J. Math. Soc. Japan
Vol. 17, No. 3, 1965

On two systems for arithmetic[1]

Dedicated to Professor Y. Akizuki on his sixtieth birthday

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(Received Aug. 31, 1963)

Takeuti-Kino [7] discussed a relation between formulas with "constructive" infinitely long expressions and the analytic predicates in Kleene [4].

We shall deal with this by the notion of "provability" instead of "representability" used in [7]. For this purpose, we shall consider two kinds of formal systems of the theory of natural numbers. One (which we denote by $\mathfrak{S}$ (or $\mathfrak{S}^*$)) is an applied second-order functional calculus with equality which differs only somewhat from the system A_ω in Grzegorczyk, Mostowski and Ryll-Nardzewski [1] (or its extension by Mostowski [6]). Another (denoted by $\mathfrak{J}$) is based on the first-order functional calculus with infinitely long expressions and the class of its formulas consists of only constructive formulas, whose nesting numbers with respect to quantifier are finite (in the sense of [7]).

As an application, we can obtain Gödel's incompleteness theorem for the system $\mathfrak{J}$.

§ 1. The systems $\mathfrak{S}$ and $\mathfrak{S}^*$.

First, we shall establish a system $\mathfrak{S}$ for the theory of natural numbers:

1.1. *Primitive* symbols are as follows: Non-logical constants 0 (zero), $*'$ (successor), $*^{\backprime}$ (predecessor), $*_1+*_2$ (addition), $*_1\cdot *_2$ (multiplication), and $\pi(*_1, *_2)$ (power; instead of $\pi(\mathrm{x}, \mathrm{y})$ we shall often write x^{y}); infinitely many distinct number-variables $\mathfrak{x}_0, \mathfrak{x}_1, \cdots$; infinitely many distinct 1-place function-variables $\varphi_0, \varphi_1 \cdots$[2]; propositional connectives $\neg$, $\vee$, $\wedge$, and $\supset$; the quantifiers $(\exists \mathfrak{x}_i)$, $(\forall \mathfrak{x}_i)$, $(\exists \varphi_i)$ and $(\forall \varphi_i)$; the equality symbol $=$; the symbol ➡; and parentheses (,).

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1) The authors wish to thank the referees for their kind advice.

2) This system contains the pairing functions: $J(\mathrm{x}, \mathrm{y})$ $(=J_2(\mathrm{x}, \mathrm{y}) = (\Delta_2^x \cdot (\Delta_2 \cdot \mathrm{y})')^{\backprime}$, $J_{k+1}(\mathrm{x}_1, \cdots, \mathrm{x}_{k+1}) = J(J_k(\mathrm{x}_1, \cdots, \mathrm{x}_k), \mathrm{x}_{k+1})$. Hence it is sufficient to have only 1-place function-variables.

1.2. *Terms* are defined inductively as follows:[3]

(1) 0 and each number-variable are terms.

(2) If s and t are terms, then s′, s‵, s+t, s·t and s^t are terms.

(3) If α is a function-variable and t is a term, then α(t) is a term.

(4) The only terms are those given by (1)-(3).

1.3. *Formulas* are defined inductively as follows:

(5) If s and t are terms, then s=t is a formula. (Call it a prime formula.)

(6) If A and B are formulas, then $\neg$A, A$\vee$B, A$\wedge$B and A$\supset$B are formulas.

(7) If A is a formula, x is a number-variable and α is a function-variable, then ($\exists$x)A, ($\forall$x)A, ($\exists\alpha$)A and ($\forall\alpha$)A are formulas.

(8) The only formulas are those given by (5)-(7).

1.4. Sequents are formal expressions of the form $\Gamma \rightarrow \Theta$, where Γ and Θ are arbitrary finite (possibly empty) sequences of formulas. The notions of numeral, closed formula and elementary (or arithmetical) formula are defined as usual. We shall denote the numeral for the natural number n by Δ_n. And we shall abbreviate

$$(\exists x)(A(x)\wedge(\forall y)(A(y)\supset x=y)) \quad \text{as} \quad (\exists!x)A(x),$$

and

$$(A\supset B)\wedge(B\supset A) \quad \text{as} \quad A\equiv B.$$

1.5. Postulates for the system $\mathfrak{S}$.

(i) Logical axiom schema: $A \rightarrow A$, where A is an arbitrary formula.

(ii) Axioms for arithmetic:

(9) $\mathfrak{x}_0'=\mathfrak{x}_1' \rightarrow \mathfrak{x}_0=\mathfrak{x}_1$

(10) $\rightarrow \neg(\mathfrak{x}_0'=0)$

(11) $\rightarrow 0^{\backprime}=0$

(12) $\rightarrow (\mathfrak{x}_0')^{\backprime}=\mathfrak{x}_0$

(13) $\rightarrow \mathfrak{x}_0+0=\mathfrak{x}_0$

(14) $\rightarrow \mathfrak{x}_0+\mathfrak{x}_1'=(\mathfrak{x}_0+\mathfrak{x}_1)'$

(15) $\rightarrow \mathfrak{x}_0\cdot 0=0$

(16) $\rightarrow \mathfrak{x}_0\cdot\mathfrak{x}_1'=\mathfrak{x}_0\cdot\mathfrak{x}_1+\mathfrak{x}_0$

(17) $\rightarrow \pi(\mathfrak{x}_0', 0)=\Delta_1$

(18) $\rightarrow \pi(\mathfrak{x}_0, \mathfrak{x}_1')=\pi(\mathfrak{x}_0, \mathfrak{x}_1)\cdot\mathfrak{x}_0$

(19) $\varphi_0(0)=0, (\forall\mathfrak{x}_0)(\varphi_0(\mathfrak{x}_0)=0\supset\varphi_0(\mathfrak{x}_0')=0) \rightarrow (\forall\mathfrak{x}_0)(\varphi_0(\mathfrak{x}_0)=0)$

(iii) Equality axioms:

3) In the following, we shall mainly use non-italic Roman letters as metamathematical ones, and the lower-case Greek letters $\alpha, \beta, \gamma, \delta$ etc., except for π and φ, as metamathematical function letters. The lower-case italic Roman letters denote natural numbers (as informal symbols). Sometimes α and β are used as informal number-theoretic functions.

(20) $\Rightarrow \mathfrak{x}_0 = \mathfrak{x}_0$

(21) $x = y, A(x) \Rightarrow A(y)$, where A(x) is an arbitrary formula and y is a variable not occuring free in A(x) and is free for x in A(x) in the sense of Kleene [2], p. 79.

(iv) The axiom (—schema) of choice at type 0:

(22) $(\forall x)(\exists y)A(x, y) \Rightarrow (\exists \alpha)(\forall x)A(x, \alpha(x))$, where α is a function-variable not occuring free in A(x, y) and is free for y in A(x, y).

(v) Rules of inference:

Structural rules of inference: As in Kleene [2].

Logical rules of inference: For the first-order functional calculus, as in Kleene [2]. Additional rules for the second-order functional calculus:

$$(\exists^f \Rightarrow)\frac{A(\alpha), \Gamma \Rightarrow \Theta}{(\exists\alpha)A(\alpha), \Gamma \Rightarrow \Theta},$$

where α is a function-variable not occuring free in the lower sequent.

$$(\Rightarrow \exists^f)\frac{\Gamma \Rightarrow \Theta, A(\beta)}{\Gamma \Rightarrow \Theta, (\exists\alpha)A(\alpha)},$$

where β is a function-variable which is free for α in $A(\alpha)$.

$$(\forall^f \Rightarrow)\frac{A(\beta), \Gamma \Rightarrow \Theta}{(\forall\alpha)A(\alpha), \Gamma \Rightarrow \Theta},$$

under the same proviso as in the rule $(\Rightarrow \exists^f)$[4].

$$(\Rightarrow \forall^f)\frac{\Gamma \Rightarrow \Theta, A(\alpha)}{\Gamma \Rightarrow \Theta, (\forall\alpha)A(\alpha)},$$

under the same proviso as in the rule $(\exists^f \Rightarrow)$.

ω-rules:

$$(\omega \Rightarrow)\frac{A(\Delta_n), \Gamma \Rightarrow \Theta \text{ for all } n}{(\exists x)A(x), \Gamma \Rightarrow \Theta}, \qquad (\Rightarrow \omega)\frac{\Gamma \Rightarrow \Theta, A(\Delta_n) \text{ for all } n}{\Gamma \Rightarrow \Theta, (\forall x)A(x)},$$

where $A(\Delta_n)$ is the formula obtained from A(x) by replacing x everywhere by Δ_n.

Cut:
$$\frac{\Gamma \Rightarrow \Theta, D \qquad D, \Lambda \Rightarrow \Pi}{\Gamma, \Lambda \Rightarrow \Theta, \Pi}.$$

$\mathfrak{S}^*$ is the system obtained from $\mathfrak{S}$ by adding the following axiom-schema (cf. Mostowski [6]):

$$(\forall x)(\exists \alpha)A(x, \alpha) \equiv (\exists \alpha)(\forall x)(\forall \beta)\{(\forall y)(\beta(y) = \alpha(\Delta_2^x \cdot \Delta_3^y)) \supset A(x, \beta)\},$$

with the trivial conditions.

If a sequent $\Gamma \rightarrow \Theta$ is deducible from the axioms by means of the rules of inference in $\mathfrak{S}$ (or $\mathfrak{S}^*$), then $\Gamma \Rightarrow \Theta$ is said to be provable in $\mathfrak{S}$ (or $\mathfrak{S}^*$) and we shall write

$$\vdash_1 \Gamma \Rightarrow \Theta \qquad (\text{or } \vdash_* \Gamma \Rightarrow \Theta).$$

In particular, if $\Gamma \Rightarrow \Theta$ is of the form $\Rightarrow A$, where A is a single formula, then

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4) It is sufficient to adopt these forms instead of e. g. the form: $\frac{\Gamma \Rightarrow \Theta,\ A(\lambda x\, t(x))}{\Gamma \Rightarrow \Theta,\ (\exists\alpha)A(\alpha)}$, because we have the axiom schema (iv) above (and the equality axiom).

we shall write simply

$$\vdash_1 \mathrm{A} \quad (\text{or } \vdash_* \mathrm{A}).$$

§ 2. The system $\mathfrak{J}$.

In this section, we shall consider a formal system with infinitely long expressions having an arithmetical structure. (Cf. Takeuti-Kino [7] and Maehara-Takeuti [5].)

2.1. *Primitive symbols* are as follows: Non-logical constants are the same as in $\mathfrak{S}$; infinitely many distinct number-variables $\mathfrak{v}_0, \mathfrak{v}_1, \mathfrak{v}_2, \cdots$; logical symbols $\neg, \vee, \wedge, \exists, \forall$; the equality symbol $=$; the symbol $\Rightarrow$; and the parentheses (,).

2.2. We shall correlate distinct odd numbers to the primitive symbols, thus:

0	$\mathfrak{v}_j$	$'$	$\backslash$	$+$	$\cdot$	π	$=$	$\neg$	$\vee$	$\wedge$	$\exists$	$\forall$
3	7^{j+1}	5	9	11	13	15	17	19	21	23	25	27,

respectively.

2.3. *Terms* are defined inductively and correlated the Gödel numbers (denoted by $\ulcorner \mathrm{t} \urcorner$ for a term t) in the following manner:

(1) 0 and $\mathfrak{v}_j$ are terms, and $\ulcorner 0 \urcorner = 3$ and $\ulcorner \mathfrak{v}_j \urcorner = 7^{j+1}$.

(2) If s and t are terms, then t', $\mathrm{t}^{\backslash}$, $\mathrm{s}+\mathrm{t}$, $\mathrm{s}\cdot\mathrm{t}$ and s^{t} are terms and

$$\ulcorner \mathrm{t}' \urcorner = 2^5 \cdot 3^{\ulcorner \mathrm{t} \urcorner}, \qquad \ulcorner \mathrm{t}^{\backslash} \urcorner = 2^9 \cdot 3^{\ulcorner \mathrm{t} \urcorner}, \qquad \ulcorner \mathrm{s}+\mathrm{t} \urcorner = 2^{11} \cdot 3^{\ulcorner \mathrm{s} \urcorner} \cdot 5^{\ulcorner \mathrm{t} \urcorner},$$

$$\ulcorner \mathrm{s}\cdot\mathrm{t} \urcorner = 2^{13} \cdot 3^{\ulcorner \mathrm{s} \urcorner} \cdot 5^{\ulcorner \mathrm{t} \urcorner} \quad \text{and} \quad \ulcorner \mathrm{s}^{\mathrm{t}} \urcorner = 2^{15} \cdot 3^{\ulcorner \mathrm{s} \urcorner} \cdot 5^{\ulcorner \mathrm{t} \urcorner}.$$

(3) The only terms are those given by (1) and (2).

2.4. *Formulas* are defined inductively and correlated the Gödel numbers[5] in the following manner. First, such formulas were defined in [7] and called "constructive".

(1) If s and t are terms, then $\mathrm{s}=\mathrm{t}$ is a formula (called a *prime* formula) and $2^{17} \cdot 3^{\ulcorner \mathrm{s} \urcorner} \cdot 5^{\ulcorner \mathrm{t} \urcorner}$ is the Gödel number of it.

(2) If $\mathfrak{A}$ is a formula and b is a Gödel number of $\mathfrak{A}$, then $\neg\mathfrak{A}$ is a formula and $2^{19} \cdot 3^b$ is a Gödel number of it.

(3) If $\mathfrak{A}_i$'s are formulas for all i and if there is a number f defining recursively $f(i)$ as a function of i such that for each i $f(i)$ is a Gödel number of $\mathfrak{A}_i$, then $\vee(\mathfrak{A}_0, \mathfrak{A}_1, \cdots)$ is a formula (abbreviated as $\bigvee_i \mathfrak{A}_i$) and $2^{21} \cdot 3^f$ is a Gödel number of it.

(4) Similarly for $\wedge(\mathfrak{A}_0, \mathfrak{A}_1, \cdots)$ with $\bigwedge_i \mathfrak{A}_i$ and $2^{23} \cdot 3^f$.

(5) If $\mathfrak{A}$ is a formula and b is a Gödel number of $\mathfrak{A}$, and if $\langle \mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots \rangle$ is a sequence of distinct variables of order-type $\leqq \omega$ such that $g(i)$ is a recur-

5) In generally, Gödel number of a formula is not uniquely determined.

sive function, then $\exists \mathfrak{v}_{g(0)}\, \mathfrak{v}_{g(1)} \cdots \mathfrak{A}$ is a formula and $2^{25}\cdot 3^{g}\cdot 5^{b}$ is a Gödel number of it, where g is a number which defines recursively the function $g(i)$.

(6) Similarly for $\forall \mathfrak{v}_{g(0)}\, \mathfrak{v}_{g(1)} \cdots \mathfrak{A}$ with $2^{27}\cdot 3^{g}\cdot 5^{b}$.

(7) The only (constructive) formulas are those given by (1)-(6).

2.5. The notions of numeral and closed formula are defined as usual. Let us denote the numeral for the natural number n by $\mathfrak{z}_n$. We shall abbreviate:

$$\bigvee(\mathfrak{A}, \mathfrak{B}, \mathfrak{B}, \cdots) \quad \text{as} \quad \mathfrak{A} \vee \mathfrak{B},$$

$$\bigwedge(\mathfrak{A}, \mathfrak{B}, \mathfrak{B}, \cdots) \quad \text{as} \quad \mathfrak{A} \wedge \mathfrak{B},$$

$$\neg\mathfrak{A} \vee \mathfrak{B} \quad \text{as} \quad \mathfrak{A} \supset \mathfrak{B},$$

and

$$(\mathfrak{A} \supset \mathfrak{B}) \wedge (\mathfrak{B} \supset \mathfrak{A}) \quad \text{as} \quad \mathfrak{A} \equiv \mathfrak{B}.$$

2.6. *Nesting number* of a formula. Let $\mathrm{n}(\mathfrak{A})$ and $\mathrm{n}'(\mathfrak{A})$ be ordinal numbers defined in [7], and called the nesting number of a formula $\mathfrak{A}$ and that with respect to quantifier, respectively. [7] proves $\mathrm{n}(\mathfrak{A}) < \omega_1$ for all constructive formula $\mathfrak{A}$, where ω_1 is the first non-constructive ordinal (in the sense of Church and Kleene).

Let the class of formulas in $\mathfrak{J}$ consist of all constructive formulas $\mathfrak{A}$ with finite $\mathrm{n}'(\mathfrak{A})$. A sequent in $\mathfrak{J}$ is defined as in § 1. It should be noted that each sequent consists of only finite number of formulas in $\mathfrak{T}$.

2.7. *Postulates* for the system $\mathfrak{J}$.

(i) Logical axiom schema: $\mathfrak{A} \rightarrow \mathfrak{A}$.

(ii) Axioms for arithmetic: Similarly as (9)-(18) in § 1.5, replacing $\mathfrak{x}_i$ there by $\mathfrak{v}_i$. They are named (9)-(18), respectively. (System $\mathfrak{J}$ has not the axiom of induction.)

(19) $\bigvee_i (\mathfrak{z}_i = \mathfrak{v}_0)$.

(iii) Equality axioms:

(20) $\mathfrak{v}_0 = \mathfrak{v}_0$.

(21) $\bigwedge_i (\mathfrak{v}_{g(i)} = \mathfrak{v}_{h(i)}),\ \mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots) \rightarrow \mathfrak{A}(\mathfrak{v}_{h(0)}, \mathfrak{v}_{h(1)}, \cdots)$, under the trivial conditions, where g and h are recursive functions. Here and after instead of $\{g\}(i)$ we write simply $g(i)$.

(iv) The additional axiom-schema:

(22) $\bigwedge_i \bigvee_j \mathfrak{A}_i(\mathfrak{z}_j) \rightarrow \exists \mathfrak{v}_{g(0)} \mathfrak{v}_{g(1)} \cdots \bigwedge_i \mathfrak{A}_i(\mathfrak{v}_{g(i)})$, where no $\mathfrak{v}_{g(i)}$ occur free in any $\mathfrak{A}_i(\mathfrak{z}_j)$ and $\mathfrak{z}_j$ is free for $\mathfrak{v}_{g(i)}$ in $\mathfrak{A}_i(\mathfrak{v}_{g(i)})$.

(v) Rules of inference.

$$(\neg \rightarrow)\frac{\Gamma \rightarrow \Theta, \mathfrak{A}}{\neg\mathfrak{A}, \Gamma \rightarrow \Theta}, \qquad (\rightarrow \neg)\frac{\mathfrak{A}, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg\mathfrak{A}}.$$

$$(\vee \rightarrow)\frac{\mathfrak{A}_i, \Gamma \rightarrow \Theta \text{ for all } i}{\bigvee_i \mathfrak{A}_i, \Gamma \rightarrow \Theta}, \qquad (\rightarrow \vee)\frac{\Gamma \rightarrow \Theta, \mathfrak{A}_n}{\Gamma \rightarrow \Theta, \bigvee_i \mathfrak{A}_i}.$$

$(\wedge \Rightarrow)$: Dually as $(\Rightarrow \vee)$, $(\Rightarrow \wedge)$: Dually as $(\vee \Rightarrow)$.

$$(\exists \Rightarrow) \quad \frac{\mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots), \Gamma \Rightarrow \Theta}{\exists \mathfrak{v}_{g(0)} \mathfrak{v}_{g(1)} \cdots \mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots), \Gamma \Rightarrow \Theta},$$

where g is recursive and $\langle \mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots \rangle$ is a sequence of distinct variables of order-type $\leqq \omega$ such that no $\mathfrak{v}_{g(i)}$ occurs free in the lower sequent.

$$(\Rightarrow \exists) \quad \frac{\Gamma \Rightarrow \Theta, \mathfrak{A}(\mathrm{t}_0, \mathrm{t}_1, \cdots)}{\Gamma \Rightarrow \Theta, \exists \mathfrak{v}_{g(0)} \mathfrak{v}_{g(1)} \cdots \mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)},$$

where g is recursive and $\langle \mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots \rangle$ is a sequence of distinct variables of order-type $\leqq \omega$, and $\langle \mathrm{t}_0, \mathrm{t}_1, \cdots \rangle$ is a recursive sequence of terms (this means: $\ulcorner \mathrm{t}_i \urcorner$ is a recursive function of i) such that it is of the same type as the former and is free for $\langle \mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots \rangle$ in $\mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{(1)}, \cdots)$.

$(\forall \Rightarrow)$: Dually as $(\Rightarrow \exists)$, $(\Rightarrow \forall)$: Dually as $(\exists \Rightarrow)$.

Structural rule of inference: $\dfrac{\Gamma \Rightarrow \Lambda}{\Theta \Rightarrow \Pi}$, where any formula occuring in Γ or Λ is contained in Θ or Π, respectively.

Cut:
$$\frac{\Gamma \Rightarrow \Lambda, \mathfrak{A} \qquad \mathfrak{A}, \Theta \Rightarrow \Pi}{\Gamma, \Theta \Rightarrow \Lambda, \Pi}.$$

The notion of "$\Gamma \Rightarrow \Theta$ is provable in $\mathfrak{J}$" is also defined as in §1. And we shall write as $\vdash_2 \Gamma \Rightarrow \Theta$. In particular, for $\vdash_2 \Rightarrow \mathfrak{A}$ we write simply $\vdash_2 \mathfrak{A}$.

§3. Statement of Theorem 1. 3.1.

THEOREM 1. *Let a be a Gödel number of a provable formula* $\mathfrak{A}$ *in* $\mathfrak{J}$ *such that* $\mathrm{n}(\mathfrak{A}) = k$ *and its proof contains no formulas in which a variable occurs both free and bound. Then*

$$\vdash_* \mathrm{Q}_k(\Delta_a, \alpha),$$

where $\mathrm{Q}_k(\mathrm{a}, \alpha)$ *is the formula obtained by formalizing the predicate* $Q_k(a, \alpha)$ *defined in* [7] *in the system* $\mathfrak{S}^*$ *(see below).*

This is a counterpart of Theorem 2 in [7].

3.2. We shall define a predicate $F(a, b, \alpha)$ for the Gödel number a of a term inductively as follows. (The notations using here are informal ones and are found in Kleene [2].)

$a = 3 \ \& \ b = 0 \cdot \rightarrow \cdot F(a, b, \alpha)$.

$a = 7^{(a)_3} \ \& \ (a)_3 = 0 \ \& \ b = \alpha((a)_3 \dot{-} 1) \cdot \rightarrow \cdot F(a, b, \alpha)$.

$a = 2^5 \cdot 3^{(a)_1} \ \& \ (Ed)(F((a)_1, d, \alpha) \ \& \ b = d+1) \cdot \rightarrow \cdot F(a, b, \alpha)$.

