J. Math. Soc. Japan
Vol. 17, No. 3, 1965

A decomposition of Markov processes

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(Received Jan. 7, 1965)

§ 1. Introduction.

Let S be a locally compact metric space and D be an open set having closure S and non-empty compact boundary $\partial D = S - D$. Let $\boldsymbol{B}(S)$ [$\boldsymbol{B}(\partial D)$] be the Borel field generated by all the closed sets in S [∂D], and $B(S)$ [$B(\partial D)$] be the space of real-valued bounded $\boldsymbol{B}(S)$-measurable [$\boldsymbol{B}(\partial D)$-measurable] functions on S [∂D]. Suppose that we are given a Markov process $X = (x_t(w), W, P_x : x \in S)$ taking values in S. Here W is a space of path functions w, and we denote the initial point by the subscript x in P_x. Precise definitions will be given in Section 2. The word 'Markov process' is used for time homogeneous Markov process in this paper. We define operators $G_\alpha : B(S) \to B(S)$ and $H_\alpha : B(\partial D) \to B(S)$ by

$$G_\alpha f(x) = E_x\Big[\int_0^\zeta e^{-\alpha t} f(x_t) dt\Big]$$

and

$$H_\alpha f(x) = E_x[e^{-\alpha\sigma} f(x_\sigma)] ,$$

where $\zeta = \zeta(w)$ is the lifetime, $\sigma = \sigma(w)$ is the first hitting time to ∂D, and E_x is the integration by P_x. We call G_α the α-order Green operator of X, and H_α the α-order hitting operator to ∂D of X. G_α [H_α] is an integral operator by a measure $G_\alpha(x, dy)$ [$H_\alpha(x, dy)$] on S [∂D], called the α-order Green measure [α-order hitting measure to ∂D]. Further, define $G_\alpha^{\min} : B(S) \to B(S)$ by

$$G_\alpha^{\min} f(x) = E_x\Big[\int_0^{\mathrm{Min}(\sigma,\zeta)} e^{-\alpha t} f(x_t) dt\Big] .$$

Then, $G_\alpha^{\min}$ is the α-order Green operator of a Markov process, which we call *the minimal part of X*. To say intuitively, we get the minimal part, killing x_t at the instant x_t reaches ∂D. Roughly speaking, the motion of X is determined by its minimal part and its behavior on the boundary. But, how can we characterize the behavior on the boundary? This is not simple, since the time spent by X on the boundary may have zero Lebesgue measure. We are concerned with this problem under some conditions.

Let m be a measure on S finite for any compact set, and let $m(\partial D) = 0$. m is fixed through this paper except in Section 6. We assume that the Markov process X satisfies Condition (A) stated in Section 2. Condition (A) requires, among others, that $G_\alpha(x, dy)$ is expressed by $g_\alpha(x, y) m(dy)$ for each x

in S and dy in a neighborhood of the boundary, and that the function $g_\alpha(x, y)$ is jointly measurable and uniquely determined for x in S and y in a neighborhood of ∂D by a certain regularity condition. The main purpose of this paper is to prove the following theorems.

THEOREM 1. *For each* $\alpha > 0$, $x \in S$, *and* $f \in B(S)$, *there is one and only one finite signed measure* $\mu_\alpha(dy, f)$ *on* ∂D *such that*

$$G_\alpha f(x) = G_\alpha^{\min} f(x) + \int_{\partial D} g_\alpha(x, y)\mu_\alpha(dy, f) \tag{1.1}$$

holds. $\mu_\alpha(dy, f)$ *depends only on the minimal part of* X. *That is, if two Markov processes satisfying Condition* (*A*) *have the same* $G_\alpha^{\min}$, *then they induce the same* $\mu_\alpha(dy, f)$.

Put $\mu(dy) = \mu_1(dy, 1)$, and let us define an operator $K^\alpha : L_\infty(\partial D, \mu) \to B(\partial D)$ by

$$K^\alpha f(x) = \int_{\partial D} g_\alpha(x, y) f(y)\mu(dy)\,, \qquad x \in \partial D\,. \tag{1.2}$$

THEOREM 2. *The Green operator of* X *is decomposed as follows:*

$$G_\alpha = G_\alpha^{\min} + H_\alpha K^\alpha \hat{H}_\alpha\,, \qquad \alpha > 0\,, \tag{1.3}$$

where $\hat{H}_\alpha$ *is an operator from* $B(S)$ *to* $L_\infty(\partial D, \mu)$ *depending only on the minimal part of* X. H_α *is also determined by the minimal part.*

THEOREM 3. *For each* $\alpha > 0$, K^α *is the* 0-*order Green operator of a Markov process on the boundary, which is obtained from* X *through time change and killing.*

We call the Markov process stated in Theorem 3 *Ueno's process on the boundary of order* α (or, simply, α-*order U-process*) *induced by* X. Let $\varphi(t, w)$ be the nonnegative continuous additive functional of X corresponding to the measure μ, namely, satisfying

$$E_x\Big[\int_0^\infty e^{-\alpha t} d\varphi(t)\Big] = \int_{\partial D} g_\alpha(x, y)\mu(dy)\,. \tag{1.4}$$

We call $\varphi(t, w)$ *the local time on the boundary induced by* X. Write $\tau(t, w)$ for the right continuous inverse of $\varphi(t, w)$. Then, $x_{\tau(t)}$ is a Markov process taking values in ∂D, (that is, the Markov process made from X through time change by $\varphi(t)$). This we call *the* 0-*order U-process induced by* X. This nomenclature will be justified in Section 5.

THEOREM 4. *Any Markov process satisfying Condition* (*A*) *is determined by its minimal part and the* 0-*order U-process induced by it.*

In proving the above theorems we use the process obtained by a killing of X and then by reversion of the direction of time scale from the killing time. Its properties needed are established in Section 3. Section 4 contains the proof of Theorems 1 and 2, while Section 5 proves Theorems 3 and 4.

In Section 6 we shall explain our motivation to the problem, make a discussion on Condition (A) and show further properties of the time-reversed process and the U-process.

The Markov process on the boundary was first introduced by Ueno [**16**] in the case of diffusion processes on a differentiable manifold. There he derived also the decomposition of type (1.3) (see (6.2)). Suppose that m is an invariant measure for the given process X, and X has the adjoint Markov process $\hat{X}$. Making use of the notion of time reversion for stationary processes, Fukushima and Ikeda (private communication) proved the formula (1.3) in this case, under the assumption that X has continuous paths. However, their definitions of K^{α} and $\hat{H}_{\alpha}$ are different from ours. Their K^{α} is defined by (1.2) with μ replaced by some a priori given measure ν on ∂D, and their $\hat{H}_{\alpha}$ is defined by $\hat{H}_{\alpha}f(y)=\int \hat{h}_{\alpha}(y, x)f(x)m(dx)$, where the α-order hitting measure $\hat{H}_{\alpha}(dy, x)$ to ∂D of $\hat{X}$ is supposed to have density $\hat{h}_{\alpha}(y, x)$ with respect to $\nu(dy)$. In their case, (1.3) is, essentially,

$$\int_{\partial D} H_{\alpha}(x, dy)g_{\alpha}(y, z)=\int_{\partial D} g_{\alpha}(x, y)\hat{H}_{\alpha}(dy, z)$$

proved by Hunt [**3**, III, (18, 3), p. 168]. Fukushima and Ikeda proved also the fact stated in our Theorem 3 [**2**, pp. 91-100]. Invariant measures, however, depend not only on the minimal part but also on the behavior on the boundary. The $\hat{H}_{\alpha}$ defined by them is not determined by the minimal part, and their results do not imply our Theorem 4.

ACKNOWLEDGEMENT. It is my pleasure to express my hearty thanks to Minoru Motoo and Tadashi Ueno. Motoo has been attacking the same problem and gave me important remarks. Ueno read the original draft and gave me valuable advices.

§ 2. Definitions and assumptions.

Let S^* be $S\cup\{\Delta\}$ where Δ is a point adjoined to S as an isolated point. Denote by W_S the set of all paths $w:[0, +\infty]\rightarrow S^*$ which satisfy the following two conditions:

(w_1) $w(t)$ is right continuous and has lefthand limits as a function of t except at $t=+\infty$.

(w_2) There exists $\zeta(w)\in[0, +\infty]$, called the lifetime of w, such that $w(t)\in S$ for $t<\zeta(w)$ and $w(t)=\Delta$ for $t\geqq\zeta(w)$.

In particular, $\lim_{t\uparrow\zeta(w)} w(t)$ exists if $\zeta(w)<+\infty$. Set $x_t=x_t(w)=w(t)$. The shifted path w_s^+ of w is defined by $x_t(w_s^+)=x_{t+s}(w)$. Let W be a subset of W_S, and suppose that W is closed under the transformations $w\rightarrow w_s^+$ for all $s>0$.

Let us denote by $\boldsymbol{B}(S^*)$ the Borel field generated by the closed sets in S^*, and by $\boldsymbol{B}_t$ the Borel field of subsets of W generated by the sets $\{x_s \in A\}$ with $s \leqq t$ and $A \in \boldsymbol{B}(S^*)$. Denote $\boldsymbol{B} = \boldsymbol{B}_{+\infty}$. Given a system of probability measures P_x on $\boldsymbol{B}$ for all $x \in S^*$, we call $X = (x_t(w), W, P_x : x \in S^*)$ a *Markov process on* S if it satisfies:

(M_1) for each fixed B in $\boldsymbol{B}$, $P_x(B)$ is $\boldsymbol{B}(S^*)$-measurable function of x;

(M_2) $P_x(x_0 = x$ or $\Delta) = 1$ for each $x \in S^*$;

(M_3) (Markov property) for every $x \in S$, every $t > 0$ and every bounded $\boldsymbol{B}$-measurable f,

$$E_x(f(w_t^+) \mid \boldsymbol{B}_t) = E_{x_t}(f)\,.$$

Often we write simply $X = (W, P_x : x \in S)$. We use the notation $P_\gamma(B) = \int_S \gamma(dx) P_x(B)$, $E_\gamma(f) = \int_W f(w) P_\gamma(dw)$, and $E_\gamma(f\,; B) = \int_B f(w) P_\gamma(dw)$ for finite signed measure γ on S.

