of unitary admissible operator fields with the "weak" topology, that is, to consider as a "dual group" of the space Ω.

The idea of the proof of this theorem is as follows.

There are four series of irreducible unitary representations of G, as we state in § 1. At first we restrict our attention to the discrete series, especially positive one. A representation in the positive discrete series is parametrized by a positive half-integer or integer m, so we denote this representation by D_m^+ and its representation space by $\mathfrak{H}_m$. A useful property of this series is that the restriction of D_m^+ to the compact subgroup of G, which consists of orthogonal matrices

$$\begin{pmatrix} \cos(\theta/2), & -\sin(\theta/2) \\ \sin(\theta/2), & \cos(\theta/2) \end{pmatrix} \qquad (-2\pi < \theta \leqq 2\pi),$$

is decomposed into the sum

$$\Sigma \oplus \rho(n) \qquad (n \geqq m,\ n+m: \text{integer}),$$

where we denote the one-dimensional representation $\{\exp(\sqrt{-1}\, n\theta), \boldsymbol{C}\}$ of this subgroup by $\rho(n)$. Since the multiplicity of $\rho(n)$, especially of $\rho(m)$ is one, the vector in $\mathfrak{H}_m$ corresponding to $\rho(m)$ is determined up to a constant factor. We fix one of normalized vectors of such a kind, and call it the *lowest vector* and denote by $\zeta(m)$.

It is shown in §2 that the Kronecker product $D_m^+ \otimes D_n^+$ is decomposed to the discrete direct sum of irreducible components, as

$$\Sigma \oplus D_p^+, \qquad (p \geqq n+m,\ p+n+m: \text{integer}),$$

and in this decomposition, the product $\zeta(m) \otimes \zeta(n)$ of lowest vectors corresponds to $c\zeta(m+n)$ in $\mathfrak{H}_{m+n}$, c being a constant of absolute value one.

Now we define *fundamental vector* v in $\mathfrak{H}_{1/2}$ as a normalized vector such that $v \otimes v$ belongs to $\mathfrak{H}_1$ in the irreducile decomposition of $\mathfrak{H}_{1/2} \otimes \mathfrak{H}_{1/2}$. Clearly, the lowest vector $\zeta(1/2)$ in $\mathfrak{H}_{1/2}$ is fundamental; moreover, for any unitary admissible operator field $\boldsymbol{T} = \{T(\omega)\}$, $T(D_{1/2}^+)\zeta(1/2)$ is also fundamental. In §4, we prove that the vectors $\{U_g(D_{1/2}^+)\zeta(1/2)\}$ simply covers the set of all fundamental vectors, that is to say, for given unitary admissible operator field $\boldsymbol{T}$, we can find unique element g in G satisfying the equation,

$$T(D_{1/2}^+)\zeta(1/2) = U_g(D_{1/2}^+)\zeta(1/2).$$

And in the same §, it is shown that there is a one-to-one correspondence between the set of vectors $\{U_g(D_{1/2}^+)\zeta(1/2)\}$ and G, and the initial topology of G coincides with the weak topology of the set of these vectors in $\mathfrak{H}_{1/2}$. The topological assertion in the theorem follows from this fact immediately.

Lastly in §5, we show that any unitary admissible operator field $\boldsymbol{T} = \{T(\omega)\}$ is completely determined by the vector $T(D_{1/2}^+)\zeta(1/2)$. Combinig the fact with the results in §4, we get $T(\omega) = U_g(\omega)$, for any ω in Ω.

In §1, we state certain elementary knowledges on irreducible unitary representations of G. And in §2, we give propositions about the Clebsch-Gordan coefficients of these representations, which is necessary in the following (Propositions 1 and 2). §3 is devoted to construct our duality theorem, and the proof of this theorem is devided into two steps, each steps of proof are given in §4, §5 respectively, as mentioned above.

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A short summary of this result has been published in [7].

§ 1. Preliminaries.

Now we quote from the works of V. Bargmann [4] and L. Pukánszky [5][1].

a) Consider in G following one-parameter subgroups:

$$R=\left\{r(\theta)=\begin{pmatrix}\cos(\theta/2), & -\sin(\theta/2)\\ \sin(\theta/2), & \cos(\theta/2)\end{pmatrix}:\ -2\pi<\theta\leqq 2\pi\right\},$$

$$D=\left\{d^j(\zeta)=(-1)^j\begin{pmatrix}\exp(\zeta/2), & 0\\ 0\ \ , & \exp(-\zeta/2)\end{pmatrix}:\ -\infty<\zeta<\infty,\ j=0,1\right\},$$

$$L=\left\{l^j(t)=(-1)^j\begin{pmatrix}\cosh(t/2), & \sinh(t/2)\\ \sinh(t/2), & \cosh(t/2)\end{pmatrix}:\ -\infty<t<\infty,\ j=0,1\right\}.$$

Let the generators of these subgroups in the Lie algebra of G be $\mathfrak{h}_R$, $\mathfrak{h}_D$, $\mathfrak{h}_L$ respectively, then there are operators $H_R(\omega)$, $H_D(\omega)$, $H_L(\omega)$ of given representation ω corresponding to these generators. That is, each operator has a dense domain in the space of representation ω and is defined by

$$H_R(\omega)v=w-\lim_{\theta\to 0}[(U_{r(\theta)}-I)/\theta]v\ ,$$

etc., where v is a vector in the domain.

b) All irreducible unitary representations of G are exhausted by following series.

1) Principal series,
   i) integral (non-spinor) representation, C_l^0 $(1/4\leqq l<\infty)$,
   ii) half-integral (spinor) representation, $C_l^{1/2}$ $(1/4<l<\infty)$,
2) Supplementary series, C_l^0 $(=E_l)$ $(0<l<1/4)$.
3) Discrete series,
   i) positive non-spinor representation, D_n^+ $(n=1, 2, \cdots)$,
   ii) positive spinor representation, D_n^+ $(n=1/2, 3/2, \cdots)$,
   iii) negative non-spinor representation, D_n^- $(n=1, 2, \cdots)$,
   iv) negative spinor representation, D_n^- $(n=1/2, 3/2, \cdots)$.
4) Identity representation, I.

These representations except 4) are constructed on the space of functions over a factor space of G, which is homeomorphic to one-dimensional euclidean space $\boldsymbol{R}$ in 1) 2), or to a half plane of complex numbers $\boldsymbol{C}$ in 3). Spinor representation is a representation in which $-e=\begin{pmatrix}-1, & 0\\ 0, & -1\end{pmatrix}$ in G corresponds

1) We make minor modifications in the notations in these papers, but the following results are deduced by simple calculation.

to $-I$, where I is the identity operator in the representation space. The non-spinor representation can be considered as a representation of the three-dimensional Lorentz group, which is the factor group of G by its centre $\{e, -e\}$.

The operator of representation is extension of a linear transformation over the representation space which carries a function $f(z)$ to $f(az+c/bz+d)(bz+d)^{\alpha}|bz+d|^{\beta}$, where $z\in\boldsymbol{R}$ in 1), 2), and $z\in\boldsymbol{C}$ and $Imz>0$ in 3).

