J. Math. Soc. Japan
Vol. 16, No. 2, 1964

4-connected differentiable 11-manifolds with certain homotopy types

By Rieko MATSUKAWA

(Received Dec. 17, 1963)

Introduction.

J. Milnor [8] and S. Smale [11] have proved that the oriented differentiable homotopy $(4k-1)$-spheres $(k>1)$ (i. e. $(4k-1)$-manifolds which have the homotopy type of the $(4k-1)$-sphere), which are boundaries of π-manifolds, are homeomorphic to the natural sphere S^{4k-1} and their diffeomorphism classes form a cyclic group $\Theta^{4k-1}(\partial\pi)$ of a finite order under the connected sum operation. It is known (cf. [8]) that in general any 7 or 11 dimensional closed (i. e. compact unbounded) oriented differentiable π-manifold always bounds a π-manifold. Thus the group Θ^7 (resp. Θ^{11}) of diffeomorphism classes of oriented differentiable homotopy 7-spheres (resp. 11-spheres) coincides with $\Theta^7(\partial\pi)$ (resp. $\Theta^{11}(\partial\pi)$) and hence homotopy 7-spheres (resp. 11-spheres) have been completely classified diffeomorphically as oriented manifolds. So it has turned out that there exist precisely 28 (resp. 992) distinct diffeomorphism classes of homotopy 7-spheres (resp. 11-spheres). (In the following we shall express this situation by saying: there exist precisely 28 (resp. 992) distinct differentiable manifolds on homotopy 7-spheres (resp. 11-spheres).)

In this paper we shall consider $(2k-2)$-connected closed oriented differentiable $(4k-1)$-manifolds which bound π-manifolds and whose $(2k-1)$-th homology groups are cyclic groups of orders n which are products of distinct prime numbers. They are all boundaries of so-called handlebodies (S. Smale [11], [12]). We shall denote the set of such manifolds with $\partial\mathcal{H}'_n(2k)$. We shall see that the homotopy type of such manifolds is uniquely determined by k and n, and shall be able to determine the numbers of differentiable manifolds of such homotopy types, when $n=p$ (a prime number).

I. Tamura [17] has proved that there exist precisely 56 differentiable 7-manifolds of the homotopy type of manifolds of $\partial\mathcal{H}'_3(4)$ and that they are obtained from the standard one by forming connected sums with elements of Θ^7 and the orientation-reversing. In the following we shall show that there exist precisely 1984 differentiable 4-connected 11-manifolds of the homotopy type of manifolds of $\partial\mathcal{H}'_p(6)$ for each prime p (resp. precisely 56 differentiable 2-connected π-manifolds of dimension 7 of the homotopy type of manifolds of

$\partial\mathcal{H}'_p(4)$) and that they are homeomorphic to each other if $p=2$ or $p\equiv 3 \pmod 4$ and there are at most two distinct topological manifolds if $p\equiv 1 \pmod 4$ and that they are obtained from the standard ones by forming connected sums with elements of Θ^{11} (resp. Θ^7) and the orientation-reversing.

The author wishes to express her sincere gratitude to Professor S. Iyanaga for his constant encouragement and valuable remarks. The author also expresses her hearty thanks to Dr. Akio Hattori and Dr. G. Fujisaki for their valuable suggesions and helpful criticisms, and to Professor N. Shimada and Professor I. Tamura for reading this paper in manuscript, and for suggesting a number of improvements.

§ 1. On handlebodies.

In this note we shall make free use of notations and results of Smale [11], [12].