$a = 2^9 \cdot 3^{(a)_1} \ \& \ (Ed)(F((a)_1, d, \alpha) \ \& \ (b = 0 \ \& \ d = 0 \cdot \vee \cdot d = b+1))$
$\cdot \rightarrow \cdot F(a, b, \alpha)$.

$$a=2^{11}\cdot 3^{(a)_1}\cdot 5^{(a)_2} \ \& \ (Ed_1d_2)(b=d_1+d_2 \ \& \ F((a)_1, d_1, \alpha) \ \& \ F((a)_2, d_2, \alpha)) \cdot\!\rightarrow\!\cdot F(a, b, \alpha).$$

$$a=2^{13}\cdot 3^{(a)_1}\cdot 5^{(a)_2} \ \& \ (Ed_1d_2)(b=d_1d_2 \ \& \ F((a)_1, d_1, \alpha) \ \& \ F((a)_2, d_2, \alpha)) \cdot\!\rightarrow\!\cdot F(a, b, \alpha).$$

$$a=2^{15}\cdot 3^{(a)_1}\cdot 5^{(a)_2} \ \& \ (Ed_1d_2)\{(b=1 \ \& \ d_2=0 \cdot\vee\cdot b=d_1^{d_2}) \ \& \ F((a)_1, d_1, \alpha) \ \& \ F((a)_2, d_2, \alpha)\} \cdot\!\rightarrow\!\cdot F(a, b, \alpha).$$

$F(a, b, \alpha)$ only as required by the above clauses.

By Kleene [3], the predicate $F(a, b, \alpha)$ is in the class Σ_1^0. Hence it is formalized in the system $\mathfrak{S}$, and the formula obtained by formalizing it we shall denote as $\mathrm{F}(\mathrm{a}, \mathrm{b}, \alpha)$, which is an elementary formula.

LEMMA 1. *If a is the Gödel number of a term, then*

$$\vdash_1 (\exists!\,\mathrm{b})\mathrm{F}(\Delta_a, \mathrm{b}, \alpha). \tag{1}$$

PROOF. Since the formula F is elementary and $(\forall\alpha)(\exists!\,\mathrm{b})\mathrm{F}(\Delta_a, \mathrm{b}, \alpha)$ is true in the principal model of $\mathfrak{S}$, we have (1) by Theorem 3.1. E in [1].

3.3. The inductive definition of the predicate $Q_k(a, \alpha)$ is given as follows (These predicates are the ones in [7])

$$\begin{aligned} Q_0(a, \alpha) \rightleftarrows (&\{a=2^{17}\cdot 3^{(a)_1}\cdot 5^{(a)_2} \ \& \ (Eb)(F((a)_1, b, \alpha) \ \& \ F((a)_2, b, \alpha))\} \\ &\vee \{a=2^{19}\cdot 3^{(a)_1} \ \& \ \bar{Q}_0((a)_1, \alpha)\} \\ &\vee \{a=2^{21}\cdot 3^{(a)_1} \ \& \ (x)(Ey)T_1((a)_1, x, y) \ \& \ (Ex)(y)(T_1((a)_1, x, y) \\ &\quad \rightarrow Q_0(U(y), \alpha))\} \\ &\vee \{a=2^{23}\cdot 3^{(a)_1} \ \& \ (x)(Ey)T_1((a)_1, x, y) \ \& \ (x)(y)(T_1((a)_1, x, y) \\ &\quad \rightarrow Q_0(U(y), \alpha))\}) \end{aligned} \tag{2}$$

$$\begin{aligned} Q_{k+1}(a, \alpha) \rightleftarrows (&\{a=2^{17}\cdot 3^{(a)_1}\cdot 5^{(a)_2} \ \& \ (Eb)(F((a)_1, b, \alpha) \ \& \ F((a)_2, b, \alpha))\} \\ &\vee \{a=2^{19}\cdot 3^{(a)_1} \ \& \ \bar{Q}_{k+1}((a)_1, \alpha)\} \\ &\vee \{a=2^{21}\cdot 3^{(a)_1} \ \& \ (x)(Ey)T_1((a)_1, x, y) \ \& \ (Ex)(y)(T_1((a)_1, x, y) \\ &\quad \rightarrow Q_{k+1}(U(y), \alpha))\} \\ &\vee \{a=2^{23}\cdot 3^{(a)_1} \ \& \ (x)(Ey)T_1((a)_1, x, y) \ \& \ (x)(y)(T_1((a)_1, x, y) \\ &\quad \rightarrow Q_{k+1}(U(y), \alpha))\} \\ &\vee \{a=2^{25}\cdot 3^{(a)_1}\cdot 5^{(a)_2} \ \& \ (x)(Ey)T_1((a)_1, x, y) \ \& \ (E\beta)(D((a)_1, \alpha, \beta) \\ &\quad \& \ Q_k((a)_2, \beta))\} \\ &\vee \{a=2^{27}\cdot 3^{(a)_1}\cdot 5^{(a)_2} \ \& \ (x)(Ey)T_1((a)_1, x, y) \ \& \ (\beta)(D((a)_1, \alpha, \beta) \\ &\quad \rightarrow Q_k((a)_2, \beta))\}), \end{aligned} \tag{3}$$

where $D(a, \alpha, \beta)$ denotes the following predicate:

(4) $$(x)((y)(u)(T_1(a, y, u) \to x \neq U(u)) \to \alpha(x) = \beta(x)).$$

Takeuti-Kino [7] proved that for each $k \geqq 0$, $Q_k(a, \alpha)$ is expressible in the form $\Sigma^1_{k+1} \cap \Pi^1_{k+1}$ under the condition: a is a Gödel number of a formula. We shall denote by $\mathrm{Q}_k(\mathrm{a}, \alpha)$ and $\mathrm{Q}'_k(\mathrm{a}, \alpha)$ the formulas obtained by formalizing in $\mathfrak{S}^*$ the Σ^1_{k+1}- and Π^1_{k+1}-predicates, respectively.

Now in the first colomn of the following table we shall list the formulas which are obtained by formalizing (in $\mathfrak{S}$) the predicates listed in the second colomn of it:

(5)

$\mathrm{B}_1(\mathrm{a})$	$a = 7^{(a)_3}$ & $(a)_3 \neq 0$	
$\mathrm{B}_2(\mathrm{a}, \mathrm{b}, \alpha)$	$b = \alpha((a)_3 \dot{-} 1)$	
$\mathrm{B}_m(\mathrm{a})$ for $m > 2$	$a = 2^m \cdot 3^{(a)_1}$ & $(a)_1 \neq 0$	
$\mathrm{B}'_m(\mathrm{a})$	$a = 2^m \cdot 3^{(a)_1} \cdot 5^{(a)_2}$ & $(a)_1 \neq 0$ & $(a)_2 \neq 0$	
$\mathrm{C}_i(\mathrm{a}, \mathrm{x})$	$(a)_i = x$	
$\mathrm{D}(\mathrm{a}, \alpha, \beta)$	$D(a, \alpha, \beta)$	(see above)
$\mathrm{U}(\mathrm{y}, \mathrm{z})$	$U(y) = z$	(see Kleene [2]).
$\mathrm{T}(\mathrm{a}, \mathrm{x}, \mathrm{y})$	$T_1(a, x, y)$	

We can show that the following formulas are provable in $\mathfrak{S}^*$ ((6) in $\mathfrak{S}$) based on the proof of Theorem 2 in [7]: For the Gödel number a of a term in $\mathfrak{J}$,

(6) $$\begin{aligned} \mathrm{F}(\Delta_a, \mathrm{b}, \alpha) \equiv & (\Delta_a = \Delta_3 \wedge \mathrm{b} = 0) \vee (\mathrm{B}_1(\Delta_a) \wedge \mathrm{B}_2(\Delta_a, \mathrm{b}, \alpha)) \vee (\mathrm{B}_5(\Delta_a) \\ & \wedge \mathrm{F}(\Delta_{(a)_1}, \mathrm{b}', \alpha)) \vee (\mathrm{B}_9(\Delta_a) \wedge [(\Delta_{(a)_1} = \Delta_3 \wedge \mathrm{b} = 0) \\ & \vee \mathrm{F}(\Delta_{(a)_1}, \mathrm{b}', \alpha)]) \\ & \vee (\mathrm{B}'_{11}(\Delta_a) \wedge (\exists \mathrm{cd})[\mathrm{F}(\Delta_{(a)_1}, \mathrm{c}, \alpha) \wedge \mathrm{F}(\Delta_{(a)_2}, \mathrm{d}, \alpha) \wedge \mathrm{b} = \mathrm{c} + \mathrm{d}]) \\ & \vee (\mathrm{B}'_{13}(\Delta_a) \wedge (\exists \mathrm{cd})[\mathrm{F}(\Delta_{(a)_1}, \mathrm{c}, \alpha) \wedge \mathrm{F}(\Delta_{(a)_2}, \mathrm{d}, \alpha) \wedge \mathrm{b} = \mathrm{c} \cdot \mathrm{d}]) \\ & \vee (\mathrm{B}'_{15}(\Delta_a) \wedge (\exists \mathrm{cd})[\mathrm{F}(\Delta_{(a)_1}, \mathrm{c}, \alpha) \wedge \mathrm{F}(\Delta_{(a)_2}, \mathrm{d}, \alpha) \wedge \mathrm{b} = \pi(\mathrm{c}, \mathrm{d})]). \end{aligned}$$

For a Gödel number a of a formula $\mathfrak{A}$ in $\mathfrak{J}$ such that $\mathrm{n}'(\mathfrak{A}) \leqq k$,

(7) $$\mathrm{Q}_k(\Delta_a, \alpha) \equiv \mathrm{Q}'_k(\Delta_a, \alpha),$$

(8) $$\begin{aligned} \mathrm{Q}_k(\Delta_a, \alpha) \equiv & (\mathrm{B}'_{17}(\Delta_a) \wedge (\exists \mathrm{b})[\mathrm{F}(\Delta_{(a)_1}, \mathrm{b}, \alpha) \wedge \mathrm{F}(\Delta_{(a)_2}, \mathrm{b}, \alpha)]) \\ & \vee (\mathrm{B}_{19}(\Delta_a) \wedge \neg \mathrm{Q}_k(\Delta_{(a)_1}, \alpha)) \\ & \vee (\mathrm{B}_{21}(\Delta_a) \wedge (\forall \mathrm{x})(\mathrm{E}\mathrm{y})\mathrm{T}(\Delta_{(a)_1}, \mathrm{x}, \mathrm{y}) \wedge (\exists \mathrm{x})(\forall \mathrm{y})\{\mathrm{T}(\Delta_{(a)_1}, \mathrm{x}, \mathrm{y}) \\ & \supset (\exists \mathrm{z})[\mathrm{U}(\mathrm{y}, \mathrm{z}) \wedge \mathrm{Q}_k(\mathrm{z}, \alpha)]\}) \\ & \vee (\mathrm{B}_{23}(\Delta_a) \wedge (\forall \mathrm{x})(\exists \mathrm{y})\mathrm{T}(\Delta_{(a)_1}, \mathrm{x}, \mathrm{y}) \wedge (\forall \mathrm{x})(\forall \mathrm{y})\{\mathrm{T}(\Delta_{(a)_1}, \mathrm{x}, \mathrm{y}) \\ & \supset (\exists \mathrm{z})[\mathrm{U}(\mathrm{y}, \mathrm{z}) \wedge \mathrm{Q}_k(\mathrm{z}, \alpha)]\}) \end{aligned}$$

$$\vee (B'_{25}(\varDelta_a) \wedge (\forall x)(\exists y)T(\varDelta_{(a)_1}, x, y) \wedge (\exists \beta)[D(\varDelta_{(a)_1}, \alpha, \beta) \wedge Q_{k-1}(\varDelta_{(a)_2}, \beta)])$$

$$\vee (B'_{27}(\varDelta_a) \wedge (\forall x)(\exists y)T(\varDelta_{(a)_1}, x, y) \wedge (\forall \beta)[D(\varDelta_{(a)_1}, \alpha, \beta) \supset Q_{k-1}(\varDelta_{(a)_2}, \beta)]),$$

where in the case $k=0$ we omit the last two disjunctive members.

And if f is a Gödel number of a recursive function of one variable (in the sense of Kleene [2]), then we have:

(9) $\vdash_1 (\forall x)(\exists ! y)T(\varDelta_f, x, y)$ and $\vdash_1 (\forall x)(\exists ! z)U(x, z)$.

(10) $\vdash_1 T(\varDelta_f, \varDelta_i, y) \rightarrow U(y, \varDelta_{f(i)})$.

(11) $\vdash_1 D(\varDelta_f, \alpha, \beta) \equiv (\forall x)[(\forall yz)(T(\varDelta_f, y, z) \supset \neg U(z, x)) \supset \alpha(x) = \beta(x)]$.

In the following, for the simplicity we shall often use e.g. $U(\delta)$, $\Gamma \rightarrow \Theta$, $P(\delta(x))$, where $U(\delta)$ is an abbreviation of $(\forall x)U(x, \delta(x))$, instead of

$$\Gamma \rightarrow \Theta, \quad (\exists y)[U(x, y) \wedge P(y)]$$

or

$$\Gamma \rightarrow \Theta, \quad (\forall y)[U(x, y) \supset P(y)].$$

Obviously we have:

(12) $\vdash_1 (\exists \delta)(\forall x)U(x, \delta(x))$ and $\vdash_1 (\exists \theta_i)(\forall x)C_i(x, \theta_i(x))$.

(In $\mathfrak{S}$, the treatments for particular primitive recursive functions are done in the same ways. But sometimes these expressions are complicated.)

§ 4. Proof of Theorem 1.

In this paragraph, we shall prove Theorem 2 below, from which Theorem 1 can be obtained as an immediate corollary.

THEOREM 2. *Let $\Lambda \rightarrow \Pi$ be a provable sequent in $\mathfrak{J}$ such that its proof contains no formulas in which a variable occurs both free and bound. Then we have*

(1) $$\vdash_* \Lambda^*(\alpha) \rightarrow \Pi^*(\alpha),$$

where, if Λ denotes a sequence of formulas $\mathfrak{A}_1, \cdots, \mathfrak{A}_m$ in $\mathfrak{J}$ whose Gödel numbers are $a_1, \cdots, a_m$, respectively, and if $n(\mathfrak{A}_i) = k_i$ $(i = 1, 2, \cdots, m)$, then $\Lambda^*(\alpha)$ denotes the following sequence of formulas of $\mathfrak{S}^*$:

$$Q_k(\varDelta_{a_1}, \alpha), \cdots, Q_{k_m}(\varDelta_{a_m}, \alpha).$$

PROOF. By the transfinite induction on the order of proof of the sequent $\Lambda \rightarrow \Pi$ in $\mathfrak{J}$[6].

6) The notion of "order of proof" is defined as usual.

Basis. The order 0.

Case 1. $\varDelta \Rightarrow \varPi$ is of the form $\mathfrak{A} \Rightarrow \mathfrak{A}$. This case is obvious.

Case 2. $\varDelta \Rightarrow \varPi$ is one of the axioms in (ii) § 2.7. E.g., let $\varDelta \Rightarrow \varPi$ be $\mathfrak{v}_0' = \mathfrak{v}_1' \Rightarrow \mathfrak{v}_0 = \mathfrak{v}_1$, and let a and b be Gödel numbers of $\mathfrak{v}_0' = \mathfrak{v}_1'$ and $\mathfrak{v}_0 = \mathfrak{v}_1$, respectively. Obviously, we obtain

$$\vdash_* \mathrm{Q}_0(\varDelta_a, \alpha) \equiv \alpha(\varDelta_0) = \alpha(\varDelta_1) \quad \text{and} \quad \vdash_* \mathrm{Q}_0(\varDelta_b, \alpha) \equiv \alpha(\varDelta_0) = \alpha(\varDelta_1)$$

by (6) and (8) § 3. Hence we have $\mathrm{Q}_0(\varDelta_a, \alpha) \Rightarrow \mathrm{Q}_0(\varDelta_b, \alpha)$. (Here and after, we often omit the symbols $\vdash_1$, $\vdash_*$ and $\vdash_2$.) For the other axioms: similarly.

Case 3. $\varDelta \Rightarrow \varPi$ is $\bigwedge_i \mathfrak{v}_{g(i)} = \mathfrak{v}_{h(i)}, \mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots) \Rightarrow \mathfrak{A}(\mathfrak{v}_{h(0)}, \mathfrak{v}_{h(1)}, \cdots)$, where g and h are recursive. Let a, b and c be Gödel numbers of those formulas, respectively; and let $\mathrm{n}'(\mathfrak{A}) = k$ and $t(i) = \ulcorner \mathfrak{v}_{h(i)} \urcorner$. Then $a = 2^{21} \cdot 3^f$, where $f(i) = \ulcorner \mathfrak{v}_{g(i)} = \mathfrak{v}_{h(i)} \urcorner$. By (8) § 3 we have

$$\text{(2)} \qquad \mathrm{U}(\delta) \Rightarrow \mathrm{Q}_0(\varDelta_a, \alpha) \equiv (\forall \mathrm{x})(\forall \mathrm{y})[\mathrm{T}(\varDelta_f, \mathrm{x}, \mathrm{y}) \supset \mathrm{Q}_0(\delta(\mathrm{y}), \alpha)], \quad \text{and for all } i$$

$$\text{(3)} \qquad \mathrm{Q}_0(\varDelta_{f(i)}, \alpha) \equiv \alpha(\varDelta_{g(i)}) = \alpha(\varDelta_{h(i)}).$$

Obviousely, for all i,

$$\text{(4)} \qquad \mathrm{U}(\delta), \mathrm{T}(\varDelta_f, \varDelta_i, \mathrm{y}), \mathrm{T}(\varDelta_f, \varDelta_i, \mathrm{y}) \supset \mathrm{Q}_0(\delta(\mathrm{y}), \alpha) \Rightarrow \mathrm{Q}_0(\varDelta_{f(i)}, \alpha).$$

By (6) § 3 for all i

$$\text{(5)} \qquad \alpha(\varDelta_{g(i)}) = \alpha(\varDelta_{h(i)}) \Rightarrow \mathrm{F}(\varDelta_{t(i)}, \alpha(\varDelta_{g(i)}), \alpha).$$

Therefore we can obtain (by above (2)-(5) and (9) § 3)

$$\mathrm{U}(\delta), \mathrm{Q}_0(\varDelta_a, \alpha) \Rightarrow (\forall \mathrm{xuv})[\mathrm{T}(\varDelta_g, \mathrm{x}, \mathrm{u}) \wedge \mathrm{T}(\varDelta_t, \mathrm{x}, \mathrm{v}) \cdot \supset \cdot \mathrm{F}(\delta(\mathrm{v}), \alpha(\delta(\mathrm{u})), \alpha)].$$

Hence by Lemma 2 below we have (using (12) § 3)

$$\mathrm{Q}_0(\varDelta_a, \alpha), \mathrm{Q}_k(\varDelta_b, \alpha) \Rightarrow \mathrm{Q}_k(\varDelta_c, \alpha).$$

LEMMA 2. *Let $\mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$ be a formula in $\mathfrak{Z}$ such that $g(i)$ is recursive, and $\mathfrak{A}(\mathrm{t}_0, \mathrm{t}_1, \cdots)$ be the formula obtained from the above formula by replacing $\mathfrak{v}_{g(i)}$ by terms t_i such that $\ulcorner \mathrm{t}_i \urcorner$ is a recursive function of i*[7]*. Further, let b and c be Gödel numbers of the formulas, respectively; and let t be a number defining recursively $\ulcorner \mathrm{t}_i \urcorner$ as a function of i. Put*

$$\text{(6)} \qquad \varOmega_{g,t}(\alpha, \delta) = (\forall \mathrm{xuv})[\mathrm{T}(\varDelta_g, \mathrm{x}, \mathrm{u}) \wedge \mathrm{T}(\varDelta_t, \mathrm{x}, \mathrm{v}) \supset \mathrm{F}(\delta(\mathrm{v}), \alpha(\delta(\mathrm{u})), \alpha)].$$

Then it holds

$$\mathrm{U}(\delta), \varOmega_{g,t}(\alpha, \delta) \Rightarrow \mathrm{Q}_k(\varDelta_b, \alpha) \equiv \mathrm{Q}_k(\varDelta_c, \alpha),$$

where $\mathrm{n}'(\mathfrak{A}) = k$.

Proof of this lemma will be given in § 5.

Case 4. $\varDelta \Rightarrow \varPi$ is $\bigvee_i \bigwedge_j \mathfrak{A}_i(\mathfrak{z}_j) \Rightarrow \exists \mathfrak{v}_{g(0)} \mathfrak{v}_{g(1)} \cdots \bigwedge_i \mathfrak{A}_i(\mathfrak{v}_{g(i)})$, where no $\mathfrak{v}_{g(m)}$ occurs

7) Of course, we assume that the substitution is free.

in any $\mathfrak{A}_i$. By our presupposition, there is $k = \operatorname{Max}_{i<\omega} n'(\mathfrak{A}_i)$. Let b be a Gödel number of $\bigwedge_i \bigvee_j \mathfrak{A}_i(\mathfrak{z}_j)$ and hence $b = 2^{23} \cdot 3^{b_1}$, where $b_1(i) = 2^{21} \cdot 3^{(b_1(i))_1}$ and for each j $\{(b_1(i))_1\}(j)$ is a Gödel number of $\mathfrak{A}_i(\mathfrak{z}_j)$. Further, let a be a Gödel number of $\exists \mathfrak{v}_{g(0)} \mathfrak{v}_{g(1)} \cdots \bigwedge_i \mathfrak{A}_i(\mathfrak{v}_{g(i)})$ and hence $a = 2^{25} \cdot 3^g \cdot 5^{a_2}$, where $a_2 = 2^{23} \cdot 3^e$ and $e(i)$ is a Gödel number of $\mathfrak{A}_i(\mathfrak{v}_{g(i)})$. First, by (8) and (9) § 3 we obtain

$$(7) \quad \begin{aligned} C_1(\theta_1), U(\delta) \Rightarrow Q_k(\Delta_b, \alpha) &\equiv (\forall xu)(T(\Delta_{b_1}, x, u) \supset Q_k(\delta(u), \alpha)) \\ &\equiv (\forall xu)(T(\Delta_{b_1}, x, u) \supset (\exists z)(\forall y)[T(\theta_1(\delta(u)), z, y) \supset Q_k(\delta(y), \alpha)]) \\ &\equiv (\forall x)(\exists u)(T(\Delta_{b_1}, x, u) \wedge (\exists z)(\forall y)[T(\theta_1((u)), z, y) \supset Q_k(\delta(y), \alpha)]) \\ &\equiv (\forall x)(\exists z)(\forall u)(T(\Delta_{b_1}, x, u) \supset (\forall y)[T(\theta_1(\delta(u)), z, y) \supset Q_k(\delta(y), \alpha)]) \\ &\equiv (\forall x)(\exists z)P(x, z, \alpha, \theta_1, \delta), \text{ where } C_1(\theta_1) \text{ denotes } (\forall x)C_1(x, \theta_1(x)) \end{aligned}$$

and $P(x, z, \alpha, \theta_1, \delta)$ denotes the formula

$$(\forall u)(T(\Delta_{b_1}, x, u) \supset (\forall y)[T(\theta_1, (\delta(u)), z, y) \supset Q_k(\delta(y), \alpha)]).$$

and

$$(8) \quad \begin{aligned} U(\delta) \Rightarrow Q_{k+1}(\Delta_a, \alpha) &\equiv (\exists \beta)(D(\Delta_g, \alpha, \beta) \wedge Q_k(\Delta_{a_2}, \beta)) \\ &\equiv (\exists \beta)(D(\Delta_g, \alpha, \beta) \wedge (\forall xy)[T(\Delta_e, x, y) \supset Q_k(\delta(y), \beta)]). \end{aligned}$$

Next, in the same way as in the proof of Lemma 2 (see § 5), we have: for all i and j.

$$U(\delta), T(\Delta_g, \Delta_i, w), T(\Delta_g, \Delta_i, w) \supset \Delta_j = \beta(\Delta_{g(j)}), Q_k(\Delta_{d(i,j)}, \beta) \Rightarrow Q_k(\Delta_{e(i)}, \beta),$$

where $d(i, j) = \{(b_1(i))_1\}(j)$. Since no $\mathfrak{b}_{g(m)}$'s occur in any $\mathfrak{A}_i$, by Lemma 3 below:

$$D(\Delta_g, \alpha, \beta) \Rightarrow Q_k(\Delta_{d(i,j)}, \beta) \equiv Q_k(\Delta_{d(i,j)}, \alpha).$$

Hence for all i and j

$$D(\Delta_g, \alpha, \beta), U(\delta), T(\Delta_g, \Delta_i, w), T(\Delta_g, \Delta_i, w) \supset \Delta_j = \beta(\delta(w)), Q_k(\Delta_{d(i,j)}, \alpha) \Rightarrow Q_k(\Delta_{e(i)}, \beta).$$

By (9) § 3

$$D(\Delta_g, \alpha, \beta), U(\delta), (\forall w)[T(\Delta_g, \Delta_i, w) \supset \Delta_j = \beta(\delta(w)), Q_k(\Delta_{d(i,j)}, \alpha) \Rightarrow Q_k(\Delta_{e(i)}, \beta),$$

and hence (by ω-rule) for all i

$$(9) \quad \begin{aligned} &D(\Delta_g, \alpha, \beta), U(\delta), (\exists z)\{(\forall w)(T(\Delta_g, \Delta_i, w) \supset z = \beta(\delta(w)) \\ &\quad \wedge (\forall y)(T(\Delta_{(b_1(i))_1}, z, y) \supset Q_k(\delta(y), \alpha))\} \Rightarrow Q_k(\Delta_{e(i)}, \beta). \end{aligned}$$

Since

$$\begin{aligned} \vdash_1 &(\forall w)[T(\Delta_g, \Delta_i, w) \supset (\forall y)(T(\Delta_{(b_1(i))_1}, \beta(\delta(w)), y) \supset Q_k(\delta(y), \alpha))] \\ &\Rightarrow (\exists z)\{(\forall w)(T(\Delta_g, \Delta_i, w) \supset z = \beta(\delta(w))) \wedge (\forall y)(T(\Delta_{(b_1(i))_1}, z, y) \supset Q_k(\delta(y), \alpha))\}, \end{aligned}$$

by (9)

$$C_1(\theta_1), D(\Delta_g, \alpha, \beta), U(\delta), (\forall w)[T(\Delta_g, \Delta_i, w) \supset (\forall u)\{T(\Delta_{b_1}, \Delta_i, u) \supset (\forall y)(T(\theta_1(\delta(u)), \beta(\delta(w)), y) \supset Q_k(\delta(y), \alpha))\}] \Rightarrow Q_k(\Delta_{e(i)}, \beta).$$

I. e.,

$$\Theta(\theta_1, \delta), D(\Delta_g, \alpha, \beta), (\forall w)[T(\Delta_g, \Delta_i, w) \supset P(\Delta_i, \beta(\delta(w)), \alpha, \theta_1, \delta)] \Rightarrow Q_k(\Delta_{e(i)}, \beta),$$

where $\Theta(\theta_1, \delta)$ denotes $C_1(\theta_1) \wedge U(\delta)$.