We don't need the precise form of the representation operator, but only the properties given in following section, so we don't state the details about representations.

c) Since the subgroup R is abelian and compact, so the restriction of each irreducible representation ω of G to R is decomposed into direct sum of multiples of one-dimensional representations $\rho_k=\{\exp(\sqrt{-1}\,k\theta), H_k(\omega)\}$. In this case the index k runs as follows, and the multiplicities are one for any k:

For 1)—i) and 2), $k=0, \pm 1, \pm 2, \cdots$.

For 1)—ii), $k=\pm(1/2), \pm(3/2), \cdots$.

For 3)—i) and ii), $k=n, n+1, n+2, \cdots$.

For 3)—iii) and iv), $k=-n, -n-1, -n-2, \cdots$.

For 4), $k=0$.

d) Now we define operators $F^+(\omega)$ and $F^-(\omega)$ by

$$F^+(\omega)=H_D(\omega)-\sqrt{-1}\,H_L(\omega),$$
$$F^-(\omega)=H_D(\omega)+\sqrt{-1}\,H_L(\omega).$$

For an irreducible representation ω, $F^+(\omega)$ (resp. $F^-(\omega)$) maps $H_k(\omega)$ onto $H_{k+1}(\omega)$ (resp. $H_{k-1}(\omega)$), when the former is not $\{0\}$. Moreover we can select a normalized vector $\zeta_k(\omega)$ in $H_k(\omega)$, satisfying conditions:

$$F^+(\omega)\zeta_k(\omega)=\sqrt{k(k+1)-q(\omega)}\,\zeta_{k+1}(\omega), \tag{1.1}$$
$$F^-(\omega)\zeta_k(\omega)=-\sqrt{k(k-1)-q(\omega)}\,\zeta_{k-1}(\omega), \tag{1.2}$$

where $q(\omega)$ is a constant depending on ω (equal to the eigenvalue of the Laplacian operator), and is given as follows:

$$\left.\begin{array}{ll}\text{For the representations of 1) and 2),} & q(\omega)=-l\\ \text{For the representations of 3),} & q(\omega)=n(n-1)\\ \text{For the representation of 4),} & q(I)=0\end{array}\right\} \tag{1.3}$$

And the system $\{\zeta_k(\omega)\}$ constitutes a complete orthonormal basis in $\mathfrak{H}(\omega)$, and $\zeta_k(\omega)$ is an eigenvector of $U_{r(\theta)}(\omega)$ corresponding to the eigenvalue $\exp(\sqrt{-1}\,k\theta)$, consequently, of $H_R(\omega)$ corresponding to $\sqrt{-1}\,k$.

e) Especially, when ω is a representation of positive discrete series D_n^+, the following equation is valid for the vector $\zeta_n(D_n^+)$,

$$F^-(D_n^+)\zeta_n(D_n^+)=0\,. \tag{1.4}$$

So we call the vector $\zeta_n(D_n^+)$ *the lowest vector* in $\mathfrak{H}(D_n^+)$. It is easy to see that the equation (1.4) determines $\zeta_n(D_n^+)$ up to a constant factor.

Hereafter we denote briefly $\zeta_k(D_n^+)$ by $\zeta(n, k)$ and $\zeta_n(D_n^+)$ by $\zeta(n)$, and the operators $F^+(\omega)$, $F^-(\omega)$ by F^+, F^-, if there is no danger of confusion.

f) For a given unitary representation $\omega_0=\{U_g^0, \mathfrak{H}_0\}$ of G, let the irreducible decomposition of ω_0 be,

$$\{U_g^0, \mathfrak{H}_0\}=\int_\Omega \oplus \{U_g(\omega), \mathfrak{H}(\omega)\}\, d\mu(\omega)\,. \tag{1.5}$$

And let the decomposition of the restriction of ω_0 to the abelian compact group R be,

$$\{U_{r(\theta)}^0, \mathfrak{H}_0\}=\sum_k \oplus \{\exp(\sqrt{-1}\, k\theta), H_k^0\}\,.$$

This gives also the eigen expansion with respect to $H_R(\omega_0)$. By reason of c) and d), the vector v in H_k^0 such that

$$F^-v=0\,, \tag{1.6}$$

must belong to the space $\mathfrak{H}(D_k^+)$ of an irreducible component in the decomposition (1.5), which is equivalent to the representation D_k^+. This situation admits us to find discrete components of the decomposition (1.5) and Clebsch-Gordan coefficients of these representations in § 2. (Cf. L. Pukánszky [**5**] Th. 1, Cor. A and Cor. B.)

g) About definitions and elementary properties of Kronecker products of two unitary representations, we refer the reader to J. Dixmier's book [**6**]. And here we remark only a certain property, which is used in the following §.

Denote by $H(\omega)$ one of the operators $H_R(\omega)$, $H_D(\omega)$ and $H_L(\omega)$ for a unitary representation ω. Then the following equality is valid for any u in $\mathfrak{H}(\omega_1)$ and v in $\mathfrak{H}(\omega_2)$:

$$H(\omega_1\otimes\omega_2)(u\otimes v)=(H(\omega_1)u)\otimes v+u\otimes(H(\omega_2)v)\,. \tag{1.7}$$

From the bilinearity of Kronecker product, the same equality is true, when we substitute H by F^+ or F^-.

§ 2. Clebsch-Gordan Coefficients.

In this §, we prove two propositions (Prop. 1 and Prop. 2) about the Clebsch-Gordan coefficients for the representations of G which are useful in the following.

Let $\omega_1=\{U_g^1, \mathfrak{H}_1\}$, $\omega_2=\{U_g^2, \mathfrak{H}_2\}$ be two given irreducible representations of

G. We take the complete orthonormal systems: $\{\zeta'_k\}=\{\zeta_k(\omega_1)\}$, $\{\zeta''_k\}=\{\zeta_k(\omega_2)\}$, in $\mathfrak{H}_1$, $\mathfrak{H}_2$ respectively, satisfying (1.1) and (1.2) as in § 1 d).

Let $\omega_1\otimes\omega_2\equiv\omega_0\equiv\{U^0_g, \mathfrak{H}_0\}$. In $\mathfrak{H}_0$, $\{\zeta'_k\otimes\zeta''_j\}$ constructs a complete orthonormal system, and from § 1 g) and (1.1) (1.2),

$$\begin{aligned} &F^+(\omega_0)(\zeta'_k\otimes\zeta''_j) \\ &=\sqrt{k(k+1)-q(\omega_1)}\,\zeta'_{k+1}\otimes\zeta''_j+\sqrt{j(j+1)-q(\omega_2)}\,\zeta'_k\otimes\zeta''_{j+1}\,, \end{aligned}\tag{2.1}$$

$$\begin{aligned} &F^-(\omega_0)(\zeta'_k\otimes\zeta''_j) \\ &=-\sqrt{k(k-1)-q(\omega_1)}\,\zeta'_{k-1}\otimes\zeta''_j-\sqrt{j(j-1)-q(\omega_2)}\,\zeta'_k\otimes\zeta''_{j-1}\,. \end{aligned}\tag{2.2}$$

By virtue of § 1 f), we decompose the space $\mathfrak{H}_0$ to the eigenspace H^0_s of $H_R(\omega_0)$ corresponding to the eigenvalue $\sqrt{-1}\,s$, and find the vector v_s in H^0_s such as $F^-v_s=0$, then the vector v_s is a lowest vector $c_s\zeta_s$ in $\mathfrak{H}(D^+_s)$ which is an irreducible component of ω_0 equivalent to the representation D^+_s in the irreducible decomposition of ω_0.

