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The kernel representation of the fractional power of the strongly elliptic operator

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Introduction.

Let A be a strongly elliptic partial differential operator of order $2m$ defined in a domain D of R^n, and let us consider the Dirichlet problem for the operator $A+\lambda I$, λ being a complex number. Then we can define the fractional power $A^{-\alpha}$ under a suitable condition on the spectrum of A. This operator is continuous from $L^2(D)$ into itself, if $\mathrm{Re}\,\alpha>0$. In the case where A is formally self-adjoint, T. Kotake and M. S. Narasimhan [2] have recently proved that $A^{-\alpha}(\mathrm{Re}\,\alpha>0)$ has the kernel representation and moreover this kernel is very regular. In this article, we want to prove the same result for not necessarily self-adjoint operator.

In § 1, we summerize some well-known facts on the Green operator attached to the Dirichlet problem in the space $L^2(D)$, and impose a condition (C) on the spectrum of A. In § 2, we express weak solutions $u\in L^2(D)$ of the equation $Au+\lambda u=f\in L^2(D)$ by means of a parametrix (formula (2.7) below), and we also express the Green kernel $K(\xi, x\,|\,\lambda)$ of the operator $A+\lambda I$ by using both the parametrix and the Green operator G_λ ((2.13)). We should mention here that these expressions have been obtained by H. G. Garnir in the case of metaharmonic functions [1] and it has played a fundamental rôle in the study of the Green kernel. In § 2 and § 3, we assumed the existence of such a parametrix $E(x, \xi\,|\,\lambda)$ with certain properties. The existence of such a parametrix will be proved in § 4.

The author wishes to express his thanks for Professor Mizohata, who suggested the utilization of parametrix on this subject.

§ 1. Green operator G_λ.

We deal with the strongly elliptic partial differential operator of order $2m$ defined in a domain D (bounded or unbounded) of R^n

(1.1) $$A=A\left(x,\frac{\partial}{\partial x}\right)=\sum_{|\nu|\leq 2m}a_\nu(x)\left(\frac{\partial}{\partial x}\right)^\nu,$$

where

$$\left(\frac{\partial}{\partial x}\right)^\nu=\left(\frac{\partial}{\partial x_1}\right)^{\nu_1}\left(\frac{\partial}{\partial x_2}\right)^{\nu_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\nu_n}.$$

The coefficients $a_\nu(x)$ are the functions of $\mathscr{B}(\tilde{D})$, where $\tilde{D}$ is a domain containing $\bar{D}$. This condition for the boundedness may be too restrictive. But we will not enter into the details of the spectral theory. The condition of strong ellipticity

(1.2) $$\text{Re}\sum_{|\nu|=2m}a_\nu(x)(iy)^\nu\geqq\gamma|y|^{2m},\quad\text{for all}\quad y\in R^n\ (\gamma:\text{const.}>0)$$

is to be fulfilled uniformly in D. From (1.2) it follows Gårding's inequality

(1.3) $$\text{Re}\,(Au,u)\geqq\frac{\gamma}{2}\|u\|_m^2-\gamma_1\|u\|^2,\ \text{for all}\ u\in\mathscr{D}_{L^2}^m(D)\ (\gamma_1:\text{ real const.})$$

where $\mathscr{D}_{L^2}^m(D)$ is the completion of $\mathscr{D}(D)$ by the norm

$$\|u\|_m^2=\int_D\sum_{|\nu|\leqq m}\left|\left(\frac{\partial}{\partial x}\right)^\nu u\right|^2dx,$$

and $(\,,)$ and $\|\ \|$ denote the inner product and the norm in $L^2(D)$ as usual.

Let λ be a complex number. The operator $A+\lambda I$ can be written

(1.4) $$A+\lambda I=\left[\frac{1}{2}(A+A^*)+\text{Re}\,\lambda\right]+i\left[\frac{1}{2i}(A-A^*)+\text{Im}\lambda\right]$$

where A^* is the formal adjoint operator of A. For the hermitian operator

(1.5) $$H=\frac{1}{2}(A+A^*)+\text{Re}\,\lambda,$$

the condition (1.2) is also satisfied by the same constant γ. Hence, if $\text{Re}\,\lambda$ is large enough, Gårding inequality for H is

(1.6) $$(Hu,u)\geqq\frac{\gamma}{2}\|u\|_m^2,\ \text{for all}\ u\in\mathscr{D}_{L^2}^m(D).$$

Of course, (Hu,u) is majorated by $C_1\|u\|_m^2$. Therefore, the norm $\sqrt{(Hu,u)}$ is equivalent to $\|u\|_m$. We can define a new inner product $(\,,)_H$ and a norm by

(1.7) $$(Hu,v)=(u,v)_H,\ \ \|u\|_H=\sqrt{(u,u)_H}$$

in $\mathscr{D}_{L^2}^m(D)$. The structure of $\mathscr{D}_{L^2}^m(D)$ is preserved invariant.

Since both A and A^* are the differential operators of order $2m$, there exists a constant C_2 such that

$$\left|\left(\left[\frac{1}{2i}(A-A^*)+\text{Im}\lambda\right]u,v\right)\right|\leqq C_2\|u\|_H\|v\|_H,\quad\text{for any}\quad u,v\in\mathscr{D}_{L^2}^m(D).$$

Therefore, the operator $\mathscr{H}$ defined by

(1.8) $$\left(\left[\frac{1}{2i}(A-A^*)+\mathrm{Im}\lambda\right]u,\, v\right)=(\mathcal{H}u,\, v)_H$$

is an hermitian continuous operator: $\mathcal{D}^m_{L^2}(D)\to\mathcal{D}^m_{L^2}(D)$. From (1.4), (1.5) and (1.8) we have

(1.9) $$(Au+\lambda u,\, v)=(u+i\mathcal{H}u,\, v)_H, \quad \text{for any} \quad u,\, v\in\mathcal{D}^m_{L^2}(D)\,,$$

and it is majorated by $(1+C_2)\| u\|_H\| v\|_H$. Hence the operator $A+\lambda I$: $\mathcal{D}^m_{L^2}(D)\to\mathcal{D}'^m_{L^2}(D)$ is continuous ($\mathcal{D}'^m_{L^2}(D)$ is the dual space of $\mathcal{D}^m_{L^2}(D)$).

