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On a fundamental theorem of Weyl-Cartan on G-structures

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§1. Introduction.

Let G be a Lie subgroup of $GL(n;\boldsymbol{R})$. A *G-structure* on an n-dimensional manifold M is by definition a subbundle of the bundle $L(M)$ of linear frames with structure group G, [4]. Consider a connection in a G-structure P on M; it is an affine connection of M. Its torsion tensor field defines at each point x of M a skew-symmetric bilinear mapping $T_x(M)\times T_x(M)\rightarrow T_x(M)$, i.e., an element of $T_x(M)\otimes\wedge^2 T_x^*(M)$, where $T_x^*(M)$ is the dual space of the tangent space $T_x(M)$. We shall say in general that a tensor field T on a manifold M is of *torsion type* if it defines a skew-symmetric bilinear mapping $T_x(M)\times T_x(M)\rightarrow T_x(M)$ at each point x of M. In other words, a tensor field T is of torsion type if and only if it is of type (1, 2) and is skew-symmetric in the lower indices. Before we state the theorem, we introduce notations for Lie algebras. We denote by V the vector space $\boldsymbol{R}^n$ (and, more generally, a vector space of dimension n over a field of characteristic 0 in later sections).

$\mathfrak{gl}(V)=$ the Lie algebra of linear transformations of V;

$\mathfrak{sl}(V)=$ the Lie algebra of linear transformations of V with trace 0;

$\mathfrak{o}(V)=$ the Lie algebra of the orthogonal group of V defined by a non-degenerate symmetric bilinear form B;

$\mathfrak{co}(V)=$ the Lie algebra of the similarity group (or the conformal group) of V defined by B;

$\mathfrak{sp}(V)=$ the Lie algebra of the symplectic group leaving a non-degenerate skew-symmetric bilinear form J invariant;

$\mathfrak{csp}(V)=$ the Lie algebra of the group of linear transformations of V leaving J invariant up to a constant factor.

For a subspace W of V,

$\mathfrak{gl}(V, W)=$ the Lie algebra of linear transformations of V leaving W invariant.

The case $\dim W=1$ is of particular interest to us. Then, with respect to a basis $e_1, \cdots, e_n$ of V such that $e_1\in W$, $\mathfrak{gl}(V, W)$ consists of matrices of the

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form

$$\begin{pmatrix} x & \xi \\ 0 & X \end{pmatrix} \qquad x \in \boldsymbol{R}, \quad X \in \mathfrak{gl}(n-1\,;\,\boldsymbol{R}).$$

For a 1-dimensional subspace W of V and a real number c, we set $\mathfrak{gl}(V, W, c)$ $=$ the subalgebra of $\mathfrak{gl}(V, W)$ consisting of matrices of the form

$$\begin{pmatrix} cTr.X & \xi \\ 0 & X \end{pmatrix} \qquad X \in \mathfrak{gl}(n-1\,;\,\boldsymbol{R}),$$

where $Tr. X$ denotes the trace of X.

We remark that $\dim \mathfrak{gl}(V, W, c) = \dim \mathfrak{gl}(V, W) - 1 = n^2 - n$ and that $\mathfrak{gl}(V, W, -1) = \mathfrak{gl}(V, W) \cap \mathfrak{sl}(V)$.

The purpose of this paper is to prove the following

THEOREM 1. *Let G be a Lie subgroup of $GL(n\,;\,\boldsymbol{R})$ and we fix a G-structure P on an n-dimensional manifold M. Assume that, for an arbitrary tensor field T of torsion type on M, there is a connection in the bundle P with torsion T. Then, for $n \geqq 3$, the Lie algebra $\mathfrak{g}$ of G must be one of the following:*

$$\mathfrak{gl}(V), \quad \mathfrak{sl}(V), \quad \mathfrak{co}(V), \quad \mathfrak{o}(V), \quad \mathfrak{gl}(V, W), \quad \mathfrak{gl}(V, W, c)$$

with $\dim W = 1$.

Conversely, let P be an arbitrary G-structure on M, where $\mathfrak{g}$ is one of the Lie algebras listed above. Then for any tensor field T of torsion type on M, there is a connection is P with torsion T.

COROLLARY. *Let G be a Lie subgroup of $GL(n\,;\,\boldsymbol{R})$ and we fix a G-structure P on an n-dimensional manifold M. Assume that, for an arbitrary tensor field T of torsion type on M, there is a unique connection in P with torsion T. Then, for $n \geqq 3$, the Lie algebra $\mathfrak{g}$ must be $\mathfrak{o}(V)$.*

Conversely, for an arbitrary G-structure P on M with $\mathfrak{g} = \mathfrak{o}(V)$ and for an arbitrary tensor field T of torsion type on M, there is a unique connection in P with torsion T.

The corollary above was obtained by Weyl, (see [**13**], [**12**; § 18]). By a completely different method, E. Cartan generalized the result of Weyl to obtain the result similar to the theorem above. More explicitly, Cartan proved that, under the additional assumption that $G \subset SL(n\,;\,\boldsymbol{R})$, $\mathfrak{g}$ must be $\mathfrak{sl}(V)$, $\mathfrak{o}(V)$, $\mathfrak{sp}(V)$ or $\mathfrak{gl}(V, W) \cap \mathfrak{sl}(V)$ with $\dim W = 1$. But our result shows that $\mathfrak{sp}(V)$ must be deleted. In [**12, 13**] Weyl shows that if G is a group satisfying the assumption in Theorem 1, then $\dim \mathfrak{g} \geqq \frac{1}{2} n(n-1)$. To classify $\mathfrak{g}$, Cartan used only this limitation on the dimension of $\mathfrak{g}$, which is a necessary but not sufficient condition. That is precisely the reason Cartan included erroneously $\mathfrak{sp}(V)$ in his list.

We also prove the following

THEOREM 2. *We fix a Lie subgroup G of $GL(n;\boldsymbol{R})$, $n\geqq 3$, and an n-dimensional manifold M which admits G-structures. Then every G-structure P on M admits a torsionfree connection if and only if the Lie algebra $\mathfrak{g}$ of G is one of the following:*

$$\mathfrak{gl}(V),\quad \mathfrak{sl}(V),\quad \mathfrak{co}(V),\quad \mathfrak{o}(V),\quad \mathfrak{gl}(V,W),\quad \mathfrak{gl}(V,W,c)$$

with $\dim W=1$.

COROLLARY. *Let G and M be as above. Then every G-structure P on M admits a unique torsionfree connection if and only if the Lie algebra $\mathfrak{g}$ of G is $\mathfrak{o}(V)$.*

The corollary above is Klingenberg's version of the theorem of Weyl-Cartan, see [**7**]. In both [**7**] and [**9**, Note 1], G is assumed to be closed. But, as we shall see, it is sufficient to assume that G is a Lie subgroup of $GL(n;\boldsymbol{R})$.