A function $\rho : W \to [0, +\infty]$ is called Markov time, if $\{\rho < t\} \in \boldsymbol{B}_t$ for each t. If ρ is a Markov time, the Borel field consisting of all B in $\boldsymbol{B}$ such that $B \cap \{\rho < t\} \in \boldsymbol{B}_t$ for all t is denoted by $\boldsymbol{B}_\rho^*$. A Markov process X is said to have the strong Markov property if, for each $x \in S$, Markov time ρ, and bounded $\boldsymbol{B}$-measurable f we have

$$E_x(f(w_\rho^+) \mid \boldsymbol{B}_\rho^*) = E_{x_\rho}(f)\,.$$

X is said to be quasi-left continuous if whenever $\{\rho_n\}$ is a sequence of Markov times satisfying $\rho_n \uparrow \rho < \zeta$ on a set B in $\boldsymbol{B}$, then

$$P_x(x_{\rho_n} \to x_\rho,\, B) = P_x(B)\,, \qquad x \in S\,.$$

A nonnegative $\boldsymbol{B}(S)$-measurable function u is called α-excessive (relative to X) if

$$E_x(e^{-\alpha t} u(x_t)\,; t < \zeta) \leqq u(x)$$

for each $t > 0$ and $x \in S$, and if the lefthand member increases to the righthand as $t \downarrow 0$. A $[0, +\infty]$-valued function $\varphi(t, w)$ of $t \geqq 0$ and $w \in W$ is called a nonnegative additive functional of X if, for each t, $\varphi(t)$ is $\boldsymbol{B}_t$-measurable and if

$$P_x(\varphi(t+s, w) = \varphi(t, w) + \varphi(s, w_t^+) \text{ for all } t, s \geqq 0) = 1$$

and

$$P_x(\varphi(t) = \varphi(\zeta - 0) \text{ for all } t \geqq \zeta) = 1$$

for each $x \in S^*$.

Let F be an open or closed subset of S. The first hitting time to F is defined by

$$\sigma_F(w) = \inf\{t : t > 0 \text{ and } x_{t-0}(w) \in F\}\,,$$

with the convention that the infimum of the empty set is $+\infty$. We write σ for $\sigma_{\partial D}$. If F is open, then σ_F is a Markov time, and it coincides with the infimum of $t>0$ such that $x_t \in F$. Let $\boldsymbol{B}(F)=\{B: B\subset F$ and $B\in \boldsymbol{B}(S)\}$. We use the space $C(F)$ of real-valued bounded continuous functions on F, the space $B(F)$ of real-valued bounded $\boldsymbol{B}(F)$-measurable functions on F. The norm is $\|f\|=\sup_x |f(x)|$. We denote $f\in C_0(F)$ $[B_0(F)]$, if f is in $C(F)$ $[B(F)]$ and vanishes outside of a compact subset of F. In case F is open, f in $C_0(F)$ or $B_0(F)$ is, if needed, considered as extended to S to vanish outside of F.

Henceforth the letter W is used for the subset of W_S consisting of all w such that

(w_3) for any $t>0$, $w(t)\in \partial D$ implies $w(t-0)\in \partial D$, and $w(t-0)\in\partial D$ implies $w(t)\in \partial D\cup\{\varDelta\}$.

Then, we have

LEMMA 2.1. *σ is a Markov time.*

PROOF. First, note that if V is an open set and if $\sigma_V(w)<\infty$, then $x_{\sigma_V}(w)$ belongs to $\bar{V}$[1]. Let us denote by V_n the set of points x satisfying $d(x, \partial D) < n^{-1}$[2]. $\sigma_n(w)\equiv\sigma_{V_n}(w)$ being monotone non-decreasing, put $\lim_{n\to\infty}\sigma_n(w)=\rho(w)$. We define σ' by $\sigma'(w)=\sigma(w)$ if $x_0(w)\notin\partial D$, and by $\sigma'(w)=0$ if $x_0(w)\in\partial D$. We prove $\rho=\sigma'$. First, $\rho\leqq\sigma'$ is obvious. Let $\rho(w)<+\infty$. If there is an N such that $\sigma_n(w)=\sigma_N(w)$ for all $n\geqq N$, then we have $x_{\sigma_n}(w)\in\partial D$, and so $\rho(w)=\sigma_n(w)=\sigma'(w)$ by (w_3). If there is no such N, then $x_{\rho-0}(w)\in\partial D$ and hence, $\rho(w)=\sigma'(w)$. Thus, σ' is a Markov time, and so is $t+\sigma'(w_t^+)$. Since $t+\sigma'(w_t^+)\downarrow\sigma(w)$ as $t\downarrow 0$, the proof is complete.

Let us define w_s^- by

$$x_t(w_s^-)=\begin{cases} x_t(w) & t<s\,, \\ \varDelta & t\geqq s\,. \end{cases}$$

Then, $w\in W$ implies $w_s^-\in W$ for each s, and $w\to w_\sigma^-$ is $\boldsymbol{B}$-measurable transformation. Given a Markov process $X=(W, P_x: x\in S)$, define $P_x^{\min}(B)=P_x(w_\sigma^-\in B)$, $B\in\boldsymbol{B}$. Then, $X^{\min}=(W, P_x^{\min}: x\in S)$ is again a Markov process, which we call *the minimal part of X*. The Green operator of $X^{\min}$ is just $G_\alpha^{\min}$ defined in Section 1.

CONDITION (A). We say that a Markov process X satisfies Condition (A), if the following hold:

(A_1) X has the strong Markov property.

(A_2) X is quasi-left continuous.

(A_3) (Conservativity) $P_x(\zeta=\infty)=1$ for each $x\in S$.

(A_4) (Regularity of ∂D relative to $X^{\min}$) $P_x^{\min}(\zeta=0)=1$ at every $x\in\partial D$.

(A_5) For each $x\in S$, $G_\alpha(x, dy)$ is absolutely continuous with respect to m.

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1) $\bar{V}$ is the closure of V.

2) d is the metric of S.

(A_6) There are an open neighborhood V_0 of ∂D and a nonnegative (possibly infinite) $\boldsymbol{B}(S)\times\boldsymbol{B}(V_0)$-measurable function $g_\alpha(x, y)$ for each $\alpha>0$, having four properties below:

$(A_{6,1})$ $g_\alpha(x, y)$ is the density of $G_\alpha(x, dy)$ in V_0 with respect to $m(dy)$.

$(A_{6,2})$ For each $\alpha>0$ and $y\in V_0$, $g_\alpha(x, y)$ is an α-excessive function of x.

$(A_{6,3})$ If $\beta>\alpha>0$, then

$$g_\alpha(x, y)=g_\beta(x, y)+(\beta-\alpha)\int_S G_\alpha(x, dz)g_\beta(z, y)$$

$$=g_\beta(x, y)+(\beta-\alpha)\int_S G_\beta(x, dz)g_\alpha(z, y)$$

holds for each $x\in S$ and $y\in V_0$.

$(A_{6,4})$ Put $\hat{G}_\alpha f(y)=\int_S f(x)m(dx)g_\alpha(x, y)$, if the right side is defined. Then, there is an $\alpha_0\geqq 0$ such that for each $f\in C_0(S)$, $\alpha>\alpha_0$, and $y\in V_0$, $\hat{G}_\alpha f(y)$ is defined and $\hat{G}_\alpha f\in C(V_0)$. For each $f\in C_0(S)$, $\alpha\hat{G}_\alpha f(y)$ tends to $f(y)$ boundedly on ∂D as α tends to infinity. For each $\alpha>\alpha_0$, there is an $f\in C_0(S)$ such that $\hat{G}_\alpha f(y)\geqq 1$ on ∂D.

(A_7) For some $\alpha_1>0$ and some finite signed measure γ_0 on S, define m_0 by

$$m_0(F)=\int_S \gamma_0(dx)G_{\alpha_1}^{\min}(x, F)\,, \qquad F\in\boldsymbol{B}(V_0)\,.$$

Then, m_0 is nonnegative on V_0, and m is absolutely continuous with respect to m_0 on $V_0-\partial D$ with continuous density $k(x)$[3].

We call $g_\alpha(x, y)$ *the α-order Green function of X*. For y in a neighborhood of ∂D, the Green function is uniquely determined by X and m. For, we need only verify it for sufficiently large α by $(A_{6,3})$, and, if $g_\alpha(x, y)$ and $g'_\alpha(x, y)$ are both Green functions of X, then $(A_{6,1})$ and $(A_{6,4})$ imply that

$$\int_S f(x)m(dx)g_\alpha(x, y)=\int_S f(x)m(dx)g'_\alpha(x, y)$$

for each f in $C_0(S)$, large α, and y near ∂D. So we have $g_\alpha(x, y)=g'_\alpha(x, y)$, noting (A_5) and using

$$\beta\int_S G_{\alpha+\beta}(x, dz)g_\alpha(z, y)\uparrow g_\alpha(x, y) \qquad \text{as} \quad \beta\to\infty \tag{2.1}$$

by $(A_{6,2})$.

Condition (A) contains rather strong regularity conditions on $\hat{G}_\alpha$. But, it says nothing about the existence of the so-called adjoint process.

An immediate consequence of the condition (A_5) is that X has no sojourn

3) $k(x)$ is necessarily nonnegative. We assume its finiteness, but do not assume its boundedness.

on the boundary: $G_\alpha(x, \partial D)=0$.

In the following three sections, we assume that $X=(W, P_x: x\in S)$ satisfies Condition (A).

§ 3. Reversion of time.

By the so-called killing procedure we can make from X the Markov process $X'=(W, P'_x: x\in S)$ whose α-order Green operator is $G'_\alpha=G_{\alpha+1}$. The construction is as follows. Let P be a probability measure on $[0, +\infty]: P(dt)=e^{-t}dt$, and put $\Omega=W\times[0, +\infty]$ and $P^{\Omega}_x=P_x\times P$. For $\omega=(w, s)\in\Omega$, define $x'_t(\omega)=x_t(w)$ if $t<s$ and $x'_t(\omega)=\Delta$ if $t\geqq s$. Define $\pi: \Omega\to W$ by $x_t(\pi(\omega))=x'_t(\omega)$, and then put $P'_x(B)=P^{\Omega}_x(\pi^{-1}(B))$ for $B\in\boldsymbol{B}$.

We have $P'_x(W^0)=1$, where $W^0=\{w: \zeta(w)<\infty\}$. For $w\in W^0$ set

$$\hat{x}_t(w)=\begin{cases} x_{\zeta(w)-t-0}(w), & 0\leqq t<\zeta(w), \\ \Delta, & t\geqq\zeta(w), \end{cases}$$

performing reversion of the direction of time. Clearly $\hat{x}_t(w)$ satisfies (w_1)—(w_3) as a function of t. Note that

$$\hat{x}_{t-0}=\lim_{s\uparrow t}x_{\zeta-s-0}=x_{\zeta-t}, \qquad 0<t\leqq\zeta. \tag{3.1}$$

If $P'_x(x_t\in dy)=q(t, x, y)m(dy)$ and $q(t, x, y)$ has some regularities, then it is proved [4, Theorem 3.6] that the process $(\hat{x}_t, P'_\nu)$ has the time homogeneous Markov property for an arbitrary initial measure ν of X' and that

$$P'_\nu(\hat{x}_{s+t}\in dx\,|\,\hat{x}_{s'}, s'\leqq s)=m(dx)\eta(x)q(t, x, \hat{x}_s)\eta(\hat{x}_s)^{-1} \tag{3.2}$$

where $\eta(y)=\int_S\nu(dx)\int_0^\infty q(t, x, y)dt$. Therefore, in order to use the time reversed process in the following sections, it is important to choose ν appropriately. It is inconvenient to confine ourselves to ordinary measures, and we use a *signed* measure in place of ν.