On the other hand, we can define a continuous linear operator C: $\mathcal{D}'^m_{L^2}(D)\to\mathcal{D}^m_{L^2}(D)$ by

(1.10) $$(f,\, v)=(Cf,\, v)_H, \quad \text{for any} \quad f\in\mathcal{D}'^m_{L^2}(D) \text{ and } v\in\mathcal{D}^m_{L^2}(D)\,.$$

Now, we can pose the Dirichlet problem as follows: For a given function $f(x)\in\mathcal{D}'^m_{L^2}(D)$ and a complex number λ, find a weak solution $u(x)\in\mathcal{D}^m_{L^2}(D)$ of the differential equation $Au+\lambda u=f$. If $\mathrm{Re}\,\lambda$ is large enough, (1.9) and (1.10) show the existence of the unique solution $u=(I+i\mathcal{H})^{-1}Cf$ (as $\mathcal{H}$ is continuous and hermitian, there exists the continuous inverse operator $(I+i\mathcal{H})^{-1}$: $\mathcal{D}^m_{L^2}(D)\to\mathcal{D}^m_{L^2}(D)$). Thus we define the Green operator

(1.11) $$G_\lambda=(I+i\mathcal{H})^{-1}C\,,$$

if it exists. From (1.3) and (1.6), we get an equality

(1.12) $$\|f\|\geqq\frac{\gamma}{2}\| G_\lambda f\|_m+(\mathrm{Re}\,\lambda-\gamma_1)\| G_\lambda f\|\,.$$

Consequently, we have an important estimate

(1.13) $$\| G_\lambda\|_{\mathcal{L}(L^2(D),L^2(D))}\leqq 1/(\mathrm{Re}\,\lambda-\gamma_1), \quad \text{for} \quad \mathrm{Re}\,\lambda>\gamma_1\,.$$

If D is bounded, the operator C is completely continuous operator: $L^2(D)\to\mathcal{D}^m_{L^2}(D)$. Therefore, G_λ is also completely continuous operator: $L^2(D)\to L^2(D)$. Accordingly, G_λ is a meromorphic function of λ and each pole of G_λ is an eigenvalue of finite multiplicity.

For any complex number λ, we define the Green operator G_λ as follows:

DEFINITION. We say that the Green operator G_λ exists, if the equation $Au+\lambda u=f$ has the unique solution $u\in\mathcal{D}^m_{L^2}(D)$ for any $f\in\mathcal{D}'^m_{L^2}(D)$. And we denote this solution by $G_\lambda f$.

In such a case, $u=G_\lambda f$ is a solution of the functional equation

(1.14) $$u+(\lambda-c)G_c u=G_c f\,,$$

if $\mathrm{Re}\,c$ is large enough. Conversely, if the equation (1.14) has a solution $u\in\mathcal{D}^m_{L^2}(D)$ for $f\in\mathcal{D}'^m_{L^2}(D)$, it satisfies $Au+\lambda u=f$. Hence these two equations are equivalent.

In this article, we impose the following condition

(C) *there exists no spectrum on the half-line:* $\lambda \geqq 0$.
In other words, there exists G_λ on the positive real axis. This condition is essential for our definition of the fractional power in § 3.

§ 2. Expressions of solutions.

In this section we will obtain an expression of the weak solution in $L^2(D)$ of the equation

$$A\left(x, \frac{\partial}{\partial x}\right)u(x) = \sum_{|\nu| \leqq 2m} a_\nu(x)\left(\frac{\partial}{\partial x}\right)^\nu u(x) = f(x) \in L^2(D) \tag{2.1}$$

by means of parametrix. Let be $a_\nu(x) \in \mathscr{B}(\tilde{D})$, where $\tilde{D}$ is an open set such that $\bar{D} \subsetneqq \tilde{D}$. The condition of ellipticity (1.2) is to be fulfilled uniformly in $\tilde{D}$. We denoted by $A' = A'\left(x, \frac{\partial}{\partial x}\right)$ the transposed operator of A. Because we only need the local expressions (expressions in a fixed compact set in D) of weak solutions and of Green kernels, without loss of generality, we may suppose that $a_\nu(x) \in \mathscr{B}(R^n)$ and the uniform ellipticity (1.2) holds in R^n as well. Otherwise we may consider a new operator $\tilde{A}\left(x, \frac{\partial}{\partial x}\right)$ instead of A as follows: Let H be any compact set in D and $H \subsetneqq U_1 \subsetneqq \bar{U}_1 \subsetneqq U_2 \subsetneqq \bar{U}_2 \subseteqq D$ ($\bar{U}_1$ and $\bar{U}_2$ are compact). And we choose a function $\chi(x) \in \mathscr{D}(U_2)$ such that $\chi(x) \equiv 1$ on U_1 and $0 \leqq \chi(x) \leqq 1$. We define

$$\tilde{A}\left(x, \frac{\partial}{\partial x}\right) = \chi(x)A\left(x, \frac{\partial}{\partial x}\right) + \{1-\chi(x)\}(1-\Delta)^m \equiv \sum_{|\nu| \leqq 2m} \tilde{a}_\nu(x)\left(\frac{\partial}{\partial x}\right)^\nu.$$

This operator $\tilde{A}$ is equal to A in H and $\tilde{a}_\nu(x) \in \mathscr{B}(R^n)$, moreover the uniform ellipticity holds in R^n. Therefore, if we construct a parametrix of $\tilde{A}$, its restriction to $H \times H$ is a parametrix of A.

Then, we assume the existence of the parametrix E of A (resp. E' of A') having the following properties:

i) E (resp. E') satisfies the equation

$$A\left(x, \frac{\partial}{\partial x}\right)E(x, \xi) = \delta_{x-\xi} - L(x, \xi) \tag{2.2}$$

$$\left(\text{resp. } A'\left(x, \frac{\partial}{\partial x}\right)E'(x, \xi) = \delta_{x-\xi} - L'(x, \xi)\right)$$

where $L(x, \xi)$ (resp. $L'(x, \xi)$) is a sufficiently smooth function of x in R^n, depending on the parameter ξ which runs through R^n.

ii) $E(x, \xi)$ (resp. $E'(x, \xi)$) is semi-regular in x and ξ at the same time and is infinitely differentiable in (x, ξ) outside of the diagonal Δ, moreover, $E(x, \xi) \in \mathscr{B}_{x,\xi}(\omega)$ for any open set ω ($\bar{\omega} \cap \Delta = \phi$).

Let D_1 be any *bounded* open set such that $\bar{D}_1 \subset D$, and put

(2.3) $\quad D_{1,\delta}=\{x \in D\,;\, \text{dis.}(x, D_1) \leqq \delta\}$ where $0<\delta(\text{fixed})<\frac{1}{2}\,\text{dis.}(\partial D, D_1)$.

Define

(2.4) $\quad \alpha_\delta(x)=\alpha_\delta(|x|) \in \mathcal{D}, \ \equiv 1$ for $|x|<\delta/2$, $\equiv 0$ for $|x|>\delta$.

(2.5) $\quad \beta(x)=\beta_{D_1}(x) \in \mathcal{D}(D), \ \equiv 1$ on a neighbourhood of $D_{1,\delta}$.

(2.6) $\quad \Phi_\delta(x, \xi)=A\Big(x, \dfrac{\partial}{\partial x}\Big)[\{1-\alpha_\delta(x-\xi)\}E(x, \xi)]$.

(2.6)′ $\quad \Phi'_\delta(x, \xi)=A'\Big(x, \dfrac{\partial}{\partial x}\Big)[\{1-\alpha_\delta(x-\xi)\}E'(x, \xi)]$.

