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On Hartogs-Osgood's theorem for Stein spaces

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§ 1. Introduction.

1. In this paper, we shall show the following fact:

Let X be a connected normal Stein space with $\dim X \geqq 2$, *K a compact subset and D an open subset of X containing K. Assume that $D-K$ is connected. Then we have the followings:* (1) *Every holomorphic function in $D-K$ can be continued holomorphically into the whole D.* (2) *Every meromorphic function in $D-K$ can be continued meromorphically into the whole D.* (3) *Every holomorphic mapping of $D-K$ into a Stein space Y can be extended to that of D into Y.* (§ 5, Theorem 3).

When X is the complex Euclidean space C^n, (1) is well-known as a Hartogs-Osgood's theorem ([**10**], [**14**]) and (2) is essentially due to E. E. Levi ([**13**]), which were proved completely by A. B. Brown ([**4**]). In the previous paper H. Fujimoto-K. Kasahara [**7**], (1) for complex manifolds was discussed. For example, (1) is true if X is a Stein manifold.

The proof of the above, which we shall conclude in § 5, is divided into two parts. A method of the global continuation (§ 2, Theorem 1) is almost similar to that given in [**7**]. In the local continuation, a Bishop's theorem ([**3**]), from which we have easily a characterization of connected normal Stein spaces (§ 3, Theorem 2), plays the essential role. In § 4, we shall discuss some properties of real analytic functions, which will be used in § 5.

In the above fact, if we take off the assumption of the connectedness of $D-K$, we have the followings: (1′) For any holomorphic function f in $D-K$, we can find a holomorphic function in D which coincides with f on a non-empty open subset of $D-K$. (2′) and (3′) are (2) and (3) modificated in the same way, respectively. The proofs are included in Theorem 1 and Lemma 4.

2. For a Stein space X which is not normal, the homological codimension of X is related to the problem. (Cf. A. Andreotti-H. Grauert [**1**].) Our main theorem of this paper is the following:

THEOREM. *Let X be a Stein space with* $\operatorname{dih} X$ (*$=$the homological codimension of X*) $\geqq 2$, *K a compact subset and D an open subset of X containing K. Assume that each irreducible component of D is also irreducible in $D-K$. Then* *we have the same* (1), (2) *and* (3) *as the above.*

We shall prove this Theorem in § 6. There, considering the normalization of X, we shall apply the above fact and use an Asami's theorem ([**2**]) and a Scheja's theorem ([**18**]).

In Theorem, if we assume $D=X$ and take off the assumption that each irreducible component of D is irreducible in $D-K$, we have the same (1′), (2′) and (3′). This (1′) is included in an Andreotti-Grauert's theorem ([**1**], Théorème 15) as a special case, and has been proved again by H. Rossi ([**17**]) in case X is normal. Our proof is the same as the proof of Theorem (§ 6) by using the following Rossi's theorem ([**17**], Theorem 6.3): On a connected normal Stein space X with $\dim X \geqq 2$, the complement of a compact set has only one unbounded connected component.

3. We make here two remarks.

By Fujimoto's theorems ([**6**]), (1) of Theorem implies that under the hypothesis of Theorem every $\mathcal{F}$-valued holomorphic function in $D-K$ admits an $\mathcal{F}$-valued holomorphic continuation in D, where $\mathcal{F}$ is a Fréchet space over C. And we have the same generalizations as in [**7**, § 8].

If X is a Stein space with dih $X=1$, Theorem is not always true. Let X be the analytic subspace $\{z_1z_4-z_2z_3=0,\ z_2^3-z_1^2z_3=0,\ z_3^3-z_4^2z_2=0\}$ of C^4. We put $K=\{(0,0,0,0)\}$ and $D=X$. X is an irreducible and locally irreducible Stein space of pure dimension 2, and $D-K$ is irreducible because it is connected. We consider the following function f: $f=\frac{z_2^2}{z_1}$ if $z_1\neq 0$, $f=\frac{z_1z_3}{z_2}$ if $z_2\neq 0$, $f=\frac{z_2z_4}{z_3}$ if $z_3\neq 0$ and $f=\frac{z_3^2}{z_4}$ if $z_4\neq 0$. Using the holomorphic mapping of C^2 onto X defined by $z_1=w_1^4$, $z_2=w_1^3w_2$, $z_3=w_1w_2^3$, $z_4=w_2^4$, we can show that f is holomorphic in $D-K$ but it can not be continued holomorphically into D.

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§ 2. The global continuation.

1. In this paper, we say merely that a curve lies in a set even if it lies in the set with the exception of end points, and we assume that neighborhoods are always open. First we give some definitions.

DEFINITION 1. Let X be a locally arcwise connected, locally compact Hausdorff space. A real-valued continuous function v on X is called *pre-admissible* if it satisfies the followings:

(i) v satisfies the maximum principle, that is, to any $p\in X$ we can find points p_ν satisfying $v(p_\nu)>v(p)$ and $\lim_{\nu\to\infty} p_\nu=p$.

(ii) To any real numbers $\rho<\rho'$, each connected component of the set

$\{p \in X \mid \rho \leqq v(p) \leqq \rho'\}$ is compact.

(iii) Any $p \in X$ has a fundamental system $\mathfrak{W}$ of neighborhoods such that, for any $W \in \mathfrak{W}$, W and $W \cap \{v > v(p)\}$ are connected.

DEFINITION 2. Let v be a pre-admissible function on X. The boundary ∂B of a relatively compact open set B is *good for* v if it satisfies the followings:

(i) Let U be a neighborhood of $p \in \partial B$ and Δ one of the sets $B \cap \{v > v(p)\}$, $\partial B \cap \{v > v(p)\}$ and $(X-B) \cap \{v < v(p)\}$. Then, p has a neighborhood V contained in U such that any point of $V \cap \Delta$ can be joined to p by a curve in $U \cap \Delta$.

(ii) There exist finitely many real numbers $\rho_0 > \rho_1 > \cdots > \rho_s$ such that, if $\rho \neq \rho_i$ for all i, every point $p \in \partial B \cap \{v = \rho\}$ has fundamental systems $\mathfrak{W}'$ and $\mathfrak{W}''$ of neighborhoods as follows:

(a) To any $W' \in \mathfrak{W}'$, $W' \cap (X-\bar{B}) \cap \{v > \rho\}$ and $W' \cap B \cap \{v > \rho\}$ are non-empty and connected.

(b) To any $W'' \in \mathfrak{W}''$, $W'' \cap (X-\bar{B})$ and $W'' \cap B$ are connected.

REMARK: Since X is locally arcwise connected, a connected open set in X is arcwise connected. In Definition 2 (i), making V small, we may take $(X-B) \cap \{v > v(p)\}$ also as Δ because of Definition 1 (iii). Furthermore, if $v(p) \neq \rho_i$ for all i, we may take $(X-B) \cap \{v \geqq v(p)\}$ also as Δ. In Definition 2 (ii) (a), we have easily that $W' \cap \partial B \cap \{v > \rho\}$ is not empty and $W' \cap (X-B) \cap \{v > \rho\}$ is arcwise connected.

DEFINITION 3. If there exists a locally homeomorphic mapping τ of a topological space $\mathcal{A}$ onto a topological space X, the pair $(\mathcal{A}, \tau, X)$ is called *a sheaf* $\mathcal{A}$ *of sets over* X. We denote by $\Gamma(U, \mathcal{A})$ the set of all continuous sections of $\mathcal{A}$ on $U \subset X$. A sheaf $\mathcal{A}$ of sets over X is called *hard* if the natural mapping $\Gamma(U, \mathcal{A}) \rightarrow \Gamma(V, \mathcal{A})$ is always injective for any pair of a connected open set $U \subset X$ and an open set $V \subset U$.