LEMMA 3.1. *There is a finite signed measure γ on S such that*

$$\int_S\gamma(dx)g_1(x, y)k(y)=1 \tag{3.3}$$

holds for m_0-almost every y in $V_0-\partial D$.

PROOF. By the resolvent equation

$$G^{\min}_\alpha-G^{\min}_\beta+(\alpha-\beta)G^{\min}_\alpha G^{\min}_\beta=0, \tag{3.4}$$

we can suppose $\alpha_1=1$ in (A_7). Since the strong Markov property implies

$$G_\alpha=G^{\min}_\alpha+H_\alpha G_\alpha, \tag{3.5}$$

we have

$$\int f(x)m_0(dx)=\int \gamma_0(dx)G_1^{\min}f(y)$$

$$=\int \gamma_0(dx)G_1f(x)-\int \gamma_0(dx)H_1(x, dy)G_1f(y).$$

Hence, we have (3.3) if we set

$$\gamma(F)=\gamma_0(F)-\int \gamma_0(dx)H_1(x, F), \qquad F\in \boldsymbol{B}(S). \tag{3.6}$$

Henceforth, γ is the signed measure in the above lemma. We describe finite dimensional distributions for the process $(\hat{x}_t, P'_\gamma)$ by making use of the quantities of X.

LEMMA 3.2. *For each* $n\geqq 0$, $0=t_0<t_1<\cdots<t_n$, $f_i\in B(S)$ $(i=0, 1, \cdots, n-1)$ *and* $f_n\in B_0(V_0)$, *we have*

$$E'_\gamma\Big[\prod_{i=0}^{n} f_i(\hat{x}_{t_i})\Big]=e^{-t_n}\int_{V_0} m_0(dx)f_n(x)E_x\Big[\prod_{i=0}^{n-1} f_i(x_{t_n-t_i})\Big]. \tag{3.7}$$

In particular,

$$E'_\gamma[f(\hat{x}_t)]=e^{-t}\int_{V_0} m_0(dx)f(x) \tag{3.8}$$

holds for each $t\geqq 0$ *and* $f\in B_0(V_0)$.

PROOF. It is sufficient to prove (3.7) for continuous f_i. We have

$$E'_x\Big[\prod_{i=0}^{n} f_i(\hat{x}_{t_i})\Big]=e^{-t_n}E_x\Big[\int_0^\infty e^{-t}\prod_{i=0}^{n} f_i(x_{t+t_n-t_i})\,dt\Big]^{4)} \tag{3.9}$$

as is seen below:

$$E'_x\Big[\prod_{i=0}^{n} f_i(\hat{x}_{t_i})\Big]=E'_x\Big[\prod_{i=0}^{n} f_i(x_{t-t_i-0});\, t_n<\zeta<\infty\Big]$$

$$=\lim_{h\downarrow 0}\sum_{j=0}^{\infty} E'_x\Big[\prod_{i=0}^{n} f_i(x_{t_n+jh-t_i});\, t_n+jh<\zeta\leqq t_n+(j+1)h\Big]$$

$$=\lim_{h\downarrow 0}\sum_{j=0}^{\infty} E_x\Big[\prod_{i=0}^{n} f_i(x_{t_n+jh-t_i})(e^{-t_n-jh}-e^{-t_n-(j+1)h})\Big]$$

$$=E_x\Big[\int_{t_n}^\infty e^{-t}\prod_{i=0}^{n} f_i(x_{t-t_i-0})\,dt\Big],$$

and we can drop -0 in the last member, since each path has at most an enumerable number of discontinuities. (3.7) follows from (3.9) combined with (3.3):

$$E'_\gamma\Big[\prod_{i=0}^{n} f_i(\hat{x}_{t_i})\Big]=e^{-t_n}E_\gamma\Big[\int_0^\infty e^{-t}dt\, f_n(x_t)E_{x_t}\Big(\prod_{i=0}^{n-1} f_i(x_{t_n-t_i})\Big)\Big]$$

$$=e^{-t_n}\int_{V_0} m_0(dx)f_n(x)\,E_x\Big[\prod_{i=0}^{n-1} f_i(x_{t_n-t_i})\Big].$$

4) This is a special case of [4, Lemma 3.3].

Define

$$\hat{\sigma}_F(w)=\inf\{t: t>0 \text{ and } \hat{x}_{t-0}(w)\in F\}$$

for F open or closed. We have, by (3.1),

(3.10) $$\hat{\sigma}_F=\inf\{\zeta-t: t<\zeta \text{ and } x_t\in F\}=\zeta-\tau_F\,,$$

where τ_F is the last hitting time to F:

$$\tau_F(w)=\sup\{t: t\geqq 0 \text{ and } x_t(w)\in F\}\,,$$

with the convention that the supremum of the empty set is $-\infty$. Clearly, $\hat{\sigma}_F$ and τ_F are $\boldsymbol{B}$-measurable, if F is an open set.

In the sequel, we frequently use the integral

(3.11) $$E'_\gamma[e^{-(\alpha-1)\hat{\sigma}_V}f(\hat{x}_0)h(\hat{x}_{\hat{\sigma}_V})\,;\,\hat{\sigma}_V<\infty]\,,$$

where $f,\ h\in B(S)$, $\alpha>0$, and V is open. (3.11) is well defined and takes a finite value. For, it is obvious for $\alpha\geqq 1$, and, in case $0<\alpha<1$, we need only note

$$E'_x(e^{-(\alpha-1)\zeta})=E_x\Big[\int_0^\infty e^{-t}dt\, e^{-(\alpha-1)t}\Big]=\frac{1}{\alpha}\,.$$

Although γ is a signed measure, the righthand member in (3.7) is nonnegative if f_i's are nonnegative. So we have

COROLLARY TO LEMMA 3.2. (3.11) *is nonnegative if f and h are nonnegative and if V is contained in V_0.*

In fact, we can suppose $h\in C(S)$, and have

$$\begin{aligned}(3.11)&=\lim_{n\to\infty}\sum_{j=1}^{\infty}E'_\gamma[e^{-(\alpha-1)j2^{-n}}f(\hat{x}_0)h(\hat{x}_{j2^{-n}})\,;\,\hat{x}_{i2^{-n}}\notin V(1\leqq i\leqq j-1),\ \hat{x}_{j2^{-n}}\in V]\\&=\lim_{n\to\infty}\sum_{j=1}^{\infty}e^{-\alpha j2^{-n}}\int_V m_0(dx)h(x)E_x[f(x_{j2^{-n}})\,;\,x_{(j-i)2^{-n}}\notin V(1\leqq i\leqq j-1)]\,,\end{aligned}$$

which is nonnegative.

Let $\{V_N: N=1, 2, \cdots\}$ be a sequence of open neighborhoods of ∂D such that the closure of V_N is a compact subset of V_{N+1} and $\{V_N\}$ exhausts V_0. Let U_N be the set of points x in V_N satisfying $d(x,\partial D)>N^{-1}$ and $k(x)<N$. Take continuous functions χ_N^* satisfying $\chi_{U_N}\leqq\chi_N^*\leqq\chi_{U_{N+1}}$[5] and set $k_N(x)=k(x)\chi_N^*(x)$.

LEMMA 3.3. *Let V be an open neighborhood of ∂D with closure contained in V_0. Then we have*

(3.12) $$\begin{aligned}&\lim_{N\to\infty}E'_\gamma[e^{-(\alpha-1)\hat{\sigma}_V}f(\hat{x}_0)(k_N\hat{G}_\beta g)(\hat{x}_{\hat{\sigma}_V})]\\&\quad=\int_S g(x)m(dx)\int_S(\delta(x,dy)+(\alpha-\beta)G_\beta(x,dy))E_y\Big[\int_{\sigma_V}^\infty e^{-\alpha t}f(x_t)dt\Big]^{6)}\end{aligned}$$

5) χ_U is the indicator function of a set U.

6) $\delta(x,dy)$ is the unit mass at a point x.

for each $\alpha>0$, $\beta>\alpha_0$, $f\in B(S)$ *and* $g\in C_0(S)$. *In particular,*

(3.13)
$$\lim_{N\to\infty} E_r'[e^{-(\alpha-1)\hat{\sigma}_V}f(\hat{x}_0)(k_N\hat{G}_\alpha g)(\hat{x}_{\hat{\sigma}_V})]$$
$$=\int_S g(x)m(dx)E_x\Big[\int_{\sigma_V}^{\infty} e^{-\alpha t}f(x_t)dt\Big].$$

PROOF. Denote by I_N the lefthand member in (3.12) before letting $N\to\infty$. Similarly to the proof of the preceding corollary, we express I_N using the finite dimensional distribution of $(\hat{x}_t, P_\nu')$, and use Lemma 3.2. Thus we have

$$I_N=\lim_{n\to\infty}\sum_{j=1}^{\infty} e^{-\alpha j2^{-n}}\int_V m_0(dx)k_N(x)\hat{G}_\beta g(x)E_x[f(x_{j2^{-n}})\,;\,x_{(j-i)2^{-n}}\notin V(1\leqq i\leqq j-1)],$$

and hence,

$$I_N=\lim_{n\to\infty}\sum_{j=1}^{\infty} e^{-\alpha j2^{-n}}\int_S m(dx)g(x)\int_0^\infty e^{-\beta t}dt$$
$$E_x[(\chi_N^*\chi_V)(x_t)E_{x_t}(f(x_{j2^{-n}})\,;\,x_{(j-i)2^{-n}}\notin V\ \ (1\leqq i\leqq j-1))],$$

which we denote by $\lim_{n\to\infty} I_N(n,\alpha,\beta)$. We have

(3.14) $\quad I_N(n,\alpha,\beta)-I_N(n,\alpha,\alpha)$
$$=(\alpha-\beta)\sum_{j=1}^{\infty} e^{-\alpha j2^{-n}}\int_S m(dx)g(x)\int_0^\infty e^{-\alpha t}dt\int_0^\infty e^{-\beta s}ds$$
$$E_x[(\chi_N^*\chi_V)(x_{s+t})E_{x_{s+t}}(f(x_{j2^{-n}})\,;\,x_{(j-i)2^{-n}}\notin V\ \ (1\leqq i\leqq j-1))].$$