Using several rules of inference

$$\Theta(\theta_1, \delta), D(\Delta_g, \alpha, \beta) \wedge (\forall vw)(T(\Delta_g, v, w) \supset P(v, \beta(\delta(w)), \alpha, \theta_1, \delta)) \Rightarrow (\exists\beta)(D(\Delta_g, \alpha, \beta) \wedge (\forall xy)[T(\Delta_e, x, y) \supset Q_k(\delta(y), \beta)]).$$

Hence by (8)

$$\Theta(\theta_1, \delta), (\exists\beta)[D(\Delta_g, \alpha, \beta) \wedge (\forall vw)(T(\Delta_g, v, w) \supset P(v, \beta(\delta(w)), \alpha, \theta_1, \delta))] \Rightarrow Q_{k+1}(\Delta_a, \alpha).$$

So, if the following sequent is provable in $\mathfrak{S}^*$:

(10) $$\Theta(\theta_1, \delta), Q_k(\Delta_b, \alpha) \Rightarrow (\exists\beta)[D(\Delta_g, \alpha, \beta) \wedge (\forall vw)(T(\Delta_g, v, w) \supset P(v, \beta(\delta(w)), \alpha, \theta_1, \delta))],$$

then (using (12) § 3) we have: $\vdash_* Q_k(\Delta_b, \alpha) \Rightarrow Q_{k+1}(\Delta_a, \alpha)$, which is the desired.

Now we shall prove (10): It can be easily shown that

$$\vdash_* P(\theta(x), z, \alpha, \theta_1, \delta) \Rightarrow (\exists z)[\{(\forall vw)(T(\Delta_g, v, w) \supset x \neq \delta(w)) \wedge z = \alpha(x)\} \vee \{(\exists vw)(T(\Delta_g, v, w) \wedge x = \delta(w)) \wedge P(\theta(x), z, \alpha, \theta_1, \delta)\}].$$

Hence we have

$$(\forall v)(\exists z)P(v, z, \alpha, \theta_1, \delta) \Rightarrow (\forall x)(\exists z)[\{(\forall vw)(T(\Delta_g, v, w) \supset x \neq \delta(w)) \wedge z = \alpha(x)\} \vee \{(\exists vw)(T(\Delta_g, v, w) \wedge x = \delta(w)) \wedge P(\theta(x), z, \alpha, \theta_1, \delta)\}].$$

By (7)

(11) $$\Theta(\theta_1, \delta), Q_k(\Delta_b, \alpha) \Rightarrow (\forall x)(\exists z)[\{(\forall vw)(T(\Delta_g, v, w) \supset x \neq \delta(w)) \wedge z = \alpha(x)\} \vee \{(\exists vw)(T(\Delta_g, v, w) \wedge x = \delta(w)) \wedge P(\theta(x), z, \alpha, \theta_1, \delta)\}].$$

On the other hand, we can obtain

$$(\forall xv)\{(\forall w)(T(\Delta_g, v, w) \supset x = \delta(w)) \supset \theta(x) = v\}, (\exists vw)(T(\Delta_g, v, w) \wedge x = \delta(w)) \supset P(\theta(x), z, \alpha, \theta_1, \delta) \Rightarrow (\forall vw)\{T(\Delta_g, v, w) \wedge x = \delta(w) \cdot \supset \cdot P(v, z, \alpha, \theta_1, \delta)\}.$$

Therefore,

$$(\forall xv)\{(\forall w)(T(\varDelta_g, v, w) \supset x = \delta(w)) \supset \theta(x) = v\}, (\forall x)(\exists z)[\{(\forall vw)(T(\varDelta_g, v, w) \supset x \neq \delta(w)) \supset z = \alpha(x)\} \wedge \{(\exists vw)(T(\varDelta_g, v, w) \wedge x = \delta(w)) \supset P(\theta(x), z, \alpha, \theta_1, \delta)\}] \rightarrow (\forall x)(\exists z)[\{(\forall vw)(T(\varDelta_g, v, w) \supset x \neq \delta(w)) \supset z = \alpha(x)\} \wedge (\forall vw)\{T(\varDelta_g, v, w) \wedge x = \delta(w) \cdot \supset \cdot P(v, z, \alpha, \theta_1, \delta)\}] .$$

From this together with (11), we obtain

$$(\exists\theta)(\forall xv)\{(\forall w)(T(\varDelta_g, v, w) \supset x = \delta(w)) \supset \theta(x) = v\}, \Theta(\theta_1, \delta), Q_k(\varDelta_b, \alpha) \rightarrow (\forall x)(\exists z)[\{(\forall vw)(T(\varDelta_g, v, w) \supset x \neq \delta(w)) \supset z = \alpha(x)\} \wedge (\forall vw)\{T(\varDelta_g, v, w) \wedge x = \delta(w) \cdot \supset \cdot P(v \cdot z \cdot \alpha, \theta_1, \delta)\}] .$$

It can be proved that the first formula of the above sequent is provable in $\mathfrak{S}$. (Use the condition: $(Ei)(g(i)=j) \rightarrow (E!i)(g(i)=j)$, which our recursive function g possesses by our supposition.) Hence by the axiom (iv) § 1.5, it holds:

$$\Theta(\theta_1, \delta), Q_k(\varDelta_b, \alpha) \rightarrow (\exists\beta)(\forall x)[\{(\forall vw)(T(\varDelta_g, v, w) \supset x \neq \delta(w)) \supset \beta(x) = \alpha(x)\} \wedge (\forall vw)\{T(\varDelta_g, v, w) \wedge x = \delta(w) \cdot \supset \cdot P(v, \beta(x), \alpha, \theta_1, \delta)\}] .$$

Therefore we have (by (11) § 3)

$$\Theta(\theta_1, \delta), Q_k(\varDelta_b, \alpha) \rightarrow (\exists\beta)[D(\varDelta_g, \alpha, \beta) \wedge (\forall vw)(T(\varDelta_g, v, w) \supset P(v, \beta(\delta(w)), \alpha, \theta_1, \delta)] .$$

This is (10).

LEMMA 3. *Let a be a Gödel number of a formula* $\mathfrak{A}$ *in* $\mathfrak{J}$, *and* $n'(\mathfrak{A}) = k$. *If no* $\mathfrak{v}_{g(i)}$ *occur in* $\mathfrak{A}$, *then it holds:*

$$\text{(12)} \qquad \vdash_* D(\varDelta_g, \alpha, \beta) \rightarrow Q_k(\varDelta_a, \alpha) \equiv Q_k(\varDelta_a, \beta) ,$$

where $g(i)$ *is recursive.*

Proof of this lemma will be given in § 5.

Case 5. $\varDelta \rightarrow \Pi$ is $\rightarrow \bigvee_i \mathfrak{z}_i = \mathfrak{v}_0$. Let $a = 2^{21} \cdot 3^f$, where for each $i, f(i) = \ulcorner \mathfrak{z}_i = \mathfrak{v}_0 \urcorner$. Then by (6) and (8) § 3 we obtain:

$$U(\delta) \rightarrow Q_0(\varDelta_a, \alpha) \equiv (\exists x)(\forall y)(T(\varDelta_f, x, y) \supset Q_0(\delta(y), \alpha))$$

$$\varDelta_i = \alpha(0) \rightarrow \varDelta_i = \alpha(0)$$

$$\varDelta_i = \alpha(0) \rightarrow Q_0(\varDelta_{f(i)}, \alpha)$$

$$\varDelta_i = \alpha(0) \rightarrow (\exists x)(\forall y)(T(\varDelta_f, x, y) \supset Q_0(\delta(y), \alpha) .$$

I. e.,

$$\varDelta_i = \alpha(0) \rightarrow Q_0(\varDelta_a, \alpha) \quad \text{for all } i.$$

Hence $(\exists x)(x = \alpha(0)) \rightarrow Q_0(\varDelta_a, \alpha)$. But $\vdash_1 (\exists x)(x = \alpha(0))$. Therefore, $\vdash_1 Q_0(\varDelta_a, \alpha)$.

Induction step in the proof of Theorem 2. We shall prove only for several cases, since the others can be similarly done.

Case 1. $\Delta \rightarrow \Pi$ is $\bigvee_i \mathfrak{A}_i, \Gamma \rightarrow \Theta$, and $\dfrac{\mathfrak{A}_i, \Gamma \rightarrow \Theta \text{ for all } i}{\bigvee_i \mathfrak{A}_i, \Gamma \rightarrow \Theta}$. By our presupposition, there exists $\underset{i<\omega}{\mathrm{Max}}\, \mathrm{n}'(\mathfrak{A}_i)$. Hence, let $k_i = \mathrm{n}'(\mathfrak{A}_i)$ and $k = \underset{i<\omega}{\mathrm{Max}}\, k_i$. We use the following lemma which will be proved in § 5.

LEMMA 4. *Let* $k = \mathrm{n}'(\mathfrak{A})$ *and* a *be a Gödel number of* $\mathfrak{A}$. *Then for every natural number* m, $\vdash_* \mathrm{Q}_k(\Delta_a, \alpha) \equiv \mathrm{Q}_{k+m}(\Delta_a, \alpha)$.

Let a be a Gödel number of $\bigvee_i \mathfrak{A}_i$ and hence $a = 2^{21} \cdot 3^f$ where $f(i)$ is a Gödel number of $\mathfrak{A}_i$ for each i. By Lemma 4 we have $\vdash_* \mathrm{Q}_{k_i}(\Delta_{f(i)}, \alpha) \equiv \mathrm{Q}_k(\Delta_{f(i)}, \alpha)$. Using $\vdash_* \mathrm{Q}_k(\Delta_{f(i)}, \alpha), \Gamma^*(\alpha) \rightarrow \Theta^*(\alpha)$ (which is obtained by the hypothesis of induction) and cut, we obtain (for all i)

$$\mathrm{U}(\delta), \mathrm{T}(\Delta_f, \Delta_i, \mathrm{y}), \mathrm{T}(\Delta_f, \Delta_i, \mathrm{y}) \supset \mathrm{Q}_k(\delta(\mathrm{y}), \alpha), \Gamma^*(\alpha) \rightarrow \Theta^*(\alpha).$$

Hence (by ω-rule and (9) § 3)

$$\mathrm{U}(\delta), (\exists \mathrm{x})(\forall \mathrm{y})(\mathrm{T}(\Delta_f, \mathrm{x}, \mathrm{y}) \supset \mathrm{Q}_k(\delta(\mathrm{y}), \alpha)), \Gamma^*(\alpha) \rightarrow \Theta^*(\alpha).$$

By (8) § 3 we obtain $\vdash_* \mathrm{Q}_k(\Delta_a, \alpha), \Gamma^*(\alpha) \rightarrow \Theta^*(\alpha)$, which is the desired.

Case 2. $\Delta \rightarrow \Pi$ is $\exists \mathfrak{v}_{g(0)} \mathfrak{v}_{g(1)} \cdots \mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots), \Gamma \rightarrow \Theta$ and $\dfrac{\mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots), \Gamma \rightarrow \Theta}{\exists \mathfrak{v}_{g(0)} \mathfrak{v}_{g(1)} \cdots \mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots), \Gamma \rightarrow \Theta}$, under the proviso in ($\exists \rightarrow$)-rule of $\mathfrak{J}$. Let $a = 2^{25} \cdot 3^g \cdot 5^b$, where b is a Gödel number of $\mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$, and $\mathrm{n}'(\mathfrak{A}) = k$. By the hypothesis of induction: $\mathrm{Q}_k(\Delta_b, \beta), \Gamma^*(\beta) \rightarrow \Theta^*(\beta)$. By that proviso and Lemma 3 stated above, we have

$$\mathrm{D}(\Delta_g, \alpha, \beta), \Gamma^*(\alpha) \rightarrow \Gamma^*(\beta) \quad \text{and} \quad \mathrm{D}(\Delta_g, \alpha, \beta), \Theta^*(\beta) \rightarrow \Theta^*(\alpha).$$

Hence $\mathrm{D}(\Delta_g, \alpha, \beta) \wedge \mathrm{Q}_k(\Delta_b, \beta), \Gamma^*(\alpha) \rightarrow \Theta^*(\alpha)$. So, by ($\exists\beta \rightarrow$)-rule and (8) § 3 we obtain $\mathrm{Q}_{k+1}(\Delta_a, \alpha), \Gamma^*(\alpha) \rightarrow \Theta^*(\alpha)$, which is desired.

Case 3. $\Delta \rightarrow \Pi$ is $\Gamma \rightarrow \Theta, \exists \mathfrak{v}_{g(0)} \mathfrak{v}_{g(1)} \cdots \mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$ and $\dfrac{\Gamma \rightarrow \Theta, \mathfrak{A}(\mathrm{t}_0, \mathrm{t}_1, \cdots)}{\Gamma \rightarrow \Theta, \exists \mathfrak{v}_{g(0)} \mathfrak{v}_{g(1)} \cdots \mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)}$, under the proviso in ($\rightarrow \exists$)-rule of $\mathfrak{J}$. Let $a = 2^{25} \cdot 3^g \cdot 5^b$, where b is a Gödel number of $\mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$, and $\mathrm{n}'(\mathfrak{A}) = k$. Further, let c be a Gödel number of $\mathfrak{A}(\mathrm{t}_0, \mathrm{t}_1, \cdots)$ and let t define recursively $\ulcorner \mathrm{t}_i \urcorner$ as a function of i. Then by the hypothesis of induction: $\Gamma^*(\alpha) \rightarrow \Theta^*(\alpha)$, $\mathrm{Q}_k(\Delta_c, \alpha)$. On the other hand, by the same method as in the proof of Lemma 2 we can obtain: $\mathrm{U}(\delta), \Omega_{g,t}(\alpha, \beta, \delta) \rightarrow \mathrm{Q}_k(\Delta_b, \beta) \equiv \mathrm{Q}_k(\Delta_c, \alpha)$, where $\Omega_{g,t}(\alpha, \beta, \delta) = (\forall \mathrm{yuw})(\mathrm{T}(\Delta_g, \mathrm{y}, \mathrm{u}) \wedge \mathrm{T}(\Delta_t, \mathrm{y}, \mathrm{w}) \cdot \supset \cdot \mathrm{F}(\delta(\mathrm{w}), \beta(\delta(\mathrm{u})), \alpha))$. Hence

$$\mathrm{U}(\delta), \mathrm{D}(\Delta_g, \alpha, \beta) \wedge \Omega_{g,t}(\alpha, \beta, \delta), \Gamma^*(\alpha) \rightarrow \Theta^*(\alpha), \mathrm{D}(\Delta_g, \alpha, \beta) \wedge \mathrm{Q}_k(\Delta_b, \beta),$$

so

$$\mathrm{U}(\delta), (\exists\beta)(\mathrm{D}(\Delta_g, \alpha, \beta) \wedge \Omega_{g,t}(\alpha, \beta, \delta)), \Gamma^*(\alpha)$$
$$\rightarrow \Theta^*(\alpha), (\exists\beta)(\mathrm{D}(\Delta_g, \alpha, \beta) \wedge \mathrm{Q}_k(\Delta_k, \beta)).$$

If we can show (13): $\vdash_1 U(\delta) \rightarrow (\exists\beta)(D(\Delta_g, \alpha, \beta) \wedge \Omega_{g,t}(\alpha, \beta, \delta))$, then we have (using (8) and (12) § 3) $\Gamma^*(\alpha) \rightarrow \Theta^*(\alpha)$, $Q_{k+1}(\Delta_a, \alpha)$, which is the desired.

PROOF OF (13): Let $P(x, z, \alpha, \delta)$ and $P(x, z, \alpha, \delta, \theta)$ be

$$\{(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u)) \supset \alpha(x) = z\}$$
$$\wedge (\forall yuw)\{T(\Delta_g, y, u) \wedge x = \delta(u)$$
$$\wedge T(\Delta_t, y, w) \cdot\supset\cdot F(\delta(w), z, \alpha)\}$$

and

$$\{(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u) \supset \alpha(x) = z\}$$
$$\wedge (\forall yuw)\{T(\Delta_g, y, u) \wedge x = \delta(u)$$
$$\wedge T(\Delta_t, \theta(x), w) \cdot\supset\cdot F(\delta(w), z, \alpha)\},$$

respectively.

Now, we have the following provable sequents:

$$(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u))$$
$$\rightarrow (\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u)) \wedge \alpha(x) = \alpha(x),$$

$$(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u))$$
$$\rightarrow (\exists z)\{(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u)) \wedge \alpha(x) = z\},$$

$$(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u))$$
$$\rightarrow (\exists z)[\{(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u)) \wedge \alpha(x) = z\}$$
$$\vee \{(\exists yu)(T(\Delta_g, y, u) \wedge x = \delta(u)) \wedge (\forall w)(T(\Delta_t, \theta(x), w)$$
$$\supset F(\delta(w), z, \alpha))\}],$$

and

$$(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u))$$
$$\rightarrow (\exists z)[\{(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u)) \supset \alpha(x) = z\}$$
$$\wedge \{(\exists yu)(T(\Delta_g, y, u) \wedge x = \delta(u) \supset (\forall w)(T(\Delta_t, \theta(x), w)$$
$$\supset F(\delta(w), z, \alpha))\}].$$

Hence

(14) $$(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u)) \rightarrow (\exists z)P(x, z, \alpha, \delta, \theta).$$

On the other hand, for all i

$$T(\Delta_g, \Delta_i, u) \wedge x = \delta(u), (T(\Delta_g, \Delta_i, u) \supset x = \delta(u)) \supset \theta(x) = \Delta_i,$$
$$T(\Delta_t, \theta(x), w), T(\Delta_t, \Delta_i, w) \supset F(\delta(w), z, \alpha) \rightarrow F(\delta(w), z, \alpha)$$

Hence for all i

$$\underbrace{T(\Delta_g, \Delta_i, u) \wedge x = \delta(u)}_{\text{(denoted by } M_1(x, \Delta_i, u, \delta),}, \underbrace{(\forall yu)\{(T(\Delta_g, y, u) \supset x = \delta(u)) \supset \theta(x) = y\}}_{M_2(x, \delta, \theta), \text{ resp.)},},$$

$$(\forall w)(T(\Delta_t, \Delta_i, w) \supset F(\delta(w), z, \alpha))$$
$$\Rightarrow (\forall w)(T(\Delta_t, \theta(x), w) \supset F(\delta(w), z, \alpha)),$$
$$M_1(x, \Delta_i, u, \delta), M_2(x, \delta, \theta), (\forall w)(T(\Delta_t, \Delta_i, w) \supset F(\delta(w), z, \alpha))$$
$$\Rightarrow (\exists z)\{(\exists yu)M_1(x, y, u, \delta) \wedge (\forall w)(T(\Delta_t, \theta(x), w) \supset F(\delta(w), z, \alpha))\}.$$

In the same way as above, we obtain

$$M_1(x, \Delta_i, u, \delta), M_2(x, \delta, \theta), (\forall w)(T(\Delta_t, \Delta_i, w) \supset F(\delta(w), z, \alpha))$$
$$\Rightarrow (\exists z)P(x, z, \alpha, \delta, \theta).$$

Hence

$$M_1(x, \Delta_i, u, \delta), M_2(x, \delta, \theta), (\exists z)(\forall w)(T(\Delta_t, \Delta_i, w) \supset F(\delta(w), z, \alpha))$$
$$\Rightarrow (\exists z)P(x, z, \alpha, \delta, \theta).$$

Since $t(i)$ is a Gödel number of a term in $\mathfrak{J}$, by Lemma 1 we have $\vdash_1 U(\delta) \Rightarrow (\exists z)(\forall w)(T(\Delta_t, \Delta_i, w) \supset F(\delta(w), z, \alpha))$.