DEFINITION 4. Let v be a pre-admissible function on X and $\mathcal{A}$ a hard sheaf of sets over X. The function v is called *admissible for* $\mathcal{A}$ if it satisfies the following: Let U be a neighborhood of $p \in X$ such that $U^+ = U \cap \{v > v(p)\}$ is connected. Then, for any $f \in \Gamma(U^+, \mathcal{A})$, p has a neighborhood V such that there exists $\tilde{f} \in \Gamma(V \cup U^+, \mathcal{A})$ satisfying $f = \tilde{f}$ on $V \cap U^+$.

2. Now, under these definitions, we can state a theorem of the global continuation as follows:

THEOREM 1. *Let X be a locally arcwise connected, locally compact Hausdorff space which has a countable basis of open sets. Let $\mathcal{A}$ be a hard sheaf of sets over X and v an admissible function for $\mathcal{A}$ on X. Let K be a compact subset and D an open subset of X containing K. Suppose that there exists an open set B which satisfies $K \subset B \Subset D$ and whose boundary ∂B is good for v. Then, for any $f \in \Gamma(D-K, \mathcal{A})$, we can find an element of $\Gamma(D, \mathcal{A})$ which* *coincides with f on a connected component of $D-K$. Particularly, if $D-K$ is connected, the natural mapping $\Gamma(D, \mathcal{A})\to\Gamma(D-K, \mathcal{A})$ is bijective.*

3. The remaining part of this section is devoted to the proof of Theorem 1, which is done by almost same way as in [7, §7]. A modification is necessary because the space X is not a manifold and we can not use the fact stated in [7, §7, (b)]. Here, we shall sketch the proof shortly and point out the places at which the proof in [7, §7] needs to be modified.

Using the same notations as in Definitions 1-4 and Theorem 1, we may assume $\rho_0=\sup_{p\in B} v(p)$ and $\rho_s=\inf_{p\in B} v(p)$. We put $\rho_{-1}=+\infty$. If $\inf_{p\in X} v(p)=-\infty$, we put $\rho_{s+\nu}=\rho_s-\nu$ $(\nu=1, 2, \cdots)$, and otherwise we put $\rho_{s+1}=\inf_{p\in X} v(p)$. We denote by $X_{\alpha,\beta}$ the set $\{p\in X\mid \alpha<v(p)<\beta\}$.

We assume at all times that a neighborhood U of $p\in X$ is so small that it satisfies the followings: (1) If $p\in X_{\rho_{i+1},\rho_i}$, then $U\subset X_{\rho_{i+1},\rho_i}$. If $v(p)=\rho_i$, then $U\subset X_{\rho_{i+1},\rho_{i-1}}$. (2) If $p\notin\partial B$, then $U\subset B$ or $U\subset\bar{B}^c$. If $p\in\partial B$, then $U\subset D-K$.

At first, we show

(α) *Let us assume $\rho_{i+1}\leqq\rho<\rho'<\rho_i$. Let G be an open subset of X_{ρ,ρ_i} such that any point of $\partial G\cap X_{\rho,\rho_i}$ has a neighborhood U satisfying $U\cap G=U\cap B$ or $U\cap G=U\cap\bar{B}^c$, and C a connected component of G. Then, $C_{\rho'}=C\cap\{v>\rho'\}$ is connected.* ([7], §7, (c)).

PROOF. Let L be the set of curves in C joining q' to q'', where q' and q'' are two points of $C_{\rho'}$. We put $\rho''=\sup_{l\in L}\inf_{p\in l} v(p)$. Suppose that $\rho''\leqq\rho'$.

To any point $p\in\bar{C}\cap\{v=\rho''\}$, we give two neighborhoods $V'(p)\Subset U'(p)$ satisfying the followings: If $p\in\partial G$, then $U'(p)\cap B\cap\{v>\rho''\}$ and $U'(p)\cap\bar{B}^c\cap\{v>\rho''\}$ are connected, and $U'(p)\cap G=U'(p)\cap B$ or $U'(p)\cap G=U'(p)\cap\bar{B}^c$ holds. If $p\in G$, then $U'(p)\cap\{v>\rho''\}$ is connected and $U'(p)\subset G$ holds. Since $\bar{C}\cap\{v=\rho''\}$ is compact, we can take finitely many points $p_1, \cdots, p_t$ satisfying $\bigcup_{\nu=1}^{t} V'(p_\nu)\supset\bar{C}\cap\{v=\rho''\}$. We write U'_ν and V'_ν instead of $U'(p_\nu)$ and $V'(p_\nu)$, respectively. We put $V_1=V'_1$. For $1\leqq\mu<\nu$, we denote by $V^\kappa_{\nu\mu}$ each connected component of $V'_\nu\cap V'_\mu$ such that the connected component of $U'_\nu\cap U'_\mu$ containing it does not intersect $G\cap\{v>\rho''\}$. We put $V_\nu=V'_\nu-\overline{\bigcup_{\mu=1}^{\nu-1}\bigcup_\kappa V^\kappa_{\nu\mu}}$ and want to show $\bigcup_{\nu=1}^{t} V_\nu\supset\bar{C}\cap\{v=\rho''\}$. Suppose that it were not true. We can take a point $p\in V'_\nu\cap\overline{(\bigcup_\kappa V^\kappa_{\nu\mu})}\cap\bar{C}\cap\{v=\rho''\}$ for suitable $1\leqq\mu<\nu\leqq t$. Let Δ be a connected component of $U'_\nu\cap U'_\mu$ containing p. Since $\Delta\cap\left(\bigcup_\kappa V^\kappa_{\nu\mu}\right)\neq\phi$, we have $\Delta\cap G\cap\{v>\rho''\}=\phi$. This implies $p\in\partial B$ and $\Delta\cap\partial B\cap\{v>\rho''\}=\phi$. Since Δ is a neighborhood of p, this contradicts Definition 2 (ii) (a).

Since $C\cap\{\rho''-\varepsilon<v\leqq\rho''\}$ is relatively compact in X for $\varepsilon>0$, we can take $l\in L$ satisfying $l\cap\{v\leqq\rho''\}\subset\bigcup_{\nu}V_{\nu}$. We have easily that $l\cap(\bigcup_{\nu}V_{\nu})$ can be replaced by arcs in $G\cap\{v>\rho''\}$. This contradicts the definition of ρ''.

q.e.d.

The following fact can be proved in the same way as in [**7**, §7, (d)], by using Definition 2 (ii) instead of (b) in [**7**, §7].

(β) *Let C be a connected component of $(X-B)\cap\{\rho\leqq v\leqq\rho_i\}$ or $(X-B)\cap\{\rho_{i+1}<v\leqq\rho_i\}$, where $\rho_{i+1}<\rho<\rho_i$. For any ρ' satisfying $\rho<\rho'<\rho_i$, the set $C\cap\{v\geqq\rho'\}$ is connected.*

(γ) *Let us assume $(X-B)\cap\{v=\rho_i\}=O^1\cup S^1$, where O^1 and S^1 are closed and disjoint. The set O^2 consists of all points p in $(X-B)\cap X_{\rho_{i+1},\rho_i}$ satisfying that the connected component of $(X-B)\cap\bar{X}_{v(p),\rho_i}$ containing p intersects O^1. We put $S^2=(X-B)\cap X_{\rho_{i+1},\rho_i}-O^1$, $O=O^1\cup O^2$ and $S=S^1\cup S^2$. Then S and O are relatively closed in $(X-B)\cap\{\rho_{i+1}<v\leqq\rho_i\}$.* ([7], §7, (e)).