Let $\tau_V^t(w)$ be the last hitting time to V before t, that is,

$$\tau_V^t(w)=\sup\{s: 0\leqq s<t \text{ and } x_s(w)\in V\},$$

and let $\tau_V^t(n,w)$ be defined by $\tau_V^t(n,w)=t-j2^{-n}$ if $x_{t-i2^{-n}}\notin V$ $(1\leqq i\leqq j-1)$ and $x_{t-j2^{-n}}\in V$, and $\tau_V^t(n,w)=-\infty$ if $x_{t-i2^{-n}}\notin V$ for all $i=1,2,\cdots$. Then,

$$I_N(n,\alpha,\alpha)$$
$$=\sum_{j=1}^{\infty}\int m(dx)g(x)\int_{j2^{-n}}^\infty e^{-\alpha t}dt\,E_x[f(x_t)(\chi_N^*\chi_V)(x_{t-j2^{-n}})\,;\,x_{t-i2^{-n}}\notin V\ \ (1\leqq i\leqq j-1)]$$
$$=\int m(dx)g(x)E_x\Big[\int_0^\infty e^{-\alpha t}f(x_t)\chi_N^*(x_{\tau_V^t(n)})\chi_{\{\tau_V^t(n)\geqq 0\}}(w)dt\Big].$$

Since, by the right continuity of paths, $\tau_V^t(n)$ is smaller than and increases to τ_V^t as $n\to\infty$, we have

$$I_N(n,\alpha,\alpha)\underset{n\to\infty}{\longrightarrow}\int m(dx)g(x)E_x\Big[\int_0^\infty e^{-\alpha t}f(x_t)\chi_N^*(x_{\tau_V^t-0})\chi_{\{\tau_V^t>0\}}dt\Big]$$
$$\underset{N\to\infty}{\longrightarrow}\int m(dx)g(x)E_x\Big[\int_0^\infty e^{-\alpha t}f(x_t)\chi_D(x_{\tau_V^t-0})\chi_{\{\tau_V^t>0\}}dt\Big].$$

Here we can omit $\chi_D(x_{\tau_V^t-0})$ since $x_{\tau_V^t-0}\in\partial D$ implies $x_{\tau_V^t}\in\partial D$ and $\tau_V^t=t$ and since

$$E_x\Big[\int_0^\infty e^{-\alpha t}f(x_t)\chi_{\partial D}(x_t)dt\Big]=0\,.$$

Hence, we have

$$\lim_{N\to\infty}\lim_{n\to\infty} I_N(n,\,\alpha,\,\alpha)=\int m(dx)g(x)E_x\Big[\int_{\sigma_V}^\infty e^{-\alpha t}f(x_t)dt\Big]\,,$$

noting that $\tau_V^t>0$ is equivalent to $\sigma_V<t$.

Similarly the righthand member in (3.14) tends to

$$(\alpha-\beta)\int m(dx)g(x)\int_0^\infty e^{-\beta s}ds\,E_x\Big[E_{x_s}\Big(\int_0^\infty e^{-\alpha t}f(x_t)\chi_N^*(x_{\tau_V^t-0})dt\Big)\Big]$$

as $n\to\infty$, and this has the limit

$$(\alpha-\beta)\int m(dx)g(x)\int_0^\infty e^{-\beta s}ds\,E_x\Big[E_{x_s}\Big(\int_{\sigma_V}^\infty e^{-\alpha t}f(x_t)\,dt\Big)\Big]$$

as $N\to\infty$. Thus the limit of I_N as $N\to\infty$ is just the righthand side of (3.12), and the proof of the lemma is complete.

Note that, we used in the above proof the property (w_3) of paths and the fact that X has no sojourn on the boundary.

We write $\hat{\sigma}$ for $\hat{\sigma}_{\partial D}$. $\hat{\sigma}$ is $\boldsymbol{B}$-measurable, as is proved similarly to Lemma 2.1.

LEMMA 3.4. *For each* $n\geqq 1$, $0=t_0<t_1<\cdots<t_n$, $f_i\in B(S)$ $(0\leqq i\leqq n-1)$ *and* $f_n\in B_0(V_0)$, *the following formula holds:*

$$E'_\Upsilon\Big[\prod_{i=0}^n f_i(\hat{x}_{t_i})\,;\;t_n<\hat{\sigma}\Big]=e^{-t_n}\int_{V_0}m_0(dx)f_n(x)E_x\Big[\prod_{i=0}^{n-1}f_i(x_{t_n-t_i})\,;\;t_n<\sigma\Big]\,. \tag{3.15}$$

PROOF. Let V_j be the open set consisting of all x such that $d(x,\partial D)<j^{-1}$. By virtue of Lemma 3.2 we have

$$\begin{aligned}
&E'_\Upsilon\Big[\prod_{i=0}^n f_i(\hat{x}_{t_i})\,;\;t_n\leqq\hat{\sigma}_{V_j}\Big]\\
&\quad=\lim_{N\to\infty}E'_\Upsilon\Big[\prod_{i=0}^n f_i(\hat{x}_{t_i})\,;\;\hat{x}_{l2^{-N}}\notin V_j(1\leqq l\leqq 2^N-1)\Big]\\
&\quad=\lim_{N\to\infty}e^{-t_n}\int m_0(dx)f_n(x)\,E_x\Big[\prod_{i=0}^{n-1}f_i(x_{t_n-t_i})\,;\;x_{(1-l2^{-N})t_n}\notin V_j(1\leqq l\leqq 2^N-1)\Big]\\
&\quad=e^{-t_n}\int m_0(dx)f_n(x)\,E_x\Big[\prod_{i=0}^{n-1}f_i(x_{t_n-t_i})\,;\;t_n\leqq\sigma_{V_j}\Big]\,.
\end{aligned}$$

By the same sort of argument as in the proof of Lemma 2.1, we can prove $(t_n<\hat{\sigma},\,\hat{x}_0\in D)=\bigcup_{j=1}^\infty(t_n\leqq\hat{\sigma}_{V_j},\,\hat{x}_0\in D)$, and similarly for $(t_n<\sigma,\,x_0\in D)$. On the other hand, $m_0(\partial D)=0$ and

$$P'_x(x_0\in\partial D)=0\,,\qquad x\in S \tag{3.16}$$

by (3.9). Hence we have (3.15), by letting $j\to\infty$.

§ 4. Proof of Theorems 1 and 2.

LEMMA 4.1. *Let μ_i $(i=1, 2)$ be finite signed measures on ∂D, and α be a positive number. Suppose that, for m-almost every x, $\int_{\partial D} g_\alpha(x, y)\mu_i(dy)$ $(i=1, 2)$ are defined and coincide. Then, $\mu_1=\mu_2$.*

PROOF. By virtue of $(A_{6,3})$ and $(A_{6,4})$

$$\iint f(x)m(dx)g_\beta(x, y)\mu_i(dy) \qquad (i=1, 2)$$

coincide for each $f\in C_0(S)$ and $\beta>\alpha_0$. Multiply β and let $\beta\to\infty$. Then we have $\mu_1=\mu_2$ since, by the assumption $(A_{6,4})$, $\beta\hat{G}_\beta f\to f$ boundedly on ∂D.

PROOF OF THEOREM 1. For every $\alpha>0$, every $f\in B(S)$, and every open neighborhood V of ∂D with closure contained in V_0, put

$$\mu_\alpha^V(dx, f)=k(x)\nu_\alpha^V(dx, f)\,, \tag{4.1}$$

where $\nu_\alpha^V(\cdot, f)$ is a finite signed measure defined by

$$\nu_\alpha^V(F, f)=E'_\gamma[e^{-(\alpha-1)\hat{\sigma}_V}f(\hat{x}_0)\,;\ \hat{x}_{\hat{\sigma}_V}\in F\cap D]\,.$$

For a while, suppose that $f\geqq 0$. Then $\mu_\alpha^V(\cdot, f)$ is nonnegative by Corollary to Lemma 3.2. For each $\beta>\alpha_0$ and $g\in C_0(S)$ we have, by Lemma 3.3,

$$\int_{V_0}\hat{G}_\beta g(x)\mu_\alpha^V(dx, f) \tag{4.2}$$
$$=\int_S g(x)m(dx)\int_S(\delta(x, dy)+(\alpha-\beta)G_\beta(x, dy))E_y\Big[\int_{\sigma_V}^\infty e^{-\alpha t}f(x_t)dt\Big]\,.$$

Since, by $(A_{6,4})$, there is a nonnegative $g\in C_0(S)$ such that $\hat{G}_\beta g(y)\geqq 2^{-1}$ in a neighborhood V'_0 of ∂D, we have

$$\mu_\alpha^V(S, f)\leqq 2\int\hat{G}_\beta g(x)\mu_\alpha^V(dx, f)\leqq 2\int g(x)m(dx)(I+|\alpha-\beta|G_\beta)G_\alpha f(x)^{7)}\,,$$

if $\bar{V}\subset V'_0$. Therefore, $\{\mu_\alpha^V(\cdot, f)\,;\ \bar{V}\subset V'_0\}$ is uniformly bounded. We can prove that $\mu_\alpha^V(\cdot, f)$ is convergent to a finite measure on ∂D in the weak star topology as V shrinks to ∂D, that is, as $d(\partial D, S-V)\to 0$. In fact, let $\{V_n^{(i)}\}$ $(i=1, 2)$ be two sequences of neighborhoods decreasing to ∂D such that $\mu_\alpha^{V_n^{(i)}}(\cdot, f)$ tends to some μ_i as $n\to\infty$. Then μ_i are finite measures on ∂D and (4.2) implies that, for each $g\in C_0(S)$,

$$\int_{\partial D}\hat{G}_\beta g(x)\mu_i(dx)=\int_S g(x)m(dx)\int_S(\delta(x, dy)+(\alpha-\beta)G_\beta(x, dy))u(y) \tag{4.3}$$

where $u(x)=E_x\Big[\int_\sigma^\infty e^{-\alpha t}f(x_t)dt\Big]$. Hence, $\mu_1=\mu_2$ by Lemma 4.1. Note that $\bar{V}$ is

7) I is the identity operator.

compact, if V is a sufficiently small neighborhood of ∂D. Let us denote the limit by $\mu_\alpha(\cdot, f)$. In order to prove that $\mu_\alpha(\cdot, f)$ satisfies (1.1), it is sufficient to verify

$$u(x)=\int_{\partial D} g_\alpha(x, y)\mu_\alpha(dy, f)\,. \tag{4.4}$$

As (4.3) is true with $\mu_\alpha(\cdot, f)$ in place of μ_i, we have

$$\int g_\beta(x, y)\mu_\alpha(dy, f)=(I+(\alpha-\beta)G_\beta)u(x)$$

for m-almost every x. Hence, $(A_{6,3})$ implies that

$$\int g_\alpha(x, y)\mu_\alpha(dy, f)=(I+(\alpha-\beta)G_\beta)u+(\beta-\alpha)G_\alpha(I+(\alpha-\beta)G_\beta)u$$

m-almost everywhere, and we have (4.4) for m-almost every x by the resolvent equation

$$G_\alpha-G_\beta+(\alpha-\beta)G_\alpha G_\beta=0\,. \tag{4.5}$$

The both sides of (4.4) being α-excessive, (4.4) holds without any exceptional point, and we have proved (1.1) for nonnegative f.