Hence for all i

$$U(\delta), M_1(x, \Delta_i, u, \delta), M_2(x, \delta, \theta) \Rightarrow (\exists z)P(x, z, \alpha, \delta, \theta).$$

By ($\exists u \Rightarrow$)-rule and ω-rule: $U(\delta), (\exists yu)M_1(x, y, u, \delta), M_2(x, \delta, \theta) \Rightarrow (\exists z)P(x, z, \alpha, \delta, \theta)$, i. e.,

(15) $\quad U(\delta), (\exists yu)(T(\Delta_g, y, u) \wedge x = \delta(u)), M_2(x, \delta, \theta) \Rightarrow (\exists z)P(x, z, \alpha, \delta, \theta).$

By (14) and (15) we have (16): $U(\delta), M_2(x, \delta, \theta) \Rightarrow (\exists z)P(x, z, \alpha, \delta, \theta)$. Now, we shall show: $U(\delta), M_2(x, \delta, \theta) \Rightarrow (\exists z)P(x, z, \alpha, \delta)$. First,

$$(T(\Delta_g, y, u) \supset x = \delta(u)) \supset \theta(x) = y, T(\Delta_g, y, u) \wedge x = \delta(u) \wedge T(\Delta_t, y, w)$$
$$\Rightarrow T(\Delta_t, \theta(x), w).$$

Hence

$$(T(\Delta_g, y, u) \supset x = \delta(u)) \supset \theta(x) = y, T(\Delta_g, y, u) \wedge x = \delta(u) \wedge T(\Delta_t, y, w),$$
$$T(\Delta_g, y, u) \wedge x = \delta(u) \wedge T(\Delta_t, \theta(x), w) \supset F(\delta(w), z, \alpha)$$
$$\Rightarrow F(\delta(w), z, \alpha),$$
$$(T(\Delta_g, y, u) \supset x = \delta(u)) \supset \theta(x) = y, T(\Delta_g, y, u) \wedge x = \delta(u)$$
$$\wedge T(\Delta_t, \theta(x), w) \cdot \supset \cdot F(\delta(w), z, \alpha) \Rightarrow T(\Delta_g, y, u)$$
$$\wedge x = \delta(u) \wedge T(\Delta_t, y, w) \cdot \supset \cdot F(\delta(w), z, \alpha),$$
$$M_2(x, \delta, \theta), (\forall yuw)\{T(\Delta_g, y, u) \wedge x = \delta(u) \wedge T(\Delta_t, \theta(x), w)$$
$$\cdot \supset \cdot F(\delta(w), z, \alpha)\} \Rightarrow (\forall yuw)\{T(\Delta_g, y, u) \wedge x = \delta(u)$$
$$\wedge T(\Delta_t, y, w) \cdot \supset \cdot F(\delta(w), z, \alpha)\}.$$
$$M_2(x, \delta, \theta), \{(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u)) \supset \alpha(x) = z\}$$
$$\wedge (\forall yuw)\{T(\Delta_g, y, u) \wedge x = \delta(u) \wedge T(\Delta_t, \theta(x), w) \cdot \supset \cdot F(\delta(w), z, \alpha)\}$$

$$\rightarrow \{(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u)) \supset \alpha(x) = z\}$$
$$\wedge (\forall yuw)\{T(\Delta_g, y, u) \wedge x = \delta(u) \wedge T(\Delta_t, y, w) \cdot\supset\cdot F(\delta(w), z, \alpha)\} .$$

i. e., $M_2(x, \delta, \theta), P(x, z, \alpha, \delta, \theta) \rightarrow P(x, z, \alpha, \delta)$. Hence

$$M_2(x, \delta, \theta), (\exists z)P(x, z, \alpha, \delta, \theta) \rightarrow (\exists z)P(x, z, \alpha, \delta) .$$

Together with (16) we obtain (17): $U(\delta), M_2(x, \delta, \theta) \rightarrow (\exists z)P(x, z, \alpha, \delta)$. From this by $(\exists \rightarrow)$ rule we have: $U(\delta), (\exists\theta)(\forall x)M_2(x, \delta, \theta) \rightarrow (\exists z)P(x, z, \alpha, \delta)$. Since we may assume that the recursive function g possesses the property: $(Ej)(g(j) = i) \rightarrow (E!j)(g(j)=i)$, we can obtain $\vdash_1 (\exists\theta)(\forall x)M_2(x, \delta, \theta)$. (Cf. the proof of (10) in this section.) Hence $U(\delta) \rightarrow (\exists z)P(x, z, \alpha, \delta)$. Therefore $U(\delta) \rightarrow (\forall x)(\exists z)P(x, z, \alpha, \delta)$, and hence $U(\delta) \rightarrow (\exists\beta)(\forall x)P(x, \beta(x), \alpha, \delta)$. That is,

$$U(\delta) \rightarrow (\exists\beta)[(\forall x)\{(\forall yu)(T(\Delta_g, y, u) \supset x \neq \delta(u)) \supset \alpha(x) = \beta(x)\}$$
$$\wedge (\forall x)(\forall yuw)\{T(\Delta_g, y, u) \wedge x = \delta(u) \wedge T(\Delta_t, y, w)$$
$$\cdot\supset\cdot F(\delta(w), \beta(x), \alpha)\}]$$

Hence

$$U(\delta) \rightarrow (\exists\beta)[D(\Delta_g, \alpha, \beta) \wedge (\forall yuw)\{T(\Delta_g, y, u)$$
$$\wedge T(\Delta_t, y, w) \cdot\supset\cdot F(\delta(w), \beta(\delta(u)), \alpha)\}] .$$

This is just (13).

Thus, the proof of Theorem 2 (and hence of Theorem 1) will be completed, when Lemmas 2–4 will be proved.

§ 5. Proofs of lemmas stated in the preceding paragraph.

5.1. Proof of Lemma 2 by the induction on the nesting number of $\mathfrak{A}$.

Basis. $\mathfrak{A}$ is a prime formula. Then $\mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$ has the form: $\mathfrak{T}_1(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots) = \mathfrak{T}_2(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$, where $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are terms (in $\mathcal{J}$) and only finite numbers of variables occur in them. And $\mathfrak{A}(t_0, t_1, \cdots)$ is of the form: $\mathfrak{T}_1(t_0, t_1, \cdots) = \mathfrak{T}_2(t_0, t_1, \cdots)$. Since $\vdash_1 B'_{17}(\Delta_b)$ and $\vdash_1 B'_{17}(\Delta_c)$, by (8) § 3 we have

$$U(\delta) \rightarrow Q_0(\Delta_b, \alpha) \equiv (\exists x)(F(\Delta_{(b)_1}, x, \alpha) \wedge F(\Delta_{(b)_2}, x, \alpha)) \cdot\wedge\cdot Q_0(\Delta_c, \alpha)$$
$$\equiv (\exists x)(F(\Delta_{(c)_1}, x, \alpha)) \wedge F(\Delta_{(c)_2}, x, \alpha)) .$$

Hence, it is sufficient to prove the following lemma:

LEMMA 5. *Let* $\langle \mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots \rangle$ *be an infinite sequence of distinct variables in* $\mathfrak{J}$, *where g is recursive, and* $\langle t_0, t_1, \cdots \rangle$ *an infinite recursive sequence of terms in* $\mathfrak{J}$ (*this means:* $\ulcorner t_i \urcorner$ *is a recursive function of i*). *Further let r and s be the Gödel numbers of terms* $\mathfrak{T}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots, \mathfrak{v}_{g(l)})$ *and* $\mathfrak{T}(t_0, t_1, \cdots, t_l)$, *respectively. If g and t define recursively the functions* $g(i)$ *and* $\ulcorner t_i \urcorner$, *respectively, then it holds*

(1) $$\vdash_1 U(\delta), \Omega_{g,t}(\alpha, \delta) \rightarrow F(\Delta_r, x, \alpha) \equiv F(\Delta_s, x, \alpha) ,$$

where $\Omega_{g,t}(\alpha, \delta)$ *denotes the following formula:*

$$(\forall yuw)(T(\Delta_g, y, u) \wedge T(\Delta_t, y, w) \cdot \supset \cdot F(\delta(w), \alpha(\delta(u)), \alpha) .$$

Proof of Lemma 5 by the induction on the construction of $\mathfrak{T}(\mathfrak{v}_{g(0)}, \cdots, \mathfrak{v}_{g(l)})$.

1°) $r=3$, i.e., the term is "0". Then $\mathfrak{T}(t_0, t_1, \cdots, t_l)$ is also "0". Hence obviously we have $\vdash_1 F(\Delta_r, x, \alpha) \equiv F(\Delta_s, x, \alpha)$.

2°) $r=7^{j+1}$, where $j \neq g(i)$ for all $i \leqq l$. Then $s=7^{j+1}$. Hence the lemma holds.

3°) $r=7^{g(i)+1}$ for some $i \leqq l$, i.e. $\mathfrak{T}(\mathfrak{v}_{g(0)}, \cdots, \mathfrak{v}_{g(l)})$ is "$\mathfrak{v}_{g(i)}$" and $\mathfrak{T}(t_0, t_1, \cdots, t_l)$ is "t_i", and hence $s=t(i)=\ulcorner t_i \urcorner$. Then

$$U(\delta), T(\Delta_g, \Delta_i, u), T(\Delta_t, \Delta_i, w), T(\Delta_g, \Delta_i, u) \wedge T(\Delta_t, \Delta_i, w) \cdot \supset \cdot F(\delta(w), \alpha(\delta(u)), \alpha)$$
$$\Rightarrow F(\Delta_{t(i)}, \alpha(\Delta_{g(i)}), \alpha) .$$

Hence by (9) §3

$$U(\delta), \Omega_{g,t}(\alpha, \delta) \Rightarrow F(\Delta_{t(i)}, \alpha(\Delta_{g(i)}), \alpha) .$$

On the other hand we have: $F(\Delta_r, x, \alpha) \equiv x = \alpha(\Delta_{g(i)})$ by (6) §3, and by Lemma 1 $\vdash_1 (\exists ! x) F(\Delta_{t(i)}, x, \alpha)$. Therefore, we obtain

$$U(\delta), \Omega_{g,t}(\alpha, \delta) \Rightarrow F(\Delta_r, x, \alpha) \equiv F(\Delta_{t(i)}, x, \alpha) \equiv F(\Delta_s, x, \alpha) .$$

4°) $r=2^{11} \cdot 3^{r_1} \cdot 5^{r_2}$. Then $\vdash_1 B'_{11}(\Delta_r)$, $\mathfrak{T}(\mathfrak{v}_{g(0)}, \cdots, \mathfrak{v}_{g(l)})$ has the form $\mathfrak{T}_1(\mathfrak{v}_{g(0)}, \cdots, \mathfrak{v}_{g(l)}) + \mathfrak{T}_2(\mathfrak{v}_{g(0)}, \cdots, \mathfrak{v}_{g(l)})$, and hence $\mathfrak{T}(t_0, \cdots, t_l)$ is of the form $\mathfrak{T}_1(t_0, \cdots, t_l) + \mathfrak{T}_2(t_0, \cdots, t_l)$, $s=2^{11} \cdot 3^{s_1} \cdot 5^{s_2}$ and $\vdash_1 B'_{11}(\Delta_s)$. By (6) §3, we have

$$U(\delta) \Rightarrow F(\Delta_r, x, \alpha) \equiv (\exists cd)(F(\Delta_{r_1}, c, \alpha) \wedge F(\Delta_{r_2}, d, \alpha) \wedge x=c+d)$$

and

$$U(\delta) \Rightarrow F(\Delta_s, x, \alpha) \equiv (\exists cd)(F(\Delta_{s_1}, c, \alpha) \wedge F(\Delta_{s_2}, d, \alpha) \wedge x=c+d) .$$

By the hypothesis of induction:

$$U(\delta), \Omega_{g,t}(\alpha, \delta) \Rightarrow F(\Delta_{r_1}, c, \alpha) \equiv F(\Delta_{s_1}, c, \alpha) \cdot \wedge \cdot F(\Delta_{r_2}, d, \alpha) \equiv F(\Delta_{s_2}, d, \alpha) .$$

From these we obtain: $U(\delta), \Omega_{g,t}(\alpha, \delta) \Rightarrow F(\Delta_r, x, \alpha) \equiv F(\Delta_s, x, \alpha)$.

5°) The other cases are also similarly treated. (q.e.d.)

Induction step in the proof of Lemma 2: Case 1. $\mathfrak{A}$ is of the form $\neg\mathfrak{B}$. This case is obvious.

Case 2. $\mathfrak{A}$ is of the form $\bigvee_i \mathfrak{B}_i$, i.e., $\mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$ has the form $\bigvee_i \mathfrak{B}_i(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$. Then $\mathfrak{A}(t_0, t_1, \cdots)$ is $\bigvee_i \mathfrak{B}_i(t_0, t_1, \cdots)$. Let $b=2^{21} \cdot 3^{b_1}$ and $c=2^{21} \cdot 3^{c_1}$, where for each i $b_1(i)$ and $c_1(i)$ are Gödel numbers of $\mathfrak{B}_i(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$ and $\mathfrak{B}_i(t_0, t_1, \cdots)$, respectively. By the hypothesis of induction:

(2) $\quad U(\delta), \Omega_{g,t}(\alpha, \delta) \Rightarrow Q_{k_i}(\Delta_{b_1(i)}, \alpha) \equiv Q_{k_i}(\Delta_{c_1(i)}, \alpha)$ for all i,

where $k_i = n'(\mathfrak{B}_i)$. But, if $k=n'(\mathfrak{A})$, then by Lemma 4 we have

(2′) $\quad U(\delta), \Omega_{g,t}(\alpha, \delta) \Rightarrow Q_k(\Delta_{b(i)}, \alpha) \equiv Q_k(\Delta_{c_1(i)}, \alpha)$ for all i.

Since

$$\mathrm{U}(\delta), \mathrm{T}(\Delta_{b_1}, \Delta_i, \mathrm{y}), \mathrm{T}(\Delta_{b_1}, \Delta_i, \mathrm{y}) \supset \mathrm{Q}_k(\delta(\mathrm{y}), \alpha) \rightarrow \mathrm{Q}_k(\Delta_{b_1(i)}, \alpha),$$

by above (2′) and (9) § 3 we obtain: for all i

$$\mathrm{U}(\delta), \Omega_{g,t}(\alpha, \delta), (\forall \mathrm{y})(\mathrm{T}(\Delta_{b_1}, \Delta_i, \mathrm{y}) \supset \mathrm{Q}_k(\delta(\mathrm{y}), \alpha))$$
$$\rightarrow (\exists \mathrm{x})(\forall \mathrm{u})(\mathrm{T}(\Delta_{c_1}, \mathrm{x}, \mathrm{u}) \supset \mathrm{Q}_k(\delta(\mathrm{u}), \alpha)).$$

By ω-rule and (8) § 3 we have

$$\mathrm{U}(\delta), \Omega_{g,t}(\alpha, \delta), \mathrm{Q}_k(\Delta_b, \alpha) \rightarrow \mathrm{Q}_k(\Delta_c, \alpha).$$

Similarly we can obtain:

$$\mathrm{U}(\delta), \Omega_{g,t}(\alpha, \delta), \mathrm{Q}_k(\Delta_c, \alpha) \rightarrow \mathrm{Q}_k(\Delta_b, \alpha).$$

Case 3. $\mathfrak{A}(\mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$ has the form $\exists \mathfrak{v}_{h(0)} \mathfrak{v}_{h(1)} \cdots \mathfrak{B}(\mathfrak{v}_{h(0)}, \mathfrak{v}_{h(1)}, \cdots; \mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$, where $(i)(j)[h(i) \neq g(j)]$. Then $\mathfrak{A}(\mathrm{t}_0, \mathrm{t}_1, \cdots)$ is of the form $\exists \mathfrak{v}_{h(0)} \mathfrak{v}_{h(1)} \cdots \mathfrak{B}(\mathfrak{v}_{h(0)}, \mathfrak{v}_{h(1)}, \cdots; \mathrm{t}_0, \mathrm{t}_1, \cdots)$. Let $b = 2^{25} \cdot 3^h \cdot 5^{b_2}$ and $c = 2^{25} \cdot 3^h \cdot 5^{c_2}$. By (8) § 3 we have:

$$\mathrm{U}(\delta) \rightarrow \mathrm{Q}_{k+1}(\Delta_b, \alpha) \equiv (\exists \beta)(\mathrm{D}(\Delta_h, \alpha, \beta) \wedge \mathrm{Q}_k(\Delta_{b_2}, \beta))$$

and

$$\mathrm{U}(\delta) \rightarrow \mathrm{Q}_{k+1}(\Delta_c, \alpha) \equiv (\exists \beta)(\mathrm{D}(\Delta_h, \alpha, \beta) \wedge \mathrm{Q}_k(\Delta_{c_2}, \beta)),$$

where $k = \mathrm{n}'(\mathfrak{B})$. By the hypothesis of induction:

$$\mathrm{U}(\delta), \Omega_{g,t}(\beta, \delta) \rightarrow \mathrm{Q}_k(\Delta_{b_2}, \beta) \equiv \mathrm{Q}_k(\Delta_{c_2}, \beta).$$

In the similar way as in the proof of Lemma 3 below, we can obtain

$$\mathrm{D}(\Delta_h, \alpha, \beta) \rightarrow \Omega_{g,t}(\alpha, \delta) \equiv \Omega_{g,t}(\beta, \delta),$$

because $(i)(j)(h(i) \neq g(j))$ and $\langle \mathrm{t}_0, \mathrm{t}_1, \cdots \rangle$ is free for $\langle \mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots \rangle$ in $\exists \mathfrak{v}_{h(0)} \mathfrak{v}_{h(1)} \cdots \mathfrak{B}(\mathfrak{v}_{h(0)}, \mathfrak{v}_{h(1)}, \cdots; \mathfrak{v}_{g(0)}, \mathfrak{v}_{g(1)}, \cdots)$ in the sense of Kleene [2] p. 79. Hence

$$\mathrm{U}(\delta), \Omega_{g,t}(\alpha, \delta), \mathrm{D}(\Delta_h, \alpha, \beta) \wedge \mathrm{Q}_k(\Delta_{b_2}, \beta) \rightarrow \mathrm{D}(\Delta_h, \alpha, \beta) \wedge \mathrm{Q}_k(\Delta_{c_2}, \beta).$$

Thus we have

$$\mathrm{U}(\delta), \Omega_{g,t}(\alpha, \delta), \mathrm{Q}_{k+1}(\Delta_b, \alpha) \rightarrow \mathrm{Q}_{k+1}(\Delta_c, \alpha).$$

Similarly,

$$\mathrm{U}(\delta), \Omega_{g,t}(\alpha, \delta), \mathrm{Q}_{k+1}(\Delta_c, \alpha) \rightarrow \mathrm{Q}_{k+1}(\Delta_b, \alpha).$$

The other two cases are also similarly treated. This completes the proof of Lemma 2.

5.2. Proof of Lemma 3 by the induction on $\mathrm{n}(\mathfrak{A})$. Basis. $\mathfrak{A}$ is a prime formula: $\mathrm{t}_1 = \mathrm{t}_2$. Then by (8) § 3 we have

$$\mathrm{Q}_0(\Delta_a, \alpha) \equiv (\exists \mathrm{x})(\mathrm{F}(\Delta_{(a)_1}, \mathrm{x}, \alpha) \wedge \mathrm{F}(\Delta_{(a)_2}, \mathrm{x}, \alpha))$$

and

$$\mathrm{Q}_0(\Delta_a, \beta) \equiv (\exists \mathrm{x})(\mathrm{F}(\Delta_{(a)_1}, \mathrm{x}, \beta) \wedge \mathrm{F}(\Delta_{(a)_2}, \mathrm{x}, \beta)).$$

Hence, it is sufficient to prove the following:

$$\vdash_1 \mathrm{D}(\Delta_g, \alpha, \beta) \Rightarrow \mathrm{F}(\Delta_c, \mathrm{x}, \alpha) \equiv \mathrm{F}(\Delta_c, \mathrm{x}, \beta),$$

where $c = \ulcorner \mathrm{t} \urcorner$ and t is a term (in $\mathfrak{Z}$) which contains no $\mathfrak{v}_{g(i)}$'s. This can be easily shown by the induction on the construction of the term t (by using (6) § 3).

Induction step. Case 1. $\mathfrak{A}$ is of the form $\neg\mathfrak{B}$. This case is obvious.

Case 2. $\mathfrak{A}$ is of the form $\bigvee_i \mathfrak{B}_i$. Let $\mathrm{n}'(\mathfrak{A}) = k$, $\mathrm{n}'(\mathfrak{B}_i) = k_i$ and $a = 2^{21} \cdot 3^f$, where $f(i)$ is a Gödel number of $\mathfrak{B}_i$. Then by the hypothesis of induction, we have

$$\mathrm{D}(\Delta_g, \alpha, \beta) \Rightarrow \mathrm{Q}_{k_i}(\Delta_{f(i)}, \alpha) \equiv \mathrm{Q}_{k_i}(\Delta_{f(i)}, \beta).$$

By Lemma 4:

$$\mathrm{D}(\Delta_g, \alpha, \beta) \Rightarrow \mathrm{Q}_k(\Delta_{f(i)}, \alpha) \equiv \mathrm{Q}_k(\Delta_{f(i)}, \beta).$$

Hence for all i.

$$\mathrm{U}(\delta), \mathrm{D}(\Delta_g, \alpha, \beta) \Rightarrow \mathrm{T}(\Delta_f, \Delta_i, \mathrm{y}) \supset \mathrm{Q}_k(\delta(\mathrm{y}), \alpha) \equiv \mathrm{T}(\Delta_f, \Delta_i, \mathrm{y}) \supset \mathrm{Q}_k(\Delta_{f(i)}, \beta),$$

and hence

$$\mathrm{U}(\delta), \mathrm{D}(\Delta_g, \alpha, \beta) \Rightarrow (\exists \mathrm{x})(\forall \mathrm{y})(\mathrm{T}(\Delta_f, \mathrm{x}, \mathrm{y}) \supset \mathrm{Q}_k(\delta(\mathrm{y}), \alpha)$$
$$\equiv (\exists \mathrm{x})(\forall \mathrm{y})(\mathrm{T}(\Delta_f, \mathrm{x}, \mathrm{y}) \supset \mathrm{Q}_k(\delta(\mathrm{y}), \beta)).$$

I. e. $\mathrm{U}(\delta), \mathrm{D}(\Delta_g, \alpha, \beta) \Rightarrow \mathrm{Q}_k(\Delta_a, \alpha) \equiv \mathrm{Q}_k(\Delta_a, \beta)$, and hence we have

$$\mathrm{D}(\Delta_g, \alpha, \beta) \Rightarrow \mathrm{Q}_k(\Delta_a, \alpha) \equiv \mathrm{Q}_k(\Delta_a, \beta).$$

Case 3. $\mathfrak{A}$ has the form $\exists \mathfrak{v}_{h(0)} \mathfrak{v}_{h(1)} \cdots \mathfrak{B}$, where

(3) $$(i)(j)[h(i) \neq g(j)].$$

Let $a = 2^{25} \cdot 3^h \cdot 5^b$, where b is a Gödel number of $\mathfrak{B}$. Then by (8) § 3

$$\mathrm{U}(\delta) \Rightarrow \mathrm{Q}_{k+1}(\Delta_a, \alpha) \equiv (\exists \beta)(\mathrm{D}(\Delta_h, \alpha, \beta) \wedge \mathrm{Q}_k(\Delta_b, \beta)),$$

where $k = \mathrm{n}'(\mathfrak{B})$. By the hypothesis of induction.