PROOF. Let us assume $\lim_{\nu\to\infty}p_{\nu}=p$ where p_{ν} and p are points in $(X-B)\cap\{\rho_{i+1}<v\leqq\rho_i\}$. If $p_{\nu}\in O$, then $p\in O$. If $p_{\nu}\in S$ and $v(p_{\nu})\geqq v(p)$, then $p\in S$. These can be proved in the same way as in [**7**, §7, (e), (i) and (ii)] by using Definition 2 and its Remark.

Suppose $p_{\nu}\in S$, $p\in O$ and $v(p_{\nu})<v(p)$. By Definition 2 (i), there exists a curve $c(t)$, $0\leqq t\leqq 1$, in $(X-B)\cap\{v<v(p)\}$, where $c(0)=p_{\nu}$ and $c(1)=p$. Putting $t_0=\sup\{t\mid \text{if } 0\leqq t'\leqq t, \text{ then } c(t')\in S\}$, we have $0<t_0<1$. In fact, $c(0)\in S$ implies $0<t_0$, and $t_0<1$ is obtained by considering the point where v attains $\min_{0\leqq t\leqq 1}v(c(t))$. Since $c(t_0)\in\bar{O}$, we have $c(t_0)\in O$. On the other hand, by the definition of O we have $v(c(t))\geqq v(c(t_0))$ for $0\leqq t\leqq t_0$, which implies $c(t_0)\in S$. This is a contradiction. q.e.d.

4. Take a section $f\in\Gamma(D-K,\mathcal{A})$. By induction, we shall continue f to K. Now, we assume the following four facts:

(I) B_i^* is an open set such that $B\subset B_i^*\Subset X$, $B_i^*\cap\{v\leqq\rho_i\}=B\cap\{v\leqq\rho_i\}$, and $B_i^*\cap\{v>\rho_{i-1}\}=B_{i-1}^*\cap\{v>\rho_{i-1}\}$.

(II) Any $p\in\partial B_i^*\cap\{v>\rho_i\}$ has a neighborhood U satisfying $U\cap B_i^*=U\cap B$.

(III) There exists a unique section $g_{\rho_i}\in\Gamma(\{D\cup B_i^*\}\cap\{v>\rho_i\},\mathcal{A})$ satisfying $g_{\rho_i}=f$ (as germs) at each point of $(D-B_i^*)\cap\{v>\rho_i\}$.

(IV) Any s.b.p.2. and any o.b.p. in $X_{\rho_i,\rho_{i-1}}$ can never be joined by a cancelable curve in $X_{\rho_i,\rho_{i-1}}$.

Terminologies used here are as follows: A point of $(B_i^*-B)\cap\partial B$ is called an s.b.p. (singular boundary point) and one of $(X-B_i^*)\cap\partial B\cap\{v>\rho_i\}$ is called an o.b.p. (ordinary boundary point). Let p be an s.b.p.. If $g_{\rho_i}\neq f$ at p, p is called an s.b.p.1.. If $g_{\rho_i}=f$ at p, p is called an s.b.p.2.. Let L be the set of curves c such that $c\cap\partial B$ consists of at most finitely many points and at each of them c runs from the interior of B to the exterior of B or conversely. Let $\mathfrak{G}$ be a non-commutative group generated by three elements {o. b. p., s. b. p. 1., s. b. p. 2.} satisfying the only relations $(\text{o. b. p.})^2=(\text{s. b. p. 1})^2=(\text{s. b. p. 2.})^2=e$, where e is the unit of $\mathfrak{G}$. Take $c\in L$. We can consider c as an element $\tilde{c}$ of $\mathfrak{G}$, that is, if $c\cap\partial B$ except end points consists of $\{q_1,\cdots,q_k\}$ in the order on c, then $\tilde{c}=q_1\cdots q_k$, and if $c\cap\partial B$ except end points is empty, then $\tilde{c}=e$. A curve c in L is called *cancelable* if $\tilde{c}=e$.

For $i=0$, if we put $B_0^*=B$, the above assumptions are true. Let us assume all of the above four for i.

5. Now, we construct B_{i+1}^*. The proof of the following is the same as in [7, §7, (f)].

(δ) $\overline{(B_i^*-B)}\cap\overline{(X-B_i^*)\cap\{v>\rho_i\}}\cap\{v=\rho_i\}=\phi$.

We decompose the set $(X-B)\cap\{v=\rho_i\}$ into the union $O^1\cup S^1$ of two disjoint sets. Take $p\in(X-B)\cap\{v=\rho_i\}$. Let U be a neighborhood of p such that $U\cap\{v>\rho_i\}$ is connected. If $p\in\overline{(B_i^*-B)}$, then $p\in S^1$. When we can assume $U\cap\{v>\rho_i\}\subset B$ by making U small, p belongs to S^1 if and only if $g_{\rho_i}\neq f$ in $U\cap\{v>\rho_i\}$, or $g_{\rho_i}=f$ in $U\cap\{v>\rho_i\}$ and p can be joined to an s. b. p. 2. in $X_{\rho_i,\rho_{i-1}}$ by a cancelable curve in $X_{\rho_i,\rho_{i-1}}$. Otherwise p belongs to O^1. The following is the same as in [7] §7 (g).

(ε) *O^1 and S^1 satisfy the assumptions of* (γ).

Applying (γ) to these O^1 and S^1, we have sets O and S as in (γ). We put $B_{i+1}^*=B_i^*\cup S$. We can easily see that B_{i+1}^* satisfies (I) and (II) except $B_{i+1}^*\Subset X$ by (γ).

We show $B_{i+1}^*\Subset X$. If C is a connected component of $(X-B)\cap\{\rho_{i+1}<v\leqq\rho_i\}$, then we have $C\subset S$ or $C\subset O$, because of (β) and the definition of O. By Definition 2 (ii), $\max_{p\in C} v(p)$ must be ρ_i. This implies that $C\cap\bar{B}_i^*\neq\phi$ if $C\subset S$. Consequently, the union of connected components of $\bar{X}_{\rho_{i+1},\rho_0+\varepsilon}$ intersecting $\bar{B}_{i+1}^*$ is equal to that intersecting $\bar{B}_i^*$, where $\varepsilon>0$. Since $\bar{B}_i^*$ is compact, we can show that the number of connected components of $\bar{X}_{\rho_{i+1},\rho_0+\varepsilon}$ intersecting $\bar{B}_i^*$ is finite. Hence, we have $B_{i+1}^*\Subset X$.

The proof of (III) in B_{i+1}^* can be obtained by the same way as in [7, Lemma 7]. (The proof of [7] Lemma 6 used there needs to be modified, but it is easy. Cf. the Proof of (α).) The proof of (IV) in B_{i+1}^* is also similar as in [7] §7 n°. 5, by using Definition 2 instead of [7] §7 (b).