The proofs of both Theorems 1 and 2 reduce to the same algebraic problem. For Theorem 2, this reduction is due to Klingenberg, [**7**]. The algebraic problem is to determine all Lie algebras $\mathfrak{g}$ of linear transformations of an n-dimensional vector space V such that a certain linear mapping $\alpha:\mathfrak{g}\otimes V^*\rightarrow V\otimes\wedge^2 V^*$ is surjective. The condition that α is surjective is equivalent to the vanishing of a certain cohomology group $H^{0,2}(\mathfrak{g})$. Since such a trivial reformulation does not simplify our problem, we shall not talk about the cohomology in this paper.

The proofs of the two corollaries reduce to the classification of $\mathfrak{g}$ such that α is bijective.

In [**7**] Klingenberg considers a complex analogue of Corollary to Theorem 2. His result may be stated as follows.

We fix a real Lie subgroup G of $GL(n;\boldsymbol{C})$ and a $2n$-dimensional almost complex manifold M admitting G-structures. Then every G-structure P on M admits a unique connection with torsion T of pure type if and only if $\mathfrak{g}$ is a real form of $\mathfrak{gl}(n;\boldsymbol{C})$.

We say that T is of pure type if $T^{\alpha}_{\beta\gamma}$ and $T^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}$ are the only nonvanishing components.

It should not be difficult to obtain complex analogues of Theorems 1 and 2. In the complex analogue of Theorem 1, the assumption should be the existence of a connection in P such that the $T^{\alpha}_{\beta\bar{\gamma}}$-components of the torsion T are prescribed. By a reasoning similar to the one in the proof of Theorem 1 and the one in Klingenberg's paper [**7**], it should follow that $\mathfrak{g}$ is irreducible and $\dim\mathfrak{g}\geqq n^2$. This necessary condition on $\mathfrak{g}$ should be strong enough to classify $\mathfrak{g}$.

§ 2. Reduction to an algebraic problem (Theorem 1).

Let G be a Lie subgroup of $GL(n;\boldsymbol{R})$ and P a G-structure on an n-dimensional manifold M. Let $\theta=(\theta^1, \cdots, \theta^n)$ be the restriction to P of the canonical form of the bundle $L(M)$ of linear frames; it is an $\boldsymbol{R}^n$-valued 1-form, (see, for instance, [**9**, p. 118]). Let $\omega=(\omega^i_j)$ be a connection form on P; it is a 1-form with values in the Lie algebra $\mathfrak{g}$ of G. The structure equations of the connection are given by (see, for instance, [**9**, p. 120])

1. $d\theta^i=-\sum_j \omega^i_j\wedge\theta^j+\Theta^i$,

where $\Theta=(\Theta^i)$ is the torsion form, and by

2. $d\omega^i_j=-\sum_k \omega^i_k\wedge\omega^k_j+\Omega^i_j$,

where $\Omega=(\Omega^i_j)$ is the curvature form. In this paper, we are only interested in the first structure equation.

Assuming that there is a connection without torsion, we fix such a connection form $\bar{\omega}=(\bar{\omega}^i_j)$. Then

$$d\theta^i=-\sum_j \bar{\omega}^i_j\wedge\theta^j .$$

We may write

$$\bar{\omega}^i_j-\omega^i_j=\sum_k S^i_{jk}\theta^k .$$

From the first structure equations of ω and $\bar{\omega}$, we obtain

$$\Theta^i=\sum_{j,k}\frac{1}{2}T^i_{jk}\theta^j\wedge\theta^k ,$$

where

$$T^i_{jk}=S^i_{jk}-S^i_{kj} .$$

If we set $V=\boldsymbol{R}^n$ and denote by V^* the dual space of V, then the Lie algebra $\mathfrak{g}$ may be considered as a subspace of $V\otimes V^*$ and, at a fixed point of M, (S^i_{jk}) can be considered as the components of a tensor belonging to $\mathfrak{g}\otimes V^*\subset V\otimes V^*\otimes V^*$. Since (T^i_{jk}) is skew-symmetric in j and k, it can be considered as an element of $V\otimes\wedge^2 V^*$. We define a linear mapping

$$\alpha:\ \mathfrak{g}\otimes V^*\to V\otimes\wedge^2 V^*$$

by

$$(S^i_{jk})\xrightarrow{\alpha}(S^i_{jk}-S^i_{kj}) .$$

Now, the following lemma of Weyl is evident.

LEMMA 2.1. *If G is a Lie subgroup of $GL(n;\boldsymbol{R})$ satisfying the assumption of Theorem* 1, *then the mapping $\alpha:\mathfrak{g}\otimes V^*\to V\otimes\wedge^2 V^*$ is surjective and, in particular,* $\dim\mathfrak{g}\geqq\frac{1}{2}n(n-1)$.

LEMMA 2.2. *If G is a Lie subgroup of $GL(n;\boldsymbol{R})$ satisfying the assumption of Corollary to Theorem* 1, *then the mapping α is bijective and, in particular,*

$\dim \mathfrak{g} = \frac{1}{2} n(n-1)$.

Conversely, we have

LEMMA 2.3. *Let P be a G-structure on M which admits a torsionfree connection. If α is surjective (resp. bijective), then for an arbitrary tensor field T of torsion type on M there exists a (resp. a unique) connection in P with torsion T.*

PROOF. We first prove the lemma locally. In other words, we assume that P admits a cross section $\sigma : M \rightarrow P$. We fix a linear mapping $\beta : V \otimes \wedge^2 V^* \rightarrow \mathfrak{g} \otimes V^*$ such that $\alpha \circ \beta$ is the identity transformation of $V \otimes \wedge^2 V^*$. Let $\bar{\omega} = (\bar{\omega}^i_j)$ be a connection form without torsion. Given a tensor field of torsion type T on M, we take a corresponding 2-form $\left(\sum_{j,k} \frac{1}{2} T^k_{jk} \theta^j \wedge \theta^k\right)$ on P. We define (S^i_{jk}) by $(S^i_{jk}) = \beta(T^i_{jk})$ along the cross section $\sigma(M)$ and then extend it to P in such a way that $\omega^i_j = \bar{\omega}^i_j - \sum_k S^i_{jk} \theta^k$ define a connection in P. (We have to extend (S^i_{jk}) to P in such a way that it is compatible with the action of the group G.) Clearly, (ω^i_j) is a desired connection form. To prove the lemma globally, we cover M with a locally finite open cover and construct a connection over each open set. Using a partition of unity subordinate to the open cover, we patch up the locally defined connections to obtain a globally defined connection with the required property. Q. E. D.

Now we look at the mapping α more closely. The kernel of α consists of tensors $(S^i_{jk}) \in \mathfrak{g} \otimes V^*$ which are symmetric in j and k. We shall denote the space of such tensors, i. e., the kernel of α, by $\mathfrak{g}_1$. Then

LEMMA 2.4. *The mapping $\alpha : \mathfrak{g} \otimes V^* \rightarrow V \otimes \wedge^2 V^*$ is surjective if and only if* $\dim \mathfrak{g}_1 = n \cdot \dim \mathfrak{g} - \frac{1}{2} n^2(n-1)$.