For general (not necessarily nonnegative) $f\in B(S)$, $\mu_\alpha^V(\cdot, f)$ is finite and tends to a finite signed measure on ∂D as well. Denoting the limit by $\mu_\alpha(\cdot, f)$, we have (1.1). Note that

$$\mu_\alpha(\cdot, c_1f_1+c_2f_2)=c_1\mu_\alpha(\cdot, f_1)+c_2\mu_\alpha(\cdot, f_2) \tag{4.6}$$

for every constant c_i and every $f_i\in B(S)$, $i=1, 2$. The uniqueness of the signed measure $\mu_\alpha(\cdot, f)$ satisfying (1.1) is an immediate consequence of Lemma 4.1.

Finally, let us prove that $\mu_\alpha(\cdot, f)$ depends only on the minimal part of X. Let g be in $C_0(D)$ and V be an open neighborhood of ∂D such that $\bar{V}\subset V_0$. Since the sequence $\{U_N\}$ introduced before Lemma 3.3 exhausts $V_0-\partial D$, we have

$$\int g(x)\mu_\alpha^V(dx, f)=E_r'[e^{-(\alpha-1)\hat{\sigma}_V}f(\hat{x}_0)(gk_N)(\hat{x}_{\hat{\sigma}_V})]$$

if N is large enough. By (3.16) we can add the restriction $\hat{\sigma}_V<\hat{\sigma}$ in the expectation on the right side, and this in turn equals

$$\lim_{n\to\infty}\sum_{j=1}^{\infty}E_r'[e^{-(\alpha-1)j2^{-n}}f(\hat{x}_0)(gk_N)(\hat{x}_{j2^{-n}})\,;$$
$$\hat{x}_{i2^{-n}}\notin V(1\leqq i\leqq j-1),\ \hat{x}_{j2^{-n}}\in V,\ j2^{-n}<\hat{\sigma}]\,.$$

Therefore

$$\int g(x)\mu_\alpha^V(dx, f)=\lim_{n\to\infty}\sum_{j=1}^{\infty}e^{-\alpha j2^{-n}}$$
$$\int_V g(x)m(dx)E_x[f(x_{j2^{-n}})\,;\ x_{i2^{-n}}\notin V\ (1\leqq i\leqq j-1),\ j2^{-n}<\sigma]$$

by Lemma 3.4. Hence, $\mu_\alpha^V(\cdot, f)$ is determined by the minimal part of X and so is $\mu_\alpha(\cdot, f)$. The proof of Theorem 1 is complete.

REMARK. The Markov process $X^{(\beta)}=(W, P_x^{(\beta)}: x\in S)$ whose α-order Green operator is $G_{\alpha+\beta}$ is made from X in a similar way as X'. We can use $X^{(\beta)}$ in constructing $\mu_\alpha(\cdot, f)$. In fact, there is a finite signed measure γ_β on S satisfying

$$\beta\int \gamma_\beta(dx)g_\beta(x, y)k(y)=1$$

m_0-almost everywhere in $V_0-\partial D$, and we have, using this γ_β,

$$\mu_\alpha(dx, f)=\lim_{V\downarrow\partial D} k(x)E_{\gamma_\beta}^{(\beta)}[e^{-(\alpha-\beta)\partial_V}f(\hat{x}_0)\,;\ \hat{x}_{\partial_V}\in dx\cap D]$$

where limit is in the weak star sense; in particular,

$$\mu_\alpha(dx, f)=\lim_{V\downarrow\partial D} k(x)E_{\gamma_\alpha}^{(\alpha)}[f(\hat{x}_0)\,;\ \hat{x}_{\partial_V}\in dx\cap D]\,.$$

We list some properties of $\mu_\alpha(\cdot, f)$.

PROPOSITION 4.1. (i) *$\mu_\alpha(\cdot, f)$ is the weak star limit of $\mu_\alpha^V(\cdot, f)$ defined by (4.1) as V shrinks to ∂D.*

(ii) *$\mu_\alpha(\cdot, f)$ is linear with respect to f.*

(iii) *$\mu_\alpha(\cdot, f)$ is nonnegative if $f\geqq 0$.*

(iv) *For each α, $\beta>0$ and $f\in B(S)$.*

$$\mu_\alpha(\cdot, f)-\mu_\beta(\cdot, f)+(\alpha-\beta)\mu_\alpha(\cdot, G_\beta^{\min}f)=0\,. \tag{4.7}$$

(v) *For every choice of α, $\beta>0$ and $f\in B(S)$, $\mu_\alpha(\cdot, f)$ is absolutely continuous with respect to $\mu_\beta(\cdot, 1)$.*

PROOF. (i), (ii) and (iii) are proved already in the proof of Theorem 1. Using $(A_{6,3})$, (1.1), (3.4) and (4.5), we have

$$\begin{aligned} G_\alpha f &= G_\beta^{\min}f+\int g_\beta(x, y)\mu_\beta(dy, f)+(\beta-\alpha)\int G_\alpha(x, dy)(G_\beta^{\min}f(y) \\ &\quad +\int g_\beta(y, z)\mu_\beta(dz, f)) \\ &= G_\alpha^{\min}f+(\alpha-\beta)G_\alpha^{\min}G_\beta^{\min}f+\int g_\alpha(x, y)\mu_\beta(dy, f)+(\beta-\alpha)G_\alpha G_\beta^{\min}f \\ &= G_\alpha^{\min}f+\int g_\alpha(x, y)\mu_\beta(dy, f)+(\beta-\alpha)\int g_\alpha(x, y)\mu_\alpha(dx, G_\beta^{\min}f)\,. \end{aligned}$$

Thus (4.7) holds. Applying (iii) to $\|f\|+f$ and $\|f\|-f$, we obtain

$$|\mu_\alpha(F, f)|\leqq \|f\|\,\mu_\alpha(F, 1)\,,\qquad F\in \boldsymbol{B}(\partial D)\,. \tag{4.8}$$

This combined with (4.7) will imply (v), and the proof is complete.

PROOF OF THEOREM 2. Recalling (v) of the above proposition, denote by

$\hat{H}_\alpha f$ the Radon-Nikodym derivative of $\mu_\alpha(\cdot, f)$ with respect to $\mu=\mu_1(\cdot, 1)$. It follows from (4.7) and (4.8) that $\hat{H}_\alpha f$ is a μ-essentially bounded function. Since

$$G_\alpha f(x)=\int g_\alpha(x, y)\mu_\alpha(dy, f)=K^\alpha \hat{H}_\alpha f(x)\,, \qquad x\in\partial D\,,$$

by (1.1), (1.2) and (A_4), we have (1.3) by (3.5). $\hat{H}_\alpha$ is determined by the minimal part of X, as $\mu_\alpha(\cdot, f)$ is.

H_α is also determined by the minimal part. For, $H_\alpha(x, dy)=\delta(x, dy)$ if $x\in\partial D$, and, if $x\in D$, then $P_x(\sigma_n<\sigma$ and $\sigma_n\uparrow\sigma)=1$ where σ_n's are the Markov times in the proof of Lemma 2.1, and we have $P_x(x_{\sigma-0}=x_\sigma, 0<\sigma<\infty)$ $=P_x(\sigma<\infty)$ by the quasi-left continuity. Note that $\sigma(w)=\sigma(w^-_{\sigma(w)})$ if $x_0(w)\in D$. Thus the proof of Theorem 2 is complete.

PROPOSITION 4.2. (i) *$\hat{H}_\alpha$ and K^α are nonnegative bounded linear operators from $B(S)$ to $L_\infty(\partial D, \mu)$ and from $L_\infty(\partial D, \mu)$ to $B(\partial D)$, respectively.*

(ii) $\hat{H}_\alpha-\hat{H}_\beta+(\alpha-\beta)\hat{H}_\alpha G_\beta^{\min}=0$.

(iii) $\hat{H}_\alpha H_\beta=\hat{H}_\beta H_\alpha$.

(iv) *If $f=f'$ m-almost everywhere, then $\hat{H}_\alpha f=\hat{H}_\alpha f'$.*

PROOF. (i) and (ii) are obvious consequences of the definition and Proposition 4.1. To prove (iii), we note first

$$H_\alpha-H_\beta+(\alpha-\beta)G_\alpha^{\min}H_\beta=0\,, \tag{4.9}$$

which is easily seen. Using (ii) and (4.9) we have

$$\hat{H}_\alpha H_\beta=\hat{H}_\alpha(H_\alpha+(\alpha-\beta)G_\beta^{\min}H_\alpha)=\hat{H}_\beta H_\alpha\,.$$

If $f=f'$ m-almost everywhere, then $G_\alpha f=G_\alpha f'$ and $G_\alpha^{\min}f=G_\alpha^{\min}f'$ by (A_5), and hence, $\mu_\alpha(\cdot, f)=\mu_\alpha(\cdot, f')$ by Theorem 1, and $\hat{H}_\alpha f=\hat{H}_\alpha f'$, completing the proof.

§ 5. Proof of Theorems 3 and 4.

The following lemma is a slight extension of a result by Nagasawa [**9**].

LEMMA 5.1. *Let U be an open set. Then*

$$g_\alpha(x, y)=E_x(e^{-\alpha\sigma_U}g_\alpha(x_{\sigma_U}, y)) \tag{5.1}$$

for each $x\in S$, $y\in U\cap\partial D$, and $\alpha>0$.