Since the system $\{\zeta'_k\otimes\zeta''_{s-k}\}_k$ is a complete orthonormal system in H_s, we can obtain the vector v_s by determining coefficients $a(s, k)$ in the equation

$$F^-(\sum_k a(s, k)\zeta'_k\otimes\zeta''_{s-k})=0\,,\tag{2.3}$$

where k runs over some set depending on ω_1 and ω_2.

From (2.2) and (2.3),

$$\begin{aligned} &\sum_k a(s, k)[-\sqrt{k(k-1)-q(\omega_1)}\zeta'_{k-1}\otimes\zeta''_{s-k} \\ &\qquad -\sqrt{(s-k)(s-k-1)-q(\omega_2)}\zeta'_k\otimes\zeta''_{s-k-1}]=0\,, \end{aligned}\tag{2.4}$$

i. e.

$$\begin{aligned} &-\sum_k [a(s, k+1)\sqrt{(k+1)k-q(\omega_1)} \\ &\qquad +a(s, k)\sqrt{(s-k)(s-k-1)-q(\omega_2)}]\,\zeta'_k\otimes\zeta''_{s-k-1}=0\,. \end{aligned}\tag{2.5}$$

By the linear independency of $\{\zeta'_k\otimes\zeta''_{s-k-1}\}$, all the coefficients of left hand must vanish, that is,

$$a(s, k+1)\sqrt{(k+1)k-q(\omega_1)}+a(s, k)\sqrt{(s-k)(s-k-1)-q(\omega_2)}=0\,.\tag{2.6}$$

Under these notations, we get the following proposition.

PROPOSITION 1. *If ω_1 is a representation D^+_m of positive discrete series and $\omega_2\neq I$, then non-zero vector $v_s(=c_s\zeta_s)$ is determined up to constant c_s for* (1) $s\geqq m+n$ $(\omega_2=D^-_n)$, (2) $m-n\geqq s>(1/2)$ $(\omega_2=D^-_n)$, (3) $s>(1/2)$ $(\omega_2=C^0_l$ *or* $C^{1/2}_l)$, *and*

$$a(s, k)=c(s, k)\langle\zeta_s, \zeta(m, k)\otimes\zeta''_{s-k}\rangle\neq 0\,,\tag{2.7}$$

for any admissible k and s, for which the corresponding vectors $\zeta(m, k)$, ζ''_{s-k} exist.

PROOF. We prove the proposition separately in each cases.

1) When $\omega_2 = D_n^+$, then the admissible pair (k, s) must satisfy conditions:

$$k \geqq m\,, \quad \text{and} \quad s-k \geqq n \qquad (k-m,\, s-k-n:\ \text{integer})\,,$$

i.e., in (2.4), k runs over the range:

$$s-n \geqq k \geqq m\,, \qquad (k-m,\, s-m-n:\ \text{integer})\,.$$

On the other hand, from (1.3), and (2.6),

$$a(s,\,k+1)\sqrt{(k-m+1)(k+m)}+a(s,\,k)\sqrt{(s-k-n)(s-k+n-1)}=0\,. \tag{2.8}$$

At the end points of range of k, where $k=m-1$ or $k=s-n$, the coefficients of $a(s, m)$ and $a(s, s-n)$ vanish, that is, (2.5) gives no condition. When k satisfies the inequalities $s-n>k\geqq m$, the both coefficients of $a(s, k)$, $a(s, k+1)$ in (2.8) are different from zero. This fact implies that if we give a non-zero value of $a(s, m)$, then all admissible $a(s, k)$'s, and therefore, the vectors v_s's are uniquely determined and are different from zero. So v_s is determined up to a constant factor. Therefore $D_m^+\otimes D_n^+$ contains D_s^+ as an irreducible component with multiplicity one, for the value of s such that

$$s \geqq m+n \qquad (s-m-n:\ \text{integer})\,.$$

2) When $\omega_2 = D_n^-$. The conditions for admissible pair (k, s) are that:

$$k \geqq m \quad \text{and} \quad s-k \leqq -n \qquad (k-m,\, s-k+n:\ \text{integer})\,,$$

i.e., in (2.4), k runs over the range:

$$k \geqq \mathrm{Max}\,(m,\, s+n) \qquad (k-m,\, s-m+n:\ \text{integer}).$$

The equation to determine $a(s, k)$ is the same as (2.8),

$$a(s,\,k+1)\sqrt{(k-m+1)(k+m)}+a(s,\,k)\sqrt{(s-k-n)(s-k+n-1)}=0\,. \tag{2.9}$$

When we consider the only end point of range of k, there are two cases.

a). If $\mathrm{Max}\,(m, s+n)=s+n$, and $\neq m$. For $k=s+n-1$, the coefficient of $\zeta(m, s+n-1)\otimes\zeta''_{-n}$ in (2.5) is

$$-a(s,\, s+n)\sqrt{(s+n-m)(s+n+m-1)}\,.$$

The condition to vanish this coefficient is evidently, that $a(s, s+n)=0$. And for the case that $k\geqq s+n$, the coefficients of $a(s, k+1)$ in (2.9) are not zero, then all $a(s, k)$ must be zero. This deduction results that if $s>m-n$, there is no vector v_s except zero, and D_s^+ don't appear as a discrete summand of $D_m^+\otimes D_n^-$.

b) If $\mathrm{Max}\,(m, s+n)=m$ $(s-m+n:\ \text{integer})$. When $k=m-1$, the coefficient of the term $a(s, m)\zeta(m, m-1)\otimes\zeta''_{s-m}$ vanishes. So this term in (2.5) gives no condition for the coefficient $a(s, m)$. Since the coefficients of $a(s, k)$ and $a(s, k+1)$ in (2.9) don't vanish when $k\geqq m$ $(k-m:\ \text{integer})$, we can determine all non-zero $a(s, k)$ from arbitrary given non-zero $a(s, m)$ inductively.

The Raabe's test for convergence assures that the series

$$\sum_{k\geqq m} a(s, k)\zeta(m, k)\otimes\zeta''_{s-k} \qquad (a(s, m)\neq 0),$$

gives a vector in H_s^0 only when

$$1/2 < s \leqq m-n \qquad (s-m+n: \text{integer}). \tag{2.10}$$

That is to say, when s satisfies (2.10), v_s is determined, and D_s^+ is contained as a discrete summand of $D_m^+\otimes D_n^-$, with multiplicity one.

3) When $\omega_2 = C_t^t$ ($t=0$ or $1/2$), i.e., a representation of the continuous series.

The only restriction for the index j in $\{\zeta''_j\}$ is that $t-j$ is an integer. Then the admissible pair (k, s) is obtained under the conditions, $k\geqq m$ ($k-m$, $s-k-t$: integer). From the same reason as in the case of 2) b), any condition for $a(s, m)$ doesn't rise from the end term of (2.5). It is easy to see that if we give arbitrary value to $a(s, m)$, then the other $a(s, k)$'s are uniquely determined, and if $a(s, m)\neq 0$, then $a(s, k)\neq 0$ for (k, s) as above. We have to check the convergence of the series

$$\sum_{k\geqq m} a(s, k)\zeta(m, k)\otimes\zeta''_{s-k}.$$

Raabe's test shows that this summation converges for

$$s > 1/2 \qquad (s-m-t: \text{integer}).$$

Then, for above s, v_s is determined up to a constant factor, i.e., D_s^+ is a component with multiplicity one of $D_m^+\otimes\omega_2$. q.e.d.