By induction, we have $g_{-\infty}\in\Gamma(D\cup B_{-\infty}^*,\mathcal{A})$ and $g_{-\infty}=f$ holds in $D-B_{-\infty}^*$ if $\inf_{p\in X} v(p)=-\infty$, and we have $g_{\rho_{s+1}}\in\Gamma(\{D\cup B_{s+1}^*\}\cap\{v>\rho_{s+1}\},\mathcal{A})$ if $\inf_{p\in X} v(p)=\rho_{s+1}$. In the latter case, we can continue $g_{\rho_{s+1}}$ to $D\cap\{v=\rho_{s+1}\}$ and we regard $g_{\rho_{s+1}}$ as an element of $\Gamma(D,\mathcal{A})$, which coincides with f in $(D-B_{s+1}^*)$

$\cap \{v > \rho_{s+1}\}$.

Assume that $D-K$ is connected. Then, the natural mapping $\Gamma(D, \mathcal{A}) \to \Gamma(D-K, \mathcal{A})$ is surjective. From Definition 1 (i), it follows that X has no compact connected component and accordingly D is connected. This implies that $\Gamma(D, \mathcal{A}) \to \Gamma(D-K, \mathcal{A})$ is injective. Thus, Theorem 1 has been proved.

§ 3. Preliminaries on complex spaces.

1. In this paper, an analytic covering, a complex space and a normal complex space mean 'analytisch verzweigte Überlagerung', 'β-Raum' and 'β_n-Raum ($=\alpha$-Raum)' in the sense of H. Grauert-R. Remmert [**9**], respectively. A Stein space is a complex space which is paracompact, holomorphically convex and holomorphically separable. A mapping τ of a topological space X into a topological space Y is called almost proper if τ is continuous and each connected component of $\tau^{-1}(K)$ is compact for each compact subset K of Y.

Let X be a purely n-dimensional Stein space. By a Bishop's theorem ([**3**, Theorem 4]), there exists an almost proper holomorphic mapping τ of X into C^n. Since X has no compact analytic set of positive dimension, τ is nowhere degenerate ('nirgends entartet'). We have easily that τ is open and locally proper (cf. R. Remmert [**15**] Satz 28 and H. Grauert [8] Satz 1).

2. The following theorem, which interests us by comparison with the characterization of an open Riemann surface as a covering surface over C^1, is an immediate consequence of the above Bishop's theorem.

THEOREM 2. *Let X be a connected Hausdorff space. Under this assumption, X is a normal Stein space if and only if there exists an almost proper mapping τ of X into C^n such that each $p \in X$ has a neighborhood U for which $\tau(U)$ is an open set in C^n and $(U, \tau | U, \tau(U))$ is an analytic covering.*

PROOF. If X is a connected normal Stein space, then X is purely n-dimensional and there exists an almost proper holomorphic mapping τ of X into C^n which is open and locally proper. Hence, we can show easily that each $p \in X$ has a neighborhood U such that $(U, \tau | U, \tau(U))$ is an analytic covering, (cf. R. Remmert-K. Stein [**16**] Satz 3). Conversely, if $\tau: X \to C^n$ satisfies the conditions, we can easily see that X is a normal complex space and τ is holomorphic and nowhere degenerate. Thus, X is K-complete and X is a countable union of compact sets (H. Grauert [8]). We denote by $\hat{K}_X$ the set $\{p \in X \mid |f(p)| \leq \sup_{q \in K} |f(q)|$ for any holomorphic function f on $X\}$. For a connected compact set K in X, we have $\hat{K}_X \subset \tau^{-1}(\widehat{\tau(K)}_{C^n})$. Since C^n is holomorphically convex and τ is almost proper, there exists a polycylinder Z containing $\widehat{\tau(K)}_{C^n}$ and the connected component U of $\tau^{-1}(Z)$ containing K is relatively compact in X.

Since $\hat{K}_X \subset \tau^{-1}(Z)$, we have $\partial U \cap \hat{K}_X = \phi$. Thus, we can construct an analytic polyhedron P containing K. We have a sequence of analytic polyhedra $\{P_\nu\}$ such that $P_\nu \Subset P_{\nu+1}$ for all ν and $\bigcup_{\nu=1}^{\infty} P_\nu = X$. Since each P_ν is a Runge domain in $P_{\nu+1}$ and a Stein space, X is a Stein space (K. Stein [**19**]).

Let $(\tilde{X}, \mu)$ be the normalization of a complex space X (H. Grauert-R. Remmert [**9**]). By a Grauert's theorem ([8] Satz A and B), we have easily that if X is a Stein space, so is $\tilde{X}$, too.

3. Let X be a complex space and $\mathcal{S}$ a coherent analytic sheaf over X. We denote by $\mathrm{dih}_p \mathcal{S}$ the homological codimension of $\mathcal{S}$ at p, whose definition and properties are given in A. Andreotti-H. Grauert [**1**] Chap. 1. (They call it 'la dimension homologique'.) By definition, we put $\mathrm{dih}\, \mathcal{S} = \min_{p \in X} \mathrm{dih}_p \mathcal{S}$, $\mathrm{dih}_p X = \mathrm{dih}_p \mathcal{O}$ and $\mathrm{dih}\, X = \mathrm{dih}\, \mathcal{O}$, where $\mathcal{O}$ is the structural sheaf of X. Here, we point out the following fact given by G. Scheja.

LEMMA 1. *Let M be an analytic subspace of an open subset D in C^n and $\mathcal{O}$ the structural sheaf of M. We assume $\mathrm{dih}_p M \geqq 2$ for a point $p \in M$. Then p has a fundamental system $\mathfrak{U}$ of neighborhoods such that, for each $U \in \mathfrak{U}$, the natural mapping $\Gamma(M \cap U, \mathcal{O}) \to \Gamma((M-\{p\}) \cap U, \mathcal{O})$ is bijective.*

PROOF. Since it is trivial if $M = D$, we may assume $n \geqq 3$ and $M \subsetneqq D$. Let $\tilde{\mathcal{O}}$ be the structural sheaf of D and $\mathcal{J}$ the sheaf of ideals of the analytic set M. We have $\mathcal{O} \cong \tilde{\mathcal{O}}/\mathcal{J}$ by definition. We have easily $\mathrm{dih}_p \mathcal{J} = \mathrm{dih}_p M + 1 \geqq 3$. Take a Stein neighborhood U of p in C^n satisfying $U \subset D$. By a Scheja's theorem ([**18**] Satz 3* Korollar, in which the notation $\mathrm{codim}_x A - \mathrm{hd}_x \mathfrak{G}$ is changed by $\mathrm{dih}_x \mathfrak{G} - \dim_x A$ according to our notations), $H^1(U, \mathcal{J}) \to H^1(U-\{p\}, \mathcal{J})$ is surjective, and hence $H^1(U-\{p\}, \mathcal{J}) = 0$. This implies that $\Gamma(U-\{p\}, \tilde{\mathcal{O}}) \to \Gamma(U \cap (M-\{p\}), \mathcal{O})$ is surjective. Since $\Gamma(U, \tilde{\mathcal{O}}) \cong \Gamma(U-\{p\}, \tilde{\mathcal{O}})$, $\Gamma(U, \tilde{\mathcal{O}}) \to \Gamma(U \cap (M-\{p\}), \mathcal{O})$ is surjective, from which the proof follows.

§4. Some properties of real analytic functions.