Though trivial, Lemma 2.4 is useful for the following reason. The space $\mathfrak{g}_1$, called "le groupe déduit" by E. Cartan [2], appears in the study of linear Lie algebras. For every irreducible linear Lie algebra $\mathfrak{g}$, we know the space $\mathfrak{g}_1$, (see [8]).

§ 3. Reducible case.

In this section we shall determine the reducible Lie algebras $\mathfrak{g}$ of linear transformations of V such that $\alpha : \mathfrak{g} \otimes V^* \rightarrow V \otimes \wedge^2 V^*$ is surjective. Since we need the result in both the real and the complex cases, we might as well assume that the coefficient field is an arbitrary field $\boldsymbol{F}$ of characteristic 0.

LEMMA 3.1. *Let $\mathfrak{g}$ be a reducible Lie algebra of linear transformations of V such that α is surjective. If W is a proper subspace of V invariant by $\mathfrak{g}$, then* $\dim W = 1$.

PROOF. Let $S\in\mathfrak{g}\otimes V^*$ and $T\in V\otimes\wedge^2 V^*$. Then S can be considered as a bilinear mapping $V\times V\to V$ such that $S(*, x)\in\mathfrak{g}$ for each $x\in V$. Similarly, T can be considered as a skew-symmetric bilinear mapping $V\times V\to V$. Then the mapping α may be expressed as follows:

$$(\alpha S)(x, y)=S(x, y)-S(y, x) \qquad \text{for} \quad x, y\in V\,.$$

For an arbitrarily given $T\in V\otimes\wedge^2 V^*$, choose $S\in\mathfrak{g}\otimes V^*$ such that $T=\alpha S$. Since $S(*, x)\in\mathfrak{g}$ for each $x\in V$ and $\mathfrak{g}$ leaves W invariant, we have $T(W, W)=(\alpha S)(W, W)\subset S(W, W)\subset W$. In order that $T(W, W)\subset W$ holds for all skew-symmetric bilinear mapping T, W must be of dimension at most 1. Q. E. D.

LEMMA 3.2. *Under the same assumption as in Lemma* 3.1, *we have either* $\mathfrak{g}=\mathfrak{gl}(V, W)$ *or* $\mathfrak{g}=\mathfrak{gl}(V, W, c)$, *provided* $\dim V\geqq 3$.

PROOF. Since every element of $\mathfrak{g}$ leaves W invariant, it induces a linear transformation of V/W. We first show that the homomorphism $\mathfrak{g}\to\mathfrak{gl}(V/W)$ is surjective, where $\mathfrak{gl}(V/W)$ denotes the Lie algebra of linear transformations of V/W. Let W' be a subspace of V complementary to W. We identify V/W with W' and $\mathfrak{gl}(V/W)$ with $\mathfrak{gl}(W')$ in a natural manner. We fix a basis w of W. Given an arbitrary linear transformation A of W' or V/W, let T be an element of $V\otimes\wedge^2 V^*$ such that

$$T(x, w)=Ax \qquad x\in W'\,.$$

Choose $S\in\mathfrak{g}\otimes V^*$ such that $T=\alpha S$. Then

$$Ax=T(x, w)=S(x, w)-S(w, x) \qquad \text{for} \quad x\in W'\,.$$

Since $S(*, x)\in\mathfrak{g}$ and $\mathfrak{g}$ leaves W invariant, we have $S(w, x)\in W$. Hence,

$$Ax\equiv S(x, w) \qquad \text{for} \quad x\in W' \quad (\text{mod } W)\,.$$

In other words, the element $S(*, w)\in\mathfrak{g}$ induces the linear transformation A of V/W. Since A is arbitrary, $\mathfrak{g}\to\mathfrak{gl}(V/W)$ is surjective.

Let $\mathfrak{h}$ be the kernel of $\mathfrak{g}\to\mathfrak{gl}(V/W)$. Every element of $\mathfrak{h}$ maps V into W. We may inject $\mathfrak{h}$ into the dual space V^* of V by sending $X\in\mathfrak{h}$ into $\xi\in V^*$ given by

$$X(x)=\langle\xi, x\rangle w \qquad \text{for} \quad x\in V\,.$$

We set

$$U=\{x\in V\,;\; X(x)=0 \text{ for all } X\in\mathfrak{h}\}\,.$$

Since $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, U is invariant by $\mathfrak{g}$. By Lemma 3.1, there are only three cases to consider: (1) $U=0$, (2) $\dim U=1$ and (3) $U=V$.

If $U=0$, then the injection $\mathfrak{h}\to V^*$ is surjective so that $\dim\mathfrak{h}=n$. Hence, $\mathfrak{g}=\mathfrak{gl}(V, W)$.

Assume $\dim U=1$ and $n\geqq 3$. Since the space spanned by U and W is invariant by $\mathfrak{g}$, Lemma 3.1 implies $U=W$. Hence, $\dim\mathfrak{h}=n-1$ and $\mathfrak{g}=\mathfrak{gl}(V, W, c)$ for some $c\in\boldsymbol{F}$.

Assume $U=V$ and $n\geqq 3$. Then $\mathfrak{h}=0$ and $\mathfrak{g}\approx\mathfrak{gl}(V/W)$. Let $\mathfrak{g}'$ and $\mathfrak{c}$ be the semi-simple part and the center of $\mathfrak{g}$ respectively so that

$$\mathfrak{g}=\mathfrak{g}'+\mathfrak{c}, \qquad \mathfrak{g}'\approx\mathfrak{sl}(V/W).$$

Since $\mathfrak{g}'$ is semi-simple, there exists a $\mathfrak{g}'$-invariant subspace W' of V complementary to W. With respect to a basis $e_1, \cdots, e_n$ for V such that $e_1\in W$ and $e_2, \cdots, e_n\in W'$, every element of $\mathfrak{g}'$ is given by a matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}.$$

Let

$$\begin{pmatrix} c & \zeta \\ 0 & I_{n-1} \end{pmatrix}$$

be the matrix representing an element of $\mathfrak{c}$. Since these two matrices must commute and A is arbitrary, we obtain $\zeta=0$ provided $n\geqq 3$. Hence, W' is invariant by $\mathfrak{g}$, in contradiction to Lemma 3.1. Q. E. D.

§4. Irreducible case (algebraically closed field).

Let $\alpha: \mathfrak{g}\otimes V^*\to V\otimes\wedge^2 V^*$ be as before. Throughout this section we shall assume:

(1) α is surjective;

(2) $\mathfrak{g}$ is irreducible;

(3) The coefficient field is algebraically closed of characteristic 0. Then $\mathfrak{g}$ is a direct sum of a semi-simple ideal $\mathfrak{g}'$ and the center $\mathfrak{c}$ of at most 1-dimension which consists of scalar multiples of the identity transformation of V, (see, for instance, [**1**, p. 79]). Let

$$\mathfrak{g}'=\mathfrak{g}_1+\cdots+\mathfrak{g}_h,$$

where $\mathfrak{g}_1, \cdots . \mathfrak{g}_h$ are the simple ideals of $\mathfrak{g}'$.