As it is shown in the proof of proposition 1, case 1), $D_{1/2}^+\otimes D_{1/2}^+$ contains D_1^+ as an irreducible component with multiplicity one. Hence we calculate Clebsh-Gordan coefficients,

$$b(s, k)\equiv\langle\zeta(1, s), \zeta(1/2, k+(1/2))\otimes\zeta(1/2, s-k-(1/2))\rangle.$$

And we obtain the following.

PROPOSITION 2.

$$\frac{b(s, 0)}{b(s, 1)} = \frac{\langle\zeta(1, s), \zeta(1/2, 1/2)\otimes\zeta(1/2, s-(1/2))\rangle}{\langle\zeta(1, s), \zeta(1/2, 3/2)\otimes\zeta(1/2, s-(3/2))\rangle} = 1. \tag{2.11}$$

PROOF. Repeated applications of (1.1) lead us to the equality

$$(F^+)^{s-1}\zeta(1) = \prod_{s-1\geqq t\geqq 1}\sqrt{t(t+1)}\,\zeta(1, s). \tag{2.12}$$

On the other hand the calculation of eigenvalue with respect to $H_R(\omega_0)$ shows,

$$\zeta(1) = c_1(\zeta(1/2)\otimes\zeta(1/2)), \tag{2.13}$$

where c_1 is a constant with absolute value one. From (1.1),

$$
\begin{aligned}
&(F^+)^{s-1}(\zeta(1/2)\otimes\zeta(1/2)) \\
&=\sum_{s-1\geqq k\geqq 0} {}_{s-1}C_k((F^+)^k\zeta(1/2)\otimes(F^+)^{s-k-1}\zeta(1/2)) \\
&=\sum_{s-1\geqq k\geqq 0} {}_{s-1}C_k(\prod_{k-1\geqq l\geqq 0}\sqrt{(l+(1/2))(l+(3/2))+(1/4)}) \qquad (2.14)\\
&\times(\prod_{s-k-2\geqq h\geqq 0}\sqrt{(h+(1/2))(h+(3/2))+(1/4)})\,\zeta(1/2,k+(1/2)) \\
&\otimes\zeta(1/2,s-k-(1/2)),
\end{aligned}
$$

where k takes the integers.

(2.12)$\sim$(2.14) give,

$$
\begin{aligned}
b(s,k)&=(\prod_{s-1\geqq t\geqq 1}\sqrt{t(t+1)})^{-1}c_1\,{}_{s-1}C_k(\prod_{k-1\geqq l\geqq 0}\sqrt{(l+(1/2))(l+(3/2))+(1/4)}) \\
&\times(\prod_{s-k-2\geqq h\geqq 0}\sqrt{(h+(1/2))(h+(3/2))+(1/4)})\neq 0\,, \qquad (2.15)\\
&(s-1\geqq k\geqq 0)\,,
\end{aligned}
$$

especially,

$$
\frac{b(s,0)}{b(s,1)}=\frac{\sqrt{(s-(1/2))(s-(3/2))+(1/4)}}{s-1}=1\,.
$$

q. e. d.

§ 3. The main theorem.

Now we give some definitions and state our duality theorem.

Let Ω be the totality of equivalence classes $\tilde{\omega}$ of irreducible unitary representations of real unimodular group G of second order, and we choose a fixed concrete representation $\omega=\{U_g(\omega),\mathfrak{H}(\omega)\}$ as a representative of the class $\tilde{\omega}$. We consider the Kronecker products of these representations and its decomposition into irreducible components. In other words, if ω_1, ω_2 are two irreducible unitary representations of G, then for any vectors u, v in $\mathfrak{H}(\omega_1)$, $\mathfrak{H}(\omega_2)$ respectively, there is a measure μ_{ω_1,ω_2} on Ω and a vector valued function $v(\omega)$ whose value at ω is in $\mathfrak{H}(\omega)$, such that

$$
u\otimes v=\int_\Omega v(\omega)d\mu_{\omega_1,\omega_2}(\omega)\,. \qquad (3.1)
$$

And for any g in G, the following equation has meaning and is valid:

$$
U_g(\omega_1)u\otimes U_g(\omega_2)v=\int_\Omega U_g(\omega)v(\omega)d\mu_{\omega_1,\omega_2}(\omega)\,. \qquad (3.2)
$$

We consider an operator field $\boldsymbol{T}=\{T(\omega)\}$ over Ω, in which $T(\omega)$ is a bounded operator in $\mathfrak{H}(\omega)$.

DEFINITION 1. *An operator field* $\boldsymbol{T}=\{T(\omega)\}$ *is called* unitary *if the all* $T(\omega)$ *are unitary.*

DEFINITION 2. *An operator field* $\boldsymbol{T}=\{T(\omega)\}$ *is called* admissible *if the following equality has meaning and is valid:*

$$T(\omega_1)u\otimes T(\omega_2)v=\int_{\Omega}T(\omega)v(\omega)\,d\mu_{\omega_1,\omega_2}(\omega)\,, \tag{3.3}$$

for arbitrary vectors u, v *in* $\mathfrak{H}(\omega_1)$, $\mathfrak{H}(\omega_2)$ *respectively and the decomposition* (3.1).

It is clear from (3.2) that the operator field

$$\boldsymbol{U}_g=\{U_g(\omega)\} \tag{3.4}$$

is unitary and admissible for any fixed g in G.

Our main purpose in the paper is to prove the converse, i.e., let $\mathfrak{R}$ be the totality of all admissible unitary operator fields, we define the product of two elements $\boldsymbol{T}=\{T(\omega)\}$ and $\boldsymbol{S}=\{S(\omega)\}$ in $\mathfrak{R}$, and the weakest topology in $\mathfrak{R}$ which makes $\langle T(\omega)u, v\rangle$ continuous for all ω, u, v. Then the mapping ϕ: $g\rightarrow U_g$ gives a homomorphism from G into $\mathfrak{R}$, and,

THEOREM. *ϕ is an isomorphism onto $\mathfrak{R}$, i.e., we can identify G with $\mathfrak{R}$ by this mapping.*

We prove this theorem in the following two steps I) and II).

I) At first we show that for given unitary admissible operator field $\boldsymbol{T}=\{T(\omega)\}$, there is a unique element g in G such that

$$T(D^+_{1/2})\zeta(1/2)=U_g(D^+_{1/2})\zeta(1/2)\,, \tag{3.6}$$

where $\zeta(1/2)$ is the lowest vector in the space on which the irreducible representation $D^+_{1/2}$ operates. (§ 4)[2].

II) Secondary, in § 5, we see if (3.6) is satisfied then for any ω in Ω and vector v in $\mathfrak{H}(\omega)$,

$$T(\omega)v=U_g(\omega)v\,. \tag{3.7}$$

In the proof of I), it is shown that the topology of G coincides with the weak topology of $\{U_g(D^+_{1/2})\zeta(1/2)\}$ as a set of vectors in the space $\mathfrak{H}(D^+_{1/2})$.

