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On some properties of a proximity

By Eiichi HAYASHI

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§1. Introduction.

Efremovich [1] defined a relation δ on a set, called a *proximity*. For a pair of subsets A and B of a point set R we usually write $A\delta B$ if A and B are proximate, otherwise $A\bar{\delta}B$. Throughout this paper we shall use the notations $(A, B)\in\delta$ and $(A, B)\notin\delta$ instead of $A\delta B$ and $A\bar{\delta}B$ respectively. (See Pervin [2].)

Efremovich required that the relation δ should satisfy the following four axioms:

Axiom 0. (Symmetry) $(A, B)\in\delta$ if and only if $(B, A)\in\delta$.

Axiom 1. Both $(A, C)\notin\delta$ and $(B, C)\notin\delta$ if and only if $(A\cup B, C)\notin\delta$.

Axiom 2. For arbitrary two points $a, b\in R$, $(\{a\}, \{b\})\in\delta$ if and only if $a=b$.

Axiom 3. (Separation) If $(A, B)\notin\delta$ then there are disjoint subsets U and V of R such that $(A, R-U)\notin\delta$ and $(B, R-V)\notin\delta$.

Efremovich [1, p. 196] showed that every proximity on a set R yields a completely regular space if one defines the topology of R as follows: a subset U of R is a neighborhood of $A\subset R$ if and only if $(A, R-U)\notin\delta$. This definition can be replaced by the following: (♯) a subset G of R is defined to be open if and only if $(\{x\}, R-G)\notin\delta$ for every $x\in G$. (See Császár et Mrówka [3, p. 195].)

In this paper we shall first (§2) define slightly different axioms from Efremovich's. In §3 we shall show that our proximity on a set yields a completely normal space. The last section 4 will be devoted to an example of our proximity on a set.

§2. Definitions and lemmas.

By a *paraproximity* on a set R we mean a relation δ for pairs of subsets of R satisfying the following axioms:

Axiom I. $(A, \phi)\notin\delta$ *for every* $A\subset R$.

Axiom II. *$(A, B\cup C)\in\delta$ if and only if $(A, B)\in\delta$ or $(A, C)\in\delta$.*

Axiom III. *For an arbitrary index set Λ, $(\bigcup_{\lambda\in\Lambda} A_\lambda, B)\in\delta$ if and only if there is an index $\mu\in\Lambda$ satisfying the relation $(A_\mu, B)\in\delta$.*

Axiom IV. *For arbitrary two points $a, b\in R$ $(\{a\}, \{b\})\in\delta$ if and only if $a=b$.*

Axiom V. *If $(A, B)\notin\delta$ and $(B, A)\notin\delta$, then there are two disjoint subsets U and V satisfying:*

$$(A, R-U)\notin\delta, \qquad (U, R-U)\notin\delta:$$

$$(B, R-V)\notin\delta, \qquad (V, R-V)\notin\delta.$$

We introduce Axiom I by the suggestion of Pervin [**2**]. We note that a paraproximity does not require the symmetry (Axiom 0) in general but it requires a new axiom III.

Before topologizing a set R we shall add lemmas which easily follow from our axioms.

LEMMA 1. *If $(A, B)\notin\delta$ then $(A, C)\notin\delta$ for any $C\subset B$.*

PROOF. Since $B=B\cup C$ and $(A, B\cup C)\notin\delta$, it follows that $(A, C)\notin\delta$ by Axiom II.

In a similar way we have the following:

LEMMA 2. *If $(A, B)\notin\delta$, then $(C, B)\notin\delta$ for any $C\subset A$.*

LEMMA 3. *If $(A, B)\notin\delta$, then $A\cap B=\phi$.*

PROOF. Suppose that there exists a point $x\in A\cap B$. Then by Lemmas 1 and 2, $(\{x\}, \{x\})\notin\delta$, contrary to Axiom IV.