PROOF. (5.1) is obvious for m-almost every y in $U\cap V_0$, since, if g vanishes outside of U, then

$$G_\alpha g(x)=E_x(e^{-\alpha\sigma_U}G_\alpha g(x_{\sigma_U}))\,.$$

By the α-excessivity of $g_\alpha(x, y)$ we have

$$g_\alpha(x, y)\geqq E_x(e^{-\alpha\sigma_U}g_\alpha(x_{\sigma_U}, y))\,.$$

Let us prove the reverse inequality for $y\in U\cap\partial D$. Denote the right side of

(5.1) by $g'_\alpha(x, y)$. Fix a point y in $U \cap \partial D$ and choose $f \in C_0(U \cap V_0)$ satisfying $0 \leqq f \leqq 1$ and $f(y)=1$. Then, we have, for each nonnegative $h \in C_0(S)$,

$$\int h(x)m(dx)g_\alpha(x, y) = \lim_{\beta\to\infty} \beta \iint h(x)m(dx)g_\alpha(x, z)f(z)m(dz)g_\beta(z, y)$$

$$\leqq \limsup_{\beta\to\infty} \beta \int h(x)m(dx)E_x\Big[e^{-\alpha\sigma_U}\int G_\alpha(x_{\sigma_U}, dz)g_\beta(z, y)\Big],$$

replacing $g_\alpha(x, z)$ by $g'_\alpha(x, z)$ and deleting $f(z)$. Since $(A_{6,3})$ implies

$$g'_\beta(x, y)+(\beta-\alpha)E_x\Big[e^{-\alpha\sigma_U}\int G_\alpha(x_{\sigma_U}, dz)g_\beta(z, y)\Big]$$

$$= g'_\beta(x, y)+(\beta-\alpha)E_x\Big[e^{-\alpha\sigma_U}\int G_\beta(x_{\sigma_U}, dz)g_\alpha(z, y)\Big]$$

and $g'_\beta(\cdot, y)$ is finite m-almost everywhere, we obtain

$$\int h(x)m(dx)g_\alpha(x, y) \leqq \limsup_{\beta\to\infty} \beta \int h(x)m(dx)E_x\Big[e^{-\alpha\sigma_U}\int G_\beta(x_{\sigma_U}, dz)g_\alpha(z, y)\Big].$$

The right side is equal to $\int h(x)m(dx)g'_\alpha(x, y)$ by (2.1), and hence $g_\alpha(\cdot, y) \leqq g'_\alpha(\cdot, y)$ m-almost everywhere. The both sides being α-excessive, this holds everywhere, and the proof is complete.

LEMMA 5.2. (i) *There is a unique (up to P_x-probability 0 for all x) nonnegative continuous additive functional $\varphi(t)$ of X such that*

$$E_x\Big[\int_0^\infty e^{-\alpha t}d\varphi(t)\Big] = \int_{\partial D} g_\alpha(x, y)\mu(dy), \qquad \alpha>0, \quad x \in S. \tag{5.2}$$

(ii) *It holds with probability 1 for each x that $\varphi(t)$ is flat on every time interval I such that $x_t \notin \partial D$, $t \in I$.*

(iii) *$P_x(\varphi(t)>0$ for each $t>0)=1$ if and only if $x \in \partial D$.*

(iv) $H_\alpha K^\alpha f(x) = \int_{\partial D} g_\alpha(x, y)f(y)\mu(dy) = E_x\Big[\int_0^\infty e^{-\alpha t}f(x_t)d\varphi(t)\Big]$ *for each $f \in B(\partial D)$.*

PROOF. Fix α for a while and denote the righthand side of (5.2) by $u(x)$. Then, u is α-excessive and we have

$$u(x) = E_x\Big[\int_\sigma^\infty e^{-\alpha t}(1+(\alpha-1)G_1^{\min}1(x_t))dt\Big]$$

by (1.1) and (4.7). Here we can replace σ by the σ' defined in the proof of Lemma 2.1. Let ρ_n be a sequence of Markov times increasing to ρ. Then we have $\rho_n+\sigma'(w^+_{\rho_n}) \to \rho+\sigma'(w^+_\rho)$, and hence $E_x(e^{-\alpha\rho_n}u(x_{\rho_n}))$ tends to $E_x(e^{-\alpha\rho}u(x_\rho))$. Hence, it follows from the result of Meyer [7, 2^e^ partie, Théorème 3.4] and Šur [14, Theorem 1] that there exists a unique nonnegative continuous additive functional satisfying (5.2) for this fixed α. Note that, by the $\boldsymbol{B}(S)$-measurability of u, we can say that $\varphi(t)$ is not only $\bigcap_{\nu}(\boldsymbol{B}_t)_\nu$-measurable[8], but $\boldsymbol{B}_t$-measurable (cf. footnote in [8, § 2]). For the $\varphi(t)$ thus obtained, the validity of (5.2) for all $\alpha>0$ has been proved in [**10**, Lemma 6.3] and (i) holds. μ being a measure on ∂D, we have (ii), since Lemma 5.1 suffices to prove [**10**, Lemma 4.1]. Write $u_1(x)=E_x\left[\int_0^\infty e^{-t}d\varphi(t)\right]$. Put $\rho=\inf\{t: \varphi(t)>0\}$, then ρ is a Markov time and we have

$$0=E_x\left[\int_0^\rho e^{-t}d\varphi(t)\right]=u_1(x)-E_x\left[e^{-\rho}u_1(x_\rho)\right]$$

$$\geqq G_1 1(x)-E_x\left[e^{-\rho}G_1 1(x_\rho)\right]=E_x\left[\int_0^\rho e^{-t}dt\right]$$

for $x\in\partial D$, since $u_1\leqq G_1 1$ on S and $u_1=G_1 1$ on ∂D. Hence $P_x(\rho=0)=1$ for $x\in\partial D$, and we have 'if' part of (iii), while 'only if' part is a consequence of (ii). The second equality in (iv) is proved in [**10**, Theorem 4.1]. The meaning of the last member in (iv) is

$$E_x\left[\int_0^\infty e^{-\alpha t}f^*(x_t)d\varphi(t)\right],$$

where f^* is an extension of f to S. But (ii) implies that this is independent of the choice of extensions. The first equality in (iv) is seen from

$$H_\alpha K^\alpha f(x)=\int_{\partial D}H_\alpha(x,dy)E_y\left[\int_0^\infty e^{-\alpha t}f(x_t)d\varphi(t)\right]$$

$$=E_x\left[\int_\sigma^\infty e^{-\alpha t}f(x_t)d\varphi(t)\right]=E_x\left[\int_0^\infty e^{-\alpha t}f(x_t)d\varphi(t)\right],$$

completing the proof.

The above proof of the 'if' part of (iii) is due to Motoo, a special case of [8, Lemma 5.5]. We call $\varphi(t)$ *the local time on the boundary induced by X.*

PROOF OF THEOREM 3. Let $\alpha>0$. Let $\tau(t)$ be the right continuous inverse of $\varphi(t)$, that is, $\tau(t,w)=\sup\{s: \varphi(s)\leqq t\}$. Put $W^1=\{w: x_{\tau(t)}\in\partial D$ for all $t<\varphi(\infty)\}$. Then we have $P_x(W^1)=1$ by Lemma 5.2 (ii). Let Ω and P^Ω be as defined at the beginning of Section 3. For $\omega=(w,s)\in W^1\times[0,+\infty]$ we define $\tilde{x}_t^{(\alpha)}(\omega)=x_{\tau(t,w)}(w)$ if $\alpha\tau(t,w)<s$ and $t<\varphi(\infty,w)$, and define $\tilde{x}_t^{(\alpha)}(\omega)=\Delta$ if otherwise. Let $W_{\partial D}$ be the set of all $w: [0,+\infty]\to(\partial D)^*=\partial D\cup\{\Delta\}$ satisfying (w_1) and (w_2) with S replaced by ∂D. Define $\pi^{(\alpha)}: W^1\times[0,+\infty]\to W_{\partial D}$ by $x_t(\pi^{(\alpha)}(\omega))=\tilde{x}_t^{(\alpha)}(\omega)$, and put $\tilde{P}_x^{(\alpha)}(B)=P^\Omega((\pi^{(\alpha)})^{-1}(B))$ for $x\in(\partial D)^*$ and $B\in\boldsymbol{B}(W_{\partial D})$, where $\boldsymbol{B}(W_{\partial D})$ is the smallest Borel field that makes all $x_t(w)$ $(w\in W_{\partial D})$ measurable. Then, $(W_{\partial D}, \tilde{P}_x^{(\alpha)}: x\in\partial D)$ is a Markov process on ∂D. It satisfies $\tilde{P}_x^{(\alpha)}(x_0=x)=1$ for each $x\in\partial D$ by Lemma 5.2 (iii), and its transition proba-

8) ν is a finite measure on S and $(\boldsymbol{B}_t)_\nu$ is the completion of $\boldsymbol{B}_t$ with respect to P_ν.

bility is

$$P_x^{\Omega}(\tilde{x}_t^{(\alpha)} \in F) = E_x(e^{-\alpha\tau(t)};\ x_{\tau(t)} \in F).$$

We denote this process by $\tilde{X}^{(\alpha)}$ and call it *the α-order U-process induced by X.* In short, $\tilde{X}^{(\alpha)}$ is the Markov process obtained from X through time change by $\varphi(t)$ and killing by $e^{-\alpha t}$. Its λ-order Green operator is

$$K_\lambda^\alpha f(x) = E_x\Big[\int_0^\infty e^{-\alpha t-\lambda\varphi(t)} f(x_t)\, d\varphi(t)\Big]. \tag{5.3}$$

As a consequence of (5.3) and Lemma 5.2 (iv), we have $K^\alpha = K_0^\alpha$, and the proof of Theorem 3 is complete.