LEMMA 4.1. *Either $\mathfrak{g}'$ is simple or $\mathfrak{g}'=\mathfrak{o}(V)$ with* $\dim V=4$, *where $\mathfrak{o}(V)$ denotes the Lie algebra of the orthogonal group with respect to a non-degenerate symmetric bilinear form.*

PROOF. We have (see, for instance, [**11**, p. 66])

$$V=V_1\otimes\cdots\otimes V_h$$

and each $\mathfrak{g}_j$, $j=1, \cdots, h$, acts irreducibly on V_j. Set

$$n_j=\dim V_j\geqq 2, \qquad r_j=\dim\mathfrak{g}_j\geqq 3.$$

Then

$$\dim\mathfrak{g}\leqq 1+\dim\mathfrak{g}'=1+r_1+\cdots+r_h.$$

By Lemma 2.1, we have

$$\frac{1}{2}n_1\cdots n_h(n_1\cdots n_h-1)\leqq 1+r_1+\cdots r_h.$$

On the other hand, $\mathfrak{g}_j \subset \mathfrak{sl}(V_j)$ and hence

$$r_j \leqq n_j^2 - 1 .$$

Hence we have

$$\frac{1}{2} n_1 \cdots n_h (n_1 \cdots n_h - 1) \leqq 1 + (n_1^2 - 1) + \cdots + (n_h^2 - 1) .$$

We may assume that $n_1 \geqq n_j \geqq 2$. Then

$$2^{h-2} n_1 (2^{h-1} n_1 - 1) \leqq h n_1^2 - h + 1 \leqq h n_1^2 .$$

It follows that if $h \geqq 3$, then

$$n_1 < 2^{h-2}/(2^{2h-3} - h) < 1 ,$$

which is a contradiction.

Consider the case $h = 2$. Then

$$\frac{1}{2} n_1 n_2 (n_1 n_2 - 1) < n_1^2 + n_2^2 .$$

If $n_1 \geqq n_2 \geqq 3$, then

$$\frac{1}{2} 3 n_1 (3 n_1 - 1) < 2 n_1^2 .$$

If follows that

$$5 n_1^2 - 3 n_1 < 0 ,$$

which is a contradiction. If $n_1 \geqq n_2 = 2$, then

$$n_1 (2 n_1 - 1) \leqq n_1^2 + 4 .$$

This implies $n_1 = 2$. We have shown that either $h = 1$ or $h = n_1 = n_2 = 2$. In the latter case, we have $r_1 \leqq 2^2 - 1$ and $r_2 \leqq 2^2 - 1$. On the other hand, $r_1 \geqq 3$ and $r_2 \geqq 3$ for $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are simple. Hence, $r_1 = r_2 = 3$. This shows that either $\mathfrak{g}'$ is simple or $\mathfrak{g}' = \mathfrak{sl}(V_1) \otimes \mathfrak{sl}(V_2)$, where $\dim V_1 = \dim V_2 = 2$. Since $\mathfrak{sl}(V_1) \otimes \mathfrak{sl}(V_2) = \mathfrak{o}(V_1 \otimes V_2)$ when $\dim V_1 = \dim V_2 = 2$, the proof of the lemma is now completed. Q. E. D.

§ 5. The case $\mathfrak{g}'$ is simple.

Throughout this section we shall assume:

(1) α is surjective;

(2) $\mathfrak{g}$ is irreducible and its semi-simple part $\mathfrak{g}'$ is simple;

(3) The coefficient field is algebraically closed of characteristic 0.

In the first half of this section, we shall use instead of (1) the following weaker assumption:

(1′) $\dim \mathfrak{g}' \geqq \frac{1}{2} n(n-1) - 1$.

We recall the formula of Weyl which expresses the degree of an irreducible representation of a Lie algebra in terms of its highest weight, (see, for instance, [**6**, **10**]). If $\mathfrak{g}'$ is an irreducible simple Lie algebra of linear transformations of V with highest weight λ, then

$$\dim V = \prod_{\alpha>0} ((\lambda+\delta, \alpha)/(\delta, \alpha)),$$

where

$$\delta = \frac{1}{2} \sum_{\alpha>0} \alpha .$$

($\prod_{\alpha>0}$ and $\sum_{\alpha>0}$ denote the product and the sum over all positive roots α.) We set

$$d(\lambda) = \prod_{\alpha>0} ((\lambda+\delta, \alpha)/(\delta, \alpha))$$

so that $\dim V = d(\lambda)$. The following lemma is evident.

LEMMA 5.1. *If λ' and λ'' are dominant integral forms, then*

$$d(\lambda'+\lambda'') > d(\lambda'), d(\lambda'') .$$

Using this lemma we can eliminate most of irreducible representations quite efficiently[1)].

(i) $\mathfrak{g}' = A_l$, $l \geqq 1$.

With respect to the following numbering of the simple roots

the degrees of the fundamental representations $\lambda_1, \cdots, \lambda_l$ are given by

$$d(\lambda_j) = \binom{l+1}{j} \qquad j = 1, \cdots, l .$$

LEMMA 5.2. *If λ is a dominant integral form of A_l such that*

$$\dim A_l + 1 > \frac{1}{2} d(\lambda)(d(\lambda)-1),$$

then λ must be one of the following:

(1) *For $l \geqq 4$, λ_1 or λ_l;*

(2) *For $l = 3$, λ_1, λ_2 or λ_3;*

(3) *For $l = 2$, λ_1 or λ_2;*

(4) *For $l = 1$, λ_1 or $2\lambda_1$.*

PROOF. (1). Assume $l \geqq 4$. First we prove

$$\dim A_l + 1 < \frac{1}{2} d(\lambda_j)(d(\lambda_j)-1) \qquad \text{for } j = 2, \cdots, l-1 .$$

Since $d(\lambda_1) < d(\lambda_2) < \cdots > d(\lambda_{l-1}) > d(\lambda_l)$ and $d(\lambda_1) = d(\lambda_l)$, $d(\lambda_2) = d(\lambda_{l-1})$, it suf-

1) In the following, we denote an irreducible representation by its highest weight. For the degrees of the fundamental representations of the simple Lie algebras, see for instance, [**5**, p. 398].

fices to prove the inequality for $j=2$. For $j=2$, $d(\lambda_2)=\frac{1}{2}l(l+1)$. Hence,

$$\begin{aligned}\frac{1}{2}d(\lambda_2)(d(\lambda_2)-1)&=\left(\frac{1}{2}\right)^3 l(l+1)(l(l+1)-2)\\ &\geqq\left(\frac{1}{2}\right)^3 l(l+1)(4(4+1)-2)\\ &>l^2+2l+1=\dim A_l+1\,.\end{aligned}$$

In view of Lemma 5.1, we have now only to consider the linear combinations of λ_1 and λ_l. Again, by Lemma 5.1, it suffices to eliminate $2\lambda_1$, $2\lambda_l$ and $\lambda_1+\lambda_l$. Since λ_1 is the natural representation of A_l on the $(l+1)$-dimensional vector space V and $2\lambda_1$ is the representation of A_l on the space $S^2(V)$ of symmetric tensors of type (2, 0), we have

$$d(2\lambda_1)=\dim S^2(V)=\frac{1}{2}(l+1)(l+2)\,.$$

A simple calculation shows that the inequality in our lemma is not satisfied. Since λ_l is the dual representation of λ_1 and $2\lambda_l$ is the representation of A_l on $S^2(V^*)$, $2\lambda_l$ may be eliminated in the same way as $2\lambda_1$. Finally, since $\lambda_1+\lambda_l$ is the adjoint representation of A_l, we have $d(\lambda_1+\lambda_l)=l^2+2l$. Hence, $\lambda_1+\lambda_l$ does not satisfy the inequality of our lemma.