By the hypothesis ii), $\Phi_\delta(x, \xi)$ (resp. $\Phi'_\delta(x, \xi)) \in \mathcal{B}_{x,\xi}(R^n \times R^n)$.

From now on, we denote the integral over R^n or D by the dual form:

$$\langle f(x), g(x)\rangle_x=\int f(x)g(x)dx\,.$$

We shall use the following lemma, which has been shown in a simple case in [**1**, p. 66].

LEMMA. *If $u(x)$ be a weak solution $\in L^2(D)$ of the equation $Au(x)=f(x) \in L^2(D)$, the equality*

(2.7) $\quad u(\xi)=\langle \alpha_\delta(x-\xi)E'(x, \xi), f(x)\rangle_x+\langle g'_\delta(x, \xi), u(x)\rangle_x$

holds as a distribution in D_1, where

$$g'_\delta(x, \xi)=\beta(x)\{\Phi'_\delta(x, \xi)+L'(x, \xi)\}\,.$$

In the same way, the equality

(2.7)′ $\quad v(\xi)=\langle \alpha_\delta(x-\xi)E(x, \xi), g(x)\rangle_x+\langle g_\delta(x, \xi), v(x)\rangle_x$

holds in D_1 for a weak solution $v(x) \in L^2(D)$ of the transposed equation $A'v(x)=g(x) \in L^2(D)$, where

$$g_\delta(x, \xi)=\beta(x)\{\Phi_\delta(x, \xi)+L(x, \xi)\}\,.$$

PROOF. Take a function $\varphi(\xi) \in \mathcal{D}_\xi(D_1)$. Then,

(2.8)
$$\begin{aligned}
&\langle\langle \beta(x)\Phi'_\delta(x, \xi), u(x)\rangle_x, \varphi(\xi)\rangle_\xi \\
&=\langle\langle \Phi'_\delta(x, \xi), \beta(x)u(x)\rangle_x, \varphi(\xi)\rangle_\xi \\
&=\langle\langle \Phi'_\delta(x, \xi), \varphi(\xi)\rangle_\xi, \beta(x)u(x)\rangle_x \\
&=\langle A'\Big(x, \frac{\partial}{\partial x}\Big)\langle\{1-\alpha_\delta(x-\xi)\}E'(x, \xi), \varphi(\xi)\rangle_\xi, \beta(x)u(x)\rangle_x\,.
\end{aligned}$$

By virtue of the semi-regularity of E' in x,

$$(2.8)=\langle A'\left(x, \frac{\partial}{\partial x}\right)\langle E'(x, \xi), \varphi(\xi)\rangle_\xi, \beta(x)u(x)\rangle_x$$
$$-\langle A'\left(x, \frac{\partial}{\partial x}\right)\langle \alpha_\delta(x-\xi)E'(x, \xi), \varphi(\xi)\rangle_\xi, \beta(x)u(x)\rangle_x .$$

We can easily show that for $x \in D$ and $\varphi(x) \in \mathcal{D}(D)$,

$$A'\left(x, \frac{\partial}{\partial x}\right)\langle E'(x, \xi), \varphi(\xi)\rangle_\xi = \varphi(x) - \langle L'(x, \xi), \varphi(\xi)\rangle_\xi .$$

Now we look at the last term of (2.8)

$$(2.9) \qquad \langle A'\left(x, \frac{\partial}{\partial x}\right)\langle \alpha_\delta(x-\xi)E'(x, \xi), \varphi(\xi)\rangle_\xi, \beta(x)u(x)\rangle_x .$$

We want to show that, taking into account of (2.1), this is equal to

$$(2.10) \qquad \langle\langle \alpha_\delta(x-\xi)E'(x, \xi), f(x)\rangle_x, \varphi(\xi)\rangle_\xi .$$

At first we observe that the distribution (in x) $\langle \alpha_\delta(x-\xi)E'(x, \xi), \varphi(\xi)\rangle_\xi$ is infinitely differentiable by virtue of the semi-regularity of E' in x. Moreover this function has its support in the set $D_{1,\delta}=\{x \in D\,; \text{dis.}(x, D_1) \leqq \delta\}$, because for any x such that $\text{dis.}(x, D_1) > \delta$, the support of the distribution (in ξ) $\alpha_\delta(x-\xi)E'(x, \xi)$ and that of $\varphi(\xi)$ do not meet. Taking into account of the fact that $\beta(x) \equiv 1$ in a neighbourhood of $D_{1,\delta}$, we have

$$(2.9) \qquad \langle\langle \alpha_\delta(x-\xi)E'(x, \xi), \varphi(\xi)\rangle_\xi, Au(x)\rangle_x$$
$$=\langle\langle \alpha_\delta(x-\xi)E'(x, \xi), \varphi(\xi)\rangle_\xi, f(x)\rangle_x .$$

Since the distribution $\alpha_\delta(x-\xi)E'(x, \xi)$ defines a continuous bilinear form on $\mathcal{D}'_x(D)\times\mathcal{D}_\xi(D_1)$, the last term is equal to (2.10). Finally we have

$$(2.8)=\langle u(x), \varphi(x)\rangle_x - \langle\langle \beta(x)L'(x, \xi), u(x)\rangle_x, \varphi(\xi)\rangle_\xi$$
$$-\langle\langle \alpha_\delta(x-\xi)E'(x, \xi), f(x)\rangle_x, \varphi(\xi)\rangle_\xi .$$

In the same way we can verify (2.7)′. Q. E. D.

The following expression is also a generalization of the formula obtained in [**1**, p. 118].

PROPOSITION 1. *If the operator A has the Green operator G, G has a kernel representation*

$$(Gf)(\xi)=\int_D K(\xi, x)f(x)dx$$

where

$$(2.11) \qquad K(\xi, x)=\alpha_\delta(x-\xi)E'(x, \xi)+\langle \alpha_\delta(\eta-x)E(\eta, x), g'_\delta(\eta, \xi)\rangle_\eta$$
$$+\langle \underset{\zeta\to\eta}{G}[g_\delta(\zeta, x)], g'_\delta(\eta, \xi)\rangle_\eta$$

in $(\xi, x) \in D_1 \times D_1$, where g_δ and g'_δ are the same as in the preceding lemma.