1. The following lemma was given by H. Fujimoto in case v is a polynomial in two variables of degree two. The method of his proof can be applied for a general case as follows:

LEMMA 2. *Let v be a real analytic, strongly plurisubharmonic function in a neighborhood G of the origin 0 in C^n, where $n \geqq 2$ and $v(0) = 0$. Then, there exists a fundamental system $\mathfrak{B}$ of connected neighborhoods of the origin satisfying the following: Let N be a thin analytic set in G, U a neighborhood of the origin and f a holomorphic function in U such that $N \cap U \subset \{z \in U \mid f(z) = 0\} \subsetneqq U \subset G$. Then, to any $V \in \mathfrak{B}$ satisfying $V \Subset U$, every curve in $V - N$ with* *both end points in* $(V-N)\cap\{v>0\}$ *is homotopic in* $V-N$ *to a curve in* $(V-N)$ $\cap\{v>0\}$.

PROOF. For a suitable system of local coordinates $z_1, \cdots, z_n$ in a neighborhood of the origin, we may assume $v=2\operatorname{Re}\varphi+\sum\alpha_{\nu\bar{\mu}}z_\nu\bar{z}_\mu+o(|z|^2)$, where $(\alpha_{\nu\bar{\mu}})$ is a positive definite Hermitian matrix and φ is one of the polynomials (i) 0, (ii) z_1 and (iii) $\sum\beta_{\nu\mu}z_\nu z_\mu$ satisfying $\beta_{nn}=0$.

For the case (i) the lemma is trivial, and therefore we assume $\varphi\neq 0$. Putting $x=(z_1, \cdots, z_{n-1})$ and $y=z_n$, we can take a neighborhood $V=V_x\times V_y$ $=\{|z_1|<\rho_1, \cdots, |z_{n-1}|<\rho_{n-1}\}\times\{|z_n|<\rho_n\}$ such that, to any $x\in V_x$, $\{y\in V_y|$ $v(x,y)>0\}$ is a non-empty connected set and $\{y\in V_y|v(x,y)<0\}$ is relatively compact in V_y. In fact, this is possible. We put $y=\xi+i\eta$. Making V small, we can take a positive number κ such that $v'\leqq v$ in V where $v'=2\operatorname{Re}\varphi$ $+\kappa\sum|z_\nu|^2$, and by Späth's theorem we have

$$v=\{\xi^2+a(x,\eta)\xi+b(x,\eta)\}q(x,\xi,\eta)$$

in V where a, b and q are real analytic functions in V satisfying $a(0)=b(0)=0$ and $q\neq 0$ in V. Since $\{y|v'(x,y)>0\}$ is the exterior of a disc whose center and radius converge to 0 if $x\to 0$, we have $\{y\in V_y|v(x,y)>0\}\neq\phi$ and $\{y\in V_y|v(x,y)<0\}\Subset V_y$ for any $x\in V_x$ by making V_x small in contrast to V_y. Regarding $v=0$ as the polynomial equation in ξ of degree two, we can show easily that $\{y\in V_y|v(x,y)>0\}$ is connected for a fixed x in V_x.

We take N, U and f as in the lemma, and assume $V\Subset U$. The function f can be represented by a Hartogs' series $\sum a_\nu(x)y^\nu$ in $\bar{V}$. Let M be the set $\{x\in\bar{V}_x|a_\nu(x)=0, \nu=0,1,2,\cdots\}$. M is an analytic set of dimension at most $n-2$.

Take a point x_0 in V_x-M. Since $f(x_0,y)$ is holomorphic in $\bar{V}_y$ and $f(x_0,y)$ $\neq 0$, $\{y\in\bar{V}_y|f(x_0,y)=0\}$ is a finite set. When it is not empty, we denote it by $\{y_1, \cdots, y_r\}$. Since $f(x_0,y)\neq 0$, in a neighborhood of each (x_0,y_i), $\{f=0\}$ can be represented by the set of zeros of a distinguished polynomial of y with the center at (x_0,y_i). We denote by $\delta_{(x_0,y_i)}(x)$ its discriminant. We may assume that $\delta_{x_0}=\prod_{i=1}^{r}\delta_{(x_0,y_i)}$ is holomorphic in a neighborhood U_{x_0} of x_0 and $\delta_{x_0}\neq 0$. Making U_{x_0} small, we take $x'\in U_{x_0}\cap\{\delta_{x_0}\neq 0\}$ and denote by $\{y'_1, \cdots, y'_s\}$ the set $\{y\in\bar{V}_y|f(x',y)=0\}$. Then, $\{f=0\}$ can be represented by a holomorphic function $y=\chi_j(x)$ of x in a neighborhood of each (x',y'_j), and for x'' sufficiently near to x' the set $\{y\in\bar{V}_y|f(x'',y)=0\}$ is contained in $\{\chi_1(x''), \cdots, \chi_s(x'')\}$. When $\{y\in\bar{V}_y|f(x_0,y)=0\}$ is empty, we take a neighborhood U_{x_0} of x_0 such that $\{y\in\bar{V}_y|f(x',y)=0\}=\phi$ for any $x'\in U_{x_0}$.

Take a curve $c(t)=(x(t),y(t))$, $0\leqq t\leqq 1$, in $V-N$ such that $v(c(0))>0$ and $v(c(1))>0$. It is shown easily that there exists a point x_0 in V_x such that $v(x_0,y)>0$ for any $y\in V_y$. Since $(V-N)\cap\{v>0\}$ is connected, we may assume

$x(0)=x(1)=x_0$. Modifying the curve a little, we may assume that $f(c(t))\neq 0$ and $x(t)$ belongs to some $U_x\cap\{\delta_x\neq 0\}$ for any t. Now, we can finish the proof by the same method as in the case of two variables ([7], p. 193). We omit the details.

2. LEMMA 3. *Let $v_0, v_1, \cdots, v_r$ be real analytic functions in a neighborhood U of the origin 0 in $\mathcal{R}^m$, where $r<m$. Let N be the set $\{x\in U\,|\,v_1(x)=\cdots=v_r(x)=0\}$ and S the set $\{x\in U\,|$ the rank of the matrix $\left(\dfrac{\partial v_i}{\partial x_j}\right)_{\substack{1\leqq i\leqq r\\ 1\leqq j\leqq m}}$ at x is less than r. We assume that $0\in\overline{N-S}$. Then, there exists a neighborhood V of the origin contained in U such that the number of stationary values of the function $v_0|(N-S)\cap V$ is at most one* (cf. [7] Proposition 8 and its Corollary).

PROOF. We denote by C the set of all points at which the rank of the matrix $\left(\dfrac{\partial v_i}{\partial x_j}\right)_{\substack{0\leqq i\leqq r\\ 1\leqq j\leqq m}}$ is less than $r+1$. We can see easily that a point x belongs to $N-S$ and is not a stationary point of $v_0|(N-S)$ if and only if $x\notin C$.

If we regard the variables x as complex variables, then N, S and C are analytic sets. In a small neighborhood of the origin in C^m, the analytic set $N\cap C$ can be decomposed into the union of finitely many irreducible components, each of which contains the origin. Let C_i be a component which meets $N-S$. (If $N\cap C\cap V\subset S$ for a neighborhood V of the origin in $\mathcal{R}^m$, then $v_0|(N-S)\cap V$ has no stationary values. We omit the case.) We denote by $\mathring{C}_i$ the set of ordinary points of C_i. Since $\overline{\mathring{C}_i}=C_i$, we have $\mathring{C}_i\cap(N-S)\neq\phi$. Take a point p in $\mathring{C}_i\cap(N-S)$. We may assume that $x_{r+1},\cdots,x_m$ is a system of local coordinates of N at p. Denoting by $u_1,\cdots,u_t$ that of $\mathring{C}_i$ at p, we have $\dfrac{\partial v_0}{\partial u_j}=\sum_{i=r+1}^{m}\dfrac{\partial v_0}{\partial x_i}\dfrac{\partial x_i}{\partial u_j}\equiv 0$ for any j in a neighborhood of p on $\mathring{C}_i$. These equations assert that v_0 must be identically constant on C_i. Thus, we have $v_0\equiv v(0)$ on $(N-S)\cap C$. Restricting variables to real numbers, we conclude the proof.