Let D^m and ∂D^m denote the m-cell and its boundary. The set $\mathcal{H}(2m, r, m)$ of handlebodies is the set of manifolds of the form $H=\chi(D^{2m}, f_1, \cdots, f_r, m)$ or simply $H=\chi(F)$, $F=(f_1, \cdots, f_r)$, where the $f_i: \partial D_i^m\times D_i^m\to\partial D^{2m}$ $(1\leqq i\leqq r)$ are imbeddings with disjoint images and H is obtained from the disjoint union $D^{2m}\cup(\bigcup_{i=1}^{r} D_i^m\times D_i^m)$ by identifying points under the f_i's and smoothing. $\mathcal{H}(m)$ denotes the disjoint union $\bigcup_{r=0}^{\infty}\mathcal{H}(2m, r, m)$ for all non-negative integers r. If W is a handlebody in $\mathcal{H}(m)$ $(m>2)$, then it is an $(m-1)$-connected compact manifold with non-vacuous $(m-2)$-connected boundary. $\partial\mathcal{H}(m)$ denotes the set of these boundaries. For two presentations $F=(f_1, \cdots, f_r)$, $F'=(f_1', \cdots, f_r')$ of $\chi(F)$, $\chi(F')$ in $\mathcal{H}(2m, r, m)$, we call them equivalent if there exists a homotopy F_t of presentations, $F_t=(f_{1t}, \cdots, f_{rt})$, $0\leqq t\leqq 1$, $F_0=F$, $F_1=F'$, where F_t for each t is a presentation and each f_{it} has a continuous differential. Let $\hat{\mathcal{H}}(2m, r, m)$ denote the set of equivalence classes of presentations fixing m, r and $\hat{\mathcal{H}}(m)$ denote the union $\bigcup_{r=0}^{\infty}\hat{\mathcal{H}}(2m, r, m)$. If F is equivalent to F' $(F\sim F')$ then $\chi(F)$ is diffeomorphic to $\chi(F')$ so they determine one element in $\mathcal{H}(m)$. Thus we have a natural projection $\pi: \hat{\mathcal{H}}(2m, r, m)\to\mathcal{H}(2m, r, m)$ and $\pi: \hat{\mathcal{H}}(m)\to\mathcal{H}(m)$.

LEMMA 1. *Any manifold W in $\mathcal{H}(m)$ for $m\equiv 6 \pmod 8$ is parallelizable.*

PROOF. The obstruction for constructing a cross-section of the tangent $2m$-frame bundle over W vanishes always, since W is an $(m-1)$-connected manifold with boundary and $\pi_{m-1}(SO(2m))$ is trivial.

Let $F=(f_1, \cdots, f_r)\in\hat{\mathcal{H}}(2m, r, m)$ be a presentation of W. The f_i's define a base for $H_m(W, D^{2m})$. Let φ_i be the inverse image of f_i under the canonical isomorphism $H_m(W)\to H_m(W, D^{2m})$. Then $\hat{\varphi}(F)$ will denote the intersection matrix $(\langle\varphi_i, \varphi_j\rangle)$. For $m>2$, $\varphi_i\in H_m(W)$ can be regarded as a homotopy class of an imbedding $\tilde{\varphi}_i: S^m\to W$ under the Hurewitz isomorphism $H_m(W)\cong\pi_m(W)$. We shall identify $HJ: \pi_{m-1}(SO(m))\to Z$ with the natural homomorphism $p_*: \pi_{m-1}(SO(m))\to\pi_{m-1}(S^{m-1})$ where $H: \pi_{2m-1}(S^m)\to Z$ is the Hopf invariant, and $J: \pi_{m-1}(SO(m))\to\pi_{2m-1}(S^m)$ is the J-homomorphism. If $T_i\in\pi_{m-1}(SO(m))$ will denote the characteristic map of the normal sphere bundle $\nu(\tilde{\varphi}_i(S^m))$ of $\tilde{\varphi}_i(S^m)$ in W, then the self-intersection number $\langle\varphi_i, \varphi_i\rangle$ of φ_i coincides with p_*T_i.

From now on we suppose $m>2$ and $m=2k$. Let $\mathcal{H}'(2m, r, m)$ denote the set of all parallelizable manifolds in $\mathcal{H}(2m, r, m)$ and $\mathcal{H}'(m)$ the disjoint union $\bigcup_{r=0}^{\infty}\mathcal{H}'(2m, r, m)$. $\mathcal{H}'(m)$ is a proper subset of $\mathcal{H}(m)$ for $m\not\equiv 6$ (mod 8) (or $k\not\equiv 3$ mod 4). Let $\hat{\mathcal{H}}'(2m, r, m)$, $\hat{\mathcal{H}}'(m)$ be the inverse images of $\mathcal{H}'(2m, r, m)$, $\mathcal{H}'(m)$ under the natural projection π, respectively.