PROOF. If we substitute Gf for u in (2.7), we have

$$(Gf)(\xi)=\langle\alpha_\delta(x-\xi)E'(x,\xi),f(x)\rangle_x+\langle g'_\delta(x,\xi),(Gf)(x)\rangle_x$$
$$=\langle\alpha_\delta(x-\xi)E'(x,\xi)+\underset{\eta\to x}{G'}[g'_\delta(\eta,\xi)],f(x)\rangle_x$$

for $\xi\in D_1$, where $\underset{\eta\to x}{G'}$ is the transposed operator: $L^2_\eta(D)\to L^2_x(D)$ of G defined by $\langle G'f,g\rangle=\langle f,Gg\rangle$. G' has the same operator norm as G. This gives

(2.11′) $$K(\xi,x)=\alpha_\delta(x-\xi)E'(x,\xi)+\underset{\eta\to x}{G'}[g'_\delta(\eta,\xi)]\,.$$

Let us show that the last term $\underset{\eta\to x}{G'}[g'_\delta(\eta,\xi)]$ is equal to

(2.12) $$\langle\alpha_\delta(x-\eta)E(\eta,x),g'_\delta(\eta,\xi)\rangle_\eta+\langle\underset{\zeta\to\eta}{G}[g_\delta(\zeta,x)],g'_\delta(\eta,\xi)\rangle_\eta\,.$$

Since this term is a weak solution $v(x)$ of

$$A'\Big(x,\frac{\partial}{\partial x}\Big)v(x)=g'_\delta(x,\xi)$$

for any fixed $\xi\in D_1$, (2.7)′ is applicable,

$$\underset{\eta\to x}{G'}[g'_\delta(\eta,\xi)]=\langle\alpha_\delta(x-\eta)E(\eta,x),g'_\delta(\eta,\xi)\rangle_\eta+\langle g_\delta(\zeta,x),\underset{\eta\to\zeta}{G'}[g'_\delta(\eta,\xi)]\rangle_\zeta\,.$$

By transposition of G' into G, we get the desired formula. Q. E. D.

REMARK. Both (2.11) and (2.11′) are expressions of the Green kernel. But we use the former because it is more favourable than the latter when we want to investigate the regularity (especially in x) of the kernel. That is to say, if we construct E and E' satisfying the conditions i) and ii), the second term of (2.12) is sufficiently smooth with respect to the parameter x, and as to the first term, the smoothness (in η) of $g'_\delta(\eta,\xi)$ is propagated to the smoothness (in x) of $\langle\alpha_\delta(x-\eta)E(\eta,x),g'_\delta(\eta,\xi)\rangle_\eta$ by virtue of the semi-regularity of E.

Finally, let us apply our formula (2.11) to the operator $A+\lambda I$ (of course we assume the properties of parametrix): Let us donote

$$G_\lambda=\{A\Big(x,\frac{\partial}{\partial x}\Big)+\lambda I\}^{-1},\quad (G_\lambda f)(\xi)=\int_D K(\xi,x\,|\,\lambda)f(x)dx\,,$$

then we have[1)]

(2.13) $$K(\xi,x\,|\,\lambda)=\alpha_\delta(x-\xi)E'(x,\xi\,|\,\lambda)+\langle\alpha_\delta(\eta-x)E(\eta,x\,|\,\lambda),g'_\delta(\eta,\xi\,|\,\lambda)\rangle_\eta$$
$$+\langle\underset{\zeta\to\eta}{G_\lambda}[g_\delta(\zeta,x\,|\,\lambda)],g'_\delta(\eta,\xi\,|\,\lambda)\rangle_\eta\,.$$

1) Using L and Φ_δ (resp. L' and Φ'_δ), we define g_δ (resp. g'_δ) as follows:

$$\{A\Big(x,\frac{\partial}{\partial x}\Big)+\lambda\}E(x,\xi\,|\,\lambda)=\delta_{x-\xi}-L(x,\xi\,|\,\lambda)\,,$$

$$\Phi_\delta(x,\xi\,|\,\lambda)=\{A\Big(x,\frac{\partial}{\partial x}\Big)+\lambda\}[\{1-\alpha_\delta(x-\xi)\}E(x,\xi\,|\,\lambda)]\,,$$

$$g_\delta(x,\xi\,|\,\lambda)=\beta(x)\{\Phi_\delta(x,\xi\,|\,\lambda)+L(x,\xi\,|\,\lambda)\}\,,\ \text{etc.}$$

§ 3. Fractional power of A.

Under the condition (C) on the spectrum of A imposed in § 1, we define the fractional power $A^{-\alpha}$ ($\mathrm{Re}\,\alpha > 0$) of A by the integral

$$A^{-\alpha} = \frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{-\alpha}(A+\lambda I)^{-1}d\lambda\,, \tag{3.1}$$

where the path Γ of integration consists of three parts; the positive real axis (from ∞ to ρ ($0<\rho\ll 1$)), the circle $|\lambda|=\rho$ (from $\lambda=\rho$ to $\lambda=\rho$ in the negative sense) and the positive real axis (from ρ to ∞). $(-\lambda)^{-\alpha}$ is equal to its principal value on the negative real axis. By the estimate (1.13) of the norm of $(A+\lambda I)^{-1}=G_\lambda$, the integral (3.1) converges and defines a continuous linear operator: $L^2(D)\to L^2(D)$.

In order to prove the kernel representation of the operator $A^{-\alpha}$, we need precise informations on $E(x,\xi\mid\lambda)$. As we shall show later, we can construct explicitely a family of parametrix $E(x,\xi\mid\lambda)$ for the operator $A+\lambda I$. Then what is essential is to elucidate, for such $E(x,\xi\mid\lambda)$, its behavior with respect to λ, when λ tends to infinity. Let us state

PROPOSITION 2. *For any given non-negative integer p, we can construct a family of parametrix $E(x,\xi\mid\lambda)$ (depending continuously on $(\xi,\lambda)\in R^n\times\Gamma$) having the following properties:*

When λ runs through the path of integration,

(1°) *$\lambda L(x,\xi\mid\lambda)$ remains bounded in $\mathcal{B}^p_{x,\xi}(R^n\times R^n)$,*

(2°) *$\lambda E(x,\xi\mid\lambda)$ remains bounded in $\mathcal{B}_{x,\xi}(\omega)$ where ω is an arbitrary open set such that $\bar{\omega}\subseteqq R^n\times R^n-\Delta$ (Δ is the diagonal set of $R^n\times R^n$),*

(3°) *$\lambda E(x,\xi\mid\lambda)$ belongs to $\mathcal{D}^{-[n/2]-1+2m}_{x,L^2}$ and remains bounded in $\mathcal{D}^{-[n/2]-1}_{x,L^2}$,*

(4°) *the mappings*

$$\varphi(\xi)\in\mathcal{D}^{k+2[n/2]+2}_{\xi}\to\lambda\langle E(x,\xi\mid\lambda),\varphi(\xi)\rangle_\xi\in\mathcal{E}^k_x \tag{3.2}$$

and

$$\psi(x)\in\mathcal{D}^{k+2[n/2]+2}_{x}\to\lambda\langle E(x,\xi\mid\lambda),\psi(x)\rangle_x\in\mathcal{E}^k_\xi \tag{3.3}$$

are equi-continuous for any positive integer k. Moreover, these mappings depend continuously on λ.

We can similarly construct $E'(x,\xi\mid\lambda)$ satisfying this proposition.

The proof of this proposition will be given in the next section.