Put $\tilde{x}_t^{(0)}(w) = x_{\tau(t,w)}(w)$ if $t < \varphi(\infty, w)$ and $\tilde{x}_t^{(0)}(w) = \varDelta$ if otherwise. Define $\pi^{(0)}: W^1 \to W_{\partial D}$ by $x_t(\pi^{(0)}(w)) = \tilde{x}_t^{(0)}(w)$ and $\tilde{P}_x^{(0)}$ by $\tilde{P}_x^{(0)}(B) = P_x((\pi^{(0)})^{-1}(B))$ for $x \in (\partial D)^*$ and $B \in \boldsymbol{B}(W_{\partial D})$. Then $\tilde{X}^{(0)} = (W_{\partial D}, \tilde{P}_x^{(0)}: x \in \partial D)$ is the Markov process obtained through time change by $\varphi(t)$, and we call $\tilde{X}^{(0)}$ *the* 0-*order U-process induced by X.* The λ-order Green operator of $\tilde{X}^{(0)}$ is

$$K_\lambda^0 f(x) = E_x\Big[\int_0^\infty e^{-\lambda\varphi(t)} f(x_t)\, d\varphi(t)\Big]. \tag{5.4}$$

PROOF OF THEOREM 4. Define G_α^λ $(\alpha > 0)$ by

$$G_\alpha^\lambda f(x) = E_x\Big[\int_0^\infty e^{-\alpha t-\lambda\varphi(t)} f(x_t)\, dt\Big]. \tag{5.5}$$

Let α, β, λ and μ be nonnegative numbers. Then we have

$$H_\alpha K_\lambda^\alpha - H_\beta K_\mu^\beta + (\lambda-\mu) H_\alpha K_\lambda^\alpha K_\mu^\beta + (\alpha-\beta) G_\alpha^\lambda H_\beta K_\mu^\beta = 0, \tag{5.6}$$

if $\alpha+\lambda > 0$ and $\beta+\mu > 0$; and

$$G_\alpha^\lambda - G_\beta^\mu + (\alpha-\beta) G_\alpha^\lambda G_\beta^\mu + (\lambda-\mu) H_\alpha K_\lambda^\alpha G_\beta^\mu = 0, \tag{5.7}$$

if α and $\beta > 0$ [**10**, Theorems 2.1 and 2.2]. As a special case we have $G_\alpha^\lambda = G_\alpha - \lambda H_\alpha K_\lambda^\alpha G_\alpha$ and

$$K_\lambda^\alpha - K_\mu^\alpha + (\lambda-\mu) K_\lambda^\alpha K_\mu^\alpha = 0, \tag{5.8}$$

and hence,

$$G_\alpha^\lambda = G_\alpha^{\min} + H_\alpha K_\lambda^\alpha \hat{H}_\alpha \tag{5.9}$$

by (1.3). It follows from (5.6) and (5.9) that

$$K_\lambda^\alpha - K_\lambda^\beta + (\alpha-\beta) K_\lambda^\alpha \hat{H}_\alpha H_\beta K_\lambda^\beta = 0.$$

Using Proposition 4.2 (iii) and letting $\alpha \downarrow 0$, we have

$$K_\lambda^0 - K_\lambda^\beta - \beta K_\lambda^0 \hat{H}_\beta H_0 K_\lambda^\beta = 0. \tag{5.10}$$

Therefore, for each $\beta > 0$ we have

$$K_\lambda^\beta = \sum_{n=0}^\infty (-\beta K_\lambda^0 \hat{H}_\beta H_0)^n K_\lambda^0 \tag{5.11}$$

if λ is sufficiently large, because $\hat{H}_\beta$ and H_0 are bounded and $\|K_\lambda^0\| \leqq \lambda^{-1}$.

Suppose that X_1 and X_2 are both Markov processes satisfying Condition (A) and that $X_1^{\min} = X_2^{\min}$ and $\tilde{X}_1^{(0)} = \tilde{X}_2^{(0)}$. Then, by virtue of Theorem 2, $H_\alpha(\alpha \geqq 0)$ and $\hat{H}_\alpha(\alpha > 0)$ are common to X_1 and X_2. Let $\beta > 0$. By (5.11), K_λ^β is common to X_1 and X_2 as well as K_λ^0, if λ is sufficiently large. Therefore K_λ^β is common to X_1 and X_2 for all $\lambda > 0$ by (5.8), and so is $K^\beta = K_0^\beta = \lim_{\lambda \to 0} K_\lambda^\beta$. Hence, the Green operator G_α is common to X_1 and X_2 by Theorem 2, which implies $X_1 = X_2$ [**5**, p. 35]. This completes the proof.

REMARK. G_α^λ is the α-order Green operator of the Markov process on S obtained from X through killing by $e^{-\lambda\varphi(t)}$. The decomposition (5.9) is the analogue to (1.3) for this process.

PROPOSITION 5.1. *For each α and $\beta > 0$ we have*

$$K^\alpha - K^\beta + (\alpha-\beta)K^\alpha \hat{H}_\alpha H_\beta K^\beta = 0\,. \tag{5.12}$$

This is a consequence of (1.3) and (5.6).

§ 6. Some comments.

A. Motivation to the problem.

Ueno [**15, 16**] proved the following results. Let D be a bounded domain with compact closure in an N-dimensional C^∞-manifold, and let its boundary ∂D be an $(N-1)$-dimensional hypersurface. Given a second order elliptic differential operator A on $D \cup \partial D$ with nonpositive coefficient in the zero order term, we assume that ∂D and the coefficients in A are sufficiently smooth. Let $Lu = 0$ be a boundary condition of Vencel' type. That is, for fixed $x \in \partial D$,

$$\begin{aligned} Lu(x) = &\sum_{i,j=1}^{N-1} \alpha_{ij}(x) \frac{\partial^2 u}{\partial \xi_i \partial \xi_j}(x) + \sum_{i=1}^{N-1} \beta_i(x) \frac{\partial u}{\partial \xi_i}(x) + \gamma(x)u(x) \\ &+ \delta(x)Au(x) + \mu(x)\frac{\partial u}{\partial n}(x) + \int_{D \cap \partial D} \Big[u(y) - u(x) \\ &- \sum_{i=1}^{N-1} \frac{\partial u}{\partial \xi_i}(x)(\xi_i(y) - \xi_i(x)) \Big] \nu_x(dy)\,, \end{aligned} \tag{6.1}$$

where ξ_i $(1 \leqq i \leqq N-1)$ are extensions to $D \cup \partial D$ of a system of coordinate functions on ∂D around x, and $\dfrac{\partial}{\partial n}$ is the inward-directed normal derivative associated with the coefficients in the second order terms in A. Now, suppose that we can find, for sufficiently many functions f on ∂D, a solution of $(\alpha - A)u = 0$ with the boundary condition $(\beta - L)u = f$. Then, under some regularity conditions on L, there exists a Markov process X whose infinitesimal generator is A with the boundary condition $Lu = 0$. Furthermore, there exists a Markov process on the boundary with infinitesimal generator $\overline{LH_\alpha}$, and the

Green operator of X is expressed as follows:

(6.2) $$G_\alpha = G_\alpha^{\min} + H_\alpha(\overline{LH_\alpha})^{-1}\overline{LG_\alpha^{\min}}.$$

Here, $\overline{LH_\alpha}$ and $\overline{LG_\alpha^{\min}}$ are certain extensions of LH_α and $LG_\alpha^{\min}$, respectively.

Let $\delta=0$, $\nu_x(D)=0$, and $\mu=1$ in (6.1). Then, whatever α_{ij}, β_i, γ and ν_x on ∂D are, $LG_\alpha^{\min}$ is reduced to $\frac{\partial}{\partial n}G_\alpha^{\min}$, and (6.2) implies that X is determined by two components—the minimal part and the Markov process on the boundary with infinitesimal generator $\overline{LH_\alpha}$. It was our aim to generalize this fact.

B. Remarks to Condition (A); Motoo's example.

We fixed a measure m in stating Condition (A). To emphasize its dependence on m, let us call it Condition (A_m). Suppose that we are given two Markov processes X_1 and X_2, which satisfy Conditions (A_{m_1}) and (A_{m_2}), respectively, and that they have the same minimal parts, and further suppose

(*) m_2 is absolutely continuous with respect to m_1 with positive continuous density $b(x)$ in a neighborhood of ∂D.

Let us denote the quantities of X_i by the subscript i. Then, we have the following:

(i) $\mu_{\alpha,2}(dx,f)=b(x)\mu_{\alpha,1}(dx,f)$;

(ii) $\hat{H}_{\alpha,1}=\hat{H}_{\alpha,2}$;

(iii) *If X_1 and X_2 induce the same* 0*-order U-processes, then* $X_1=X_2$.

PROOF. (ii) is an obvious consequence of (i), and (iii) is proved from (ii) similarly to Section 5. To verify (i), take g and V as in the last paragraph of the proof of Theorem 1. Then,

$$\int g(x)\mu_{\alpha,2}^V(dx,f)=\lim_{n\to\infty}\sum_{j=1}^{\infty}e^{-\alpha j2^{-n}}\int_V g(x)b(x)m_1(dx)E_x\Big[f(x_{j2^{-n}});$$

$$x_{i2^{-n}}\notin V(1\leqq i\leqq j-1),\, j2^{-n}<\sigma\Big]=\int g(x)b(x)\mu_{\alpha,1}^V(dx,f),$$

from which (i) follows.

Motoo remarked me that (iii) is not necessarily valid unless we assume (*). The following example is due to him. Let S be the real line R^1 and $D=R^1-\{0\}$, $\partial D=\{0\}$. Let X_1 be a diffusion with infinitesimal generator $D_{v_1}D_{u_1}$ of Feller [1] such that $u_1(-\infty)=-\infty$ and $u_1(+\infty)=+\infty$, and let X_2 be the diffusion with infinitesimal generator $D_{v_2}D_{u_2}$ where

$$u_2(x)=\begin{cases} u_1(x), & x\leqq 0,\\ u_1(0)+c(u_1(x)-u_1(0)), & x>0,\end{cases}$$

and

$$v_2(x)=\begin{cases} v_1(x), & x\leqq 0,\\ v_1(0)+c^{-1}(v_1(x)-v_1(0)), & x>0,\end{cases}$$

c being a positive constant. Denote by m_1 $[m_2]$ the measure induced by v_1 $[v_2]$. Then, X_1 and X_2 satisfy Conditions (A_{m1}) and (A_{m2}), respectively, and it is obvious that $X_1^{\min}=X_2^{\min}$. Moreover, we can prove $P_{x,1}(\varphi_1(\infty)=\infty)=P_{x,2}(\varphi_2(\infty)=\infty)=1$, $x\in R^1$, for their local times on the boundary[9]. Hence, their 0-order U-processes are both at a standstill at 0. X_1 and X_2 are, however, different from each other if $c\neq 1$. In this example, X_1 and X_2 induce also the same 1-order U-processes by virtue of (5.10), since $\hat{H}_{1,1}H_0f(0)=\hat{H}_{1,2}H_0f(0)=f(0)$. Though m_1 and m_2 are mutually absolutely continuous, the density is discontinuous at 0, and hence, (*) does not hold.