LEMMA 2. *Let $F=(f_1, \cdots, f_r)$ be an element in $\hat{\mathcal{H}}(2m, r, m)$. F belongs to $\hat{\mathcal{H}}'(2m, r, m)$ if and only if T_i $(1\leqq i\leqq r)$ are in the kernel of $i_*: \pi_{m-1}(SO(m))\to\pi_{m-1}(SO(m+1))$ induced by the natural inclusion map.*

PROOF. For $W=\pi(F)$, the only possible obstruction for constructing a cross-section of the tangent $2m$-frame bundle over W is in $H^m(W, \pi_{m-1}(SO(2m)))$. So F belongs to $\hat{\mathcal{H}}'(2m, r, m)$ if and only if a cross-section is extendable over $\tilde{\varphi}_i(S^m)$ $(1\leqq i\leqq r)$. The restriction over $\tilde{\varphi}_i(S^m)$ of the tangent $2m$-frame bundle over W, is the $SO(2m)$-bundle associated with the Whitney sum $\tau(\tilde{\varphi}_i(S^m))\oplus\nu(\tilde{\varphi}_i(S^m))$ where $\tau(\tilde{\varphi}_i(S^m))$ is the tangent sphere bundle over $\tilde{\varphi}_i(S^m)$. $\tau(\tilde{\varphi}_i(S^m))\oplus\nu(\tilde{\varphi}_i(S^m))$ is trivial if and only if T_i is in the kernel of i_* since the Whitney sum of $\tau(\tilde{\varphi}_i(S^m))$ and trivial line bundle is trivial. This completes the proof.

Thus for $F\in\hat{\mathcal{H}}'(2m, r, m)$, $\langle\varphi_i, \varphi_i\rangle$ is even since T_i belongs to the image of the boundary homomorphism $\partial: \pi_m(S^m)\to\pi_{m-1}(SO(m))$ and the image of $HJ\partial$ consists of even elements.

Let $\hat{Q}(r)$ be the set of all r by r symmetric integral matrices whose diagonal entries are all even. $Q(r)$ denotes the set of equivalence classes of $\hat{Q}(r)$ and $\pi_1: \hat{Q}(r)\to Q(r)$ the projection. If we put $\hat{\varphi}'(F)=\hat{\varphi}(F)$ for $F\in\hat{\mathcal{H}}'(2m, r, m)$, φ_i's define a transformation $\hat{\varphi}': \hat{\mathcal{H}}'(2m, r, m)\to\hat{Q}(r)$. Let φ' be the induced transformation by $\hat{\varphi}'$ such that the diagram is commutative:

$$\begin{array}{ccc} \hat{\mathcal{H}}'(2m, r, m) & \xrightarrow{\hat{\varphi}'} & \hat{Q}(r) \\ \pi\downarrow & & \pi_1\downarrow \\ \mathcal{H}'(2m, r, m) & \xrightarrow{\varphi'} & Q(r). \end{array}$$

THEOREM 3. *φ' is bijective for $m=2k$ $(k>1)$.*

REMARK. Let $\hat{Q}_m(r)$ be the set of all r by r integral matrices, antisymmetric if m is odd, symmetric if m is even and furthermore whose diagonal entries are all even if m is even except in case $m=4$, or 8. In these cases, we can define the transformations $\hat{\varphi}:\hat{\mathcal{H}}(2m, r, m)\to\hat{Q}(r)$, $\varphi:\mathcal{H}(2m, r, m)\to Q(r)$, respectively, whose restrictions over $\hat{\mathcal{H}}'(2m, r, m)$ and $\mathcal{H}'(2m, r, m)$ coincide with $\hat{\varphi}'$ and φ' for $m=2k$, and it is shown that φ is surjective. S. Smale proved that φ is bijective for $m=3, 7$ and remarked without proof that it is also valid for $m\equiv 6 \pmod 8$ (cf. [**12**]).

To prove this theorem, it suffices to show ([**12**], Th. 3.1 and Remark about it) that $\hat{\varphi}':\hat{\mathcal{H}}'(2m, r, m)\to\hat{Q}(r)$ is bijective.

We restate here some results by C. T. C. Wall [**19**]: The complete invariants for $F=(f_1, \cdots, f_r)$ in $\hat{\mathcal{H}}(2m, r, m)$ are $c_{ij}=\langle\varphi_i, \varphi_j\rangle$ $(1\leqq i, j\leqq r)$ and $\alpha(\varphi_i)\in\pi_{m-1}(SO(m))$ $(1\leqq i\leqq r)$ where $\varphi_1, \cdots, \varphi_r$ are corresponding homology classes of $\chi(F)$ to $f_1, \cdots, f_r$ and $\alpha(\varphi_i)$ is the characteristic map T_i of the normal sphere bundle $\nu(\tilde{\varphi}_i(S^m))$ of $\tilde{\varphi}_i(S^m)$ in $\chi(F)$. Furthermore if we regard $H_m(\chi(F))\to\pi_{m-1}(SO(m))$ as a correspondence, we have the following relations:

$$HJ\alpha(\varphi_i)=\langle\varphi_i, \varphi_i\rangle \qquad (1\leqq i\leqq r),$$
$$\alpha(x+y)=\alpha(x)+\alpha(y)+\langle x, y\rangle\partial\iota,$$

where x, y are elements in $H_m(\chi(F))$, ι is a generator of $\pi_m(S^m)$ and ∂ is the boundary homomorphism $\pi_m(S^m)\to\pi_{m-1}(SO(m))$. For $m=2k$, $i_*\oplus HJ$: $\pi_{2k-1}(SO(2k))\to\pi_{2k-1}(SO(2k+1))\oplus Z$ is injective since we have $HJ\partial\iota=2$ by choosing a suitable orientation. So by Lemma 2, we can adopt invariants $\langle\varphi_i, \varphi_i\rangle$ in place of $\alpha(\varphi_i)=T_i$ $(1\leqq i\leqq r)$ for F in $\hat{\mathcal{H}}'(2m, r, m)$. Clearly $\hat{\varphi}'$ is surjective and for F, F' in $\hat{\mathcal{H}}'(2m, r, m)$, F is equivalent to F' if and only if $\hat{\varphi}'(F)$ coincides with $\hat{\varphi}'(F')$.

REMARK. Let $\tau=\{T, \pi^{4k-1}, S^{2k}, S^{2k-1}\}$ be the tangent sphere bundle over S^{2k} and $\bar{\tau}=\{\bar{T}, \bar{\pi}^{4k}, S^{2k}, D^{2k}\}$ the $2k$-cell bundle associated with τ. The total spaces T and $\bar{T}$ have differentiable structures naturally induced from their bundle structures. The characteristic map of τ and hence of $\bar{\tau}$ is a generator of the kernel of the homomorphism $i_*:\pi_{2k-1}(SO(2k))\to\pi_{2k-1}(SO(2k+1))$ (N. E. Steenrod [**14**], § 23). It follows from this and Lemma 2, that $\bar{T}$ is parallelizable and hence T is a manifold in $\partial\mathcal{H}'(4k, 1, 2k)$. Since $\varphi'(\bar{T})$ is the matrix defined by the image of the characteristic map of τ under the projection p_*, the matrix $\varphi'(\bar{T})$ is (2) of rank 1 by choosing a suitable orientation of $\bar{T}$.

§ 2. The invariant $\bar{\lambda}$.

Let W be a handlebody in $\mathcal{H}(m)$ $(m=2k)$ and M be its boundary. By the exact homology sequence of (W, M) and the Poincaré-Lefschetz duality, we have non-trivial part

$$0\to H_m(M)\to H_m(W)\to H_m(W, M)\to H_{m-1}(M)\to 0\,,$$

where first three groups are free abelian.

Let ϕ denote the quadratic form over the group $H_m(W)$ defined by the formula $x\to\langle x, x\rangle$ $(x\in H_m(W))$. The signature of this form ϕ will be denoted by $I(W)$. Clearly ϕ defines a matrix A of $\hat{Q}(r)$, where r is the Betti number of $H_m(W)$ by choosing a base of $H_m(W)$ over Z and $I(W)$ is the signature of A, i. e. the number of positive eigenvalues minus the number of negative ones, considering A as a matrix in real coefficients.

LEMMA 4. *The residue class of $I(W)$ modulo $2^{2k+2}(2^{2k-1}-1)$ is a diffeomorphy invariant of a rational sphere M (i. e. $H_{m-1}(M, Q)=0$) for odd $k>1$.*

PROOF. We suppose M in $\partial\mathscr{H}(2k)$, and we suppose that M is the boundary of two oriented $(2k-1)$-connected manifolds W_1 and W_2 in $\mathscr{H}(2k)$. Let V be the closed oriented differentiable $4k$-manifold obtained from W_1 and $-W_2$ by pasting together the common boundary. As is easily seen, V is $(2k-1)$-connected and hence the i-th Pontrjagin class $p_i(V)$ of V vanishes for $i<k$. Therefore the index theorem

$$I(V)=\frac{2^{2k}(2^{2k-1}-1)}{(2k)!}B_k p_k(V)[V]$$

(Hirzebruch [**5**]) and the fact that $\hat{A}$-genus

$$\hat{A}(V)=-\frac{1}{2(2k)!}B_k p_k(V)[V]$$

is an even integer (Borel-Hirzebruch [**2**]), where $[V]$ denotes the fundamental class of $H_{4k}(V)$, imply

$$I(V)\equiv 0 \pmod{2^{2k+2}(2^{2k-1}-1)}\,.$$

Since $I(V)=I(W_1)-I(W_2)$ we have

$$I(W_1)\equiv I(W_2) \mod 2^{2k+2}(2^{2k-1}-1)\,.$$

This completes the proof.