§ 4. **Fundamental vector.** (The step I.)

In this section we treat the representations $D^+_{1/2}$ and D^+_1 only. So we denote $D^+_j=\{U_g(D^+_j), \mathfrak{H}(D^+_j)\}$ by $\omega_j=\{U^j_g, \mathfrak{H}_j\}$ simply.

DEFINITION 3. *A normalized vector v in $\mathfrak{H}_{1/2}$ is called fundamental if $v\otimes v$ in $\mathfrak{H}_{1/2}\otimes\mathfrak{H}_{1/2}$ belongs to $\mathfrak{H}_1$ in the decomposition* (3.1).

The lowest vector $\zeta(1/2)$ in $\mathfrak{H}_{1/2}$ is fundamental by (2.13). Moreover, for any unitary admissible operator field $\boldsymbol{T}=\{T(\omega)\}$, $T(\omega_{1/2})\zeta(1/2)$ is also a fundamental vector. In fact, from (2.13) and (3.3),

2) We can characterize the representations of discrete series using the law of decomposition (3.1), but we don't state it here.

$$T(\omega_{1/2})\zeta(1/2)\otimes T(\omega_{1/2})\zeta(1/2)=c^{-1}T(\omega_1)\zeta(1) .$$

Since $T(\omega_1)$ makes the space $\mathfrak{H}_1$ invariant in the decomposition (3.1), the right-hand side of the above equality belongs to $\mathfrak{H}_1$.

Now we deduce a necessary condition that a vector v is fundamental. Let v be a vector in $\mathfrak{H}_{1/2}$, then v is expanded by the complete orthonomal system $\{\zeta(1/2, k+(1/2))\}$ of weight vectors in $\mathfrak{H}_{1/2}$ as follows:

LEMMA 1. *If*

$$v=\sum_{k\geqq 0}a_k\zeta(1/2, k+(1/2)) , \qquad (k: \textit{integers})$$

is fundamental, then a_k *satisfies*

$$c(s)b(s, k)=a_k a_{s-k-1} , \tag{4.1}$$

for some $c(s)$.

PROOF. We have

$$\begin{aligned} v\otimes v &= \sum_{k,l}a_k a_l\zeta(1/2, k+(1/2))\otimes\zeta(1/2, l+(1/2)) \\ &= \sum_s\Big(\sum_{s-1\geqq k\geqq 0}a_k a_{s-k-1}\zeta(1/2, k+(1/2))\otimes\zeta(1/2, s-k-(1/2))\Big) . \end{aligned}$$

This equality gives the decomposition of $v\otimes v$ to the sum of the eigenvectors of H_R corresponding to the eigen value $\sqrt{-1}\,s$. Furthermore suppose v be fundamental, then each components of $v\otimes v$ must belong to $\mathfrak{H}_1$, i.e.:

$$\sum_{s-1\geqq k\geqq 0}a_k a_{s-k-1}\zeta(1/2, k+(1/2))\otimes\zeta(1/2, s-k-(1/2))\in\mathfrak{H}_1 .$$

Since the only eigenvector of H_R corresponding to $\sqrt{-1}\,s$ in $\mathfrak{H}_1$ is of the form $c(s)\zeta(1, s)$, we have

$$\sum_{s-1\geqq k\geqq 0}a_k a_{s-k-1}\zeta(1/2, k+(1/2))\otimes\zeta(1/2, s-k-(1/2))=c(s)\zeta(1, s) .$$

On the other hand we have from (2.15),

$$c(s)\zeta(1, s)=c(s)\sum_{s-1\geqq k\geqq 0}b(s, k)\zeta(1/2, k+(1/2))\otimes\zeta(1/2, s-k-(1/2)) .$$

And because of the linear independency, the corresponding coefficients in the above equalities must be equal, that is,

$$c(s)b(s, k)=a_k a_{s-k-1} , \qquad (b(s, k)\neq 0) . \qquad \text{q. e. d.}$$

Now we show the followings:

LEMMA 2. *Let* $\{a_k\}_{k\geqq 0}$ *be a sequence of complex numbers satisfying the next condition* (C).

(C) *If* $a_{k_0}a_{s-k_0}=0$ *for some* k_0, *then* $a_k a_{s-k}=0$ *for any admissible* k.

Under this assumption if a pair (h_1, h_2) *such that* $a_{h_1}=0$, *and* $a_{h_2}\neq 0$ *exists, then for all* $l\geqq h_1-h_2$, $a_l=0$.

PROOF. For $s \geqq h_1$, $a_{h_1} a_{s-h_1} = 0$, the assumption gives $a_k a_{s-k} = 0$ for these s and admissible k. On the other hand $a_{h_2} \neq 0$, so $a_k = 0$ for $s-k=h_2$, i.e., for $k = s-h_2 \geqq h_1 - h_2$. q. e. d.

LEMMA 3. *Under the same assumption as in lemma 2, if there is a term* $a_{k_0} = 0$, *then* $a_k = 0$ *for* $k \geqq 1$.

PROOF. If $\{a_k\} = \{0\}$, it is trivial, so we suppose for some h_2, $a_{h_2} \neq 0$. Let $h_1 = \text{Min.}\{h : a_h = 0\}$, then from lemma 1, $a_k = 0$ for any $k \geqq h_1 - h_2$. But by the definition of h_1, $h_2 = 0$ follows. That is $a_k = 0$ for $k \geqq 1$. q. e. d.

Now we apply these lemmas to the coefficients a_k of a non-zero fundamental vector:

$$v = \sum_{k \geqq 0} a_k \zeta(1/2, k+(1/2)),$$

then there are only two possibilities, one of which is

$$a_k = 0 \qquad \text{for} \quad k \geqq 1,$$

and the other is

$$a_k \neq 0 \qquad \text{for all} \quad k.$$

In the first case we have $v = a_0 \zeta(1/2)$.

In the latter case $c(s)$ in (4.1) is not zero for all s, and this implies that:

$$b(s, k-1)/b(s, k) = a_{k-1} a_{s-k} / a_k a_{s-k-1} \qquad (s \geqq k),$$

especially,

$$1 = b(s, 0)/b(s, 1) = a_0 a_{s-1} / a_1 a_{s-2},$$

i. e.,

$$a_s = (a_1/a_0)^s a_0.$$

Therefore in this case we have the equality

$$v = a_0 \sum_{k \geqq 0} (a_1/a_0)^k \zeta(1/2, k+(1/2)).$$

Condition for convergence of this vector is

$$|a_1/a_0| < 1.$$

Denote

$$z = (a_1/a_0),$$

then this condition can be written as follows:

$$|z| < 1. \tag{4.2}$$

And the equality

$$1 = \|v\|^2 = |a_0|^2 \sum_{k \geqq 0} |z|^{2k} = |a_0|^2 (1-|z|^2)^{-1},$$

implies

$$|a_0| = (1-|z|^2)^{1/2}. \tag{4.3}$$

Conversely if we give a pair (a_0, z) of complex numbers satisfying the conditions (4.2) and (4.3) then

$$v = a_0 \sum_{k \geqq 0} z^k \zeta(1/2,\ k+(1/2))$$

is a fundamental vector. Consequently, all fundamental vectors correspond in one-to-one way to a pair (a_0, z) satisfying (4.2), (4.3). It is evident that weak topology of $\{v\}$ in $\mathfrak{H}$ is stronger than the natural topology of $\{(a_0, z)\}$ in $\boldsymbol{C} \times \boldsymbol{C}$.