(2) Assume $l=3$. As above, we have

$$d(2\lambda_1)=d(2\lambda_3)=10\,,\qquad d(\lambda_1+\lambda_3)=15\,.$$

Using the formula of Weyl, we have also

$$d(\lambda_2)=6\,,\qquad d(\lambda_1+\lambda_2)=20\,,\qquad d(\lambda_2+\lambda_3)=20\,,\qquad d(2\lambda_2)=20\,.$$

By Lemma 5.1, we see that λ must be λ_1, λ_2 or λ_3.

(3) Assume $l=2$. As in the proof of (1), we have

$$d(2\lambda_1)=d(2\lambda_2)=6\,,\qquad d(\lambda_1+\lambda_2)=8\,.$$

Hence, λ must be either λ_1 or λ_2.

(4) Assume $l=1$. By the formula of Weyl, we have $d(2\lambda_1)=3$ and $d(3\lambda_1)=4$. Hence, λ must be either λ_1 or $2\lambda_1$. Q. E. D.

(ii) $\mathfrak{g}'=B_l$, $l\geqq 2$.

With respect to the following numbering of the simple roots

the degrees of the fundamental representations $\lambda_1,\cdots,\lambda_l$ are given by

$$d(\lambda_j)=\binom{2l+1}{j}\qquad\text{for }\ j=1,\cdots,l-1\,,$$

$$d(\lambda_l)=2^l\,.$$

LEMMA 5.3. *If λ is a dominant integral form of B_l such that*

$$\dim B_l+1 \geqq \frac{1}{2} d(\lambda)(d(\lambda)-1)\,,$$

then λ must be one of the following:

(1) *For $l \geqq 3$, λ_1;*

(2) *For $l=2$, λ_1 or λ_2.*

PROOF. (1) Assume $l \geqq 3$. Since $d(\lambda_2)=2l^2+l$, we have

$$\begin{aligned}\frac{1}{2} d(\lambda_2)(d(\lambda_2)-1) &= \frac{1}{2}(2l^2+l)(2l^2+l-1)\\ &> \frac{1}{2}(2l^2+l)(2^3+2-1)\\ &> 2l^2+l+1=\dim B_l+1\,.\end{aligned}$$

Since $d(\lambda_2)<d(\lambda_3)< \cdots <d(\lambda_{l-1})$, we have

$$\frac{1}{2} d(\lambda_j)(d(\lambda_j)-1) > \dim B_l+1 \qquad \text{for} \quad j=2, \cdots, l-1\,.$$

We have

$$\frac{1}{2} d(\lambda_l)(d(\lambda_l)-1) = 2^{l-1}(2^l-1) > 2l^2+l+1 = \dim B_l+1\,.$$

We shall show that $2\lambda_1$ does not satisfy the inequality of our lemma. We first observe that

$$\begin{aligned}(\lambda_1+\delta, \alpha_1)/(\delta, \alpha_1) &= 1+(\lambda_1, \alpha_1)/(\delta, \alpha_1)=2\,,\\ (2\lambda_1+\delta, \alpha_1)/(\delta, \alpha_1) &= 1+2(\lambda_1, \alpha_1)/(\delta, \alpha_1)=3\,.\end{aligned}$$

From the formula of Weyl, it follows that

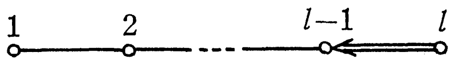
$$d(2\lambda_1) \geqq \frac{3}{2} d(\lambda_1)\,. \qquad (\text{In fact, } d(2\lambda_1)=2l^2+3l.)$$

Since $d(\lambda_1)=2l+1$, we see easily that $2\lambda_1$ does not satisfy the inequality of our lemma.

(2) Assume $l=2$. As in the proof of (1), we can eliminate $2\lambda_1$ easily. Since $2\lambda_2$ is the adjoint representation of B_l we have $d(2\lambda_2)=10$, thus eliminating $2\lambda_2$. Using the formula of Weyl, we obtain also $d(\lambda_1+\lambda_2)=16$, thus eliminating $\lambda_1+\lambda_2$. Q. E. D.

(iii) $\mathfrak{g}'=C_l \geqq 3$.

With respect to the following numbering of the simple roots



the degrees of the fundamental representations $\lambda_1, \cdots, \lambda_l$ are given by

$$d(\lambda_j)=\binom{2l}{j}-\binom{2l}{j-2}=\frac{l+1-j}{l+1}\binom{2l+2}{j} \qquad \text{for} \quad j=1, \cdots, l\,.$$

LEMMA 5.4. *If λ is a dominant integral form of C_l, $l>3$, such that*

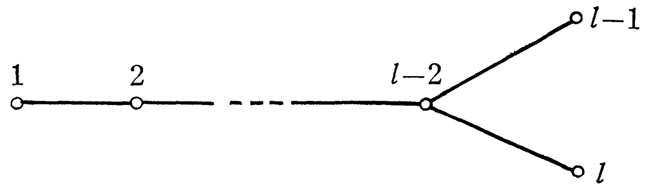
$$\dim C_l+1 \geqq \frac{1}{2} d(\lambda)(d(\lambda)-1),$$

then $\lambda=\lambda_1$.

PROOF. Since $d(\lambda_2)=(l-1)(2l+1)$ and $\dim C_l=2l^2+l$, we see that λ_2 does not satisfy the inequality of our lemma. It is also easy to verify that $d(\lambda_2)<d(\lambda_j)$ for $j=3, \cdots, l$. Consequently, $\lambda_3, \cdots, \lambda_l$ do not satisfy the inequality of our lemma. Since $2\lambda_1$ is the adjoint representation of C_l, we have $d(2\lambda_1)=2l^2+l$, thus eliminating $2\lambda_1$. Q. E. D.

(iv) $\mathfrak{g}'=D_l$, $l \geqq 4$.