COROLLARY. *If we construct E and E' such that both $L(\zeta,x\mid\lambda)\in\mathcal{B}^p_{x,\zeta}$ and $L'(\eta,\xi\mid\lambda)\in\mathcal{B}^{2p+2[n/2]+2}_{\eta,\xi}$ are bounded, then, from the definitions of g_δ and g'_δ in the footnote 1), it follows*

(5°) *$g'_\delta(\eta,\xi\mid\lambda)$ remains bounded in $\mathcal{D}^{p+2[n/2]+2}_\eta(D)$ with all its derivatives in ξ up to order p.*

(6°) $g_\delta(\zeta, x \mid \lambda)$ *remains bounded in* $\mathcal{D}^0_\zeta(D)$ (*a fortiori in* $L^2_\zeta(D)$) *with all its derivatives in x up to order p.*

(7°) $\langle \lambda \underset{\zeta\to\eta}{G_\lambda}[g_\delta(\zeta, x \mid \lambda)], g'_\delta(\eta, \xi \mid \lambda)\rangle_\eta \in \mathcal{E}^p_{x,\xi}(D_1\times D_1)$ *remains bounded as* λ *tends to infinity.*

(8°) *Take any compact set* $H \subseteqq D$ *and a bounded open set* ω ($\bar{\omega} \subseteqq D$). *Suppose that* $H \cup \omega \subseteqq D_1$ (*see* § 2). *Then* $\langle g'_\delta(\eta, \xi \mid \lambda), \varphi(\xi)\rangle_\xi \in \mathcal{D}^{p+2[n/2]+2}_\eta(D)$ *and* $\langle \lambda \underset{\zeta\to\eta}{G_\lambda}[g_\delta(\zeta, x \mid \lambda)], \langle g'_\delta(\eta, \xi \mid \lambda), \varphi(\xi)\rangle_\xi\rangle_\eta \in \mathcal{E}^p_x(\omega)$ *remain bounded respectively if* $\varphi(\xi)$ *run through any bounded set in* $\mathcal{D}^0_\xi(H)$.

(9°) *Let* ω *be as in* (8°). *Then* $\langle \chi(\eta), g'_\delta(\eta, \xi \mid \lambda)\rangle_\eta \in \mathcal{E}^p_\xi(\omega)$ *remains bounded if* $\chi(\eta)$ *is in any bounded set in* $\mathcal{D}^0_\eta(D)$.

THEOREM. *The operator* $A^{-\alpha}$ ($\mathrm{Re}\,\alpha > 0$) *has a kernel representation*

$$(A^{-\alpha}f)(\xi) = \int_D K^{(\alpha)}(\xi, x)f(x)dx, \quad \textit{for all } f(x) \in L^2(D) \tag{3.4}$$

where the kernel

$$K^{(\alpha)}(\xi, x) = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\alpha}K(\xi, x \mid \lambda)d\lambda \tag{3.5}$$

is very regular (cf. L. Schwartz [4]): *more precisely*

i) $K^{(\alpha)}(\xi, x)$ *is infinitely differentiable in* $(\xi, x) \in D\times D-\varDelta$.

ii) *This kernel maps continuously* $\mathcal{D}_x(D)$ *into* $\mathcal{E}_\xi(D)$ *and* $\mathcal{E}'_x(D)$ *into* $\mathcal{D}'_\xi(D)$ *and moreover, if* $f(x) \in \mathcal{E}'_x(D)$ *is infinitely differentiable in an open set, then* $\langle K^{(\alpha)}(\xi, x), f(x)\rangle_x$ *is also infinitely differentiable there (in* ξ).

REMARK. We can verify that the mappings

$$\psi(x) \in \mathcal{D}^{k+2[n/2]+2}_x(D) \to \langle K^{(\alpha)}(\xi, x), \psi(x)\rangle_x \in \mathcal{E}^k_\xi(D) \tag{3.6}$$

and

$$\varphi(\xi) \in \mathcal{D}^{k+2[n/2]+2}_\xi(D) \to \langle K^{(\alpha)}(\xi, x), \varphi(\xi)\rangle_\xi \in \mathcal{E}^k_x(D) \tag{3.7}$$

are continuous for any integer $k \geqq 0$.

PROOF OF THE THEOREM. In view of (2.13), the right hand side of (3.5) consists of three terms

$$K^{(\alpha)}(\xi, x) = \sum_{i=1}^{3} \frac{1}{2\pi i}\int_\Gamma \frac{(-\lambda)^{-\alpha}}{\lambda} F_i(\xi, x \mid \lambda)d\lambda$$

where, for $(x, \xi) \in D_1\times D_1 - \varDelta$ (D_1 is any compact set contained in D),

$$F_1(\xi, x \mid \lambda) = \lambda\alpha_\delta(x-\xi)E'(x, \xi \mid \lambda),$$

$$F_2(\xi, x \mid \lambda) = \langle \lambda\alpha_\delta(\eta-x)E(\eta, x \mid \lambda), g'(\eta, \xi \mid \lambda)\rangle_\eta,$$

and

$$F_3(\xi, x \mid \lambda) = \langle \lambda \underset{\zeta\to\eta}{G_\lambda}[g_\delta(\zeta, x \mid \lambda)], g'_\delta(\eta, \xi \mid \lambda)\rangle_\eta,$$

(the definitions of g_δ and g'_δ are given in the footnote 1)).

At first we investigate the differentiability of the kernel. If we construct E and E' such that both $L(\zeta, x \mid \lambda) \in \mathcal{B}^{p}_{x,\zeta}$ and $L'(\eta, \xi \mid \lambda) \in \mathcal{B}^{2p+2[n/2]+2}_{\eta,\xi}$ are bounded, then each term $F_i(\xi, x \mid \lambda) \in \mathcal{B}^{p}_{x,\xi}(K)$ remains bounded and depends continuously on λ, where K is any compact set in $D_1 \times D_1 - \Delta$. To show this, we use repeatedly (1°)~(9°) of the Prop. 2 and its Cor.

In fact, the boundedness of $F_1(\xi, x \mid \lambda) \in \mathcal{B}_{x,\xi}(K)$ (a fortiori $\in \mathcal{B}^{p}_{x,\xi}(K)$) follows immediately from (2°) of Prop. 2.

Concerning $F_2(\xi, x \mid \lambda)$, this is, by (4°) and (5°), bounded and p-times continuously differentiable $\mathcal{E}^{p}_{x}(D_1)$-valued function of $\xi \in D_1$, namely, $F_2(\xi, x \mid \lambda) \in \mathcal{E}^{p}_{x,\xi}(D_1 \times D_1)$ remains bounded.

And $F_3(\xi, x \mid \lambda)$ is bounded in $\mathcal{E}^{p}_{x,\xi}(D_1 \times D_1)$ by (7°).

Consequently, $F_i(\xi, x \mid \lambda) \in \mathcal{E}^{p}_{x,\xi}(K)$ remain bounded. Hence, because of the summability of $(-\lambda)^{-\alpha}/\lambda$, the integral (3.5) converges in $\mathcal{B}^{p}_{x,\xi}(K)$. Since the integer $p \geqq 0$ and the compact set $K \subseteqq D_1 \times D_1 - \Delta \subseteqq D \times D - \Delta$ are arbitrary, this integral converges in $\mathcal{E}_{x,\xi}(D \times D - \Delta)$.