From the definition of a_k,

$$a_0 = \langle v, \zeta(1/2) \rangle\,,$$

$$z = (a_1/a_0) = \langle v, \zeta(1/2, 3/2) \rangle / \langle v, \zeta(1/2) \rangle\,.$$

In the remaining part of this paragraph, we show that arbitrary fundamental vector v is the form of $U_g^{1/2}\zeta(1/2)$ for some g in G. For this purpose it is sufficient to show that any pair (a_0, z) satisfying (4.2) and (4.3) is given by a certain g in G as follows:

$$a_0 = a_0(g) \equiv \langle U_g^{1/2}\zeta(1/2), \zeta(1/2) \rangle\,, \tag{4.4}$$

$$z = z(g) \equiv \langle U_g^{1/2}\zeta(1/2), \zeta(1/2, 3/2) \rangle / \langle U_g^{1/2}\zeta(1/2), \zeta(1/2) \rangle\,. \tag{4.5}$$

This fact is verified by the calculations of the matrix elements of the operator $U_g^{1/2}$. That is, the element g is written in the form $r(\theta)d^0(t)r(\varphi)$, and in this decomposition,

$$\begin{aligned} a_0(g) &= \langle U_{r(\theta)} U_{d^0(t)} U_{r(\varphi)} \zeta(1/2), \zeta(1/2) \rangle \\ &= \exp(\sqrt{-1}\,(\theta+\varphi)/2) \langle U_{d^0(t)} \zeta(1/2), \zeta(1/2) \rangle\,, \\ z(g) &= \frac{\langle U_{r(\theta)} U_{d^0(t)} U_{r(\varphi)} \zeta(1/2), \zeta(1/2, 3/2) \rangle}{\langle U_{r(\theta)} U_{d^0(t)} U_{r(\varphi)} \zeta(1/2), \zeta(1/2) \rangle} \\ &= \exp(\sqrt{-1}\,\theta) \frac{\langle U_{d^0(t)} \zeta(1/2), \zeta(1/2, 3/2) \rangle}{\langle U_{d^0(t)} \zeta(1/2), \zeta(1/2) \rangle}\,, \end{aligned}$$

where

$$-2\pi < \theta \leqq 2\pi\,, \qquad -\pi < \varphi \leqq \pi\,, \qquad 0 \leqq t < \infty\,.$$

In this formula we can determine θ, φ, t independently. So if we show that the values $|z(d^0(t))|$ cover the interval $[0, 1)$, then (4.4), (4.5) are proved. And by reason of (4.3), it is equivalent to show the range of $|a_0(d^0(t))|$ covers the interval $(0, 1]$. Now we shall calculate the matrix element

$$a_k(g) = \langle U_g^{1/2}\zeta(1/2), \zeta(1/2, k+(1/2)) \rangle\,.$$

Clearly

$$a_k(r(\theta) g r(\varphi)) = \exp(\sqrt{-1}\,((1+2k)\theta+\varphi)/2) a_k(g)\,. \tag{4.6}$$

Considering the definition of H_D, H_L, F^- and the relation $F^-(\omega_{1/2})\zeta(1/2) = 0$, we get

$$\begin{aligned} 0 &= \langle U_g^{1/2} F^- \zeta(1/2), \zeta(1/2, k+(1/2)) \rangle \\ &= \langle U_g^{1/2}(H_D + \sqrt{-1}\,H_L)\zeta(1/2), \zeta(1/2, k+(1/2)) \rangle \end{aligned}$$

$$= \lim_{t\to 0}(1/t)\langle U_g^{1/2}(U_{d^0(t)}^{1/2}-I)\zeta(1/2),\ \zeta(1/2,\ k+(1/2))\rangle$$

$$+\sqrt{-1}\lim_{\xi\to 0}(1/\xi) < U_g^{1/2}(U_{l^0(\xi)}^{1/2}-I)\zeta(1/2),\ \zeta(1/2,\ k+(1/2))\rangle$$

$$=(d/dt)(\langle U_{gd^0(t)}^{1/2}\zeta(1/2),\ \zeta(1/2,\ k+(1/2))\rangle)|_{t=0}$$

$$+\sqrt{-1}(d/d\xi)(\langle U_{gl^0(\xi)}^{1/2}\zeta(1/2),\ \zeta(1/2,\ k+(1/2))\rangle)|_{\xi=0}\,.$$

Put

$$\boldsymbol{F}^-(f(g))=(d/dt)(f(gd^0(t)))|_{t=0}+\sqrt{-1}(d/d\xi)(f(gl^0(\xi)))|_{\xi=0}\,, \tag{4.7}$$

then the above equation is

$$\boldsymbol{F}^-(a_k(g))=0\,. \tag{4.8}$$

We solve the equations (4.6) and (4.8). From (4.6), it is enough to solve the equation (4.8) when $g=d^0(t_0)$. Under this restriction, the first term in the right-hand side of (4.7) is ordinary derivation with respect to t. For the calculation of the second term, decompose the element $d^0(t_0)l^0(\xi)$ in G, as follows:

$$d^0(t_0)l^0(\xi)=r(\theta)d^0(t)r(\varphi)\,,\qquad (-\pi<\varphi\leqq\pi)\,.$$

In this decomposition, θ, t, φ can be regarded as functions of ξ. Differentiating both sides with respect to ξ, we get

$$d^0(t_0)(d/d\xi)(l^0(\xi))=((d/d\xi)r(\theta))d^0(t)r(\varphi)$$
$$+r(\theta)((d/d\xi)d^0(t))r(\varphi)+r(\theta)d^0(t)((d/d\xi)r(\varphi))\,.$$

Put $\xi=0$, then $r(\theta)=r(\varphi)=\text{identity}$, and $d^0(t)=d^0(t_0)$, so

$$d^0(t_0)\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}=(d\theta/d\xi)\Big|_{\xi=0}\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix}d^0(t_0)$$
$$+(dt/d\xi)\Big|_{\xi=0}d^0(t_0)\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}+(d\varphi/d\xi)\Big|_{\xi=0}d^0(t_0)\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix},$$

i.e.,

$$\begin{aligned}\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}&=(d\theta/d\xi)\Big|_{\xi=0}d(-t_0)\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix}d^0(t_0)\\ &\quad+(dt/d\xi)\Big|_{\xi=0}\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}+(d\varphi/d\xi)\Big|_{\xi=0}\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix}\\ &=(d\theta/d\xi)\Big|_{\xi=0}\begin{pmatrix}0 & -(\exp(-t))/2\\ (\exp t)/2 & 0\end{pmatrix}\\ &\quad+(dt/d\xi)\Big|_{\xi=0}\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}+(d\varphi/d\xi)\Big|_{\xi=0}\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix}.\end{aligned} \tag{4.9}$$

(4.9) means

$$(dt/d\xi)|_{\xi=0}=0\,,\quad (d\theta/d\xi)|_{\xi=0}=1/\sinh t\,,\quad (d\varphi/d\xi)|_{\xi=0}=-\cosh t/\sinh t\,.$$

By means of these equalities and (4.6), (4.8) reduces to

$$\boldsymbol{F}^{-}(a_k(d^0(t))) = (d/dt)a_k(d^0(t)) + \sqrt{-1}\{-(\sqrt{-1}/2)(\cosh t/\sinh t)$$
$$+\sqrt{-1}((1+2k)/2\sinh t)\}a_k(d^0(t)) = (d/dt)a_k(d^0(t))$$
$$+\{(\cosh t-(1+2k))/2\sinh t\}a_k(d^0(t)) = 0\,.$$

That is, $a_k(d^0(t))$ must be equal to $c_k(\cosh(t/2))^{-k-1}(\sinh(t/2))^k$, where c_k is a non-zero constant.