§5. The case of normal Stein spaces.

1. Let X and Y be complex spaces. We denote by $\mathcal{O}(X)$ the structural sheaf of X, by $\mathcal{M}(X)$ the sheaf of germs of meromorphic functions over X and by $\mathcal{H}(X, Y)$ the sheaf of germs of holomorphic mappings of X into Y over X. If X is a locally irreducible complex space, these are hard sheaves of sets over X.

LEMMA 4. *Let X be a connected normal Stein space of dimension $n\geqq 2$ and Y a Stein space. Then, there exists an admissible function v for $\mathcal{O}(X)$, $\mathcal{M}(X)$ and $\mathcal{H}(X, Y)$ such that, for any compact set K and its neighborhood D, we can construct an open set B with the good boundary ∂B for v satisfying*

$K \subset B \Subset D$.

As a corollary of Lemma 4 and Theorem 1, we have immediately

THEOREM 3. *Let X be a connected normal Stein space of dimension $n \geqq 2$ and Y a Stein space. Let K be a compact subset of X and D an open subset such that $K \subset D \Subset X$ and $D-K$ is connected. Then, the natural mappings $\Gamma(D, \mathcal{O}(X)) \to \Gamma(D-K, \mathcal{O}(X))$, $\Gamma(D, \mathcal{M}(X)) \to \Gamma(D-K, \mathcal{M}(X))$ and $\Gamma(D, \mathcal{H}(X, Y)) \to \Gamma(D-K, \mathcal{H}(X, Y))$ are bijective.*

2. Now, we begin the proof of Lemma 4. Let τ be an almost proper holomorphic mapping of X into C^n given in Theorem 2. We denote by $\{z_1, \cdots, z_n\}$ a fixed system of coordinates of C^n. We put $\tilde{v} = \sum_{\nu=1}^{n} |z_\nu|^2$ and $v = \tilde{v} \circ \tau$. We want to show that this v satisfies our purpose.

We first prove that v is pre-admissible. Since τ is open and almost proper, v satisfies (i) and (ii) of Definition 1 trivially. For the proof of (iii), we take a point $p \in X$ and put $\tau(p) = z_0$. Let W be a neighborhood of p such that $\tau^{-1} \circ \tau(p) \cap W = \{p\}$ and $(W, \tau | W, \tau(W))$ is an analytic covering. We denote by N' the minimal critical set of the analytic covering and put $N = \tau^{-1}(N') \cap W$. Applying Lemma 2 to $\tilde{v}$, we construct a fundamental system $\mathfrak{B}$ of neighborhoods of z_0. We take $V' \in \mathfrak{B}$ satisfying $V' \subset \tau(W)$ and the conclusion of Lemma 2 for N'. Since V' is connected and $\tau^{-1} \circ \tau(p) \cap W = \{p\}$, $V = \tau^{-1}(V') \cap W$ is connected. Since $V-N$ is an unramified, unlimited and connected covering over $V'-N'$, $V \cap \{v > v(p)\}$ is connected by Lemma 2. Thus, v is pre-admissible.

Now, we show that v is admissible for $\mathcal{O}(X)$ and $\mathcal{M}(X)$. We use the same notations as above. Take a neighborhood U of p such that $U^+ = U \cap \{v > v(p)\}$ is connected and $f \in \Gamma(U^+, \mathcal{O}(X))$ (resp. $\Gamma(U^+, \mathcal{M}(X))$). We may assume $W \subset U$. We put $W^+ = W \cap \{v > v(p)\}$ and $'W^+ = \tau(W) \cap \{\tilde{v} > v(p)\}$. Since $(W^+, \tau | W^+, 'W^+)$ is also an analytic covering, there exists a polynomial $P(w; z) = w^k + a_1(z) w^{k-1} + \cdots a_k(z)$ whose coefficients are holomorphic (resp. meromorphic) in $'W^+$ such that $P(f(q); \tau(q)) = 0$ holds in W^+. We may assume that the discriminant $d(z)$ is not identically zero. By the Hartogs' (resp. Levi's) theorem of continuity ([**10**], [**13**]), $a_i(z)$ and $d(z)$ can be continued to z_0. Accordingly, making W small, we may assume that all $a_i(z)$ and $d(z)$ are holomorphic (resp. meromorphic) in $\tau(W)$. The discriminant of $P(w; z)$ in $\tau(W)$ is also $d(z)$. Let S' be the set $\{z \in \tau(W) | d(z) = 0\}$ (resp. $\{z \in \tau(W) | d(z) = 0\} \cup \{z \in \tau(W) | z$ is a singularity of some $a_i(z)$ or $d(z)\}$). We may assume that the neighborhood V' satisfies the conclusion of Lemma 2 for S'. We put $S = \tau^{-1}(S') \cap V$. Since $V-S$ is an unramified, unlimited and connected covering over $V'-S'$, f in $V \cap \{v > v(p)\}$ can be continued holomorphically to $V-S$ as a solution of the polynomial equation $P = 0$. By Lemma 2, this continuation is single-valued. By the theorem of removable singularities, we can regard f as a holomorphic (resp. meromorphic) function in V. (Throughout this proof, see H. Grauert-R. Remmert [**9**], 259-261 and 266-270). Thus, v is admissible for $\mathcal{O}(X)$ and $\mathcal{M}(X)$.

Using the same notations, we prove that v is admissible for $\mathcal{H}(X, Y)$. Let μ be a holomorphic mapping of U^+ into Y. We put $\Delta=\{y\in Y \mid$ there exists a sequence p_ν contained in U^+ satisfying $\lim_{\nu\to\infty} p_\nu=p$ and $\lim_{\nu\to\infty}\mu(p_\nu)=y\}$. Since v is admissible for $\mathcal{O}(X)$, Δ is non-empty because of the holomorphic convexity of Y, and moreover Δ consists of one and only one point y because of the holomorphic separability of Y. We may regard a neighborhood of y as an analytic subspace M of the unit polycylinder P in C^m. Furthermore, we may assume $\mu(U^+)\subset M$, because of $\Delta=\{y\}$. We put $\mu=(f_1, \cdots, f_m)$ where $f_i \in\Gamma(U^+, \mathcal{O}(X))$. From the admissibility of v for $\mathcal{O}(X)$, we may assume $f_i \in\Gamma(V, \mathcal{O}(X))$. Moreover, we may assume that V is contained in the envelope of holomorphy of U^+. (In fact, this is true if $\bar{V}'$ is contained in a Stein neighborhood of z_0 which is contained in the envelope of holomorphy of $'W^+$.) Then, since $|f_i|<1$ in U^+, we have $|f_i|<1$ on V. We denote by $\tilde{\mu}$ the mapping $(f_1, \cdots, f_m)$ of V into P. Since M can be represented by the set of common zeros of holomorphic functions in P, we have easily $\tilde{\mu}(V)\subset M$, and hence $\tilde{\mu}$ is a holomorphic mapping of V into Y for which $\tilde{\mu}=\mu$ holds on $V\cap U^+$. Thus, v is admissible for $\mathcal{H}(X, Y)$.