LEMMA 4. *$(R-x, \{x\})\notin\delta$ for any point $x\in R$.*

PROOF. Let $R-x=\bigcup_{\lambda\in\Lambda} y_\lambda$. Then $y_\lambda\neq x$ for all λ. Suppose that $(R-x, \{x\})\in\delta$ or equivalently $(\bigcup_{\lambda\in\Lambda} y_\lambda, \{x\})\in\delta$. Then there is an index μ satisfying $(\{y_\mu\}, \{x\})\in\delta$ by Axiom III. From Axiom IV follows $y_\mu=x$ which is a contradiction.

We now remark the following:

1) In Axiom V, we may choose U and V such that $(U, V)\notin\delta$ and $(V, U)\notin\delta$. In fact, it follows that $V\subset R-U$ since U and V are disjoint, and hence from $(U, R-U)\notin\delta$ and Lemma 1 follows that $(U, V)\notin\delta$. In the same way we can prove $(V, U)\notin\delta$.

2) In Axiom V we may deduce that $A\subset U$ and $B\subset V$. In fact, let us suppose that U does not contain A and so there is a point $x\in A-U$. Because $x\in A$, $x\in R-U$ and $(A, R-U)\notin\delta$, it follows that $(\{x\}, \{x\})\notin\delta$ from Lemmas 1 and 2. This is a contradiction. Similarly we have $B\subset V$.

§3. The main theorem.

We shall now topologize the set R as follows:

(*) *A set U is open if and only if* $(U, R-U)\notin\delta$.

We note that the definition (*) is equivalent to the preceding definition (♯). To show this, let U be open in the definition (*). Because of the hypothesis $(U, R-U)\notin\delta$ and Lemma 2 it follows that $(\{x\}, R-U)\notin\delta$ for every $x\in U$. Hence U is also open in the definition (♯). Conversely let U be open in (♯). Putting $U=\bigcup_{\lambda} x_{\lambda}$, $(\{x_{\lambda}\}, R-U)\notin\delta$ for every λ. From Axiom III follows $(U, R-U)=(\bigcup_{\lambda} x_{\lambda}, R-U)\notin\delta$. Therefore U is open in (*).

The main result of this paper is the following:

THEOREM 1. *Let R be a set with a paraproximity δ satisfying Axioms* I-V. *Then the set R is a completely normal space if R is topologized by* (*).

We shall call this space R a *paraproximity space.*

PROOF. First we show that R is a topological space. By Axiom I, $(R, R-R)=(R, \phi)\notin\delta$; this means that the whole space R is open. Let U and V be open: $(U, R-U)\notin\delta$ and $(V, R-V)\notin\delta$. By Lemma 2 it follows that $(U\cap V, R-U)\notin\delta$ and $(U\cap V, R-V)\notin\delta$. Hence $(U\cap V, R-(U\cap V))=(U\cap V, (R-U)\cup(R-V))\notin\delta$, by Axiom II. Consequently $U\cap V$ is open. Next suppose that U_{λ} is open, that is $(U_{\lambda}, R-U_{\lambda})\notin\delta$ for every $\lambda\in\Lambda$. By Axiom III, $(\bigcup_{\lambda} U_{\lambda}, R-U_{\lambda})\notin\delta$ for every λ and so $(\bigcup_{\lambda} U_{\lambda}, R-\bigcup_{\lambda} U_{\lambda})\notin\delta$ by Lemma 1. This proves that $\bigcup_{\lambda} U_{\lambda}$ is open. Consequently the finite intersection and arbitrary union of open sets are also open.

From Lemma 4 it is easy to show that R satisfies the T_1 separation axiom or equivalently that for every point x of R, $R-x$ is open.

It remains only to show that any subset of R satisfies the T_4 separation axiom. To this end it is sufficient to verify that if A and B are separated in the T_1 space R (i. e., $\bar{A}\cap B=\phi$ and $A\cap\bar{B}=\phi$), there are disjoint open sets U and V such that $A\subset U$ and $B\subset V$. Now $(R-\bar{A}, \bar{A})\notin\delta$, because $R-\bar{A}$ is open. Since A and B are separated, it follows that $R-\bar{A}\supset B$ and so $(B, \bar{A})\notin\delta$ by Lemma 2. Hence by Lemma 1 it follows that $(B, A)\notin\delta$. Similarly $(A, B)\notin\delta$. As a direct consequence of Axiom V and the foregoing remark 2), we can find the required open sets U and V. This completes the proof of Theorem 1.