If M and W are manifolds in $\partial\mathscr{H}'(2k)$ and $\mathscr{H}'(2k)$ for even $k\geqq 2$, and furthermore if M is a rational sphere, we have the following for such a pair (M, W) by the integrality of $\hat{A}$-genus for a $4k$-manifold with $w_2=0$.

LEMMA 4′. *The residue class of $I(W)$ modulo $2^{2k+1}(2^{2k-1}-1)$ is a diffeomorphy invariant of M.*

DEFINITION. The residue class of $I(W)$ mod $2^{2k+1}(2^{2k-1}-1)a_k$ will be denoted by $\bar{\lambda}(M)$ for a rational sphere $M\in\partial\mathscr{H}(2k)$ with odd $k>1$, for $M\in\partial\mathscr{H}'(2k)$ with even $k\geqq 2$, respectively, where a_k is 2 for odd k and 1 for even k.

REMARK. It is easily seen by our definition and $I_k=2^{2k+1}(2^{2k-1}-1)a_k$ for

$k=2, 3, 4$ and 5 (cf. Milnor [8], Lemmas 3.5, 3.6 and Toda [18]), where I_k is the greatest common divisor of indices for all closed almost parallelizable $4k$-manifolds, that $\bar{\lambda}$ coincides with 8 times the Milnor invariant λ' for homotopy $(4k-1)$-spheres which bound π-manifolds [8]. Furthermore this invariant was adopted by Tamura for $k=2$ and for a certain special type of M [17].

From now on, we consider $M \in \partial\mathcal{H}'(2k)$ such that $H_{2k-1}(M)$ is a cyclic group of order $n=p_1 \cdots p_s$, having mutually distinct prime factors p_i $(1 \leqq i \leqq s)$ and $W \in \mathcal{H}'(2k)$ with such boundary. The following lemma can be proved analogously as in [17], Lemma 6.

LEMMA 5. *The determinant of the matrix of the quadratic from* ϕ *over* $H_{2k}(W)$ *is* $\pm n$ *corresponding to* $H_{2k-1}(\partial W) \cong Z/nZ$, *where* $n=p_1 \cdots p_s$.

§ 3. On quadratic forms.

In this section, we shall state some results from the theory of quadratic forms.

Let $\hat{Q}^n(r)$ denote the set of all matrices in $\hat{Q}(r)$ (i.e. the set of all r by r symmetric integral matrices whose diagonal entries are all even) whose determinants are $\pm n$. Let $\bar{Q}^n(r)$ denote the set of all indefinite matrices in $\hat{Q}^n(r)$. Let $\hat{Q}^n$, $\hat{Q}$, $\bar{Q}^n$, $\bar{Q}$ denote the disjoint unions $\bigcup_{r=0}^{\infty} \hat{Q}^n(r)$, $\bigcup_n \hat{Q}^n$, $\bigcup_{r=0}^{\infty} \bar{Q}^n(r)$, $\bigcup_n \bar{Q}^n$ respectively, where n runs over 1 and all positive integers which have mutually distinct prime factors. Let $r(A)$ be the rank of a matrix A, $\det A$ the determinant of A and $I(A)$ the signature.

Z_p and F_p for a finite or the infinite prime p, will denote the ring of p-adic integers and its quotient field, i.e. the field of p-adic numbers. Z_∞ and F_∞ are both the field of real numbers. (From now on, we shall mean by a prime p, a finite prime or the infinite prime ∞.) Furthermore $c_p(A)$ for a prime p will denote the Hasse's symbol of the quadratic form corresponding to A (cf. Jones [6], Chap. II, 11). For non-zero numbers a, b in F_p, $(a, b)_p$ will denote the Hilbert's symbol, i.e. 1 or -1 according as $ax^2+by^2=1$ has or has not a solution in F_p (cf. [6], Chap. II, 10).