Next, we prove the semi-regularities, that is, the continuities of the mappings (3.6) and (3.7). Take any compact set $H \subseteqq D$ and a bounded open set ω ($\bar{\omega} \subseteqq D$). Suppose that $H \cup \bar{\omega} \subseteqq D_1$ (see § 2). It is enough to show that each of

$$\varphi(\xi) \in \mathcal{D}^{k+2[n/2]+2}_{\xi}(H) \to \langle F_i(\xi, x \mid \lambda), \varphi(\xi) \rangle_{\xi} \in \mathcal{E}^{k}_{x}(\omega) \quad (i=1, 2, 3)$$

and

$$\psi(x) \in \mathcal{D}^{k+2[n/2]+2}_{x}(H) \to \langle F_i(\xi, x \mid \lambda), \psi(x) \rangle_{x} \in \mathcal{E}^{k}_{\xi}(\omega) \quad (i=1, 2, 3)$$

is equi-continuous. For this, we apply the Prop. 2 and its Cor. for some $p \geqq k$.

For $i=1$, it is trivial by (4°) of Prop. 2.

Then

$$\varphi(\xi) \to \langle F_2(\xi, x \mid \lambda), \varphi(\xi) \rangle_{\xi} = \langle \lambda \alpha_\delta(\eta - x) E(\eta, x \mid \lambda), \langle g'_\delta(\eta, \xi \mid \lambda), \varphi(\xi) \rangle_{\xi} \rangle_{\eta}$$

is equi-continuous because of (8°).

Next,

$$\psi(x) \to \langle F_2(\xi, x \mid \lambda), \psi(x) \rangle_{x} = \langle \langle \lambda \alpha_\delta(\eta - x) E(\eta, x \mid \lambda), \psi(x) \rangle_{x}, g'_\delta(\eta, \xi \mid \lambda) \rangle_{\eta}$$

is equi-continuous. From (4°), if we put $\chi(\eta) = \langle \lambda \alpha_\delta(\eta - x) E(\eta, x \mid \lambda), \psi(x) \rangle_{x}$, this is bounded in $\mathcal{D}^{k}_{\eta}(D) \subseteqq \mathcal{D}^{0}_{\eta}(D)$. Then by (9°), $\langle F_2(\xi, x \mid \lambda), \psi(x) \rangle_{x} = \langle \chi(\eta), g'_\delta(\eta, \xi \mid \lambda) \rangle_{\eta} \in \mathcal{E}^{k}_{\xi}(\omega)$ is bounded.

Finally for $i=3$.

$$\varphi(\xi) \to \langle F_3(\xi, x \mid \lambda), \varphi(\xi) \rangle_{\xi} = \langle \lambda \underset{\zeta \to \eta}{G_\lambda}[g_\delta(\zeta, x \mid \lambda)], \langle g'_\delta(\eta, \xi \mid \lambda), \varphi(\xi) \rangle_{\xi} \rangle_{\eta}$$

is equi-continuous by (8°).

Quite similarly, $\psi(x)\to\langle F_3(\xi, x\mid\lambda), \psi(x)\rangle_x$ is equi-continuous.

Because $H\subseteqq D$ and $\bar{\omega}\subseteqq D$ are arbitrary, the semi-regularities have been proved. We have shown that the kernel $K^{(\alpha)}(\xi, x)$ is regular and infinitely differentiable outside of the diagonal. Consequently, this kernel is very regular. Q. E. D.

COROLLARY. *For any complex number* α *(not necessarily* $\operatorname{Re}\alpha>0$*), the operator* $A^{-\alpha}$ *has a very regular kernel* $K^{(\alpha)}(\xi, x)$*, moreover, this kernel is an entire function of* α *outside of the diagonal* Δ.

PROOF. Let be $\operatorname{Re}\alpha\leqq 0$ and h be an integer >0 such that $\operatorname{Re}\alpha+h>0$. For any $\psi(x)\in\mathcal{D}(D)$, $A^{-\alpha}\psi(\xi)$ is written as follows:

$$A^{-\alpha}\psi(\xi)=A^{-\alpha-h}(A^h\psi)(\xi) = \langle K^{(\alpha+h)}(\xi, x), A\left(x, \frac{\partial}{\partial x}\right)^h\psi(x)\rangle_x. \tag{3.8}$$

This expression has a sense for any $\psi(x)\in\mathcal{E}'_x(D)$. Therefore, the regularities of the operator $A^{-\alpha}$ are evident from the preceding theorem.

If the support of $\psi(x)$ does not contain the point ξ, we have

$$A^{-\alpha}\psi(\xi)=\langle A'\left(x, \frac{\partial}{\partial x}\right)^h K^{(\alpha+h)}(\xi, x), \psi(x)\rangle_x.$$

Hence $A^{-\alpha}$ is represented by a kernel

$$K^{(\alpha)}(\xi, x)=A'\left(x, \frac{\partial}{\partial x}\right)^h K^{(\alpha+h)}(\xi, x) \tag{3.9}$$

outside of the diagonal Δ. Let us show the analyticity in α. If $\operatorname{Re}\alpha>0$, the summability of

$$-(-\lambda)^{-\alpha}\log(-\lambda)K(\xi, x\mid\lambda)=\frac{d}{d\alpha}\{(-\lambda)^{-\alpha}K(\xi, x\mid\lambda)\}$$

in $\mathcal{E}_{x,\xi}(D\times D-\Delta)$ has been already proved. Therefore, $\alpha\to K^{(\alpha)}(\xi, x)$ $\in\mathcal{E}_{x,\xi}(D\times D-\Delta)$ is holomorphic in the half plane $\operatorname{Re}\alpha>0$, and moreover, in view of (3.9), this is entire. Q. E. D.

§ 4. Proof of Proposition 2.

Now we construct a family of parametrix satisfying Prop. 2 according to [3]. In this section we regard ξ not as an independent variable but as a parameter which runs through R^n, and λ is another parameter as before. We use the space Ω^s (s: any integer) of distributions

$$\Omega^s=\{f(x)\in\mathcal{S}'_x;\ x^\nu f(x)\in\mathcal{D}^{s+|\nu|}_{L^2}(R^n)\ \text{ for all multi-index }\ \nu\geqq 0\}. \tag{4.1}$$

Ω^s is a Fréchet space. We know some properties of Ω^s, that is to say,

(a) if $s>s'$, the inclusion mapping: $\Omega^s\to\Omega^{s'}$ is continuous.

(b) The mapping: $f(x)\rightarrow x_i f(x)$ $(i=1,2,\cdots,n)$ from Ω^s into Ω^{s+1} is continuous.