The proof of this theorem implies the following:

COROLLARY. *Let R be a set with a relation δ satisfying Axioms* I, II *and* III. *If the topology of R is defined by* (*), *then R is a topological space. Moreover if a relation δ satisfies Axioms* I-IV, *then R is a T_1 space.*

In a proximity space R, $\bar{A}\cap\bar{B}\neq\phi$ implies $(A, B)\in\delta$. The converse implication holds if a space R is a compact proximity space (Efremovich [**1**, p.

198]). In this connection we have the following:

THEOREM 2. *Let* (R, δ) *be a paraproximity space. Then* $(A, B) \in \delta$ *implies* $A \cap \bar{B} \neq \phi$.

PROOF. Assume that $A \cap \bar{B} = \phi$, and so $A \subset R - \bar{B}$. If we choose all open sets O_λ which contain the closed set $\bar{B}$, then $\bigcap_\lambda \bar{O}_\lambda = \bar{B}$ by the regularlity of R. Therefore $A \subset R - \bar{B} = R - \bigcap_\lambda \bar{O}_\lambda = \bigcup_\lambda (R - \bar{O}_\lambda)$. Since all sets $R - \bar{O}_\lambda$ are open, $(R - \bar{O}_\lambda, \bar{O}_\lambda) \notin \delta$ for all λ. Then by Axiom III $(\bigcup_\lambda (R - \bar{O}_\lambda), \bar{O}_\lambda) \notin \delta$ for all λ. Consequently it follows from Lemmas 1 and 2 that $(A, B) \notin \delta$. This contradicts our assumption $(A, B) \in \delta$.

COROLLARY. *Let* (R, δ) *be a paraproximity space. Let* x *be a point of* R *and* A *be a subset of* R. *Then* $(A, \{x\}) \in \delta$ *if and only if* $x \in A$. *If* $(\{x\}, A) \in \delta$, *then* $x \in \bar{A}$.

§ 4. An example.

Finally when a space R is completely normal, we may introduce a paraproximity δ in R. Our next method is similar to Pervin [2].

THEOREM 3. *If* R *is a completely normal space and the relation* δ *is defined by setting* "$(A, B) \in \delta$ *if and only if* $A \cap \bar{B} \neq \phi$", *then* δ *is a paraproximity for* R. (Of course, $\bar{B}$ is the closure of B for the original topology of R.)

PROOF. Axiom I: For any $A \subset R$, $A \cap \bar{\phi} = \phi$ and so $(A, \phi) \notin \delta$. We can easily prove that δ satisfies Axioms II and III. Axiom IV: For any point $a \in R$ it follows that $a \cap \bar{a} = a \neq \phi$ which means $(\{a\}, \{a\}) \in \delta$. Conversely if $a \cap \bar{b} \neq \phi$ then $a = b$ since $b = \bar{b}$. Axiom V: Suppose that $(A, B) \notin \delta$ and $(B, A) \notin \delta$. Because $A \cap \bar{B} = \phi$, $\bar{A} \cap B = \phi$ and R is completely normal, there are two disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Since $\overline{R-U} = R - U$, $A \cap \overline{R-U} = A \cap (R-U) = \phi$ and so $(A, R-U) \notin \delta$. Similarly $(B, R-V) \notin \delta$. We can easily deduce that $(U, R-U) \notin \delta$ and $(V, R-V) \notin \delta$.

COROLLARY. *If* R *is a topological space and* δ *is defined as above, then* δ *satisfies Axioms* I, II *and* III. *Moreover if* R *is a* T_1 *space, then* δ *satisfies Axioms* I-IV.

Aichi Gakugei University

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