C. Remarks to the time-reversed process in Section 3.

If $0=t_0<t_1<\cdots<t_n$, $f_i\in B(S)$ $(0\leqq i\leqq n-1)$ and f_n, $g\in B_0(V_0)$, then we have

$$E'_{\gamma}\Big[\Big(\prod_{i=0}^{n}f_i(\hat{x}_{t_i})\Big)\int_{t_n}^{\infty}e^{-(\alpha-1)t}g(\hat{x}_t)dt\Big]=\lim_{N\to\infty}E'_{\gamma}\Big[e^{-(\alpha-1)t_n}\Big(\prod_{i=0}^{n}f_i(\hat{x}_{t_i})\Big)h_N(\hat{x}_{t_n})\Big], \tag{6.3}$$

where $h_N(y)=\int_{V_0}g(x)m(dx)g_\alpha(x,y)k(y)$ if $y\in V_0-\partial D$ and if this integral is smaller than N, and $h_N(y)=0$ if otherwise. In fact, (6.3) is a consequence of Lemma 3.2, the both sides being equal to

$$\int_{V_0}g(x)m_0(dx)E_x\Big[\int_{t_n}^{\infty}e^{-\alpha t}\prod_{i=0}^{n}f_i(x_{t-t_i})dt\Big].$$

Suppose $V_0=S$ in this paragraph. Then (6.3) suggests that the process $(\hat{x}_t, P'_\gamma)$ has the time homogeneous Markov property, whose $(\alpha-1)$-order Green measure is $m_0(dy)g_\alpha(y,x)k(x)$. Its transition probability is $m_0(dy)q(t,y,x)k(x)$, that is, $m(dy)k(y)^{-1}q(t,y,x)k(x)$, if we can write $q(t,x,y)m(dy)$ for the transition probability of X'. This is in accordance with (3.2) combined with (3.3). By the technique in the proof of Lemma 3.3, one can prove that

$$\lim_{N\to\infty}E'_{\gamma}\Big[f(\hat{x}_0)\int_{\hat{\sigma}_V}^{\infty}e^{-(\alpha-1)t}(gk_N)(\hat{x}_t)dt\Big]=\int_S g(x)m(dx)E_x\Big[\int_{\sigma_V}^{\infty}e^{-\alpha t}f(x_t)dt\Big] \tag{6.4}$$

for each f and $g\in B(S)$. Combining (3.13) and (6.4), we see that a kind of strong Markov property holds for the process $(\hat{x}_t, P'_\gamma)$. One can also see that

$$\lim_{N\to\infty}E'_{\gamma}\Big[f(\hat{x}_0)\int_0^{\hat{\sigma}_V}e^{-(\alpha-1)t}(gk_N)(\hat{x}_t)dt\Big]=\int_S g(x)m(dx)E_x\Big[\int_0^{\sigma_V}e^{-\alpha t}f(x_t)dt\Big]. \tag{6.5}$$

(6.5) suggests that the minimal part of the process $(\hat{x}_t, P'_\gamma)$ is determined only by the minimal part of X, which is proved in Lemma 3.4 in a different manner.

Further, it is to be remarked that (3.13) is an analogue to Hunt's formula

9) Motoo proved that if X satisfies Condition (A) and if $P_x(\sigma<\infty)=1$ for each $x\in S$, then $P_x(\varphi(\infty)=\infty)=1$, $x\in S$ [8, Theorem 5.2].

[3, III (18.3)].

D. Further properties of U-processes.

Suggested by Neveu [11], Fukushima and Ikeda observed the following facts [2, pp. 93-99]. Though their definitions of $\hat{H}_\alpha$ and K^α differ from ours, their results remain valid in our case. Define $Z_\alpha = \alpha \hat{H}_\alpha H_0$, for $\alpha > 0$. Then Z_α is a bounded nonnegative operator $B(\partial D) \to L_\infty(\partial D, \mu)$ and satisfies $Z_\alpha - Z_\beta = (\alpha - \beta)\hat{H}_\alpha H_\beta$. Therefore, we have

$$K_\lambda^\alpha - K_\mu^\beta + (\lambda - \mu) K_\lambda^\alpha K_\mu^\beta + K_\lambda^\alpha (Z_\alpha - Z_\beta) K_\mu^\beta = 0$$

by (5.6) and (5.9). The range $\mathfrak{R}$ and the null space $\mathfrak{N}$ of K_λ^α are both independent of α and λ. We define an operator $Q^{(\alpha)}$ from $\mathfrak{R}$ to $B(\partial D)/\mathfrak{N}$ by $Q^{(\alpha)} = \lambda - (K_\lambda^\alpha)^{-1}$, which is independent of λ. $Q^{(\alpha)}$ is the generator in the sense of [5] of the α-order U-process, and we have

$$Q^{(\alpha)} = Q^{(0)} - Z_\alpha .$$

The strong Markov property of U-processes is verified similarly to Volkonskiĭ [17, pp. 154-157]. They are quasi-left continuous, as is proved by Motoo [8, Theorem 6.12].

Choosing arbitrary $\beta > 0$ and $f \in B(S)$ satisfying $\inf_{x \in S} f(x) > 0$, put $\mu^\#(\cdot) = \mu_\beta(\cdot, f)$. $\mu^\#$ can be used instead of μ. $\varphi^\#(t)$ thus obtained is connected with $\varphi(t)$ as follows: there is a function a in $B(\partial D)$ such that $\inf_{x \in \partial D} a(x) > 0$ and

$$\varphi^\#(t) = \int_0^t a(x_s) d\varphi(s) . \tag{6.6}$$

For, there is a constant c such that $\mu \leqq c\mu^\#$ and $\mu^\# \leqq c\mu$, and we can choose a version a of $\frac{\mu^\#(dx)}{\mu(dx)}$ satisfying $c^{-1} \leqq a \leqq c$.

E. The case that $E_x^{\min}(\sigma)$ is bounded.

Put $h(x) = E_x^{\min}(\sigma) = G_0^{\min} 1(x)$, and assume that h is bounded. Then, the definition of $\mu_\alpha(\cdot, f)$ is naturally extended to the case $\alpha = 0$. Namely, we can prove

LEMMA 6E.1. *For each f in $B(S)$, $\mu_\alpha(\cdot, f)$ converges in weak star to a finite signed measure on ∂D as $\alpha \downarrow 0$. The statements* (ii)—(iv) *in Proposition* 4.1 *remain valid even if α or β is zero.*

PROOF. It follows from the boundedness of h and (4.7) that $\{\mu_\alpha(\cdot, f): \alpha > 0\}$ is uniformly bounded. If $f \geqq 0$, then $\mu_\alpha(\cdot, f)$ increases as α decreases. Hence, the first half of the lemma is true. (4.7) with $\alpha = 0$ is obvious. Since $\hat{H}_\alpha$ can be considered as an integral operator, $\mu_\alpha(dx, G_\beta^{\min} f) = \hat{H}_\alpha G_\beta^{\min} f(x) \mu(dx) \to \hat{H}_\alpha G_0^{\min} f(x) \mu(dx)$ in weak star as $\beta \downarrow 0$, Therefore we have (4.7) with $\beta = 0$. (ii), (iii) and (iv) are easily proved.

Put $\mu_0(\cdot, 1) = \mu^\#(\cdot)$. Then μ and $\mu^\#$ are mutually absolutely continuous.

Define

$$\hat{H}^{\#}_{\alpha}f(x)=\frac{\mu_{\alpha}(dx, f)}{\mu^{\#}(dx)}$$

for $\alpha \geqq 0$ and $f \in B(S)$, and

$$K^{\#\alpha}f(x)=\int_{\partial D} g_{\alpha}(x, y)f(y)\mu^{\#}(dy)$$

for $\alpha > 0$ and $f \in B(\partial D)$. Then we have $\hat{H}^{\#}_{\alpha}1(x) \uparrow \hat{H}^{\#}_{0}1(x)=1$ $\mu^{\#}$-almost everywhere as $\alpha \downarrow 0$. And further, there is a positive constant c_{α} such that $\hat{H}^{\#}_{\alpha}1(x) \geqq c_{\alpha}$ $\mu^{\#}$-almost everywhere, since

$$|\mu_0(F, f)| \leqq \|f\|(1+\alpha\|h\|)\mu_{\alpha}(F, 1), \qquad F \in \boldsymbol{B}(\partial D),$$

by (4.7) with $\beta=0$. Let us call the process obtained through time change by $\varphi^{\#}(t)$ in the following lemma the 0-order U$^{\#}$-process induced by X. Then, *Theorems* 2, 3, 4, *Propositions* 4.2 *and* 5.1 *remain true if we replace* $\hat{H}_{\alpha}$ *by* $\hat{H}^{\#}_{\alpha}$, K^{α} *by* $K^{\#\alpha}$, *and* 0-*order* U-*process by* 0-*order* U$^{\#}$-*process. The formulas in Propositions* 4.2 *and* 5.1 *hold for* α *and* β *including zero.*

LEMMA 6E.2. *Lemma* 5.2 (i) *is valid if* μ *is replaced by* $\mu^{\#}$ *in* (5.2). *The additive functional* $\varphi^{\#}(t)$ *thus found is connected with* $\varphi(t)$ *in such a way that*

$$\varphi(t)=\int_0^t H^{\#}_1 1(x_s)d\varphi^{\#}(s).$$

PROOF. Using (1.1) and (4.7) with $\beta=0$, we have

$$\int g_{\alpha}(x, y)\mu^{\#}(dy)=E_x\Big[\int_{\sigma}^{\infty} e^{-\alpha t}(1+\alpha h(x_t))dt\Big].$$

Thus the existence and the uniqueness of $\varphi^{\#}(t)$ is proved similarly to Lemma 5.2 (i). Further we have

$$\int g_{\alpha}(x, y)\mu(dy)=\int g_{\alpha}(x, y)\hat{H}^{\#}_1 1(y)\mu^{\#}(dy)$$
$$=E_x\Big[\int_0^{\infty} e^{-\alpha t}\hat{H}^{\#}_1 1(x_t)d\varphi^{\#}(t)\Big],$$

completing the proof.

F. Example of process with Condition (A).

Let a domain D and an elliptic differential operator A be as at the beginning of this section, and suppose that the zero order term of A vanishes identically. Let m be the Riemannian volume measure connected with the coefficients of the second order terms of A. Consider the reflecting boundary condition $\dfrac{\partial u}{\partial n}=0$. Then, the Markov process X associated satisfies Condition (A). The Green function is constructed by S. Ito [**6**]. In the condition $(A_{6,4})$ we can take $V_0=D\cup\partial D$ and $\alpha_0=\mathrm{Max}(\mathrm{Max}_{x\in S}\,\hat{c}(x), 0)$, where $\hat{c}$ is the coefficient in the zero order term of the formal adjoint of A. In (A_7) we can take $\gamma_0 = m$. Then $k(x)$ tends to infinity as x approaches ∂D.

In this example, the author **[12]** stated that X is transformed to the Markov process $\tilde{X}$ on the boundary with infinitesimal generator $\overline{\frac{\partial}{\partial n} H_0}$ through time change by a certain additive functional $\mathfrak{t}(t, w)$ defined in **[13]**. The connection between $\mathfrak{t}(t, w)$ and the local time on the boundary $\varphi(t, w)$ in this paper is that there is a positive continuous function a on ∂D satisfying (6.6) with $\varphi^{\#}(t)$ replaced by $\mathfrak{t}(t)$. Hence, $\tilde{X}$ and the 0-order U-process $\tilde{X}^{(0)}$ are transformed to each other by time change.

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