(c) The mapping: $f(x)\rightarrow \dfrac{\partial}{\partial x_i}f(x)$ $(i=1,2,\cdots,n)$ from Ω^s into Ω^{s-1} is continuous.

(d) The mapping: $(a(x), f(x))\rightarrow a(x)f(x)$ from $\mathcal{B}\times\Omega^s$ into Ω^s is continuous.

(e) If ω is an open set in R^n and $0\notin\bar{\omega}$, the restriction mapping: $f(x)\rightarrow f_\omega(x)$ from Ω^s into $\mathcal{B}(\omega)$ is continuous.

(f) If the Fourier image $\hat{S}(y)$ of $S\in\mathcal{S}'_x$ is a continuous function of y and $(1+|y|)^{\sigma+|\nu|}\left(\dfrac{\partial}{\partial y}\right)^\nu\hat{S}(y)\in\mathcal{B}_y^0$ for all $\nu\geqq 0$, the mapping: $f(x)\rightarrow S*f(x)$ from Ω^s into $\Omega^{s+\sigma}$ is continuous. More exactly, if we introduce the Fréchet space $\hat{\mathcal{B}}^\sigma$ (σ: any integer) of distributions

$$\hat{\mathcal{B}}^\sigma=\{S\in\mathcal{S}'_x\,;\,(1+|y|)^{\sigma+|\nu|}\left(\frac{\partial}{\partial y}\right)^\nu\hat{S}(y)\in\mathcal{B}_y^0 \text{ for all multi-index } \nu\geqq 0\}\,, \tag{4.2}$$

the mapping: $(S, f(x))\rightarrow S*f(x)$ from $\hat{\mathcal{B}}^\sigma\times\Omega^s$ into $\Omega^{s+\sigma}$ is continuous.

(g) (Sobolev) The inclusion mapping: $\Omega^{[n/2]+k+1}\rightarrow\mathcal{B}^k$ is continuous, (for the proof of these see [3]).

We define

$$P\left(\xi\,|\,\lambda\,;\frac{\partial}{\partial x}\right)=\sum_{|\nu|=2m}a_\nu(\xi)\left(\frac{\partial}{\partial x}\right)^\nu+1+\lambda\,, \tag{4.3}$$

$$Q\left(x, \frac{\partial}{\partial x}\,;\xi\right)=\sum_{|\nu|=2m}\{a_\nu(\xi)-a_\nu(x+\xi)\}\left(\frac{\partial}{\partial x}\right)^\nu-\sum_{|\nu|<2m}a_\nu(x+\xi)\left(\frac{\partial}{\partial x}\right)^\nu+1\,, \tag{4.4}$$

$$f_0(\xi\,|\,\lambda)=P\left(\xi\,|\,\lambda\,;\frac{\partial}{\partial x}\right)^{-1}\delta_x=\frac{1}{(2\pi)^n}\int\frac{e^{i\langle x,y\rangle}}{P(\xi\,|\,\lambda\,;iy)}dy\,, \tag{4.5}$$

$$f_i(\xi\,|\,\lambda)=P\left(\xi\,|\,\lambda\,;\frac{\partial}{\partial x}\right)^{-1}Q\left(x,\frac{\partial}{\partial x}\,;\xi\right)f_{i-1}(\xi\,|\,\lambda),\ i=1,2,\cdots,\text{ and} \tag{4.6}$$

$$E(x,\xi\,|\,\lambda)=\tau_\xi[f_0(\xi\,|\,\lambda)+\cdots+f_j(\xi\,|\,\lambda)]\,. \tag{4.7}$$

Then we have

$$\left\{A\left(x,\frac{\partial}{\partial x}\right)+\lambda\right\}E(x,\xi\,|\,\lambda)=\delta_{x-\xi}-\tau_\xi\left[Q\left(x,\frac{\partial}{\partial x}\,;\xi\right)f_j(\xi\,|\,\lambda)\right], \tag{4.8}$$

where τ_ξ is the translation operator: $\tau_\xi[\varphi(x)]=\varphi(x-\xi)$. We prove step by step that (2°), (3°) and (4°) of Prop. 2 are fulfilled by this $E(x,\xi\,|\,\lambda)$ whatever j may be, and that (1°) is satisfied if we take $j\geqq 2[n/2]+1+2m+p$.

(A) For $q=0$ or 1, the mapping $P\left(\xi\,|\,\lambda\,;\dfrac{\partial}{\partial x}\right)^{-1}\in\mathcal{L}(\Omega^s, \Omega^{s+2m(1-q)})$ is equicontinuous of order λ^{-q}, namely, there exist constants $C_{s,q,\nu}>0$ for all multi-index ν, such that

$$\|x^\nu P\left(\xi\,|\,\lambda\,;\frac{\partial}{\partial x}\right)^{-1}\varphi(x)\|_{\mathcal{D}_{L^2}^{s+2m(1-q)+|\nu|}}\leqq C_{s,q,\nu}\lambda^{-q}\sum_{0\leqq\mu\leqq\nu}\|x^\mu\varphi(x)\|_{\mathcal{D}_{L^2}^{s+|\mu|}} \tag{4.9}$$

holds for all $\varphi(x)\in\Omega^s$. And this mapping is infinitely differentiable in $(\xi, \lambda)\in R^n\times\Gamma$. This means that $\lambda^{-q}P\Big(\xi\mid\lambda;\frac{\partial}{\partial x}\Big)^{-1}$ remains bounded in the space $\hat{\mathcal{B}}^{-2m(1-q)}$.

PROOF. It is evident in view of (f). However, we give the proof because this is an important part of our paper.

We can write by Plancherel,

$$\begin{aligned}
&\| x^{\nu}P\Big(\xi\mid\lambda;\frac{\partial}{\partial x}\Big)^{-1}\varphi(x)\|^2_{\mathcal{D}^{s+2m(1-q)+|\nu|}_{L^2}}\\
&\leqq \text{const.}\int (1+|y|)^{2s+2|\nu|+4m(1-q)}\left|\Big(\frac{\partial}{\partial y}\Big)^{\nu}\frac{\hat{\varphi}(y)}{P(\xi\mid\lambda;iy)}\right|^2 dy\\
&\leqq \text{const.}\sum_{0\leqq\mu\leqq\nu}\int |P_\mu(\xi\mid\lambda;y)|^2(1+|y|)^{2s+2|\mu|}\left|\Big(\frac{\partial}{\partial y}\Big)^{\mu}\hat{\varphi}(y)\right|^2 dy\\
&\leqq \text{const.}\sum_{0\leqq\mu\leqq\nu}(\sup_{\xi,y}|P_\mu(\xi\mid\lambda;y)|)^2\,\|x^{\mu}\varphi(x)\|^2_{\mathcal{D}^{s+|\mu|}_{L^2}}
\end{aligned}$$

where $P_\mu(\xi\mid\lambda;y)=(1+|y|)^{|\nu|-|\mu|+2m(1-q)}\Big(\frac{\partial}{\partial y}\Big)^{\nu-\mu}\frac{1}{P(\xi\mid\lambda;iy)}$.