3. We construct here an open set B.

Let p be a point in K and let $B_1\Subset D$ be a neighborhood of p such that $(B_1, \tau|B_1, 'B_1)$ is an analytic covering satisfying $\tau^{-1}\circ\tau(p)\cap B_1=\{p\}$, where $'B_1=\tau(B_1)$. We may assume that there exists a holomorphic function f in B_1 such that an equation $P(f(q); \tau(q))=0$ holds on B_1 where $P(w; z)$ is an irreducible polynomial in w whose coefficients are holomorphic functions in $'B_1$ and whose degree is equal to the number of sheets of the covering. Furthermore, we may assume that, after a suitable affine transformation of the coordinates, we have new coordinates $z_1', \cdots, z_n'$ which satisfy $\tau(p)=0$ and the followings:

$'B_1$ is a polycylinder $Z_1\times\cdots\times Z_n$, where $Z_i=\{|z_i'|<\varepsilon_i\}$. There exist distinguished polynomials $\Delta_i(z_i'; z_{i+1}', \cdots, z_n')$ in z_i' having their centers at 0 whose coefficients are holomorphic in $Z_{i+1}\times\cdots\times Z_n$ such that every solution of $\Delta_i(z_i'; z_{i+1}', \cdots, z_n')=0$ belongs to Z_i for any $(z_{i+1}'. \cdots, z_n')\in Z_{i+1}\times\cdots\times Z_n$ $(i=1, 2, \cdots, r)$. The set of zeros of the discriminant of Δ_i is contained in the set $\Delta_{i+1}=0$ $(i=0, 1, 2, \cdots, r-1$: Here we put $z_0'=w$ and $\Delta_0=P)$. The degree of Δ_r is one.

Denoting by N_j the set

$$\{z'\in 'B_1 \mid \Delta_1=\cdots=\Delta_j=0,\ \Delta_{j+1}\neq 0\} \qquad (j=1, 2, \cdots, r;\ \text{provided } \Delta_{r+1}=1),$$

we can see that N_j is a locally analytic set without singularities and $\{z_{j+1}', \cdots, z_n'\}$ is a system of local coordinates at any point of N_j (cf. K. Kasahara

[12] Lemma 3). Similarly, the same is true for $\{P=\varDelta_1=\cdots=\varDelta_j=0, \varDelta_{j+1}\neq 0\}$. Here, we make a remark; if $\tau^{-1}(z)\cap B_1=\{q_1, \cdots, q_m\}$ for $z\in N_j$, then, for $z'\in N_j$ near z, $\tau^{-1}(z')\cap B_1$ consists of just m points and therefore each q_i has a neighborhood which contains one and only one point of $\tau^{-1}(z')\cap B_1$. This can be proved by a Kasahara's lemma ([12], Lemma 2) and the fact that B_1 is the normalization of the analytic set $\{P=0\}$ by the mapping which maps $q\in B_1$ to $(f(q), \tau(q))\in\{P=0\}$.

Let $'B_2$ and $'B_3$ be concentric open balls by the coordinates z with the center at $\tau(p)$ satisfying $'B_3\Subset{}'B_2\Subset{}'B_1$. We put $B_i=\tau^{-1}('B_i)\cap B_1$ $(i=2, 3)$. We now find points p_ν in K and corresponding sets $B_i^{(\nu)}$, $'B_i^{(\nu)}$ $(i=1, 2, 3; \nu=1, 2, \cdots, l)$ such that $\bigcup_{\nu=1}^{l} B_3^{(\nu)}\supset K$. We denote the corresponding z', r, N_j by $z^{(\nu)}$, $r^{(\nu)}$, $N_j^{(\nu)}$, respectively. By Proposition 10 in [7], we can take open balls $'B^{(\nu)}$ which satisfy $'B_3^{(\nu)}\Subset{}'B^{(\nu)}\Subset{}'B_2^{(\nu)}$ and the following: We denote by S_ν the boundary of $'B^{(\nu)}$ and by $S_{i_1,\cdots,i_k}$ the set $S_{i_1}\cap\cdots\cap S_{i_k}$. Then, any $S_{i_1,\cdots,i_k}$ and any $S_{i_1,\cdots,i_k}\cap N_j^{(\nu)}$ are empty or regular surfaces.

We put $B^{(\nu)}=\tau^{-1}('B^{(\nu)})\cap B_1^{(\nu)}$ and $B=\bigcup_{\nu=1}^{l} B^{(\nu)}$. B is an open set and satisfies $K\subset B\Subset D$.

4. We shall prove that ∂B is good for v.

To a point $p\in\partial B$, we may assume $p\in\bigcap_{\nu=1}^{k}\partial B^{(\nu)}$ and $p\notin\bigcup_{\nu=k+1}^{l}\bar{B}^{(\nu)}$. We take a neighborhood W of p such that $W\subset\bigcap_{\nu=1}^{k}B_1^{(\nu)}$, $W\cap(\bigcup_{\nu=k+1}^{l}\bar{B}^{(\nu)})=\phi$ and $(W, \tau|W, 'W)$ is an analytic covering satisfying $W\cap\tau^{-1}\circ\tau(p)=\{p\}$, where $'W=\tau(W)$. We can make W sufficiently small. We put $'B={}'B^{(1)}\cup\cdots\cup{}'B^{(k)}$.

We prove the condition (i) of Definition 2. Let $'\varDelta$ be one of the sets $'B\cap\{\tilde{v}>v(p)\}$, $\partial'B\cap\{\tilde{v}>v(p)\}$ and $'B^c\cap\{\tilde{v}<v(p)\}$. We can take a neighborhood $'V$ of $\tau(p)$ contained in $'W$ such that any point of $'V\cap{}'\varDelta$ can be joined to $\tau(p)$ by a curve in $'W\cap{}'\varDelta$, (cf. [7] Proposition 9). We put $V=\tau^{-1}('V)\cap W$, which satisfies the condition.

To prove the condition (ii), we first show that the set R of all stationary values of the functions $\tilde{v}|S_{i_1,\cdots,i_r}$ and $\tilde{v}|S_{i_1,\cdots,i_r}\cap N_j^{(\nu)}\cap{}'B_2^{(\nu)}$ is finite. (When the domain of the function is one point, we regard it as a stationary point.) By Lemma 3 and the compactness of $S_{i_1,\cdots,i_r}$, the functions $\tilde{v}|S_{i_1,\cdots,i_r}$ have finitely many stationary values. Suppose that $\{\rho_\lambda\}$ were an infinite set of stationary values of $\tilde{v}|S_{i_1,\cdots,i_r}\cap N_j^{(\nu)}\cap{}'B_2^{(\nu)}$. Let z_λ be stationary points such that $\tilde{v}(z_\lambda)=\rho_\lambda$. Considering a suitable subsequence, we may assume $\lim_{\lambda\to\infty}\rho_\lambda=\rho$ and $\lim_{\lambda\to\infty}z_\lambda=z$. We may assume that z_λ and z belongs to the same $S_{i_1,\cdots,i_r}\cap\overline{N_j^{(\nu)}}\cap\overline{'B_2^{(\nu)}}$. Applying Lemma 3 to $\tilde{v}$ and $S_{i_1,\cdots,i_r}\cap\overline{N_j^{(\nu)}}$ in a neighborhood of z, we have a contradiction.