When $k=0$, put $t=0$, then $c_0 = a_0(d^0(0)) = \|\zeta(1/2)\|^2 = 1$. If t increases to infinity, then $a_0(d^0(t))$ decreases to zero. So by the connectedness of $\{d^0(t)\}$ and continuity of $a_0(d^0(t))$, the range of $a_0(d^0(t))$ covers the interval $(0, 1]$. This proves the existence of g in G corresponding to a fundamental vector v.

Next we consider the case $k=1$. In this case,

$$a_1(d^0(t)) = c_1(\cosh(t/2))^{-2}(\sinh(t/2))\,,$$
$$z(d^0(t)) = a_1(d^0(t))/a_0(d^0(t)) = c_1\sinh t/(\cosh t+1)\,,$$

i. e.,

$$a_0(g) = a_0(r(\theta)d^0(t)r(\varphi)) = \exp(\sqrt{-1}(\theta+\varphi)/2)(1+\cosh t)^{-1/2}\,, \tag{4.10}$$

$$z(g) = z(r(\theta)d^0(t)r(\varphi)) = \exp(\sqrt{-1}\,\theta)(1+\cosh t)^{-1}c_1\sinh t\,. \tag{4.11}$$

The classical theory of non-euclidean space shows that (4.10) and (4.11) gives a homeomorphism between G and the set of pairs $(a_0(g), z(g))$ in $\boldsymbol{C}\times\boldsymbol{C}$. That is, the topology of G is weaker than the weak topology of $\{U_g^{1/2}\zeta(1/2)\}$ in $\mathfrak{H}_{1/2}$. But it is evident from the continuity of the representation that the latter topology is weaker than the former. Then the both topologies must coincide.

§ 5. Determination of admissible operator field. (The step II.)

Now we proceed to the second step of the proof (cf. § 3).

First we study the representations of positive discrete series.

LEMMA 4. *Let $\boldsymbol{T}=\{T(\omega)\}$ be a given unitary admissible operator field and g in G, satisfying the relation*

$$T(D_{1/2}^+)\zeta(1/2) = U_g(D_{1/2}^+)\zeta(1/2)\,, \tag{5.1}$$

then for any $n \geqq m$

$$T(D_m^+)\zeta(m, n) = U_g(D_m^+)\zeta(m, n)\,. \tag{5.2}$$

PROOF. i) Since the lowest vector in $D_{1/2}^+\otimes D_m^+$ is unique up to a constant factor of absolute value one, we have

$$\zeta(1/2)\otimes\zeta(m) = c_m^{-1}\zeta(m+(1/2))\,, \qquad |c_m| = 1\,.$$

By the induction on m, we get

$$T(D_m^+)\zeta(m) = T(D_m^+)(c_{m-(1/2)}\zeta(1/2)\otimes\zeta(m-1/2))$$
$$= c_{m-(1/2)}(T(D_{1/2}^+)\zeta(1/2)\otimes T(D_{m-(1/2)}^+)\zeta(m-(1/2)) \tag{5.3}$$

$$= c_{m-(1/2)}(U_g(D^+_{1/2})\zeta(1/2) \otimes U_g(D^+_{m-(1/2)})\zeta(m-(1/2))$$
$$= U_g(D^+_m)\zeta(m)\,,$$

for all m. Therefore (5.2) is true in the case of $m=n$.

ii) We define the subspace V_k of $\mathfrak{H}(D^+_m)$ as follows:

$$V_1 = \{v:\ U_g(D^+_{1/2})\zeta(1/2) \otimes v \in \mathfrak{H}(D^+_{m+(1/2)})\}\,. \tag{5.4}$$

And for $k>1$, we define inductively

$$V_k = \{v:\ U_g(D^+_{1/2})\zeta(1/2) \otimes v \in \sum_{k \geqq s \geqq 1} \mathfrak{H}(D^+_{m+s-(1/2)}),\ v \perp V_s \text{ for } k>s \geqq 1\}\,. \tag{5.5}$$

Then the dimension of V_k is one for any k, and the vectors in V_k are of the form

$$v = c(k)U_g(D^+_m)\zeta(m, m+k-1)\,, \tag{5.6}$$

where $c(k)$ is a complex number.

In fact, let the expansion of a vector v in V_k be

$$v = \sum_{j \geqq 0} a_j U_g(D^+_m)\zeta(m, m+j)\,,$$

in which $\{U_g(D^+_m)\zeta(m, m+j)\}_j$ is a complete orthonormal system in $\mathfrak{H}(D^+_m)$. Then,

$$U_g(D^+_{1/2})\zeta(1/2) \otimes v = \sum_{j \geqq 0} a_j U_g(D^+_{1/2})\zeta(1/2) \otimes U_g(D^+_m)\zeta(m, m+j)\,,$$

so the component of this vector in $U_g(D^+_{1/2} \otimes D^+_m)\boldsymbol{H}_{m+s+(1/2)}$ is

$$U_g(D^+_{1/2})\zeta(1/2) \otimes (a_s U_g(D^+_m)\zeta(m, m+s))$$
$$= a_s U_s(D^+_{1/2} \otimes D^+_m)(\zeta(1/2) \otimes \zeta(m, m+s))\,.$$

Here, $\boldsymbol{H}_{m+s+(1/2)}$ is the eigenspace of H_R corresponding to the eigenvalue $\sqrt{-1}\,(m+s+(1/2))$ in $\mathfrak{H}(D^+_{1/2} \otimes D^+_m)$ (cf. § 1 and § 2).

From the results of § 2, the component of the vector $U_g(D^+_{1/2} \otimes D^+_m) \times (\zeta(1/2) \otimes \zeta(m, m+s))$ in the space $\mathfrak{H}(D^+_{m+s+(1/2)})$ is

$$\langle \zeta(1/2) \otimes \zeta(m, m+s), \zeta(m+s+(1/2)) \rangle\, U_g(D^+_{m+s+(1/2)})\zeta(m+s+(1/2)) \neq 0\,.$$

By the orthogonal relation of $\{U_g(D^+_m)\boldsymbol{H}_k\}$, we see that the vector $U_g(D^+_{1/2})\zeta(1/2) \otimes v$ belongs to $\sum_{k \geqq s \geqq 1} \oplus \mathfrak{H}_{m+s-(1/2)}$ if and only if $a_j = 0$ for $j \geqq k$, i.e.,

$$v = \sum_{k-1 \geqq s \geqq 0} c_s U_g(D^+_m)\zeta(m, m+s)\,.$$

By putting $k=1$, (5.6) is shown for V_1.