(4) $$\mathrm{D}(\Delta_g, \gamma, \theta) \Rightarrow \mathrm{Q}_k(\Delta_b, \gamma) \equiv \mathrm{Q}_k(\Delta_b, \theta).$$

Since by the supposition (3) we can obtain

$$\mathrm{U}(\delta) \Rightarrow (\forall \mathrm{xyuw})[\mathrm{T}(\Delta_h, \mathrm{x}, \mathrm{u}) \wedge \mathrm{T}(\Delta_g, \mathrm{y}, \mathrm{w}) \cdot \supset \cdot \delta(\mathrm{u}) \neq \delta(\mathrm{w})],$$

we can show that the following sequents are provable in $\mathfrak{S}$:

$$\mathrm{U}(\delta) \Rightarrow (\forall \mathrm{x})(\exists \mathrm{z})[\{(\forall \mathrm{uvw})(\mathrm{T}(\Delta_h, \mathrm{u}, \mathrm{v}) \supset \mathrm{x} \neq \delta(\mathrm{v}) \cdot \wedge \cdot \mathrm{T}(\Delta_g, \mathrm{u}, \mathrm{w})$$
$$\supset \mathrm{x} \neq \delta(\mathrm{w})) \wedge \mathrm{z} = \alpha(\mathrm{x})\} \vee \{(\exists \mathrm{uv})(\mathrm{T}(\Delta_h, \mathrm{u}, \mathrm{v}) \wedge \mathrm{x} = \delta(\mathrm{v})) \wedge \mathrm{z} = \gamma(\mathrm{x})\}$$
$$\vee \{(\exists \mathrm{uw})(\mathrm{T}(\Delta_g, \mathrm{u}, \mathrm{w}) \wedge \mathrm{x} = \delta(\mathrm{w})) \wedge \mathrm{z} = \beta(\mathrm{x})\}],$$

$$\mathrm{U}(\delta) \Rightarrow (\forall \mathrm{x})(\exists \mathrm{z})[\{(\forall \mathrm{uvw})(\mathrm{T}(\Delta_h, \mathrm{u}, \mathrm{v}) \supset \mathrm{x} \neq \delta(\mathrm{v}) \cdot \wedge \cdot \mathrm{T}(\Delta_g, \mathrm{u}, \mathrm{w})$$
$$\supset \mathrm{x} \neq \delta(\mathrm{w})) \supset \mathrm{z} = \alpha(\mathrm{x})\} \wedge \{(\exists \mathrm{uv})(\mathrm{T}(\Delta_h, \mathrm{u}, \mathrm{v}) \wedge \mathrm{x} = \delta(\mathrm{v})) \supset \mathrm{z} = \gamma(\mathrm{x})\}$$
$$\wedge \{(\exists \mathrm{uw})(\mathrm{T}(\Delta_g, \mathrm{u}, \mathrm{w}) \wedge \mathrm{x} = \delta(\mathrm{w})) \supset \mathrm{z} = \beta(\mathrm{x})\}],$$

$$\text{(5)}\quad U(\delta)\Rightarrow(\exists\theta)(\forall x)[\{(\forall uvw)(T(\Delta_h, u, v)\supset x\neq\delta(v)\cdot\wedge\cdot T(\Delta_g, u, w)$$
$$\supset x\neq\delta(w))\supset\theta(x)=\alpha(x)\}\wedge\{(\exists uv)(T(\Delta_h, u, v)\wedge x=\delta(v))$$
$$\supset\theta(x)=\gamma(x)\}\wedge\{(\exists uw)(T(\Delta_g, u, w)\wedge x=\delta(w))\supset\theta(x)=\beta(x)\}].$$

We shall abbreviate the formula in succedent of the last sequent by $(\exists\theta)M(\alpha, \beta, \gamma, \theta, \delta)$ or even we write simply $(\exists\theta)M$. Then we can obtain:

$$U(\delta), M(\alpha, \beta, \gamma, \theta, \delta), D(\Delta_g, \alpha, \beta), D(_h, \alpha, \gamma)\Rightarrow D(\Delta_g, \gamma, \theta)\wedge D(\Delta_h, \beta, \theta).$$

Hence by (4)

$$U(\delta), M, D(\Delta_g, \alpha, \beta), D(\Delta_h, \alpha, \gamma)\wedge Q_k(\Delta_b, \gamma)\Rightarrow D(\Delta_h, \beta, \theta)\wedge Q_k(\Delta_b, \theta).$$

By $(\Rightarrow\exists\theta)$, $(\exists\theta\Rightarrow)$, (5), $(\exists\gamma\Rightarrow)$ and (12) § 3 we obtain

$$D(\Delta_g, \alpha, \beta), Q_{k+1}(\Delta_a, \alpha)\Rightarrow Q_{k+1}(\Delta_a, \beta).$$

Similarly.

$$D(\Delta_g, \alpha, \beta), Q_{k+1}(\Delta_a, \beta)\Rightarrow Q_{k+1}(\Delta_a, \alpha).$$

Case 4. $\mathfrak{A}$ has the form $\forall \mathfrak{v}_{h(0)}\mathfrak{v}_{h(1)}\cdots\mathfrak{B}$, where (3) holds. Then

$$U(\delta)\Rightarrow Q_{k+1}(\Delta_a, \alpha)\equiv(\forall\theta)(D(\Delta_h, \alpha, \theta)\supset Q_k(\Delta_b, \theta)),$$

where $a=2^{27}\cdot 3^h\cdot 5^b$, b is a Gödel number of $\mathfrak{B}$ and $n'(\mathfrak{B})=k$. As in Case 3 we can obtain

$$\text{(6)}\quad U(\delta)\Rightarrow(\exists\theta)(\forall x)[\{(\forall uw)(T(\Delta_h, u, w)\supset x\neq\delta(w))\supset\theta(x)=\alpha(x)\}$$
$$\wedge\{(\exists uw)(T(\Delta_h, u, w)\wedge x=\delta(w))\supset\theta(x)=\gamma(x)\}]$$

and

$$U(\delta), D(\Delta_g, \alpha, \beta), D(\Delta_h, \beta, \gamma), N(\alpha, \gamma, \theta, \delta)\Rightarrow D(\Delta_g, \gamma, \theta)\wedge D(\Delta_h, \alpha, \theta),$$

where $(\exists\theta)N(\alpha, \gamma, \theta, \delta)$ denotes the formula occuring in the succedent of the sequent (6). Hence by (4)

$$U(\delta), D(\Delta_g, \alpha, \beta), N(\alpha, \gamma, \theta, \delta), D(\Delta_h, \alpha, \theta)\supset Q_k(\Delta_b, \theta)\Rightarrow D(\Delta_h, \beta, \gamma)\supset Q_k(\Delta_b, \gamma).$$

By $(\forall\theta\Rightarrow)$. $(\exists\theta\Rightarrow)$, (6) and $(\Rightarrow\forall\gamma)$ we obtain

$$U(\delta), D(\Delta_g, \alpha, \beta), Q_{k+1}(\Delta_a, \alpha)\Rightarrow Q_{k+1}(\Delta_a, \beta).$$

Hence $D(\Delta_g, \alpha, \beta), Q_{k+1}(\Delta_a, \alpha)\Rightarrow Q_{k+1}(\Delta_a, \beta)$. This completes the proof of Lemma 3.

5.3. Proof of Lemma 4 by the induction on $k=n'(\mathfrak{A})$ can be readily done. Thus, all the lemmas used in the proof of Theorem 2 have been proved.

§ 6. Statement of Theorem 3.

In this paragraph, we shall deal with a counterpart of Theorem 3 in [7], which asserts that a predicate expressible in n-function-quantifier form is "representable" by a constructive formula $\mathfrak{A}$ such that $n'(\mathfrak{A})=n$. For this purpose, we shall quote several definitions from [7].

Formulas in $\mathfrak{S}$ in which each variable belongs to at most one quantifier and no variable occurs in both free and bound, we shall call *regular*. Let t be a term and A be a regular formula in $\mathfrak{S}$. Let q be a number which defines recursively the function $\lambda ij(2^i(2j+1)\dot{-}1)$. We shall write $q(i,j)$ instead of $\{q\}(i,j)$. Further, let J_m and K_m (or simply, J and K) denote sequence of distinct numbers $\langle j_1, \cdots, j_m\rangle$ and sequence of numbers $\langle k_1, \cdots, k_m\rangle$ $(m \geqq 0)$, respectively. We shall define formulas $[i\,;\mathrm{t}]^J_K$ and $[\mathrm{A}]^J_K$ in $\mathfrak{J}$, where i is a number and no $\mathfrak{x}_{j_1}, \cdots, \mathfrak{x}_{j_m}$ occur as bound variables in A, as follows:

(1) If t is 0, then $[i\,;\mathrm{t}]^J_K$ is $\mathfrak{z}_i=0$.

(2) If t is $\mathfrak{x}_j$, where $j \notin J$, then $[i\,;\mathrm{t}]^J_K$ is $\mathfrak{z}_i=\mathfrak{v}_{q(0,j)}$.

(3) If t is $\mathfrak{x}_{j_l}$, where $1 \leqq l \leqq m$, then $[i\,;\mathrm{t}]^J_K$ is $\mathfrak{z}_i=\mathfrak{z}_{k_l}$.

(4) If t is $\varphi_j(\mathrm{t}_1)$, then $[i\,;\mathrm{t}]^J_K$ is $\bigvee_h(\mathfrak{z}_i=\mathfrak{z}_{q(j+1,h)} \wedge [h\,;\mathrm{t}_1]^J_K)$.

(5) If t has one of the following forms: t_1', $\mathrm{t}_1^{\backprime}$, $\mathrm{t}_1+\mathrm{t}_2$, $\mathrm{t}_1\cdot\mathrm{t}_2$ and $\pi(\mathrm{t}_1,\mathrm{t}_2)$ then $[i\,;\mathrm{t}]^J_K$ is

$$\bigvee_h(\mathfrak{z}_i=\mathfrak{z}_h' \wedge [h\,;\mathrm{t}_1]^J_K)^{8)}, \qquad \bigvee_h(\mathfrak{z}_i=\mathfrak{z}_h^{\backprime} \wedge [h\,;\mathrm{t}_1]^J_K),$$

$$\bigvee_{h_1}\bigvee_{h_2}(\mathfrak{z}_i=\mathfrak{z}_{h_1}+\mathfrak{z}_{h_2} \mathring{\wedge} [h_1\,;\mathrm{t}_1]^J_K \mathring{\wedge} [h_2\,;\mathrm{t}_2]^J_K)^{9)},$$

$$\bigvee_{h_1}\bigvee_{h_2}(\mathfrak{z}_i=\mathfrak{z}_{h_1}\cdot\mathfrak{z}_{h_2} \mathring{\wedge} [h_1\,;\mathrm{t}_1]^J_K \mathring{\wedge} [h_2\,;\mathrm{t}_2]^J_K)$$

or

$$\bigvee_{h_1}\bigvee_{h_2}(\mathfrak{z}_i=\pi(\mathfrak{z}_{h_1},\mathfrak{z}_{h_2}) \mathring{\wedge} [h_1\,;\mathrm{t}_1]^J_K \wedge]h_2\,;\mathrm{t}_2]^J_K),$$

according as t is t_1', $\mathrm{t}_1^{\backprime}$, $\mathrm{t}_1+\mathrm{t}_2$, $\mathrm{t}_1\cdot\mathrm{t}_2$ or $\pi(\mathrm{t}_1,\mathrm{t}_2)$.

(6) If A is of the form $\mathrm{t}_1=\mathrm{t}_2$, then $[\mathrm{A}]^J_K$ is $\bigvee_i([i\,;\mathrm{t}_1]^J_K \wedge [i\,;\mathrm{t}_2]^J_K)$.

(7) If A has one of the forms: $\neg\mathrm{A}_1$, $\mathrm{A}_1\vee\mathrm{A}_2$, $\mathrm{A}_1\wedge\mathrm{A}_2$ and $\mathrm{A}_1\supset\mathrm{A}_2$, then $[\mathrm{A}]^J_K$ is: $\neg[\mathrm{A}_1]^J_K$, $[\mathrm{A}_1]^J_K\vee[\mathrm{A}_2]^J_K$, $[\mathrm{A}_1]^J_K\wedge[\mathrm{A}_2]^J_K$ or $\neg[\mathrm{A}_1]^J_K\vee[\mathrm{A}_2]^J_K$.

(8) If A has one of the forms: $(\exists\mathfrak{x}_j)\mathrm{A}_1$ and $(\forall\mathfrak{x}_j)\mathrm{A}_1$, then $[\mathrm{A}]^J_K$ is $\bigvee_k[\mathrm{A}_1]^{J,j}_{K,k}$ or $\bigwedge_k[\mathrm{A}_1]^{J,j}_{K,k}$, where "$J, j$" denotes the sequence $\langle j_1, \cdots, j_m, j\rangle$.

(9) If A has one of the forms: $(\exists\varphi_j)\mathrm{A}_1$ and $(\forall\varphi_j)\mathrm{A}_1$, then $[\mathrm{A}]^J_K$ is

$$\exists\mathfrak{v}_{q(j+1,0)}\mathfrak{v}_{q(j+1,1)}\cdots[\mathrm{A}_1]^J_K \quad \text{or} \quad \forall\mathfrak{v}_{q(j+1,0)}\mathfrak{v}_{q(j+1,1)}\cdots[\mathrm{A}_1]^J_K.$$

In case J and K are empty, we write $[\mathrm{A}]$ instead of $[\mathrm{A}]^J_K$. The following lemma was proved in [7].

LEMMA 6. *For each regular formula* A *in* $\mathfrak{S}$, $[\mathrm{A}]^J_K$ *is a formula in* $\mathfrak{J}$ *(in which no variable occurs in both free and bound).*

Then we can obtain the following

8) $\mathfrak{z}_0=0$, $\mathfrak{z}_{j+1}=\mathfrak{z}_j'$.

9) We abbreviate $\wedge(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{C}, \mathfrak{C}, \cdots)$ as $\mathfrak{A}\mathring{\wedge}\mathfrak{B}\mathring{\wedge}\mathfrak{C}$.

THEOREM 3. *If* A *is a provable regular formula in* $\mathfrak{S}$, *then* [A] *is provable in* $\mathfrak{J}$.

§ 7. Lemmas used in the proof of Theorem 3.

7.1. LEMMA 7. *If* t *is a term of* $\mathfrak{S}$, *then* $\vdash_2 \bigvee_h [h\,;\mathrm{t}]^J_K$.

Proof by the induction on the construction of t.

Case 1. t is 0. This case is obvious.

Case 2. t is $\mathfrak{x}_j$, where $j \neq j_l$ $(1 \leqq l \leqq m)$. Then $[h\,;\mathrm{t}]^J_K$ is $\mathfrak{z}_h = \mathfrak{v}_{q(0,j)}$. By the axiom (21) of $\mathfrak{J}$ (see § 2.7) we have $\vdash_2 \bigvee_h (\mathfrak{z}_h = \mathfrak{v}_0)$. Hence $\vdash_2 \bigvee_h (\mathfrak{z}_h = \mathfrak{v}_{q(0,j)})$.

Case 3. t is $\mathfrak{x}_{j_l}$, where $1 \leqq l \leqq m$. Then $[h\,;\mathrm{t}]^J_K$ is $\mathfrak{z}_h = \mathfrak{z}_{k_l}$. But it holds $\vdash_2 \bigvee_h (\mathfrak{z}_h = \mathfrak{z}_{k_l})$. Hence $\vdash_2 \bigvee_h [h\,;\mathrm{t}]^J_K$.

Case 4. t is t_1'. Then $[h\,;\mathrm{t}]^J_K$ is $\bigvee_i (\mathfrak{z}_h = \mathfrak{z}_i' \wedge [i\,;\mathrm{t}_1]^J_K)$. By the hypothesis of induction we have $\vdash_2 \bigvee_i [i\,;\mathrm{t}_1]^J_K$. Hence we can obtain $\vdash_2 \bigvee_h \bigvee_i (\mathfrak{z}_h = \mathfrak{z}_i' \wedge [i\,;\mathrm{t}_1]^J_K)$.

For the case that t has one of the forms t_1', $\mathrm{t}_1+\mathrm{t}_2$, $\mathrm{t}_1 \cdot \mathrm{t}_2$ and $\pi(\mathrm{t}_1, \mathrm{t}_2)$: Similarly above.

Case 5. t is $\varphi_j(\mathrm{t}_1)$. Then $[h\,;\mathrm{t}]^J_K$ is $\bigvee_k (\mathfrak{z}_h = \mathfrak{v}_{q(j+1,k)} \wedge [k\,;\mathrm{t}_1]^J_K)$. We shall write $\mathfrak{v}_k^*$ for $\mathfrak{v}_{q(j+1,k)}$. Then we have the following proof of $\bigvee_h [h\,;\mathrm{t}]^J_K$:

$$\dfrac{\begin{matrix}\text{By the hyp. ind.}\\ \Rightarrow \bigvee_k [k\,;\mathrm{t}_1]^J_K\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\text{By using (19) § 2.7}\\ \Rightarrow \bigvee_h (\mathfrak{z}_h = \mathfrak{v}_k^*)\end{matrix} \qquad \dfrac{\dfrac{\mathfrak{z}_k = \mathfrak{v}_k^*, [k\,;\mathrm{t}_1]^J_K \Rightarrow \mathfrak{z}_k = \mathfrak{v}_k^* \wedge [k\,;\mathrm{t}_1]^J_K \text{ for all } h, k}{\mathfrak{z}_h = \mathfrak{v}_k^*, [k\,;\mathrm{t}_1]^J_K \Rightarrow \bigvee_h \bigvee_k (\mathfrak{z}_h = \mathfrak{v}_k^* \wedge [k\,;\mathrm{t}_1]^J_K) \text{ for all } h, k}}{\bigvee_h (\mathfrak{z}_h = \mathfrak{v}_k^*), [k\,;\mathrm{t}_1]^J_K \Rightarrow \bigvee_h [h\,;\mathrm{t}]^J_K \text{ for all } k}}{[k\,;\mathrm{t}_1]^J_K \Rightarrow \bigvee_h [h\,;\mathrm{t}]^J_K \text{ for all } k}}{\bigvee_k [k\,;\mathrm{t}_1]^J_K \Rightarrow \bigvee_h [h\,;\mathrm{t}]^J_K}}{\Rightarrow \bigvee_h [h\,;\mathrm{t}]^J_K}$$

This completes the proof of Lemma 7.

7.2. LEMMA 8. *If* T(t) *is a term of* $\mathfrak{S}$, *then*

$$\vdash_2 [i\,;\mathrm{T}(\mathrm{t})]^J_K \equiv \bigvee_h ([h\,;\mathrm{t}]^J_K \wedge [i\,;\mathrm{T}(\Delta_h)]^J_K)\,.$$

PROOF. First of all, we must show that the expression on the righthand side is a formula in $\mathfrak{J}$. For this, it is sufficient to show that there is a partial recursive function $\psi^m(a, t, i, h, J, K)$ such that if t and a are Gödel numbers of terms t and T(t) of $\mathfrak{S}$, respectively (under a suitable Gödel numbering of terms (and formulas) in $\mathfrak{S}$), then $\psi^m(a, t, i, h, J, K)$ is defined and its value is a Gödel number of formula $[i\,;\mathrm{T}(\Delta_h)]^J_K$ of $\mathfrak{J}$. This fact can be proved by the same method as in [7] which uses the Kleene's recursion theorem. Here, the proof is omitted.

Now we shall show the above equivalence: For simplicity we shall omit J and K, and furthermore, (here and after) we shall often use the notation for "chains of equivalences" in Kleene [2] pp. 117-8.

Case 1. T(t) does not contain t. If we write T for T(t), we can obtain (by Lemma 7):

$$\vdash_2 \bigvee_h ([h; \mathrm{t}] \vee [i; \mathrm{T}(\Delta_h)]) \equiv (\bigvee_h [h; \mathrm{t}]) \wedge [i; \mathrm{T}] \equiv [i; \mathrm{T}].$$

Case 2. T(t) is t. Then we have

$$\vdash_2 \bigvee_h ([h; \mathrm{t}] \wedge [i; \mathrm{T}(\Delta_h)]) \equiv \bigvee_h ([h; \mathrm{t}] \wedge [i; \Delta_h])$$
$$\equiv \bigvee_h ([h; \mathrm{t}] \wedge \mathfrak{z}_i = \mathfrak{z}_h) \equiv [i; \mathrm{t}] \wedge \mathfrak{z}_i = \mathfrak{z}_i \equiv [i; \mathrm{t}]$$
$$\equiv [i; \mathrm{T(t)}],$$

by using the following facts:

1°) If $i \neq h$, then $\vdash_2 \neg(\mathfrak{z}_i = \mathfrak{z}_h)$

2°) $\vdash_2 \vee(\mathfrak{A}, \mathfrak{B}_1 \wedge \neg\mathfrak{B}_1, \mathfrak{B}_2 \wedge \neg\mathfrak{B}_2, \cdots) \equiv \mathfrak{A}$, and

3°) $\vdash_2 [i; \Delta_h] \equiv \mathfrak{z}_i = \mathfrak{z}_h$ (by the mathematical induction on h).

Case 3. T(t) is one of the forms: $\mathrm{T}_1(\mathrm{t})'$, $\mathrm{T}_1(\mathrm{t})^{\backprime}$, $\mathrm{T}_1(\mathrm{t})+\mathrm{T}_2(\mathrm{t})$, $\mathrm{T}_1(\mathrm{t}) \cdot \mathrm{T}_2(\mathrm{t})$ and $\pi(\mathrm{T}_1(\mathrm{t}), \mathrm{T}_2(\mathrm{t}))$. Let T(t) be $\mathrm{T}_1(\mathrm{t})+\mathrm{T}_2(\mathrm{t})$. Then we have:

$$\vdash_2 [i; \mathrm{T(t)}] \equiv \bigvee_{r_1} \bigvee_{r_2} (\mathfrak{z}_i = \mathfrak{z}_{r_1} + \mathfrak{z}_{r_2} \wedge [r_1; \mathrm{T}_1(\mathrm{t})] \wedge [r_2; \mathrm{T}_2(\mathrm{t})])$$
$$\equiv \bigvee_{r_1} \bigvee_{r_2} \{\mathfrak{z}_i = \mathfrak{z}_{r_1} + \mathfrak{z}_{r_2} \wedge \bigvee_{h_1} ([r_1; \mathrm{T}_1(\Delta_{h_1})] \wedge [h_1; \mathrm{t}])$$
$$\wedge \bigvee_{h_2} ([r_2; \mathrm{T}_2(\Delta_{h_2})] \wedge [h_2; \mathrm{t}])\},$$

by the hyp. ind., $\equiv \bigvee_{r_1} \bigvee_{r_2} \bigvee_h \{\mathfrak{z}_i = \mathfrak{z}_{r_1} + \mathfrak{z}_{r_2} \wedge ([r_1; \mathrm{T}_1(\Delta_h)] \mathring{\wedge} [r_2; \mathrm{T}_2(\Delta_h)] \mathring{\wedge} [h; \mathrm{t}])\}$ (since we obtain $\vdash_2 [h_1; \mathrm{t}] \wedge [h_2; \mathrm{t}] \rightarrow \mathfrak{z}_{h_1} = \mathfrak{z}_{h_2}$ by Lemma 10)

$$\equiv \bigvee_h \{[h; \mathrm{t}] \wedge \bigvee_{r_1} \bigvee_{r_2} (\mathfrak{z}_i = \mathfrak{z}_{r_1} + \mathfrak{z}_{r_2} \mathring{\wedge} [r_1; \mathrm{T}_1(\Delta_h)] \mathring{\wedge} [r_2; \mathrm{T}_2(\Delta_h)]\}$$
$$\equiv \bigvee_h \{[h; \mathrm{t}] \wedge [i; \mathrm{T}_1(\Delta_h) + \mathrm{T}_2(\Delta_h)]\} \equiv \bigvee_h ([h; \mathrm{t}] \wedge [i; \mathrm{T}(\Delta_h)]).$$

Case 4. T(t) is $\varphi_j(\mathrm{T}_1(\mathrm{t}))$. Then $\vdash_2 [i; \mathrm{T(t)}] \equiv \bigvee_r (\mathfrak{z}_i = \mathfrak{v}_r \wedge [r; \mathrm{T}_1(\mathrm{t})])$ (where we write simply $\mathfrak{v}_r$ instead of $\mathfrak{v}_{q(j+1,r)}$) $\equiv \bigvee_r \{\mathfrak{z}_i = \mathfrak{v}_r \wedge \bigvee_h ([h; \mathrm{t}] \wedge [r; \mathrm{T}_1(\Delta_h)])\}$ (by the hyp. ind.) $\equiv \bigvee_h \{[h; \mathrm{t}] \wedge [i; \varphi_j(\mathrm{T}_1(\Delta_h))]\} \equiv \bigvee_h \{[h; \mathrm{t}] \wedge [i; \mathrm{T}(\Delta_h)]\}$.