With respect to the following numbering of the simple roots



the degrees of the fundamental representations $\lambda_1, \cdots, \lambda_l$ are given by

$$d(\lambda_j)=\binom{2l}{j} \qquad \text{for } j=1, \cdots, l-2,$$

$$d(\lambda_{l-1})=d(\lambda_l)=2^{l-1}.$$

LEMMA 5.5. *If λ is a dominant integral form of D_l, $l \geqq 4$, such that*

$$\dim D_l+1 \geqq \frac{1}{2} d(\lambda)(d(\lambda)-1),$$

then λ is one of the following:

(1) *For $l \geqq 5$, λ_1;*

(2) *For $l=4$, λ_1, λ_3 or λ_4.*

PROOF. (1) Assume $l \geqq 5$. Since $d(\lambda_2)=l(2l-1)$ and $\dim D_l=2l^2-l$, we see that λ_2 does not satisfy the inequality of our lemma. Since $d(\lambda_2)<d(\lambda_j)$ for $j=3, \cdots, l-2$, it follows that $\lambda_3, \cdots, \lambda_{l-2}$ do not satisfy the inequality of our lemma. Since $d(\lambda_{l-1})=d(\lambda_l)=2^{l-1}$, it follows that, for $l \geqq 5$, λ_{l-1} and λ_l do not satisfy the inequality of our lemma. The argument used in the proof of (2) of Lemma 5.3 eliminates also $2\lambda_1$. (In fact, $d(2\lambda_1)=(l+1)(2l-1)$.)

(2) Assume $l=4$. Then $d(\lambda_1)=d(\lambda_3)=d(\lambda_4)=8$ and $d(\lambda_2)=28$. Again, the argument used in the proof of (2) of Lemma 5.3 eliminates $2\lambda_1$, $2\lambda_3$ and $2\lambda_4$. Using the formula of Weyl, we obtain $d(\lambda_1+\lambda_3)=d(\lambda_3+\lambda_4)=d(\lambda_4+\lambda_1)=56$, thus eliminating $\lambda_1+\lambda_3$, $\lambda_3+\lambda_4$ and $\lambda_4+\lambda_1$. Q. E. D.

(v) $\mathfrak{g}'=E_6, E_7, E_8, F_4, G_2$.

If we denote by r the dimension of $\mathfrak{g}'$ and by d the minimum degree of the fundamental representations of $\mathfrak{g}'$, then

E_6: $r=78$, $d=27$;
E_7: $r=133$, $d=56$;
E_8: $r=248$, $d=248$;
F_4: $r=52$, $d=26$;
G_2: $r=14$, $d=7$.

We obtain immediately the following

LEMMA 5.6. *If $\mathfrak{g}'$ is an exceptional simple Lie algebra, there is no dominant integral form λ such that*

$$\dim \mathfrak{g}'+1 \geqq \frac{1}{2} d(\lambda)(d(\lambda)-1).$$

Finally, we shall prove

LEMMA 5.7. *Under the assumptions* (1), (2) *and* (3) *stated in the beginning of this section, $\mathfrak{g}$ must be one of the following:*

$$\mathfrak{gl}(V), \quad \mathfrak{sl}(V), \quad \mathfrak{co}(V) \quad or \quad \mathfrak{o}(V).$$

PROOF. We first examine the case $\mathfrak{g}'=A_l$ using Lemma 5.2. If the representation is λ_1 or λ_l, then $\mathfrak{g}=\mathfrak{gl}(V)$ or $\mathfrak{g}=\mathfrak{sl}(V)$ according as $\mathfrak{g}$ has a center or not. If $\mathfrak{g}'=A_3$ with representation λ_2, then $\mathfrak{g}$ is $\mathfrak{co}(V)$ or $\mathfrak{o}(V)$ with $\dim V=6$ according as $\mathfrak{g}$ has a center or not. If $\mathfrak{g}'=A_1$ with representation $2\lambda_1$, then $\mathfrak{g}$ is $\mathfrak{co}(V)$ or $\mathfrak{o}(V)$ with $\dim V=3$ according as $\mathfrak{g}$ has a center or not.

We shall examine the case $\mathfrak{g}'=B_l$ using Lemma 5.3. If the representation is λ_1, then $\mathfrak{g}$ is $\mathfrak{co}(V)$ or $\mathfrak{o}(V)$ according as $\mathfrak{g}$ has a center or not. If $\mathfrak{g}'=B_2$ with representation λ_2, then $\mathfrak{g}'=C_2$ with representation λ_1. This case is included in the next case.

We shall examine the case $\mathfrak{g}'=C_l$ using Lemma 5.4. If the representation is λ_1, then $\mathfrak{g}$ is $\mathfrak{csp}(V)$ or $\mathfrak{sp}(V)$ according as $\mathfrak{g}$ has a center or not. Using Lemma 2.4, we shall completely eliminate this case. Whether $\mathfrak{g}=\mathfrak{csp}(V)$ or $\mathfrak{g}=\mathfrak{sp}(V)$, the kernel of α, $\mathfrak{g}_1$, is known to be isomorphic with the space of symmetric trilinear mappings $V\times V\times V\to \boldsymbol{F}$. Hence, $\dim \mathfrak{g}_1=n(n+1)(n+2)/6$, where $n=\dim V$. Since $\dim \mathfrak{csp}(V)=\frac{1}{2}n(n+1)+1$ and $\dim \mathfrak{sp}(V)=\frac{1}{2}n(n+1)$, the equality in Lemma 2.4 does not hold.

We shall finally examine the case $\mathfrak{g}'=D_l$ using Lemma 5.5. If the representation is λ_1, then $\mathfrak{g}$ is $\mathfrak{co}(V)$ or $\mathfrak{o}(V)$ according as $\mathfrak{g}$ has a center or not. For D_4, the three representations λ_1, λ_3 and λ_4 give the same Lie algebra $\mathfrak{o}(V)$ with $\dim V=8$. Hence, $\mathfrak{g}$ is $\mathfrak{co}(V)$ or $\mathfrak{o}(V)$ according as $\mathfrak{g}$ has a center or not.

Q. E. D.

§ 6. Irreducible case (arbitrary field).

Throughout this section we shall assume:

(1) $\alpha : \mathfrak{g}\otimes V^* \to V\otimes \wedge^2 V$ is surjective;

(2) $\mathfrak{g}$ is irreducible;

(3) The coefficient field $\boldsymbol{F}$ is of characteristic 0.