Now $(\partial/\partial y)^{\nu-\mu}(1/P)$ is expressed by the sum of a finite number of the terms: (polynomial in y of order $\leqq 2m(j-1)-|\nu|+|\mu|)/P^j$, $j=1,2,\cdots$, where the polynomial does not contain λ and its coefficients $\in\mathcal{B}_\xi$. We put for $\lambda>1$, $\lambda'=\sqrt[2m]{\lambda}$. Then, $|P|\geqq \operatorname{Re}P\geqq\gamma|y|^{2m}+\lambda=\lambda\{\gamma|y'|^{2m}+1\}$, where $y'=y/\lambda'$. We see that the polynomial is majorated by const. $(1+|y|)^{2m(j-1)-|\nu|+|\mu|}$, a fortiori, this is majorated by const. $(1+|y'|)^{2m(j-1)-|\nu|+|\mu|}\lambda'^{2m(j-1)-|\nu|+|\mu|}$. Hence $\sup_{\xi,y}|P_\mu(\xi\mid\lambda;y)|$ is estimated by const. $\lambda'^{-2mq}=$ const. λ^{-q}.

Since the reasoning is the same for its derivatives in (ξ,λ), we omit it. The repeated use of (A) yields:

$$\lambda^q f_i(\xi\mid\lambda)\in\Omega^{-[n/2]-1+2m(1-q)+i}\quad(q=0\text{ or }1)\tag{4.10}$$

remains bounded with all its derivatives in ξ and depends continuously on λ because $\delta\in\Omega^{-[n/2]-1}$.

(2°) and (3°) of Prop. 2 follow immediately from this.

(1°) follows from (4.8)

$$L(x,\xi\mid\lambda)=\tau_\xi\Big[Q\Big(x,\frac{\partial}{\partial x};\xi\Big)f_j(\xi\mid\lambda)\Big]$$

taking account of

$$\Big(\frac{\partial}{\partial x}\Big)^{\nu}\Big(\frac{\partial}{\partial\xi}\Big)^{\nu'}L(x,\xi\mid\lambda)=\tau_\xi\Big[\Big(\frac{\partial}{\partial x}\Big)^{\nu}\Big(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial x}\Big)^{\nu'}\Big\{Q\Big(x,\frac{\partial}{\partial x};\xi\Big)f_j(\xi\mid\lambda)\Big\}\Big].$$

(B) Proof of (4°): As $\lambda^q f_i(\xi\mid\lambda)$ is bounded in $\Omega^{-[n/2]+2(1-q)m-1}$ with all its derivatives in ξ, (4.11) $f_i(x, \xi \mid \lambda) = (1-\Delta_x)^{-q'} \lambda^q f_i(\xi \mid \lambda) \quad (q' = [n/2]+1+m(q-1)\,;\ q=0 \text{ or } 1)$

is bounded in $\mathcal{B}_x^0$ with all its derivatives in ξ. Let us show that

(4.12) $\lambda^q \langle \tau_\xi[f_i(\xi \mid \lambda)], \varphi(\xi) \rangle_\xi = (1-\Delta_x)^{q'} \langle f_i(x', x-x' \mid \lambda), \varphi(x-x') \rangle_{x'}$

for any $\varphi(\xi) \in \mathcal{D}_\xi^{k+2[n/2]+2}$. We consider a dual form for $\Psi(x) \in \mathcal{E}_x'^k$

(4.13)
$$\begin{aligned} &\langle \Psi(x), \lambda^q \langle \tau_\xi[f_i(\xi \mid \lambda)], \varphi(\xi) \rangle_\xi \rangle_x \\ &= \langle \langle \Psi(x+\xi), \lambda^q f_i(\xi \mid \lambda) \rangle_x, \varphi(\xi) \rangle_\xi \\ &= \langle \langle \Psi(x+\xi), (1-\Delta_x)^{q'} f_i(x, \xi \mid \lambda) \rangle_x, \varphi(\xi) \rangle_\xi \\ &= \langle \langle (1-\Delta_x)^{q'} \Psi(x+\xi), f_i(x, \xi \mid \lambda) \rangle_x, \varphi(\xi) \rangle_\xi\,. \end{aligned}$$

Since, $(1-\Delta_x)^{q'} \Psi(x+\xi) = (1-\Delta_\xi)^{q'} \Psi(x+\xi)$,

$$(4.13) = \langle \Psi(x+\xi), (1-\Delta_\xi)^{q'} \{ f_i(x, \xi \mid \lambda) \varphi(\xi) \} \rangle_{\xi, x}\,.$$

We regard $\Psi(x+\xi)$ as a distribution in ξ translated by $-x$, namely $\Psi(x+\xi) = \tau_{-x}[\Psi(\xi)]$. Then,

$$\begin{aligned} (4.13) &= \langle \Psi(\xi), \tau_x[(1-\Delta_\xi)^{q'} \{ f_i(x, \xi \mid \lambda) \varphi(\xi) \}] \rangle_{\xi, x} \\ &= \langle \Psi(\xi), (1-\Delta_\xi)^{q'} \langle f_i(x, \xi - x \mid \lambda), \varphi(\xi - x) \rangle_x \rangle_\xi\,. \end{aligned}$$

Consequently,

$$\begin{aligned} &\langle \Psi(x), \lambda^q \langle \tau_\xi[f_i(\xi \mid \lambda)], \varphi(\xi) \rangle_\xi \rangle_x \\ &\quad = \langle \Psi(\xi), (1-\Delta_\xi)^{q'} \langle f_i(x, \xi - x \mid \lambda), \varphi(\xi - x) \rangle_x \rangle_\xi\,. \end{aligned}$$

This shows (4.12). By Leibniz, the boundedness of

$$\begin{aligned} &(1-\Delta_x)^{q'} \langle f_i(x', x-x' \mid \lambda), \varphi(x-x') \rangle_{x'} \\ &= \sum_{|\nu|+|\nu'| \leq 2q'} c_{\nu,\nu'} \left\langle \left(\frac{\partial}{\partial x} \right)^{\nu'} f_i(x', x-x' \mid \lambda), \left(\frac{\partial}{\partial x} \right)^{\nu} \varphi(x-x') \right\rangle_{x'} \end{aligned}$$

in $\mathcal{E}_x^k$ is obvious.

Similarly we have for $\psi(x) \in \mathcal{D}_x^{k+2[n/2]+2}$

$$\lambda^q \langle \tau_\xi[f_i(\xi \mid \lambda)], \psi(x) \rangle_x = \langle f_i(x, \xi \mid \lambda), (1-\Delta_x)^{q'} \psi(x+\xi) \rangle_x\,.$$

Its boundedness in $\mathcal{E}_\xi^k$ is also obvious.

These mappings are continuous in λ because $f_i(x, \xi \mid \lambda) \in \mathcal{B}_x^0$ (and all its derivatives in ξ) are continuous.

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