This completes the proof of Lemma 8.

7.3. Let $\mathrm{A}(\mathfrak{x}_{p_1}, \cdots, \mathfrak{x}_{p_n})$ be a formula of $\mathfrak{S}$ and $\mathrm{t}_1, \cdots, \mathrm{t}_n$ terms of $\mathfrak{S}$ such that each t_i either a numeral or a number-variable. For such a term t we define $[\mathrm{t}]_K^J$ as follows:

$$(1)\qquad [\mathrm{t}]_K^J \overset{\mathrm{Def}}{=} \begin{cases} \mathfrak{z}_i & \text{if t is } \Delta_i\,, \\ \mathfrak{z}_{k_l} & \text{if t is } \mathfrak{x}_{j_l},\ \text{where } j_l \in J\,, \\ \mathfrak{v}_{q(0,j)} & \text{if t is } \mathfrak{x}_j,\ \text{where } j \notin J\,. \end{cases}$$

Then we have (2): $\vdash_2 [h\,;\,\mathrm{t}]_K^J \equiv \mathfrak{z}_h = [\mathrm{t}]_K^J$. Further, suppose $\mathrm{t}_1, \cdots, \mathrm{t}_n$ is free in $\mathrm{A}(\mathfrak{x}_{p_1}, \cdots, \mathfrak{x}_{p_n})$.

LEMMA 9. *Under the above provisos, for every J and K (such that $p_1, \cdots, p_n \notin J$) $[\mathrm{A}]_K^J([\mathrm{t}_1]_K^J, \cdots, [\mathrm{t}_n]_K^J)$—that which is obtained from $[\mathrm{A}(\mathfrak{x}_{p_1}, \cdots, \mathfrak{x}_{p_n})]_K^J$ by replacing $\mathfrak{v}_{q(0,p_1)}, \cdots, \mathfrak{v}_{q(0,p_n)}$ by $[\mathrm{t}_1]_K^J, \cdots, [\mathrm{t}_n]_K^J$, respectively—is a formula of $\mathfrak{Z}$, and it holds:*

$$(3)\qquad \vdash_2 [\mathrm{A}(\mathrm{t}_1, \cdots, \mathrm{t}_n)]_K^J \equiv [\mathrm{A}]_K^J([\mathrm{t}_1]_K^J, \cdots, [\mathrm{t}_n]_K^J)\,.$$

PROOF. For the sake of simplicity we shall deal with only the case $n=1$ and write t for t_1, x_p for x_{p_1}. That $[\mathrm{A}]_K^J([\mathrm{t}]_K^J)$ is a formula of $\mathfrak{Z}$ is proved by the same method as in [7] pp. 184-187. Hence, it is sufficient to prove (3) by an induction of form corresponding to the inductive definition of $\mathrm{A}(\mathfrak{x}_p)$. We shall show only few cases.

Case 1. A is a prime formula. Then A(t) is of the form $\mathrm{T}_1(\mathrm{t}) = \mathrm{T}_2(\mathrm{t})$. By using Lemma 8 and (2) we can obtain:

$$\begin{aligned} \vdash_2 [\mathrm{A}(\mathfrak{x}_p)]_K^J &\equiv [\mathrm{T}_1(\mathfrak{x}_p) = \mathrm{T}_2(\mathfrak{x}_p)]_K^J \\ &\equiv \bigvee_i \bigvee_h \{[h\,;\,\mathfrak{x}_p]_K^J \mathbin{\mathring{\wedge}} [i\,;\,\mathrm{T}_1(\Delta_h)]_K^J \mathbin{\mathring{\wedge}} [i\,;\,\mathrm{T}_2(\Delta_h)]_K^J\} \\ &\equiv \bigvee_i \bigvee_h \{\mathfrak{z}_h = \mathfrak{v}_{q(0,p)} \mathbin{\mathring{\wedge}} [i\,;\,\mathrm{T}_1(\Delta_h)]_K^J \mathbin{\mathring{\wedge}} [i\,;\,\mathrm{T}_2(\Delta_h)]_K^J\}\,. \end{aligned}$$

Similarly we have

$$\vdash_2 [\mathrm{A}(\mathrm{t})]_K^J \equiv \bigvee_i \bigvee_h \{\mathfrak{z}_h = [\mathrm{t}]_K^J \mathbin{\mathring{\wedge}} [i\,;\,\mathrm{T}_1(\Delta_h)]_K^J \mathbin{\mathring{\wedge}} [i\,;\,\mathrm{T}_2(\Delta_h)]_K^J\}\,.$$

Hence

$$\vdash_2 [\mathrm{A}(\mathrm{t})]_K^J \equiv [\mathrm{A}]_K^J([\mathrm{t}]_K^J)\,.$$

Case 2. A(t) is of the form $(\exists \mathfrak{x}_j)\mathrm{A}_1(\mathrm{t})$, where t is not $\mathfrak{x}_j$. Then

$$\vdash_2 [\mathrm{A}(\mathrm{t})]_K^J \equiv \bigvee_k [\mathrm{A}_1(\mathrm{t})]_{K,k}^{J,j} \equiv \bigvee_k [\mathrm{A}_1]_{K,k}^{J,j}([\mathrm{t}]_{K,k}^{J,j}) \quad \text{(by the hyp. ind.)}$$

$$\equiv \bigvee_k [\mathrm{A}_1]_{K,k}^{J,j}([\mathrm{t}]_K^J), \text{ because t is not } \mathfrak{x}_j.$$

$$\vdash_2 [\mathrm{A}(\mathfrak{x}_p)]_K^J \equiv \bigvee_k [\mathrm{A}_1]_{K,k}^{J,j}(\mathfrak{v}_{q(0,p)})\,.$$

$$[\mathrm{A}]_K^J([\mathrm{t}]_K^J) \overset{\mathrm{Def}}{=} \mathrm{Sub}_{[\mathrm{t}]_K^J}^{v_{q(0,p)}} [\mathrm{A}(\mathfrak{x}_p)]_K^J\,.$$

Hence we obtain $\vdash_2 [\mathrm{A}(\mathrm{t})]_K^J \equiv [\mathrm{A}]_K^J([\mathrm{t}]_K^J)$.

Case 3. A(t) is of the form $(\exists \varphi_j)\mathrm{A}_1(\mathrm{t})$. Then

$$\vdash_2 [A(t)]^J_K \equiv \exists \mathfrak{v}_{q(j+1,0)}\mathfrak{v}_{q(j+1,1)} \cdots [A_1(t)]^J_K$$

$$\equiv \exists \mathfrak{v}_{q(j+1,0)}\mathfrak{v}_{q(j+1,1)} \cdots [A_1]^J_K([t]^J_K) \text{ by the hyp. ind.}$$

$$\vdash_2 [A(\mathfrak{x}_p)]^J_K \equiv \exists \mathfrak{v}_{q(j+1,0)}\mathfrak{v}_{q(j+1,1)} \cdots [A_1]^J_K(\mathfrak{v}_{q(0,p)}) .$$

Hence $\vdash_2 [A(t)]^J_K \equiv [A]^J_K([t]^J_K)$.

7.4. LEMMA 10. *If* t *is a term of* $\mathfrak{S}$, *then*

$$\vdash_2 [k\,;\,t]^J_K \Rightarrow [h\,;\,t]^J_K \equiv \mathfrak{z}_h = \mathfrak{z}_k . \tag{4}$$

Proof by an induction of form corresponding to the inductive definition of term t.

Case 1. t is 0 or $\mathfrak{x}_j$. Let t be $\mathfrak{x}_j$. Then $[k\,;\,t]^J_K$ is $\mathfrak{z}_k = \mathfrak{v}_{q(0,j)}$ or $\mathfrak{z}_k = \mathfrak{z}_{k_l}$, according as $j \notin J$ or $j = j_l \in J$. Hence. obviously (4) holds.

Case 2. Let t be $t_1 + t_2$. Then $[k\,;\,t]^J_K$ is $\bigvee_{h_1}\bigvee_{h_2}(\mathfrak{z}_k = \mathfrak{z}_{h_1} + \mathfrak{z}_{h_2} \mathring{\wedge} [h_1\,;\,t_1]^J_K \mathring{\wedge} [h_2\,;\,t_2]^J_K)$. Now for all natural numbers h_1, h_2 we have

$$\mathfrak{z}_h = \mathfrak{z}_{h_1} + \mathfrak{z}_{h_2}, [h_1\,;\,t_1]^J_K, [h_2\,;\,t_2]^J_K, \mathfrak{z}_h = \mathfrak{z}_k \Rightarrow \mathfrak{z}_h = \mathfrak{z}_{k_1} + \mathfrak{z}_{k_2} \mathring{\wedge} [h_1 : t_1]^J_K \mathring{\wedge} [h_2\,;\,t_2]^J_K .$$

Hence by rules $(\Rightarrow \vee)$ and $(\vee \Rightarrow)$

$$[k\,;\,t_1 + t_2]^J_K, \mathfrak{z}_h = \mathfrak{z}_k \Rightarrow [h\,;\,t]^J_K .$$

Therefore, $[k\,;\,t]^J_K \Rightarrow \mathfrak{z}_h = \mathfrak{z}_k \supset [h\,;\,t]^J_K$. Conversely, by the hyp. ind., we have

$$\mathfrak{z}_k = \mathfrak{z}_{h_1} + \mathfrak{z}_{h_2}, [h_1\,;\,t]^J_K, [h_2\,;\,t_2]^J_K, \mathfrak{z}_h = \mathfrak{z}_{r_1} + \mathfrak{z}_{r_2}, [r_1\,;\,t_1]^J_K, [r_2\,;\,t_2]^J_K$$

$$\Rightarrow \mathfrak{z}_{r_1} = \mathfrak{z}_{h_1} \wedge \mathfrak{z}_{r_2} = \mathfrak{z}_{h_2} \text{ for all } h_1,\ h_2,\ r_1 \text{ and } r_2 .$$

Hence $\Gamma \Rightarrow \mathfrak{z}_k = \mathfrak{z}_h$, where Γ denotes the antecedent of the above sequent. Therefore

$$[k\,;\,t_1 + t_2]^J_K, [h\,;\,t_1 + t_2]^J_K \Rightarrow \mathfrak{z}_k = \mathfrak{z}_h .$$

Hence $[k\,;\,t]^J_K \Rightarrow [h\,;\,t]^J_K \supset \mathfrak{z}_k = \mathfrak{z}_h$.

Case 3. t is of the form $\varphi_l(t_1)$. What is to be proved is:

$$\bigvee_i (\mathfrak{z}_k = \mathfrak{v}_{q(l+1,i)} \wedge [i\,;\,t_1]^J_K \Rightarrow \bigvee_j (\mathfrak{z}_h = \mathfrak{v}_{q(l+1,j)} \wedge [j\,;\,t_1]^J_K) \equiv \mathfrak{z}_k = \mathfrak{z}_h .$$

This sequent can be easily proved by using following fact:

$$\vdash_2 \mathfrak{z}_i = \mathfrak{z}_j \Rightarrow \mathfrak{v}_{q(l+1,i)} = \mathfrak{v}_{q(l+1,j)} \text{ for all } i,\ j . \tag{5}$$

7.5. LEMMA 11. $\vdash_2 \bigvee_h (\mathfrak{B}(\mathfrak{v}_{q(j,k)}) \wedge \mathfrak{z}_h = \mathfrak{z}_k) \equiv \mathfrak{B}(\mathfrak{v}_{q(j,k)})$.

PROOF. This lemma can be easily proved by using (5) and the equality axiom in $\mathfrak{Z}$. Or, directly:

$$\vdash_2 \mathfrak{B}(\mathfrak{v}_{q(j,k)}) \wedge \mathfrak{z}_k = \mathfrak{z}_k \Rightarrow \mathfrak{B}(\mathfrak{v}_{q(j,k)}) . \tag{6}$$

And, if $h \neq k$, then

$$\frac{\dfrac{\mathfrak{z}_h = \mathfrak{z}_k \rightarrow}{\mathfrak{z}_h = \mathfrak{z}_k \rightarrow \mathfrak{B}(\mathfrak{v}_{q(j,k)})}}{\mathfrak{B}(\mathfrak{v}_{q(j,k)}) \wedge \mathfrak{z}_h = \mathfrak{z}_k \rightarrow \mathfrak{B}(\mathfrak{v}_{q(j,k)})}\,. \tag{7}$$

By (6) and (7) $\vdash_2 \mathfrak{B}(\mathfrak{v}_{q(j,k)}) \wedge \mathfrak{z}_h = \mathfrak{z}_k \rightarrow \mathfrak{B}(\mathfrak{v}_{q(j,k)})$ for all h. Hence $\vdash_2 \bigvee_h (\mathfrak{B}(\mathfrak{v}_{q(j,k)}) \wedge \mathfrak{z}_h = \mathfrak{z}_k) \rightarrow \mathfrak{B}(\mathfrak{v}_{q(j,k)})$. The converse is obvious.

7.6. LEMMA 12. *Let* $\mathrm{T}(\varphi_l(\mathfrak{x}_j))$ *be a term of* $\mathfrak{S}$. *Then in* $\mathfrak{J}$, $[i\,;\,\mathrm{T}(\varphi_l(\mathfrak{x}_j))]^{J,j}_{K,k}$ *is equivalent to the formula obtained from* $[i\,;\,\mathrm{T}(\mathfrak{x}_p)]^{J,j}_{K,k}$ *(where* $\mathfrak{x}_p$ *does not occur in* $\mathrm{T}(\varphi_l(\mathfrak{x}_j))$ *and* $p \notin J$*) by replacing* $\mathfrak{v}_{q(0,p)}$ *by* $\mathfrak{v}_{q(l+1,k)}$.

PROOF. By Lemma 8 we have (omitting J and K)

$$\begin{aligned}\vdash_2 [i\,;\,\mathrm{T}(\varphi_l(\mathfrak{x}_j))]^j_k &\equiv \bigvee_h ([h\,;\,\mathfrak{x}_j]^j_k \wedge [i\,;\,\mathrm{T}(\varDelta_h)]^j_k \\ &\equiv \bigvee_h (\bigvee_r (\mathfrak{z}_h = \mathfrak{v}_{q(l+1,r)} \wedge [r\,;\,\mathfrak{x}_j]^j_k) \wedge [i\,;\,\mathrm{T}(\varDelta_h)]^j_k) \\ &\equiv \bigvee_h (\bigvee_r (\mathfrak{z}_h = \mathfrak{v}_{q(l+1,r)} \wedge \mathfrak{z}_r = \mathfrak{z}_k) \wedge [i\,;\,\mathrm{T}(\varDelta_h)]^j_k) \\ &\equiv \bigvee_h (\mathfrak{z}_h = \mathfrak{v}_{q(l+1,k)} \wedge [i\,;\,\mathrm{T}(\varDelta_h)]^j_k)\,,\end{aligned}$$

by Lemma 11. And

$$\begin{aligned}\vdash_2 [i\,;\,\mathrm{T}(\mathfrak{x}_p)]^j_k &\equiv \bigvee_h ([h\,;\,\mathfrak{x}_p]^j_k \wedge [i\,;\,\mathrm{T}(\varDelta_h)]^j_k) \\ &\equiv \bigvee_h (\mathfrak{z}_h = \mathfrak{v}_{q(0,p)} \wedge [i\,;\,\mathrm{T}(\varDelta_h)]^j_k)\,.\end{aligned}$$

Hence the assertion of Lemma 12 holds.

LEMMA 13. $\varphi_l(\mathfrak{x}_j)$ *is a term of* $\mathfrak{S}$ *free for* $\mathfrak{x}_p$ *in* $\mathrm{A}(\mathfrak{x}_p)$. $[\mathrm{A}]^{J,j}_{K,k}(\mathfrak{v}_{q(l+1,k)})$ *is defined similarly as in Lemma* 9. *Then*

$$\vdash_2 [\mathrm{A}(\varphi_l(\mathfrak{x}_j))]^{J,j}_{K,k} \equiv [\mathrm{A}]^{J,j}_{K,k}(\mathfrak{v}_{q(l+1,k)})\,. \tag{8}$$

Proof by an induction of form corresponding to the inductive definition of the formula $\mathrm{A}(\mathfrak{x}_p)$.

Case 1. $\mathrm{A}(\mathfrak{x}_p)$ is prime, i. e., of the form $\mathrm{T}_1(\mathfrak{x}_p) = \mathrm{T}_2(\mathfrak{x}_p)$. Then

$$[\mathrm{A}(\mathfrak{x}_p)]^{J,j}_{K,k} \equiv \bigvee_i ([i\,;\,\mathrm{T}_1(\mathfrak{x}_p)]^{J,j}_{K,k} \wedge [i\,;\,\mathrm{T}_2(\mathfrak{x}_p)]^{J,j}_{K,k})$$

and

$$[\mathrm{A}(\varphi_l(\mathfrak{x}_j))]^{J,j}_{K,k} \equiv \bigvee_i ([i\,;\,\mathrm{T}_1(\varphi_l(\mathfrak{x}_j))]^{J,j}_{K,k} \wedge [i\,;\,\mathrm{T}_2(\varphi_l(\mathfrak{x}_j))]^{J,j}_{K,k})\,.$$

Hence by Lemma 12 (8) holds.

Case 2. $\mathrm{A}(\mathfrak{x}_p)$ is of the form $(\exists \mathfrak{x}_m)\mathrm{A}_1(\mathfrak{x}_p)$, where $m \neq j$ and $m \neq p$. Then

$$[\mathrm{A}(\mathfrak{x}_p)]^{J,j}_{K,k} \equiv \bigvee_n [\mathrm{A}_1(\mathfrak{x}_p)]^{J,m,j}_{K,n,k} \tag{9}$$

and

$$\mathrm{A}[(\varphi_l(\mathfrak{x}_j))]^{J,j}_{K,k} \equiv \bigvee_n [\mathrm{A}_1(\varphi_l(\mathfrak{x}_j))]^{J,m,j}_{K,n,k}\,. \tag{10}$$

By the hyp. ind. we have

(11) $$[A_1(\varphi_l(\mathfrak{x}_j))]^{J,m,j}_{K,n,k} \equiv [A_1]^{J,m,j}_{K,n,k}(\mathfrak{v}_{q(l+1,k)})\,.$$

Hence by (9)-(11), (8) holds for this case.

7.7. LEMMA 14. *Let* t *be a term of* $\mathfrak{S}$ *free for* $\mathfrak{x}_j$ *in* $A(\mathfrak{x}_j)$. *Then*

(12) $$\vdash_2 [k\,;\,\mathrm{t}]^J_K \Rightarrow [A(\mathrm{t})]^J_K \equiv [A(\mathfrak{x}_j)]^{J,j}_{K,k} \quad \textit{for all } k\,.$$

PROOF. Case 1. $A(\mathfrak{x}_j)$ is prime. Let $A(\mathfrak{x}_j)$ be $T_1(\mathfrak{x}_j) = T_2(\mathfrak{x}_j)$. Then by Lemma 8

$$[T_1(\mathfrak{x}_j) = T_2(\mathfrak{x}_j)]^{J,j}_{K,k} \equiv \bigvee_i \bigvee_h (\mathfrak{z}_h = \mathfrak{z}_k \mathring{\wedge} [i\,;\,T_1(\Delta_h)]^J_K \mathring{\wedge} [i\,;\,T_2(\Delta_h)]^J_K)$$

$$[T_1(\mathrm{t}) = T_2(\mathrm{t})]^J_K \equiv \bigvee_i \bigvee_h ([h\,;\,\mathrm{t}]^J_K \mathring{\wedge} [i\,;\,T_1(\Delta_h)]^J_K \mathring{\wedge} [i\,;\,T_2(\Delta_h)]^J_K)\,.$$

By Lemma 10, $[k\,;\,\mathrm{t}]^J_K \Rightarrow \mathfrak{z}_h = \mathfrak{z}_k \equiv [k\,;\,\mathrm{t}]^J_K$. Hence (1) holds for our case.