We shall prove

LEMMA 6.1. *If* $\dim V \geqq 3$, *then* $\mathfrak{g}$ *is one of the following:*

$$\mathfrak{gl}(V),\ \mathfrak{sl}(V),\ \mathfrak{co}(V) \ \text{or}\ \mathfrak{o}(V).$$

PROOF. Let $\bar{\boldsymbol{F}}$ be the algebraic closure of $\boldsymbol{F}$ and we set $\bar{V}=V\otimes\bar{\boldsymbol{F}}$ and $\bar{\mathfrak{g}}=\mathfrak{g}\otimes\bar{\boldsymbol{F}}$. Then the mapping $\alpha:\bar{\mathfrak{g}}\otimes\bar{V}\rightarrow\bar{V}\otimes\wedge^2\bar{V}^*$ is surjective. We shall show that $\bar{\mathfrak{g}}$ is irreducible. If $\bar{V}=W_1+\cdots+W_k$ is the decomposition into the $\bar{\mathfrak{g}}$-irreducible subspaces, then $\dim W_1=\cdots=\dim W_k$. Since $\dim\bar{V}\geqq 3$, this contradicts Lemma 3.1. By Lemmas 4.1 and 5.7, $\bar{\mathfrak{g}}$ must be one of the following:

$$\mathfrak{gl}(\bar{V}),\ \mathfrak{sl}(\bar{V}),\ \mathfrak{co}(\bar{V}) \ \text{or}\ \mathfrak{o}(\bar{V}).$$

Considering the dimensions of $\mathfrak{g}$ and $\bar{\mathfrak{g}}$, we see that if $\bar{\mathfrak{g}}=\mathfrak{gl}(\bar{V})$ (resp. $\bar{\mathfrak{g}}=\mathfrak{sl}(\bar{V})$), then $\mathfrak{g}=\mathfrak{gl}(V)$ (resp. $\mathfrak{g}=\mathfrak{sl}(V)$).

We shall show that if $\bar{\mathfrak{g}}=\mathfrak{o}(\bar{V})$, then $\mathfrak{g}=\mathfrak{o}(V)$. Let B be a non-degenerate symmetric bilinear form on $\bar{V}$ which defines $\mathfrak{o}(\bar{V})$. Taking a basis for V, we express B as a non-degenerate symmetric matrix with entries from $\bar{\boldsymbol{F}}$. Multiplying by a non-zero element of $\bar{\boldsymbol{F}}$ if necessary, we may assume that at least one of the entries in the matrix B is 1. Let $\boldsymbol{F}'$ be a finite Galois extension of $\boldsymbol{F}$ containing all entries of the matrix B. Let Γ be the Galois group of $\boldsymbol{F}'$ and set

$$B'=\sum_{\gamma\in\Gamma}\gamma(B).$$

Then B' is a symmetric bilinear form defined on V and is invariant by $\mathfrak{g}$. Since B has 1 as an entry, B' has m as an entry, where m is the order of the group Γ. In particular, B' is nonzero. Since $\mathfrak{g}$ is irreducible and leaves B' invariant, B' is non-degenerate. Let $\mathfrak{o}(V)$ be the Lie algebra of the orthogonal group defined by B'. Then $\mathfrak{g}\subset\mathfrak{o}(V)$. On the other hand, $\dim\mathfrak{g}=\dim\bar{\mathfrak{g}}=\dim\mathfrak{o}(\bar{V})=\dim\mathfrak{o}(V)$. Hence, $\mathfrak{g}=\mathfrak{o}(V)$.

It follows now trivially that if $\bar{\mathfrak{g}}=\mathfrak{co}(\bar{V})$, then $\mathfrak{g}=\mathfrak{co}(V)$. Q. E. D.

§7. The final step of the proof.

By Lemmas 3.2 and 6.1 we know that if G is a group satisfying the assumption in Theorem 1, that is, if $\alpha:\mathfrak{g}\otimes V^*\rightarrow V\otimes\wedge^2 V^*$ is surjective, then $\mathfrak{g}$ must be one of the following:

$$\mathfrak{gl}(V),\ \mathfrak{sl}(V),\ \mathfrak{co}(V),\ \mathfrak{o}(V),\ \mathfrak{gl}(V,W) \ \text{or}\ \mathfrak{gl}(V,W,c)$$

where $\dim W=1$. We shall show that, conversely, if $\mathfrak{g}$ is one of the Lie algebras listed above, then α is surjective. In view of Lemma 2.4, it suffices to prove the following lemma.

LEMMA 7.1. *Let* $\mathfrak{g}_1$ *be the kernel of* α *as in* § 2. *Then we have*

(1) *If* $\mathfrak{g}=\mathfrak{gl}(V)$, *then* $\dim \mathfrak{g}_1=\frac{1}{2}n^2(n+1)$;

(2) *If* $\mathfrak{g}=\mathfrak{sl}(V)$, *then* $\dim \mathfrak{g}_1=\frac{1}{2}n^2(n+1)-n$;

(3) *If* $\mathfrak{g}=\mathfrak{co}(V)$, *then* $\dim \mathfrak{g}_1=n$;

(4) *If* $\mathfrak{g}=\mathfrak{o}(V)$, *then* $\dim \mathfrak{g}_1=0$;

(5) *If* $\mathfrak{g}=\mathfrak{gl}(V, W)$ *with* $\dim W=1$, *then* $\dim \mathfrak{g}_1=\frac{1}{2}n(n^2-n+2)$;

(6) *If* $\mathfrak{g}=\mathfrak{gl}(V, W, c)$ *with* $\dim W=1$, *then* $\dim \mathfrak{g}_1=\frac{1}{2}n^2(n-1)$.

PROOF. As we have already stated in § 2, (1), (2), (3) and (4) are known, (see, for instance, [8]). We shall prove (5). We denote by $\mathfrak{gl}(V)_1$ (resp. $\mathfrak{gl}(V, W)_1$) $\mathfrak{g}_1$ for $\mathfrak{g}=\mathfrak{gl}(V)$ (resp. $\mathfrak{g}=\mathfrak{gl}(V, W)$). Let $S=(S^i_{jk})$ be an element of $\mathfrak{gl}(V)_1$ with respect to a basis $e_1, \cdots, e_n$ such that $e_1 \in W$. Then S belongs to $\mathfrak{gl}(V, W)_1$ if and only if

$$S^i_{1k}=0 \quad \text{for} \quad i=2, \cdots, n \text{ and } k=1, \cdots, n.$$

These $n(n-1)$ conditions are independent. Hence, $\dim \mathfrak{gl}(V, W)_1=\dim \mathfrak{gl}(V)_1 -n(n-1)=\frac{1}{2}n(n^2-n+2)$. Similarly, $\mathfrak{gl}(V, W, c)_1$ is defined by

$$S^i_{1k}=0 \quad \text{and} \quad S^1_{1k}=c\sum_{i=2}^{n} S^i_{ik}$$
$$\text{for} \quad i=2, \cdots, n \text{ and } k=1, \cdots, n.$$

Hence, $\dim \mathfrak{gl}(V, W, c)=\dim \mathfrak{gl}(V, W)_1-n^2=\frac{1}{2}n^2(n-1)$. Q. E. D.

This completes the proof of the first half of Theorem. We shall now prove the second half of Theorem. In view of Lemma 2.3, it is sufficient to prove the following lemma.

LEMMA 7.2. *Every G-structure admits a torsionfree connection if* $\mathfrak{g}$ *is one of the following:*

$$\mathfrak{gl}(V), \quad \mathfrak{sl}(V), \quad \mathfrak{co}(V), \quad \mathfrak{o}(V), \quad \mathfrak{gl}(V, W) \quad or \quad \mathfrak{gl}(V, W, c)$$

where $\dim W=1$.