We shall restate here a result from Theorems 29 and 45 in [6].

LEMMA 6. *Given a positive integer* r, *a non-zero integer* d, *a set of values* 1 *or* -1 *for* $c_p(A)$ *for all primes and an integer* I *whose absolute value is not greater than* r, *there is a symmetric integral matrix* A *with rank* r, *determinant* d, *and with Hasse's symbols of the given values and signature* I, *if and only if the following conditions hold:*

(1) $c_p(A)=1$ *for a finite prime* p *not dividing* $2d$.

(2) $\prod c_p(A)=1$, *the product extending over all primes.*

(3) *If* $r=1$, $c_p(A)=(-1, -d)_p$ *for all prime* p.

(4) *If* $r=2$, $c_p(A)=1$ *for all prime* p, *for which* $-d$ *is a square.*

(5) $\frac{1}{2}(r-I)\equiv c_\infty(A)+\frac{1}{2}\{1+(-1, d)_\infty\}$ (mod 4).

Furthermore, there is a matrix A whose diagonal entries are all even if

(6) *r is even and* $d\equiv(-1)^{r/2}$ (mod 4).

Let U, U' denote the matrices $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$, respectively.

LEMMA 7. *An integer I is the signature of a matrix A in* $\hat{Q}^n$ ($n=1$ *or* $n=p_1\cdots p_s$ ($s\geqq 1$) *having mutually distinct prime factors*) *if and only if one of the following conditions is satisfied:*

(1) $I\equiv 0$ (mod 8) *for* $n=1$ (*Milnor*),

(2) $I\equiv\pm 1$ (mod 8) *for* $n=2$,

(3) $I\equiv 0$ (mod 4) *for* $n>1$ *and* $n\equiv 1$ (mod 4),

(4) $I\equiv 2$ (mod 4) *for* $n\equiv 3$ (mod 4), and

(5) $I\equiv 1$ (mod 2) *for* $n>2$ *and* $n\equiv 2$ (mod 4).

PROOF. First we shall show that I is even if and only if n is odd. For any matrix A, we have $r(A)\equiv I(A)$ (mod 2) so it suffices to show that r is even if and only if n is odd. Since an odd integer n is a unit in Z_2, a matrix A in $\hat{Q}^n(r)$ is equivalent to $A_1=\text{diag.}(U,\cdots,U)=\begin{pmatrix}U & & \\ & \ddots & \\ & & U\end{pmatrix}$ or $A_2=\text{diag.}(U,\cdots,U,U')$ with rank r as Z_2-matrices (cf. [6], Theorems 33a, 36) so that $r(A)$ is even. For even n, i.e. $n\equiv 2$ (mod 4), a matrix A in $\hat{Q}^n(r)$ is equivalent to diag. $(A_1,(2k))$ or diag. $(A_2,(2k))$ with rank r where k is a unit in Z_2 (cf. [6], Th. 33) so that $r(A)$ is odd.

The existence of a matrix in $\hat{Q}(r)$ with n and I satisfying (3) or (4) follows from Lemma 6.

Let A be a matrix with signature 0 mod 4 (resp. 2 mod 4). A is equivalent to A_1 or A_2 in Z_2 if and only if $\det A=(-1)^{(r-I)/2}n$ equals $(\det A_1)\sigma^2$ or $(\det A_2)\sigma^2$ for a suitable unit σ in Z_2 (cf. [6], Th. 36) if and only if n equals 1 mod 8 or 5 mod 8 (resp. n equals 7 mod 8 or 3 mod 8).

For $n\equiv 2$ (mod 4) we proceed as follows. Let V denote the matrix

$$\begin{pmatrix}2 & 1 & & & & & & \\ 1 & 2 & 1 & & -1 & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & -1 & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & 1\\ & & & & & & 1 & 2\end{pmatrix}$$

of rank 8, determinant 1 and signature 8 (cf. [7]).