Case 2. $A(\mathfrak{x}_j)$ is $\neg A_1(\mathfrak{x}_j)$. By the hyp. ind.

$$[k\,;\,\mathrm{t}]^J_K \Rightarrow [A_1(\mathrm{t})]^J_K \equiv [A_1(\mathfrak{x}_j)]^{J,j}_{K,k}$$

Hence

$$[k\,;\,\mathrm{t}]^J_K \Rightarrow \neg[A_1(\mathrm{t})]^J_K \equiv \neg[A_1(\mathfrak{x}_j)]^{J,j}_{K,k}\,.$$

Thus,

$$[k\,;\,\mathrm{t}]^J_K \Rightarrow [\neg A_1(\mathrm{t})]^J_K \equiv [\neg A_1(\mathfrak{x}_j)]^{J,j}_{K,k}\,.$$

Case 3. $A(\mathfrak{x}_j)$ is $A_1(\mathfrak{x}_j) \chi A_2(\mathfrak{x}_j)$. By the hyp. ind. $[k\,;\,\mathrm{t}]^J_K \Rightarrow [A_n(\mathrm{t})]^J_K \equiv [A(\mathfrak{x}_j)]^{J,j}_{K,k}$ $n = 1, 2$.

$$[k\,;\,\mathrm{t}]^J_K \Rightarrow [A_1(\mathrm{t})]^J_K \chi [A_2(\mathrm{t})]^J_K \equiv [A_1(\mathfrak{x}_j)]^{J,j}_{K,k} \chi [A_2(\mathfrak{x}_j)]^{J,j}_{K,k}\,.$$

Hence

$$[k\,;\,\mathrm{t}]^J_K \Rightarrow [A_1(\mathrm{t}) \chi A_2(\mathrm{t})]^J_K \equiv [A_1(\mathfrak{x}_j) \chi A_2(\mathfrak{x}_j)]^{J,j}_{K,k}\,.$$

Case 4. $A(\mathfrak{x}_j)$ is $(\exists\mathfrak{x}_n)A_1(\mathfrak{x}_j, \mathfrak{x}_n)$ or $(\forall\mathfrak{x}_n)A_1(\mathfrak{x}_j, \mathfrak{x}_n)$. By the hyp. ind.

$$[k\,;\,\mathrm{t}]^{J,n}_{K,m} \Rightarrow [A_1(\mathrm{t}, \mathfrak{x}_n)]^{J,n}_{K,m} \equiv [A_1(\mathfrak{x}_j, \mathfrak{x}_n)]^{J,n,j}_{K,m,k}\,.$$

Since t does not contain $\mathfrak{x}_n$ (by our stipulation),

$$[k\,;\,\mathrm{t}]^{J,n}_{K,m} \equiv [k\,;\,\mathrm{t}]^J_K\,.$$

Hence $[k\,;\,\mathrm{t}]^J_K \Rightarrow \bigvee\limits_m [A_1(\mathrm{t}, \mathfrak{x}_n)]^{J,n}_{K,m} \equiv \bigvee\limits_m [A_1(\mathfrak{x}_j, \mathfrak{x}_n)]^{J,n,j}_{K,m,k}$.

$$[k\,;\,\mathrm{t}]^J_K \Rightarrow [(\exists\mathfrak{x}_n)A_1(\mathrm{t}, \mathfrak{x}_n)]^J_K \equiv [(\exists\mathfrak{x}_n)A_1(\mathfrak{x}_j, \mathfrak{x}_n)]^{J,j}_{K,k}\,.$$

Similarly

$$[k\,;\,\mathrm{t}]^J_K \Rightarrow [(\forall\mathfrak{x}_n)A_1(\mathrm{t}, \mathfrak{x}_n)]^J_K \equiv [(\forall\mathfrak{x}_n)A_1(\mathfrak{x}_j, \mathfrak{x}_n)]^{J,j}_{K,k}\,.$$

Case 5. $A(\mathfrak{x}_j)$ is $(\exists\varphi_l)A_1(\mathfrak{x}_j, \varphi_l)$ or $(\forall\varphi_l)A_1(\mathfrak{x}_j, \varphi_l)$. By the hyp. ind.

$$[k\,;\,\mathrm{t}]^J_K \Rightarrow [A_1(\mathrm{t}, \varphi_l)]^J_K \equiv [A_1(\mathfrak{x}_j, \varphi_l)]^{J,j}_{K,k}\,.$$

Hence

$$[k\,;\,\mathrm{t}]^J_K \Rightarrow \exists\mathfrak{v}_{q(l+1,0)}\mathfrak{v}_{q(l+1,1)} \cdots [A_1(\mathrm{t}, \varphi_l)]^J_K \equiv \exists\mathfrak{v}_{q(l+1,0)}\mathfrak{v}_{q(l+1,1)} \cdots [A_1(\mathfrak{x}_j, \varphi_l)]^{J,j}_{K,k}\,.$$

Thus,

$$[k\,;\,\mathrm{t}]^J_K \Rightarrow [(\exists\varphi_l)A_1(\mathrm{t}, \varphi_l)]^J_K \equiv [(\exists\varphi_l)A_1(\mathfrak{x}_j, \varphi_l)]^{J,j}_{K,k}\,.$$

§ 8. Proof of Theorem 3.

Let Λ be a finite sequence of formulas $\mathrm{A}_1, \cdots, \mathrm{A}_n$ of $\mathfrak{S}$. Then we shall denote the sequence $[\mathrm{A}_1]_K^J, \cdots, [\mathrm{A}_n]_K^J$ of formulas in $\mathfrak{J}$ by $[\Lambda]_K^J$. We shall prove the following theorem from which Theorem 3 can be derived as a corollary:

THEOREM 4. If the sequent $\Lambda \rightarrow \Pi$ is provable in $\mathfrak{S}$ and no variable both occurs free and bound in it, then $[\Lambda]_K^J \rightarrow [\Pi]_K^J$ is provable in $\mathfrak{J}$.

Proof by the transfinite induction on the order of the proof of $\Lambda \rightarrow \Pi$ in $\mathfrak{S}$.

Basis. The order is 0. Case 1. $\Lambda \rightarrow \Pi$ is $\mathrm{A} \rightarrow \mathrm{A}$, where A is a formula of $\mathfrak{S}$. This case is obvious.

Case 2. $\Lambda \rightarrow \Pi$ is one of the axioms for arithmetic, i.e., one of (9)-(19) in § 1.5. We shall show only two cases: (i) $\Lambda \rightarrow \Pi$ is $\mathfrak{x}_0' = \mathfrak{x}_1' \rightarrow \mathfrak{x}_0 = \mathfrak{x}_1$. Then $[\Lambda]_K^J \rightarrow [\Pi]_K^J$ is $\bigvee_i ([i\,;\mathfrak{x}_0']_K^J \wedge [i\,;\mathfrak{x}_1']_K^J) \rightarrow \bigvee_i ([i\,;\mathfrak{x}_0]_K^J \wedge [i\,;\mathfrak{x}_1]_K^J)$. In $\mathfrak{J}$, this sequent is equivalent to one of the following sequents:

$$\bigvee_i (\mathfrak{z}_i = \mathfrak{v}'_{q(0,0)} \wedge \mathfrak{z}_i = \mathfrak{v}'_{q(0,1)}) \rightarrow \bigvee_i (\mathfrak{z}_i = \mathfrak{v}_{q(0,0)} \wedge \mathfrak{z}_i = \mathfrak{v}_{q(0,1)}),$$

hence to

$$\mathfrak{v}'_{q(0,0)} = \mathfrak{v}'_{q(0,1)} \rightarrow \mathfrak{v}_{q(0,0)} = \mathfrak{v}_{q(0,1)},$$

$$\text{or}\quad \mathfrak{v}'_{q(0,0)} = \mathfrak{z}'_{k2} \rightarrow \mathfrak{v}_{q(0,0)} = \mathfrak{z}_{k2},$$

$$\text{or}\quad \mathfrak{z}'_{k1} = \mathfrak{v}'_{q(0,1)} \rightarrow \mathfrak{z}_{k1} = \mathfrak{v}_{q(0,1)},$$

$$\text{or}\quad \mathfrak{z}'_{k1} = \mathfrak{z}'_{k2} \rightarrow \mathfrak{z}_{k1} = \mathfrak{z}_{k2},$$

according as $0, 1 \notin J$, $0 \notin J$ & $1 \in J$, $0 \in J$ & $1 \notin J$ or $0, 1 \in J$. (If $i \in J$ (for $i=0$ or 1), then assume i is j_{i+1}.) It is obvious that these sequents are all provable in $\mathfrak{J}$. (ii) $\Lambda \rightarrow \Pi$ is

$$\varphi_0(0) = 0, (\forall \mathfrak{x}_0)(\varphi_0(\mathfrak{x}_0) = 0 \supset \varphi_0(\mathfrak{x}_0') = 0) \rightarrow (\forall \mathfrak{x}_0)(\varphi(\mathfrak{x}_0) = 0).$$

Since the formulas occuring in the sequent contain no free number-variable, we may assume J and K are empty. It is sufficient to show that

$$(1)\qquad \vdash_2 \bigvee_h (\mathfrak{v}_{q(1,h)} = 0 \wedge \mathfrak{z}_h = 0), \bigwedge_k \{(\mathfrak{v}_{q(1,k)} = 0 \wedge \mathfrak{z}_h = \mathfrak{z}_k) \supset \bigvee_h (\mathfrak{v}_{q(1,h)} = 0 \wedge \mathfrak{z}_h = \mathfrak{z}_k')\}$$
$$\rightarrow \bigwedge_k \bigvee_h (\mathfrak{v}_{q(1,h)} = 0 \wedge \mathfrak{z}_h = \mathfrak{z}_k).$$

Let $\mathfrak{B}(\mathfrak{z}_k)$ be $\bigvee_h (\mathfrak{v}_{q(1,h)} = 0 \wedge \mathfrak{z}_h = \mathfrak{z}_k)$. Then (1) is expressed as follows:

$$(1')\qquad \vdash_2 \mathfrak{B}(\mathfrak{z}_0), \bigwedge_k (\mathfrak{B}(\mathfrak{z}_k) \supset \mathfrak{B}(\mathfrak{z}_k')) \rightarrow \bigwedge_k \mathfrak{B}(\mathfrak{z}_k).$$

By the mathematical induction on n, we shall prove:

$$(2)\qquad \vdash_2 \mathfrak{B}(\mathfrak{z}_0), \bigwedge_k (\mathfrak{B}(\mathfrak{z}_k) \supset \mathfrak{B}(\mathfrak{z}_k')) \rightarrow \mathfrak{B}(\mathfrak{z}_n)$$

for all n.

Basis. $n=0$. Obvious.

Induction Step Obviously:

$$\dfrac{\dfrac{\text{By the hyp. ind.}}{\mathfrak{B}(\mathfrak{z}_0),\ \bigwedge_k(\mathfrak{B}(\mathfrak{z}_k)\supset\mathfrak{B}(\mathfrak{z}_k'))\Rightarrow\mathfrak{B}(\mathfrak{z}_n)} \qquad \dfrac{\mathfrak{B}(\mathfrak{z}_n),\ \mathfrak{B}(\mathfrak{z}_n)\supset\mathfrak{B}(\mathfrak{z}_k')\Rightarrow\mathfrak{B}(\mathfrak{z}_n')}{\mathfrak{B}(\mathfrak{z}_n),\ \bigwedge_k(\mathfrak{B}(\mathfrak{z}_k)\supset\mathfrak{B}(\mathfrak{z}_k'))\Rightarrow\mathfrak{B}(\mathfrak{z}_{n+1})}}{\mathfrak{B}(\mathfrak{z}_0),\ \bigwedge_k(\mathfrak{B}(\mathfrak{z}_k)\supset\mathfrak{B}(\mathfrak{z}_k'))\Rightarrow\mathfrak{B}(\mathfrak{z}_{n+1})\,.}$$

Hence for all n, we have (2). Therefore, (1′) holds good.

Case 3. $\Lambda\Rightarrow\Pi$ is an equality axiom. Let $\Lambda\Rightarrow\Pi$ be $\mathfrak{x}_0=\mathfrak{x}_1$, $\mathrm{A}(\mathfrak{x}_0)\Rightarrow\mathrm{A}(\mathfrak{x}_1)$. For example, suppose $0\in J$, $j_1=0$ and $1\in J$. (The other cases are also similarly done.) Then by Lemma 9, it is sufficient to prove:

$$\vdash_2 \mathfrak{z}_{k_1}=\mathfrak{v}_{q(0,1)},\ [\mathrm{A}]^J_K(\mathfrak{z}_{k_1})\Rightarrow[\mathrm{A}]^J_K(\mathfrak{v}_{q(0,1)})\,.$$

It holds by the equality axioms in $\mathfrak{J}$.

Case 4. $\Lambda\Rightarrow\Pi$ is an instance of the axiom-shceme (22) in § 1.5. Let $\Lambda\Rightarrow\Pi$ be $(\forall\mathfrak{x}_j)(\exists\mathfrak{x}_h)\mathrm{A}(\mathfrak{x}_j,\mathfrak{x}_h)\Rightarrow(\exists\varphi_l)(\forall\mathfrak{x}_j)\mathrm{A}(\mathfrak{x}_j,\varphi_l(\mathfrak{x}_j))$, where φ_l does not occur in $\mathrm{A}(\mathfrak{x}_j,\mathfrak{x}_h)$. If we write $\mathfrak{A}(\mathfrak{z}_k,\mathfrak{z}_r)$ for $[\mathrm{A}(\mathfrak{x}_j,\mathfrak{x}_h)]^{J,j,h}_{K,k,r}$, then $[\mathrm{A}(\mathfrak{x}_j,\varphi_l(\mathfrak{x}_j))]^{J,j}_{K,k}$ is equivalent to the formula $\mathfrak{A}(\mathfrak{z}_k,\mathfrak{v}_{q(l+1,k)})$, by Lemma 13. If we write simply $\mathfrak{A}(\mathfrak{v}_{q(0,j)},\mathfrak{v}_{q(0,h)})$ for $[\mathrm{A}(\mathfrak{x}_j,\mathfrak{x}_h)]^J_K$ (where $j,h\notin J$), then by Lemmas 9 and 13 we obtain

$$\vdash_2[\mathrm{A}(\mathfrak{x}_j,\mathfrak{x}_h)]^{J,j,h}_{K,k,r}\equiv\mathfrak{A}(\mathfrak{z}_k,\mathfrak{z}_r)$$

and $\vdash_2[\mathrm{A}(\mathfrak{x}_j,\varphi_l(\mathfrak{x}_j))]^{J,j}_{K,k}\equiv\mathfrak{A}(\mathfrak{z}_k,\mathfrak{v}_{q(l+1,k)})$. Now by an axiom of $\mathfrak{J}$ (i.e. (22) § 2.7)

$$\vdash_2\bigwedge_k\bigvee_r\mathfrak{A}(\mathfrak{z}_k,\mathfrak{z}_r)\Rightarrow\exists\mathfrak{v}_{q(l+1,0)}\mathfrak{v}_{q(l+1,1)}\cdots\bigwedge_k\mathfrak{A}(\mathfrak{z}_k,\mathfrak{v}_{q(l+1,k)})\,.$$

Hence we have

$$\vdash_2[\Lambda]^J_K\Rightarrow[\Pi]^J_K\,.$$

Induction step.

Case 1. The last-applicated rule of inference in the proof of $\Lambda\Rightarrow\Pi$ is one of structural rules of inference. This case is obvious.

Case 2. That rule of inference is one of rules of inference in the propositional calculus. This case is also obvious.

Case 3. $\Lambda\Rightarrow\Pi$ is $(\exists\mathfrak{x}_j)\mathrm{A}(\mathfrak{x}_j)$, $\Gamma\Rightarrow\Theta$ and $\dfrac{\mathrm{A}(\mathfrak{x}_j),\Gamma\Rightarrow\Theta}{(\exists\mathfrak{x}_j)\mathrm{A}(\mathfrak{x}_j),\Gamma\Rightarrow\Theta}$, where $\mathfrak{x}_j$ does not occur in Γ and Θ.

By the hyp. ind. we have:

$$\vdash_2[\mathrm{A}(\mathfrak{x}_j)]^{J,j}_{K,k},\ [\Gamma]^{J,j}_{K,k}\Rightarrow[\Theta]^{J,j}_{K,k}\quad\text{for all } k\,.$$

By Lemma 9 and by the fact that $\mathfrak{x}_j$ does not occur in Γ and Θ,

$$\vdash_2[\mathrm{A}]^{J,j}_{K,k}(\mathfrak{z}_k),\ [\Gamma]^J_K\Rightarrow[\Theta]^J_K\quad\text{for all } k\,.$$

Hence

$$\vdash_2\bigvee_k[\mathrm{A}]^{J,j}_{K,k}(\mathfrak{z}_k),\ [\Gamma]^J_K\Rightarrow[\Theta]^J_K\,.$$

That is,

$$\vdash_2[(\exists\mathfrak{x}_j)\mathfrak{A}(\mathfrak{x}_j)]^J_K,\ [\Gamma]^J_K\Rightarrow[\Theta]^J_K\,.$$

Case 4. $\Lambda \to \Pi$ is $\Gamma \to \Theta, (\exists \mathfrak{x}_j)\mathrm{A}(\mathfrak{x}_j)$ and $\dfrac{\Gamma \to \Theta, \mathrm{A(t)}}{\Gamma \to \Theta, (\exists \mathfrak{x}_j)\mathrm{A}(\mathfrak{x}_j)}$, where t is an arbitrary term free for $\mathfrak{x}_j$ in $\mathrm{A}(\mathfrak{x}_j)$. By Lemma 14

$$[k\,;\mathrm{t}]^J_K, [\mathrm{A(t)}]^J_K \to [\mathrm{A}(\mathfrak{x}_k)]^{J,j}_{K,k} \quad \text{for all } k\,.$$

Hence

$$\bigvee_k [k\,;\mathrm{t}]^J_K, [\mathrm{A(t)}]^J_K \to \bigvee_k [\mathrm{A}(\mathfrak{x}_j)]^{J,j}_{K,k}\,.$$

By Lemma 7

$$[\mathrm{A(t)}]^J_K \to \bigvee_k [\mathrm{A}(\mathfrak{x}_j)]^{J,j}_{K,k}\,.$$

On the other hand, by the hyp. ind.

$$[\Gamma]^J_K \to [\Theta]^J_K, [\mathrm{A(t)}]^J_K\,.$$

Hence

$$[\Gamma]^J_K \to [\Theta]^J_K, \bigvee_k [\mathrm{A}(\mathfrak{x}_j)]^{J,j}_{K,k}\,.$$

That is,

$$[\Gamma]^J_K \to [\Theta]^J_K, [(\exists \mathfrak{x}_j)\mathrm{A}(\mathfrak{x}_j)]^J_K\,.$$

Case 5. $\Lambda \to \Pi$ is $(\exists \varphi_j)\mathrm{A}(\varphi_j), \Gamma \to \Theta$ and $\dfrac{\mathrm{A}(\varphi_j), \Gamma \to \Theta}{(\exists \varphi_j)\mathrm{A}(\varphi_j), \Gamma \to \Theta}$, where φ_j does not occur in Γ and Θ. Then by the hyp. ind.

$$[\mathrm{A}(\varphi_j)]^J_K, [\Gamma]^J_K \to [\Theta]^J_K\,.$$

Hence

$$\exists \mathfrak{v}_{q(j+1,0)} \mathfrak{v}_{q(j+1,1)} \cdots [\mathrm{A}(\varphi_j)]^J_K, [\Gamma]^J_K \to [\Theta]^J_K\,.$$

(Notice that: Since φ_j does not occur in Γ and Θ, each of $\mathfrak{v}_{q(j+1,0)}, \mathfrak{v}_{q(j+1,1)}, \cdots$ does not occur in $[\Gamma]^J_K$ and $[\Theta]^J_K$. Hence the $(\exists \to)$ rule is applicable to the first sequent.)

Case 6. $\Lambda \to \Pi$ is $\Gamma \to \Theta, (\exists \varphi_j)\mathrm{A}(\varphi_j)$ and $\dfrac{\Gamma \to \Theta, \mathrm{A}(\varphi_j)}{\Gamma \to \Theta, (\exists \varphi_j)\mathrm{A}(\varphi_j)}$. By the hyp. ind.

$$[\Gamma]^J_K \to [\Theta]^J_K, [\mathrm{A}(\varphi_j)]^J_K\,.$$

Hence

$$[\Gamma]^J_K \to [\Theta]^J_K, \exists \mathfrak{v}_{q(j+1,0)} \mathfrak{v}_{q(j+1,1)} \cdots [\mathrm{A}(\varphi_j)]^J_K\,.$$

That is,

$$[\Gamma]^J_K \to [\Theta]^J_K, [(\exists \varphi_j)\mathrm{A}(\varphi_j)]^J_K\,.$$

(Notice that: $[\mathrm{A}(\varphi_j)]^J_K$ can be expressed in the form $\mathfrak{A}(\mathfrak{v}_{q(j+1,0)}, \mathfrak{v}_{q(j+1,1)}, \cdots)$.) Obviously, the sequence of natural numbers $\langle \mathfrak{v}_{q(j+1,0)}, \mathfrak{v}_{q(j+1,1)}, \cdots \rangle$ is recursive. Hence our $(\to \exists)$ rule is applicable to the first sequent.

Case 7. $\Lambda \to \Pi$ is $(\exists \mathfrak{x}_j)\mathrm{A}(\mathfrak{x}_j), \Gamma \to \Theta$ and $\dfrac{\mathrm{A}(\Delta_n), \Gamma \to \Theta \text{ for all } n}{(\exists \mathfrak{x}_j)\mathrm{A}(\mathfrak{x}_j), \Gamma \to \Theta}$, where $\mathrm{A}(\Delta_n)$ is the formula of $\mathfrak{S}$ obtained from $\mathrm{A}(\mathfrak{x}_j)$ by replacing $\mathfrak{x}_j$ by Δ_n. Then by the hyp. ind.:

$$[\mathrm{A}(\Delta_k)]^J_K,\ [\Gamma]^J_K \Rightarrow [\Theta]^J_K \quad \text{for all } k.$$

By Lemma 9

$$[\mathrm{A}]^J_K(\delta_k),\ [\Gamma]^J_K \Rightarrow [\Theta]^J_K \quad \text{for all } k.$$

Hence

$$[\mathrm{A}(\mathfrak{x}_j)]^{J,j}_{K,k},\ [\Gamma]^J_K \Rightarrow [\Theta]^J_K \quad \text{for all } k.$$

and

$$\bigvee_k [\mathrm{A}(\mathfrak{x}_j)]^{J,j}_{K,k},\ [\Gamma]^J_K \Rightarrow [\Theta]^J_K .$$

That is,

$$[(\exists \mathfrak{x}_j)\mathrm{A}(\mathfrak{x}_j)]^J_K,\ [\Gamma]^J_K \Rightarrow [\Theta]^J_K .$$

This completes the proof of Theorem 4.

§ 9. An application of Theorem 1.

In this section, by Theorem 1 above and Theorem 5 below we shall show that there exists a (constructive) formula such that neither it nor its negation is provable in $\mathfrak{J}$, using a result in Mostowski [**6**].

THEOREM 5. *Let* A *be an arbitrary regular formula of* $\mathfrak{S}^*$, *a be a Gödel number of the formula* [A] *in* $\mathfrak{J}$ *and* $n = \mathrm{n}([\mathrm{A}])$. *Put*

$$\Phi_i(\alpha, \delta, \mathrm{x}) = (\forall \mathrm{y})\{\mathrm{T}(\Delta_{S^1_1(q,0)}, \Delta_i, \mathrm{y}) \supset \alpha(\delta(\mathrm{y})) = \mathrm{x}\}$$

$$\Psi_i(\alpha, \delta, \beta) = (\forall \mathrm{yz})\{(\Delta_q, \Delta_{i+1}, \mathrm{y}, \mathrm{z}) \supset \alpha(\delta(\mathrm{z})) = \beta(\mathrm{y})\},$$

where q is the number explained in § 6, *and* T(a, x, y, z) *is a formula which strongly represents the predicate* $T_2(a, x, y, z)$ (*cf. Kleene* [**2**]) *in the system* $\mathfrak{S}$. *Further, let* $\mathfrak{x}_{i_1}, \cdots, \mathfrak{x}_{i_r}$ *and* $\varphi_{l_1}, \cdots, \varphi_{l_s}$ *be the number- and function-variables which occur free in* A, *respectively. Then the sequent*

$$\mathrm{U}(\delta),\ \{\Phi_i(\alpha, \delta, \mathfrak{x}_i)\}_{i=i_1,\cdots,i_r},\ \{\Psi_i(\alpha, \delta, \varphi_i)\}_{i=l_1,\cdots,l_s} \Rightarrow \mathrm{A} \equiv \mathrm{Q}_n(\Delta_a, \alpha)$$

is provable in $\mathfrak{S}^*$.

First, we shall prove the following lemmas:

LEMMA 15. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be formula in* $\mathfrak{J}$, *a and b Gödel numbers of* $\mathfrak{A}$ *and* $\mathfrak{B}$, *respectively. Suppose* $\vdash_2 \mathfrak{A} \equiv \mathfrak{B}$. *Then*

$$\vdash_* \mathrm{Q}_{n_1}(\Delta_a, \alpha) \equiv \mathrm{Q}_{n_2}(\Delta_b, \beta),$$

where $n_1 = \mathrm{n}(\mathfrak{A})$ *and* $n_2 = \mathrm{n}(\mathfrak{B})$.