On the other hand, from the last condition of (5.5), V_k must be orthogonal to all V_s for $k \gneqq s \geqq 1$. The induction on k leads us to (5.6), and this is equivalent to the condition

$$\dim V_k = 1\,.$$

It is easy to see that both $T(D^+_m)\zeta(m, m+k)$ and $U_g(D^+_m)\zeta(m, m+k)$ belong to V_{k-1}. (5.1) and the induction on k leads us to the equality

$$T(D_m^+)\zeta(m, n) = c(m, n)U_g(D_m^+)\zeta(m, n), \qquad (n \geqq m),$$

where $c(m, n)$ is a non-zero constant depending on m and n. But this constant is equal to one. In fact, we consider the vector

$$\begin{aligned} U_g(D_{1/2}^+)\zeta(1/2) \otimes T(D_m^+)\zeta(m, n) &= T(D_{1/2}^+)\zeta(1/2) \otimes T(D_m^+)\zeta(m, n) \\ &= T(D_{1/2}^+ \otimes D_m^+)(\zeta(1/2) \otimes \zeta(m, n)), \end{aligned} \tag{5.7}$$

where

$$\zeta(1/2) \otimes \zeta(m, n) = \sum_{s \leqq n+(1/2)} \langle \zeta(1/2) \otimes \zeta(m, n), \zeta(s, n+(1/2)) \rangle \times \zeta(s, n+(1/2)), \qquad \text{(cf. § 2)}.$$

Then

$$\begin{aligned} &T(D_{1/2}^+ \otimes D_m^+)(\zeta(1/2) \otimes \zeta(m, n)) \\ &\quad = \sum_{s \leqq n+(1/2)} \langle \zeta(1/2) \otimes \zeta(m, n), \zeta(s, n+(1/2)) \rangle T(D_s^+)\zeta(s, n+(1/2)). \end{aligned}$$

The left hand of (5.7) is

$$\begin{aligned} &U_g(D_{1/2}^+)\zeta(1/2) \otimes T(D_m^+)\zeta(m, n) \\ &\quad = c(m, n)U_g(D_{1/2}^+)\zeta(1/2) \otimes U_g(D_m^+)\zeta(m, n) \\ &\quad = c(m, n)U_g(D_{1/2}^+ \otimes D_m^+)(\zeta(1/2) \otimes \zeta(m, n)) \\ &\quad = c(m, n) \sum_{s \leqq n+(1/2)} \langle \zeta(1/2) \otimes \zeta(m, n), \zeta(s, n+(1/2)) \rangle U_g(D_s^+) \\ &\qquad \times \zeta(s, n+(1/2)). \end{aligned}$$

By (2.7), the comparison of coefficients of the term $T(D_{n+(1/2)}^+)\zeta(n+(1/2))$, which is equal to $U_g(D_{n+(1/2)}^+)\zeta(n+(1/2))$ (cf. (5.3)), leads to

$$c(m, n) = 1, \qquad \text{for all } m \text{ and } n. \qquad \text{q.e.d.}$$

LEMMA 5. *For a given unitary admissible operator field $\boldsymbol{T} = \{T(\omega)\}$ and some g in G, suppose*

$$T(D_m^+)\zeta(m, n) = U_g(D_m^+)\zeta(m, n), \qquad \textit{for all } m \textit{ and } n.$$

Then for any irreducible unitary representation ω, we have

$$T(\omega) = U_g(\omega). \tag{5.8}$$

I. e.,

$$\boldsymbol{T} = \boldsymbol{U}_g. \tag{5.9}$$

PROOF. Any v in $\mathfrak{H}(\omega)$ can be expanded by the orthogonal basis $\{\zeta_k(\omega)\}$ of weight vectors, that is,

$$v = \sum_k a_k \zeta_k(\omega).$$

Consider the Kronecker product $D_m^+ \otimes \omega$, then

$$\zeta(m, n) \otimes v = \sum_k a_k \zeta(m, n) \otimes \zeta_k(\omega).$$

The results of § 2 show that for sufficiently large m, $D_m^+ \otimes \omega$ contains D_s^+ as a discrete irreducible component and

$$\langle \zeta(s), \zeta(m, n) \otimes v \rangle = \sum_k \bar{a}_k \langle \zeta(s), \zeta(m, n) \otimes \zeta_k(\omega) \rangle$$
$$= \bar{a}_{s-n} \langle \zeta(s), \zeta(m, n) \otimes \zeta_{s-n}(\omega) \rangle .$$

As $c_s \equiv \langle \zeta(s), \zeta(m, n) \otimes \zeta_{s-n}(\omega) \rangle$ is not zero, so

$$\bar{a}_{s-n} = (c_s)^{-1} \langle \zeta(s), \zeta(m, n) \otimes v \rangle .$$

Putting

$$v = T(\omega) v_0 ,$$

in this equality, we have

$$\begin{aligned} \bar{a}_{s-n} &= (c_s)^{-1} \langle \zeta(s), \zeta(m, n) \otimes T(\omega) v_0 \rangle \\ &= (c_s)^{-1} \langle \zeta(s), T(D_m^+ \otimes \omega)((T(D_m^+))^{-1} \zeta(m, n) \otimes v_0) \rangle \\ &= (c_s)^{-1} \langle (T(D_s^+))^{-1} \zeta(s), ((T(D_m^+))^{-1} \zeta(m, n) \otimes v_0) \rangle \\ &= (c_s)^{-1} \langle U_{g^{-1}}(D_s^+) \zeta(s), (U_{g^{-1}}(D_m^+) \zeta(m, n) \otimes v_0) \rangle \\ &= (c_s)^{-1} \langle \zeta(s), U_g(D_m^+ \otimes \omega)(U_{g^{-1}}(D_m^+) \zeta(m, n) \otimes v_0) \rangle \\ &= (c_s)^{-1} \langle \zeta(s), \zeta(m, n) \otimes U_g(\omega) v_0 \rangle , \end{aligned} \tag{5.10}$$

from the assumption of admissibility of the operator field $\boldsymbol{T}$. The right hand of (5.10) is equal to the coefficient of $U_g(\omega) v_0$, that is to say,

$$T(\omega) v_0 = \sum_k a_k \zeta_k(\omega) = U_g(\omega) v_0 .$$

This proves (5.8) and (5.9). q. e. d.

The main theorem follows immediately from the results of § 4 and lemmas 4 and 5.

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References

[1] L. Pontrjagin, Topological groups, Princeton Univ. Press, 1939

[2] T. Tannaka, Ueber den Dualitätsatz der nichtkommutativen topologischen Gruppen, Tôhoku Math. J., **45** (1939), 1-12.

[3] Harish-Chandra, Lie algebras and the Tannaka duality theorem, Ann. of Math., **51** (1950), 299-330.

[4] V. Bargmann, Irreducible unitary representations of the Lorentz group, Ann. of Math., **48** (1947), 568-640.

[5] L. Pukánszky, On the Kronecker products of irreducible representations of the 2×2 real unimodular group. I, Trans. of Amer. Math. Soc., **100** (1961), 116-152.

[6] J. Dixmier, Les algèbres d'operateurs dans l'espace Hilbertien, Gauthier-Villars, 1957.

[7] N. Tatsuuma, A duality theorem for the real unimodular group of second order, Proc. Japan Acad., **40** (1964), 638-642.