By considering $-A$, diag. $(A, V,\cdots, V)$ or diag. $(A, -V,\cdots,-V)$ if necessary, it suffices to consider (2) for $I=1, 3$. In case $n=2$, there is a matrix with signature 1 (e. g. $(2)\in\hat{Q}^2(1)$). Let A be a matrix with signature 3 in $\hat{Q}^2(r)$. A is equivalent to $B_1=\text{diag.}(A_1, (2k))$ or $B_2=\text{diag.}(A_2, (2k))$ with rank r as Z_2-matrices so we have $c_2(A)=c_2(B_1)$ or $c_2(A)=c_2(B_2)$ (cf. [**6**], Th. 12). By the product formula of Hasse's symbols (Lemma 6(2)), $c_\infty(A)$ must be equal to $c_2(A)$ so that $r=r(A)$ equals 1 (mod 4). On the other hand a matrix $A'=\text{diag.}(A, U)$ of rank $(r+2)$ has also the required properties (i. e. $\det A'=\pm 2$, $I(A')=3$). This contradicts with the condition for the rank.

For $n>2$ and $n\equiv 2 \pmod 4$, i. e. $n=2q$ (q: odd), there are matrices B, C in $\hat{Q}$ with determinant 2, q. We may suppose $I(B)=1$ and $I(C)\equiv 0$ or $2 \pmod 4$ according to $q\equiv 1$ or $3 \pmod 4$. Then $A=\text{diag.}(B, C)$, $A'=\text{diag.}(-B, C)$ are matrices with required properties. This completes the proof.

We shall denote with $c_n(A)$ the product $\prod_{i=1}^{s} c_{p_i}(A)$ for a positive integer $n=p_1\cdots p_s$ ($s\geqq 1$) having mutually distinct prime factors and a matrix A in $\hat{Q}(r)$.

LEMMA 8. *For a matrix A in $\hat{Q}(r)$ with determinant $\pm n$, $c_n(A)$ is uniquely determined by n, $r=r(A)$ and $I=I(A)$.*

PROOF. $c_\infty(A)$ is determined by r and I: $c_\infty(A)=1$ if and only if $(r-I)/2\equiv 1, 2 \pmod 4$ (cf. Lemma 6. (5)). So the lemma follows from Lemma 6, if it is shown that $c_2(A)$ depends only upon n, r and I for odd n. In fact, A is equivalent to A_1 or A_2 according to conditions for n, I (cf. the proof of Lemma 7) so we have $c_2(A)=c_2(A_1)$ or $c_2(A)=c_2(A_2)$. Clearly both $c_2(A_1)$ and $c_2(A_2)$ depend only upon the rank r.

REMARK. For even n, say $n=2q$ (q: odd), $c_2(A)$ also depends only upon n, r and I. In fact, A is equivalent to $B_1=\text{diag.}(A_1, (2k))$ or $B_2=\text{diag.}(A_2, (2k))$ as Z_2-matrices according to $(-1)^{(I-1)/2}q\equiv k \pmod 8$ or $(-1)^{(I-1)/2}q\equiv 5k \pmod 8$. If we calculate

$$c_2(B_i)=c_2(A_i)(-1, -2k)_2(-1, -1)_2(\det A_i, 2k)_2$$

$(i=1, 2)$, we have

$$c_2(A)=(-1)^{(m-1)(m-2)/2} \qquad \text{for} \quad (-1)^{(I-1)/2}q\equiv 1 \pmod 4$$

and

$$c_2(A)=(-1)^{(m-1)m/2} \qquad \text{for} \quad (-1)^{(I-1)/2}q\equiv 3 \pmod 4$$

where m denotes $(r-1)/2$. On the other hand, for any odd prime p dividing n, $c_p(A)$ can be either 1 or -1 so far as they satisfy the condition for $c_n(A)$.

LEMMA 9. *If two matrices A, B in $\hat{Q}^n$ for $n=p_1\cdots p_s$ ($s\geqq 1$) satisfying the conditions $r(A)=r(B)$, $I(A)=I(B)$ and*

$$(*) \qquad c_{p_i}(A)=c_{p_i}(B) \qquad \textit{for} \quad i\leqq s-1,$$

then A is equivalent to B in Z_p for a finite prime p not dividing n and A is equivalent to B in F_p for $p=p_i$ ($1\leqq i\leqq s$). (The condition (*) on Hasse's symbol is trivial for $s=1$. Cf. Milnor [7] for $n=1$, Tamura [17] for $n=3$.)

PROOF. $r(A)=r(B)$ and $I(A)=I(B)$ imply $\det A=\det B$ and $c_\infty(A)=c_\infty(B)$. So $c_p(A)=c_p(B)$ holds for all prime p by Lemmas 6 and 8. Thus this lemma follows from Theorems 15, 36 in [6].

For convenience' sake we shall write $n=p_1\cdots p_s$ also for $n=1$ $(s=0)$. Then Lemma 9 is valid for $s=0$.