LEMMA 16. *Let* $\mathfrak{x}_{i_1}, \cdots, \mathfrak{x}_{i_r}$ *and* $\varphi_{l_1}, \cdots, \varphi_{l_i}$ *be the number- and function-variables occuring in a term* t *of* $\mathfrak{S}$, *respectively. Then*

$$\mathrm{U}(\delta),\ \{\Phi_i(\alpha, \delta, \mathfrak{x}_i)\}_{i=i_1,\cdots,i_r},\ \{\Psi_i(\alpha, \delta, \varphi_i)\}_{i=l_1,\cdots,l_s} \Rightarrow \Delta_j = \mathrm{t} \equiv \mathrm{Q}_0(\Delta_a, \alpha)$$

is provable in $\mathfrak{S}^*$, *where a is a Gödel number of the formula* $[j\,;\mathrm{t}]$ *in* $\mathfrak{J}$.

LEMMA 17. *Let f be a Gödel number of a recursive function such that* $(Ei)(f(i) = j) \Rightarrow (E\,!\,i)(f(i) = j)$. *Then*

$$\vdash_1 (\exists\gamma)\{D(\Delta_f, \alpha, \gamma) \wedge (\forall xy)(T(\Delta_f, x, y) \supset \gamma(\delta(y)) = \beta(x))\}.$$

PROOF OF LEMMA 15: By Theorem 1, $\vdash_* Q_n(\Delta_c, \alpha)$, where c is a Gödel number of $\mathfrak{A} \equiv \mathfrak{B}$ in $\mathfrak{J}$ and $n = \mathrm{Max}(n_1, n_2)$. It can be shown that

$$\vdash_* Q_n(\Delta_c, \alpha) \equiv (Q_{n_1}(\Delta_a, \alpha) \equiv Q_{n_2}(\Delta_b, \alpha)).$$

Hence we have

$$\vdash_* Q_n(\Delta_a, \alpha) \equiv Q_n(\Delta_b, \alpha).$$

PROOF OF LEMMA 16: Case 1. t contains no function variables. Then we can obtain $\vdash_2 [i; t] \equiv \mathfrak{z}_i = [t]$, where $[t]$ is the term in $\mathfrak{J}$ obtained from t (in $\mathfrak{S}$) by replacing $\mathfrak{x}_{i_k}$ (for $k = 1, 2, \cdots, r$) by $\mathfrak{v}_{q(0, i_k)}$ and the other symbols in t by the corresponding to ones in $\mathfrak{J}$. Hence by (8) § 3 and above Lemma 15 we have

$$U(\delta), \{\Phi_i(\alpha, \delta, \mathfrak{x}_i)\}_{i=i_1,\cdots,i_r} \Rightarrow \Delta_j = t \equiv Q_0(\Delta_a, \alpha).$$

Case 2. t is $\varphi_l(t_1)$. We must show that

$$U(\delta), \{\Phi_i(\alpha, \delta, \mathfrak{x}_i)\}_{i=i_1,\cdots,i_r}, \{\Psi_i(\alpha, \delta, \varphi_i)\}_{i=l_1,\cdots,l_s,l} \Rightarrow \Delta_j = t \equiv Q_0(\Delta_a, \alpha),$$

where $\mathfrak{x}_{i_1}, \cdots, \mathfrak{x}_{i_r}$ and $\varphi_{l_1}, \cdots, \varphi_{l_s}$ are the variables occuring in t_1. By the def. of $[j; t]$, it is $\bigvee_k (\mathfrak{z}_j = \mathfrak{v}_{q(l+1,k)} \wedge [k; t_1])$. Now let $a = 2^{21} \cdot 3^f$, where $f(k)$ is a Gödel number of $\mathfrak{z}_i = \mathfrak{v}_{q(l+1,k)} \wedge [k, t]$. By the hyp. ind. we have

(1) $\quad U(\delta), \{\Phi_i(\alpha, \delta, \mathfrak{x}_i)\}_{i=i_1,\cdots,i_r}, \{\Psi_i(\alpha, \delta, \varphi_i)\}_{i=l_1,\cdots,l_s} \Rightarrow \Delta_k = t \equiv Q_0(\Delta_{g(k)}, \alpha),$

where $g(k)$ is a Gödel number of $[k; t_1]$. (We may assume g is recursive.) The antecedent of (1) we shall abbreviate as $\Xi(\alpha, \delta)$ or simply Ξ. Further, let $f(k) = 2^{23} \cdot 3^{(f(k))_1}$, where $\{(f(k))_1\}(0)$ is the Gödel number of $\mathfrak{z}_j = \mathfrak{v}_{q(l+1,k)}$, and for $m = 0, 1, 2, \cdots$ $\{(f(k))_1\}(m+1)$ is a Gödel number of $[k; t_1]$. By (8) and (10) § 3 we have:

(2) $\quad U(\delta) \Rightarrow Q_0(\Delta_a, \alpha) \equiv (\exists x)(\forall y)\{T(\Delta_f, x, y) \supset Q_0(\delta(y), \alpha)\},$

(3) $\quad U(\delta), T(\Delta_f, \Delta_k, y), T(\Delta_f, \Delta_k, y) \supset Q_0(\delta(y), \alpha) \Rightarrow Q_0(\Delta_{f(k)}, \alpha)$ for all k,

(4) $\quad U(\delta) \Rightarrow Q_0(\Delta_{f(k)}, \alpha) \equiv (\forall uz)\{T(\Delta_{(f(k))_1}, u, z) \supset Q_0(\delta(z), \alpha)\},$

$$U(\delta), T(\Delta_{(f(k))_1}, 0, z), T(\Delta_{(f(k))_1}, 0, z) \supset Q_0(\delta(z), \alpha) \Rightarrow Q_0(\Delta_{\{(f(k))_1\}(0)}, \alpha),$$

and for all m

(5) $\quad U(\delta), T(\Delta_{(f(k))_1}, \Delta_{m+1}, z), T(\Delta_{(f(k))_1}, \Delta_{m+1}, z) \supset Q_0(\delta(z), \alpha)$

$$\Rightarrow Q_0(\Delta_{\{(f(k))_1\}(m+1)}, \alpha).$$

Hence by above (1) and (6), (9) § 3, for all k

$$\Xi, (\forall uz)\{T(\Delta_{(f(k))_1}, u, z) \supset Q_0(\delta(z), \alpha)\} \Rightarrow \Delta_k = t_1 \wedge \Delta_j = \alpha(\Delta_{q(l+1,k)}),$$

i.e.,

$$\Xi, Q_0(\Delta_{f(k)}, \alpha) \Rightarrow \Delta_k = t_1 \wedge \Delta_j = \alpha(\Delta_{q(l+1,k)}) \quad \text{for all } k.$$

By this

$$\Xi,\ \mathrm{T}(\Delta_q, \Delta_l, \Delta_k, \mathrm{z}),\ \mathrm{T}(\Delta_q, \Delta_l, \Delta_k, \mathrm{z})$$
$$\supset \alpha(\delta(\mathrm{z})) = \varphi_l(\Delta_k),\ \mathrm{Q}_0(\Delta_{f(k)}, \alpha) \Rightarrow \Delta_j = \varphi_l(\mathrm{t}_1)\,.$$

Hence for all k,

$$\Xi,\ \Psi_l(\alpha, \delta, \varphi_l),\ \mathrm{Q}_0(\Delta_{f(k)}, \alpha) \Rightarrow \Delta_j = \mathrm{t}\,.$$

From this together with (2) and (3), we can easily obtain:

$$\Xi,\ \Psi_l(\alpha, \delta, \varphi_l),\ \mathrm{Q}_0(\Delta_a, \alpha) \Rightarrow \Delta_j = \mathrm{t}\,,$$

and hence

$$\Xi,\ \Psi_l(\alpha, \delta, \varphi_l) \Rightarrow \mathrm{Q}_0(\Delta_a, \alpha) \supset \Delta_j = \mathrm{t}\,.$$

Conversely, from (1) and (5)

(6) $$\Xi,\ \Delta_k = \mathrm{t}_1 \Rightarrow (\forall \mathrm{z})\{\mathrm{T}(\Delta_{(f(k))_1}, \Delta_{m+1}, \mathrm{z}) \supset \mathrm{Q}_0(\delta(\mathrm{z}), \alpha)\} \quad \text{for all } m\,.$$

Since

$$\mathrm{T}(\Delta_q, \Delta_l, \Delta_k, \mathrm{z}),\ \mathrm{T}(\Delta_q, \Delta_l, \Delta_k, \mathrm{z}) \supset \alpha(\delta(\mathrm{z})) = \varphi_l(\Delta_k) \Rightarrow \alpha(\Delta_{q(l+1,k)}) = \varphi_l(\Delta_k)\,,$$

we have

$$\Psi_l(\alpha, \delta, \varphi_l),\ \Delta_j = \varphi_l(\mathrm{t}_1),\ \Delta_k = \mathrm{t}_1 \Rightarrow \Delta_j = \alpha(\Delta_{q(l+1,k)})\,,$$

and hence

$$\mathrm{U}(\delta),\ \Psi_l(\alpha, \delta, \varphi_l),\ \Delta_j = \varphi_l(\mathrm{t}_1),\ \Delta_k = \mathrm{t}_1 \Rightarrow (\forall \mathrm{z})\{\mathrm{T}(\Delta_{(f(k))_1}, \Delta_0, \mathrm{z}) \supset \mathrm{Q}_0(\delta(\mathrm{z}), \alpha)\}\,.$$

Together with (6), by ω-rule and (4),

$$\Xi,\ \Psi_l(\alpha, \delta, \varphi_l),\ \Delta_j = \varphi_l(\mathrm{t}_1),\ \Delta_k = \mathrm{t}_1 \Rightarrow \mathrm{Q}_0(\Delta_{f(k)}, \alpha)\,, \quad \text{for all } k\,.$$

By (3), for all k

$$\Xi,\ \Psi_l(\alpha, \delta, \varphi_l),\ \Delta_j = \varphi_l(\mathrm{t}_1),\ \Delta_k = \mathrm{t}_1 \Rightarrow (\exists \mathrm{x})(\forall \mathrm{y})\{\mathrm{T}(\Delta_f, \mathrm{x}, \mathrm{y}) \supset \mathrm{Q}_0(\delta(\mathrm{y}), \alpha)\}\,.$$

Using ($\exists \Rightarrow$), $\vdash_1 (\exists \mathrm{x})(\mathrm{x} = \mathrm{t}_1)$ and (2), we obtain

$$\Xi,\ \Psi_l(\alpha, \delta, \varphi_l),\ \Delta_j = \mathrm{t} \Rightarrow \mathrm{Q}_0(\Delta_a, \alpha)\,,$$
$$\Xi,\ \Psi_l(\alpha, \delta, \varphi_l) \Rightarrow \Delta_j = \mathrm{t} \supset \mathrm{Q}_0(\Delta_a, \alpha)\,.$$

Case 3. t is $\mathrm{t}_1 + \mathrm{t}_2$. We must prove: $\Xi \Rightarrow \Delta_j = \mathrm{t}_1 + \mathrm{t}_2 \equiv \mathrm{Q}_0(\Delta_a, \alpha)$, where $\mathfrak{x}_{i_1}, \cdots, \mathfrak{x}_{i_r}$ and $\varphi_{l_1}, \cdots, \varphi_{l_s}$ are the variables occuring in t_1 or t_2 (and a is a Gödel number of $[j\,;\mathrm{t}_1 + \mathrm{t}_2]$). By the hyp. ind.

(7) $$\Xi \Rightarrow \Delta_{j_1} = \mathrm{t}_1 \equiv \mathrm{Q}_0(\Delta_b, \alpha) \cdot \wedge \cdot \Delta_{j_2} = \mathrm{t}_2 \equiv \mathrm{Q}_0(\Delta_c, \alpha)\,,$$

where b and c are Gödel numbers of $[j_1, \mathrm{t}_1]$ and $[j_2, \mathrm{t}_2]$, respectively. Now $[j\,;\mathrm{t}_1 + \mathrm{t}_2]$ is $\bigvee_{j_1} \bigvee_{j_2} (\mathfrak{z}_j = \mathfrak{z}_{j_1} + \mathfrak{z}_{j_2} \mathring{\wedge} [j_1\,;\mathrm{t}_1] \mathring{\wedge} [j_2\,;\mathrm{t}_2])$. Let a be its Gödel number. Then $a = 2^{21} \cdot 3^{f_1}$, $f_1(j_1)$ is a Gödel number of $\bigvee_{j_2} (\mathfrak{z}_j = \mathfrak{z}_{j_1} + \mathfrak{z}_{j_2} \mathring{\wedge} [j_1\,;\mathrm{t}_1] \mathring{\wedge} [j_2\,;\mathrm{t}_2])$ and hence is of the form $2^{21} \cdot 3^{f_2(j_1)}$, where $f_2(j_1) = (f_1(j_1))_1$. Further, $\{f_2(j_1)\}(j_2)$ is a Gödel number of $(\mathfrak{z}_j = \mathfrak{z}_{j_1} + \mathfrak{z}_{j_2} \mathring{\wedge} [j_1\,;\mathrm{t}_1] \mathring{\wedge} [j_2\,;\mathrm{t}_2])$ and hence is of the form $2^{23} \cdot 3^{f_3(j_1, j_2)}$, where $f_3(j_1, j_2) = (\{f_2(j_1)\}(j_2))_1$. And $\{f_3(j_1, j_2)\}(0)$ is a Gödel number of $\mathfrak{z}_j = \mathfrak{z}_{j_1} + \mathfrak{z}_{j_2}$, $\{f_3(j_1, j_2)\}(1)$ a Gödel number of $[j_1; t_1]$ and $\{f_3(j_1, j_2)\}(m+2)$ a Gödel number of $[j_2; t_2]$ for all m. By (7) we obtain

$$\Xi \Rightarrow Q_0(\Delta_{\{f_3(j_1,j_2)\}(1)}, \alpha) \equiv \Delta_{j_1} = t_1 ,$$

and for all m

$$\Xi \Rightarrow Q_0(\Delta_{\{f_3(j_1,j_2)\}(m+2)}, \alpha) \equiv \Delta_{j_2} = t_2 .$$

Further, we have $Q_0(\Delta_{\{f_3(j_1,j_2)\}(0)}, \alpha) \equiv \Delta_{j_1} = \Delta_1 + \Delta_{j_2}$. From these, by using ω-rule and by (8), (9), (10) § 3 we can obtain:

$$\Xi \Rightarrow Q_0(\Delta_a, \alpha) \equiv \Delta_j = t_1 + t_2 .$$

The other cases are also similarly treated. Hence Lemma 16 has been proved.

PROOF OF LEMMA 17. This is done in the same way as in the proof of (10) § 4.

PROOF OF THEOREM 5. Case 1. A is prime. Then A has the form $t_1 = t_2$, and hence [A] is $\bigvee_j ([j; t_1] \wedge [j; t_2])$. Let a be a Gödel number of [A], and hence $a = 2^{21} \cdot 3^f$, where $f(j)$ is a Gödel number of $[j; t_1] \wedge [j; t_2]$ and hence $f(j) = 2^{23} \cdot 3^{(f(j))_1}$; $\{(f(j))_1\}(0)$ is a Gödel number of $[j; t_1]$ and $\{(f(j))_1\}(m+1)$ a Gödel number of $[j; t_2]$ for all m. Hence by Lemma 16 (we shall write $g(j, m)$ instead of $\{(f(j))_1\}(m)$) $\Xi \rightarrow \Delta_j = t_1 \equiv Q_0(\Delta_{g(j,0)}, \alpha)$; $\Xi \Rightarrow \Delta_j = t_2 \equiv Q_0(\Delta_{g(j,m+1)}, \alpha)$ for all m. From these, in the similar way as in the preceding paragraphs, we can obtain:

$$\Xi \Rightarrow A \equiv Q_0(\Delta_a, \alpha) .$$

Case 2. A has the form $\neg A_1$. This case is obvious, because we have $\vdash_2 [\neg A_1] \equiv \neg [A_1]$ and $\vdash_* U(\delta) \Rightarrow Q_n(\Delta_a, \alpha) \equiv \neg Q_n(\Delta_b, \alpha)$, where b is a Gödel number of $[A_1]$.

Case 3. A has the form $A_1 \vee A_2$. Then [A] is $\bigvee([A_1], [A_2], [A_2], \cdots)$. Let $a = 2^{21} \cdot 3^f$, where $f(0)$ is a Gödel number of $[A_1]$ and $f(i+1)$ a Gödel number of $[A_2]$ for all i. By the hyp. ind. and by Lemma 4, for all i $\Xi \Rightarrow A_1 \equiv Q_n(\Delta_{f(0)}, \alpha) \cdot \wedge \cdot A_2 \equiv Q_n(\Delta_{f(i+1)}, \alpha)$, where $n = n([A]) \geqq \mathrm{Max}(n([A_1]), n([A_2]))$. From this, in the similar way as in the preceding paragraphs, we can obtain:

$$\Xi \Rightarrow A_1 \vee A_2 \equiv Q_n(\Delta_a, \alpha) .$$

Case 4. A is of the form $(\exists \mathfrak{x}_i) A_1(\mathfrak{x}_i)$. Then by Lemma 9, $\vdash_2 [A] \equiv \bigvee_k [A_1](\mathfrak{z}_k)$. Let b be a Gödel number of $\bigvee_k [A_1](\mathfrak{z}_k)$ and hence $b = 2^{21} \cdot 3^f$, where $f(k)$ is a Gödel number of $[A_1](\mathfrak{z}_k)$. By Lemma 15

(8) $\qquad Q_n(\Delta_a, \alpha) \equiv Q_n(\Delta_b, \alpha)$, where $n = n([A_1]) = n([A])$.

By the hyp. ind. and by Lemmas 9 and 15 (since $\vdash_2 [A_1(\Delta_k)] \equiv [A_1](\mathfrak{z}_k)$)

(9) $\qquad \Xi \Rightarrow A_1(\Delta_k) \equiv Q_n(\Delta_{f(k)}, \alpha)$, where $\mathfrak{x}_{i_1}, \cdots, \mathfrak{x}_{i_r}$ and $\varphi_{l_1}, \cdots, \varphi_{l_s}$ are the variables occuring free in $A_1(\Delta_k)$ (and hence also in $(\exists \mathfrak{x}_i) A_1(\mathfrak{x}_i)$). So,

$$\Xi, A_1(\Delta_k) \Rightarrow T(\Delta_f, \Delta_k, y) \supset Q_n(\delta(y), \alpha).$$

Hence for all k

$$\Xi, A_1(\Delta_k) \Rightarrow (\exists x)(\forall y)\{T(\Delta_f, x, y) \supset Q_n(\delta(y), \alpha)\}.$$

By ω-rule

$$\Xi, (\exists \mathfrak{x}_i)A_1(\mathfrak{x}_i) \Rightarrow Q_n(\Delta_a, \alpha).$$

On the other hand, from (9) we can obtain

$$\Xi, Q_n(\Delta_b, \alpha) \Rightarrow (\exists \mathfrak{x}_i)A_1(\mathfrak{x}_i).$$

Hence, by (8) we have $\Xi \Rightarrow (\exists \mathfrak{x}_i)A_1(\mathfrak{x}_i) \equiv Q_n(\Delta_a, \alpha)$.

Case 5. A is of the form $(\exists \varphi_l)A_1(\varphi_l)$. Then by the def. of [A], it is $\exists \mathfrak{v}_{q(l+1,0)}\mathfrak{v}_{q(l+1,1)} \cdots [A_1(\varphi_l)]$. We must show:

$$\Xi(\alpha, \delta) \Rightarrow (\exists \varphi_l)A_1(\varphi_l) \equiv Q_{n+1}(\Delta_a, \alpha),$$

where a is a Gödel number of [A] and $n+1 = \mathrm{n}([A])$.

By the hyp. ind.

(10) $$\Xi(\gamma, \delta), \Psi_l(\gamma, \delta, \varphi_l) \Rightarrow A_1(\varphi_l) \equiv Q_n(\Delta_b, \gamma),$$

where b is a Gödel number of $[A_1(\varphi_l)]$. Hence

(11) $$\Xi(\gamma, \delta), \Psi_l(\gamma, \delta, \varphi_l), D(\Delta_{S_1^1(q,l+1)}, \alpha, \gamma), A(\varphi_l)$$

$$\Rightarrow (\exists \gamma)\{D(\Delta_{S_1^1(q,l+1)}, \alpha, \gamma) \wedge Q_n(\Delta_b, \gamma)\}. \quad (\text{i.e. } Q_{n+1}(\Delta_a, \alpha))$$

On the other hand, since the ranges of the function $\lambda xq(i, x)$ for $i \neq l+1$ are disjointed from the range of $\lambda xq(l+1, x)$, we can prove

$$D(\Delta_{S_1^1(q,l+1)}, \alpha, \gamma) \Rightarrow \Xi(\alpha, \delta) \equiv \Xi(\gamma, \delta),$$

in the similar way as in the proof of Lemma 3. Therefore by (11)

$$\Xi(\alpha, \delta), \Psi_l(\gamma, \delta, \varphi_l), D(\Delta_{S_1^1(q,l+1)}, \alpha, \gamma), A_1(\varphi_l) \Rightarrow Q_{n+1}(\Delta_a, \alpha),$$

and hence

$$\Xi(\alpha, \delta), (\exists \gamma)\{\Psi_l(\gamma, \delta, \varphi_l) \wedge D(\Delta_{S_1^1(q,l+1)}, \alpha, \gamma)\}, A_1(\varphi_l) \Rightarrow Q_{n+1}(\Delta_a, \alpha).$$

By Lemma 17 and $(\exists \varphi_l \Rightarrow)$ we can obtain

$$\Xi(\alpha, \delta), (\exists \varphi_l)A_1(\varphi_l) \Rightarrow Q_{n+1}(\Delta_a, \alpha).$$

Conversely, by using (10) and by Lemma 17, we can show that:

$$\Xi(\alpha, \delta), Q_{n+1}(\Delta_a, \alpha) \Rightarrow (\exists \varphi_l)A_1(\varphi_l).$$

Thus, we have proved Case 5.

The other cases are also similarly treated. Hence Theorem 5 is established.

THEOREM 6. *There is a formula of $\mathfrak{J}$ such that neither it nor its negation is provable in $\mathfrak{J}$.*

PROOF. Let A be a closed formula of $\mathfrak{S}$ which is undecidable in $\mathfrak{S}^*$. (See

A. Mostowski [**6**].) Then [A] is the desired formula of $\mathfrak{J}$. For, if [A] were provable in $\mathfrak{J}$, then by Theorem 1 we have

$$\vdash_* Q_n(\varDelta_a, \alpha),$$

where a is a Gödel number of [A] and $n([A]) = n$. On the other hand, by Theorem 5 (notice: A contains no free variables): $\vdash_* A \equiv Q_n(\varDelta_a, \alpha)$. Hence it would be $\vdash_* A$. This contradicts unprovability of A. Next, suppose $\neg$[A] were provable in $\mathfrak{J}$. Then we had $\vdash_* Q_n(\varDelta_b, \alpha)$, where b is a Gödel number of $\neg$[A]. Since

$$\vdash_* Q_n(\varDelta_b, \alpha) \equiv Q_n(\varDelta_c, \alpha),$$

where c is a Gödel number of [$\neg$A], and by Theorem 5 $\vdash_* \neg A \equiv Q_n(\varDelta_c, \alpha)$, we would have $\vdash_* \neg A$. This contradicts unprovability of $\neg$A. (Q. E. D.)

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