Suppose $v(p) \notin R$. If $\tau(p)$ belongs to no $N_j^{(\nu)} \cap 'B^{(\nu)}$, we can take a neighborhood of p on which τ is a homeomorphism. This implies the conditions (a) and (b). We assume $p \in N_j^{(1)} \cap 'B^{(1)}$. Since $S_{1,\dots,k} \cap \{\tilde{v} = v(p)\} \cap N_j^{(1)}$ is a regular surface on $N_j^{(1)}$, in any neighborhood of p, $N_j^{(1)}$ intersects all of $'\bar{B}^c$, $'B$, $'\bar{B}^c \cap \{\tilde{v} > v(p)\}$ and $'B \cap \{\tilde{v} > v(p)\}$. If we make W small, $\tau^{-1}(z) \cap W$ contains one and only one point for any $z \in N_j^{(1)} \cap 'W$. Taking a neighborhood $'U \subset 'W$ of $\tau(p)$ such that $'U \cap '\bar{B}^c$, $'U \cap 'B$, $'U \cap '\bar{B}^c \cap \{\tilde{v} > v(p)\}$ and $'U \cap 'B \cap \{\tilde{v} > v(p)\}$ are connected, we can verify easily that $\tau^{-1}('U) \cap W$ satisfies (a) and (b). Thus, Lemma 4 has been proved.

§ 6. The general case.

1. At first, we recall the definition of weakly holomorphic function on a complex space. Let X be a complex space and $\mathring{X}$ the set of all ordinary points of X. A function f on $\mathring{X}$ is called weakly holomorphic (schwach holomorph) on X if f is holomorphic on $\mathring{X}$ and for every $p \in X$ f is bounded in a neighborhood of p. Let us denote the normalization of X by $(\tilde{X}, \mu)$. A function f on $\mathring{X}$ is weakly holomorphic on X if and only if there exists a holomorphic function $\tilde{f}$ on $\tilde{X}$ such that $\tilde{f} = f \circ \mu$ holds on $\mu^{-1}(\mathring{X})$. For a weakly holomorhic function f on X, we define the set $S_N(f)$ as follows: A point $p \in X$ does not belong to $S_N(f)$ if and only if there exist a neighborhood U of p and a holomorphic function $\hat{f}$ in U such that $f = \hat{f}$ holds on $U \cap \mathring{X}$. The following was given by T. Asami ([2], Theorem 1).

LEMMA 5. *If f is a weakly holomorphic function on a complex space X, then $S_N(f)$ is an analytic set in X.*

PROOF. We give here a shorter proof. Since $S_N(f)$ is closed, we show that it is locally analytic. Take $p \in S_N(f)$. We may assume that X is an analytic subspace of an open set G in C^n. By a Hitotumatu's theorem ([**11**], Theorem 3), we have a neighborhood U of p in C^n and a meromorphic function $\dfrac{\psi}{\varphi}$ in U whose trace on $X \cap U$ is $f | X \cap U$, where φ and ψ are holomorphic functions in U. Let $\mathcal{J}$ be the sheaf of ideals of the analytic set X in G. We denote by $(\mathcal{J}, \varphi)$ the analytic sheaf generated by $\mathcal{J}$ and φ, and by $(\mathcal{J}, \varphi, \psi)$ that generated by $\mathcal{J}$, φ and ψ. They are coherent analytic sheaves over U, and therefore the quotient sheaf $\mathcal{Q} = (\mathcal{J}, \varphi, \psi)/(\mathcal{J}, \varphi)$ is also coherent. We have easily $S_N(f) \cap U = \{q \in U | \mathcal{Q}_q \neq 0\}$, which asserts Lemma 5 (H. Cartan [4], lemme 1 of Exposé X).

LEMMA 6. *Let $K(X)$ be the set of meromorphic functions on a complex space X. We denote by $(\tilde{X}, \mu)$ the normalization of X. Then, the natural mapping $\mu^*: K(X) \to K(\tilde{X})$ induced by μ is bijective.*

This is an immediate corollary of the Hitotumatu theorem stated above. A quotient of weakly holomorphic functions is a quotient of holomorphic functions locally.

2. Now, we can prove Theorem stated in § 1. Let X be a Stein space with $\operatorname{dih} X \geqq 2$, K a compact set and D an open set of X containing K. We assume that each irreducible component of D is also irreducible in $D-K$. This means that, for each connected component $\tilde{D}_i$ of $\tilde{D}$, $\tilde{D}_i-\tilde{K}$ is also connected, where $(\tilde{X}, \mu)=$ the normalization of X, $\tilde{D}=\mu^{-1}(D)$ and $\tilde{K}=\mu^{-1}(K)$.

(1) Take a holomorphic function f in $D-K$. Applying Theorem 3 to $f\circ\mu$ in $\tilde{D}_i-\tilde{K}$ for each i, we have a weakly holomorphic function $\tilde{f}$ in D satisfying $\tilde{f}=f$ in $D-K$. Since $S_N(f)\cap(D-K)=\phi$, it is a compact analytic set and therefore it is discrete. Thus, Lemma 1 asserts that $S_N(f)$ is empty. Hence, $\tilde{f}$ is holomorphic in D.

(2) Take a meromorphic function f in $D-K$. By Theorem 3, we have a meromorphic continuation $\hat{f}$ of $f\circ\mu$ to $\tilde{D}$. By Lemma 6, we have a meromorphic function $\tilde{f}$ in D such that $\hat{f}=\tilde{f}\circ\mu$ on $\tilde{D}$. It is the continuation of f to D.

(3) Take a holomorphic mapping λ of $D-K$ into a Stein space Y. By Theorem 3, we have a holomorphic continuation $\hat{\lambda}$ of $\lambda\circ\mu$ to $\tilde{D}$.

Suppose $\hat{\lambda}(\tilde{p}_1)\neq\hat{\lambda}(\tilde{p}_2)$, where $\mu^{-1}(p)=\{\tilde{p}_1, \cdots, \tilde{p}_k\}$ for $p\in D$. Then, we have $p\in K$. Since Y is Stein, we have a holomorphic function f in Y satisfying $f(\hat{\lambda}(\tilde{p}_1))\neq f(\hat{\lambda}(\tilde{p}_2))$. We can regard $f\circ\hat{\lambda}$ as a weakly holomorphic function $\tilde{f}$ on D, and $f\circ\lambda$ on $D-K$ has a holomorphic continuation to D by (1), which coincides with $\tilde{f}$. This implies $f(\hat{\lambda}(\tilde{p}_\nu))=\tilde{f}(p)$ for all ν, which is a contradiction. Thus, we can define a mapping $\tilde{\lambda}: D\rightarrow Y$ as $\tilde{\lambda}=\hat{\lambda}\circ\mu^{-1}$. We have easily that $\tilde{\lambda}$ is continuous. Take $p\in K$ and put $y=\tilde{\lambda}(p)$. We can regard a neighborhood of y as an analytic subspace M of an open set in C^m and take a neighborhood U of p satisfying $\tilde{\lambda}(U)\subset M$. The mapping $\tilde{\lambda}\,|\,U$ is written by a pair $(f_1, \cdots, f_m)$ of weakly holomorphic functions in U. In the same way as above, we can see that the set $\{p\in D\,|\,\tilde{\lambda}$ is not holomorphic at $p\}$ is an analytic set contained in K, and therefore it is discrete. Applying Lemma 1, we have that it is empty and $\tilde{\lambda}$ is holomorphic in D.

Thus, Theorem has been proved.

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