PROOF. If $\mathfrak{g}=\mathfrak{o}(V)$, then there is a unique torsionfree connection (i. e., the so-called Levi-Civita connection) in every G-structure. Hence, if $\mathfrak{g}=\mathfrak{o}(V)$, then every G-structure admits a torsionfree connection. This takes care of $\mathfrak{gl}(V)$, $\mathfrak{sl}(V)$, and $\mathfrak{co}(V)$. Our lemma for $\mathfrak{g}=\mathfrak{gl}(V, W)$ or $\mathfrak{g}=\mathfrak{gl}(V, W, c)$ follows from the following two lemmas.

LEMMA 7.3. *If* $\mathfrak{g}=\mathfrak{gl}(V, W)$ *or* $\mathfrak{g}=\mathfrak{gl}(V, W, c)$ *with* $\dim W=1$, *then every G-structure is integrable.*

LEMMA 7.4. *Every integrable G-structure admits a torsionfree connection.*

We recall that a G-structure P on M is said to be *integrable* if every point of M has a coordinate neighborhood U with local coordinate system $x^1, \cdots, x^n$ such that the cross section $(\partial/\partial x^1, \cdots, \partial/\partial x^n)$ of $L(M)$ over U is really a cross section of P over U [2].

We first prove Lemma 7.4. We cover M with a locally finite family of coordinate neighborhoods U with local coordinate system $x^1, \cdots, x^n$ such that $(\partial/\partial x^1, \cdots, \partial/\partial x^n)$ is a cross section of the given G-structure P. Over each U, we take a flat connection in P. Using a partition of unity subordinate to $\{U\}$ we patch up these locally defined flat connections to obtain a globally defined torsionfree connection.

We shall now prove Lemma 7.3[2]. Assume $\mathfrak{g} = \mathfrak{gl}(V, W)$ with $\dim W = 1$. Then a G-structure on M is a 1-dimensional distribution (which is always involutive) and hence integrable. Assume $\mathfrak{g} = \mathfrak{gl}(V, W, c)$ with $\dim W = 1$. Let P be a G-structure on M. From what we have just proved for $\mathfrak{gl}(V, W)$, it follows that every point of M has a coordinate neighborhood U with local coordinate system $x^1, \cdots, x^n$ such that $(f(\partial/\partial x^1), \partial/\partial x^2, \cdots, \partial/\partial x^n)$ is a cross section of P, where f is a function on U which is different from zero everywhere. If we set

$$y = \int \frac{dx^1}{f},$$

then $y, x^2, \cdots, x^n$ is a desired local coordinate system. Q.E.D.

§ 8. Proof of Theorem 2.

Let G be a Lie subgroup of $GL(n; \boldsymbol{R})$ and M an n-dimensional manifold admitting G-structures. Assuming that every G-structure on M admits a torsionfree connection, we shall show that the linear mapping $\alpha : \mathfrak{g} \otimes V^* \to V \otimes \wedge^2 V^*$ defined in § 2 is surjective.

As in § 2, let $\theta = (\theta^1, \cdots, \theta^n)$ be the canonical form on the bundle $L(M)$ of linear frames over M. Let U be a neighborhood of a point o of M and $\sigma : U \to L(M)$ a local cross section. We set

$$\varphi^i = \sigma^*(\theta^i) \qquad i = 1, \cdots, n.$$

Then $\varphi^1, \cdots, \varphi^n$ are linearly independent 1-forms on U. We define (L^i_{jk}) by

$$d\varphi^i = \sum_{j,k} \frac{1}{2} L^i_{jk} \varphi^j \wedge \varphi^k \qquad i = 1, \cdots, n,$$

where

$$L^i_{jk} = -L^i_{kj}.$$

2) Let G and G', $G' \subset G$, be two Lie subgroups of $GL(n; \boldsymbol{R})$ with the same Lie algebra $\mathfrak{g}$. Let P' be a G'-structure and P the G-structure containing P'. Then P is integrable if and only if P' is integrable.

LEMMA 8.1. *Given a set of numbers* (T^i_{jk}) *with* $T^i_{jk}=-T^i_{kj}$, *there exist a* G*-structure* P *on* M *and a local cross section* $\sigma: U\rightarrow P$ *such that*

$$T^i_{jk}=L^i_{jk}(o)\,.$$

PROOF. Let $\bar{P}$ be an arbitrary G-structure on M and $\bar{\sigma}: U\rightarrow\bar{P}$ a local cross section. We set

$$\bar{\varphi}^i=\bar{\sigma}^*(\theta^i)$$

$$d\bar{\varphi}^i=\sum_{j,k}\frac{1}{2}\bar{L}^i_{jk}\bar{\varphi}^j\wedge\bar{\varphi}^k\,,\qquad \bar{L}^i_{jk}=-\bar{L}^i_{kj}\,.$$

Let $a=(a^i_j)$ be a mapping of M into $GL(n;\boldsymbol{R})$ such that

$$a^i_j(o)=\delta^i_j\,,$$

$$a^i_j(x)=\delta^i_j\qquad\text{for}\quad x\in M-U\,,$$

and otherwise arbitrary for the moment. We define a local cross section $\sigma: U\rightarrow L(M)$ by

$$\sigma(x)=\bar{\sigma}(x)a(x)^{-1}\qquad\text{for}\quad x\in U$$

so that

$$\varphi^i=\sum_j a^i_j\bar{\varphi}^j\,.$$

Since $a^i_j(o)=\delta^i_j$, we obtain the equality $T^i_{jk}=L^i_{jk}(o)$ by choosing a^i_j in such a way that

$$\sum_{j,k}\frac{1}{2}(T^i_{jk}-\bar{L}^i_{jk})\bar{\varphi}^j\wedge\bar{\varphi}^k=\sum_j da^i_j\wedge\bar{\varphi}^j\qquad\text{at}\quad o,$$

which is clearly possible. To complete the proof, we define P by

$$P|M-U=\bar{P}|M-U\,,$$

$$P|U=\{\sigma(x)s\,;\;x\in U,\;s\in G\}\,.$$

Q. E. D.

Let P and σ be as in Lemma 8.1. Let $\omega=(\omega^i_j)$ be a torsionfree connection in P and we define (γ^i_{jk}) by

$$\sigma^*(\omega^i_j)=\sum_k\gamma^i_{jk}\varphi^k\,.$$

Then the first structure equation yields

$$L^i_{jk}=\gamma^i_{jk}-\gamma^i_{kj}\,.$$

Since $(L^i_{jk}(o))$ is an arbitrary element of $V\otimes\wedge^2V^*$ by Lemma 8.1 and since $(\gamma^i_{jk}(o))$ is an element of $\mathfrak{g}\otimes V^*$, it follows that the mapping $\alpha:\mathfrak{g}\otimes V^*\rightarrow V\otimes\wedge^2V^*$ is surjective.

Now, Theorem 2 follows from Lemmas 3.2, 6.1 and 7.2.

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