Now we consider a lattice L in an r-dimensional vector space over the field of rational numbers such that the matrix $A=(a_{ij})$ determined by the inner product $a_{ij}=\langle\omega_i, \omega_j\rangle$ of a basis $\omega_1, \cdots, \omega_r$ of L over Z, belongs to $\hat{Q}^n(r)$. In general for any matrix A in $\hat{Q}(r)$ $(r>0)$, there is a lattice L and its basis $\{\omega_i\}$ over Z having A as the matrix $(\langle\omega_i, \omega_j\rangle)$. In fact if we choose $F=(f_1, \cdots, f_r)\in\hat{\varphi}^{-1}A\in\hat{\mathscr{H}}(4k, r, 2k)$ $(k>1)$ $(F=\hat{\varphi}^{-1}A$ if $k\equiv 3 \pmod 4)$ and $\pi(F)=W$, then $\varphi_1, \cdots, \varphi_r$ corresponding to $f_1, \cdots, f_r$ (cf. § 1) form a basis of $H_{2k}(W, Q)$ where Q is the field of rational numbers. If we define the inner product of φ_i, φ_j by their intersection number $\langle\varphi_i, \varphi_j\rangle$, the lattice $H_{2k}(W)=H_{2k}(W, Z)$ and a basis $\varphi_1, \cdots, \varphi_r$ over Z have the required property. Let L_A denote the lattice corresponding to A in this manner. For any positive integer n having distinct prime factors (as for $n=1$ and 3, [7], [17]), L_A is always maximal for $A\in\hat{Q}^n$ ([3], Sätze 9.3, 12.3). Furthermore, if $(L_A)_p$ for a finite prime p dividing n denotes the p-adic extension of L_A the norm $n(L_A)_p$ of $(L_A)_p$ coincides with the ideal (p) in Z_p if $r(A)=1$, Z_p if $r(A)\geqq 2$. Thus $I(A)=I(B)$, $r(A)=r(B)$ and $(*)$ imply that $(L_A)_p$ is isomorphic to $(L_B)_p$ as Z_p-lattices for p dividing n ([3], Satz 9.6) and hence $(L_A)_q$ is isomorphic to $(L_B)_q$ as Z_q-lattices for all finite prime q by Lemma 9. Thus the following lemma follows from a theorem of Eichler (cf. [4], Satz 3).

LEMMA 10. *The absolute value $n=p_1\cdots p_s$ $(s\geqq 0)$ of the determinant, the rank r, the signature I and Hasse's symbols $\{c_{p_i}\}$ $(i\leqq s-1)$ form a complete system of invariants for equivalence classes of matrices in $\bar{Q}$ of rank $r\geqq 3$ ($\bar{Q}$ is the set of symmetric indefinite integral matrices whose determinants have distinct prime factors, and whose diagonal entries are all even).*

For two matrices A, B in $\hat{Q}^n$, we shall call them *weakly* equivalent $(A\underset{w}{\sim}B)$ if there are non-negative integers s, t such that $\mathrm{diag.}(A, \underbrace{U, \cdots, U}_{s})$ is equivalent to $\mathrm{diag.}(B, \underbrace{U, \cdots, U}_{t})$. Any A in $\hat{Q}$ is weakly equivalent to an indefinite matrix $\mathrm{diag.}(A, U)$ of rank $\geqq 3$, so we have $n=p_1\cdots p_s$ $(s\geqq 0)$, I and $\{c_{p_i}\}$ $(i\leqq s-1)$ as a complete system of invariants for weak equivalence classes in $\hat{Q}$.

For a finite prime p, we shall denote with $\boldsymbol{Q}^p$ the set of weak equivalence classes of $\hat{Q}^p$ and $\boldsymbol{A}$ its element. (For a fixed p, the signature I is the only invariant for weak equivalence classes.) We shall call a matrix A *of a reduced type* in $\boldsymbol{A}$ if $r(A)$ is the least of $r(B)$ for B in $\boldsymbol{A}$. A matrix of a reduced type is not necessarily unique.

Let U_1, U_2^p, U_3^p and U_4^p be matrices of reduced types of signatures 1, 2, 0 and 4, and of determinants 2, p, $-p$ and p, respectively, e. g.

$$U_1=(2),$$

$$U_2^p=\begin{pmatrix} 2 & 1 \\ 1 & 2(t+1) \end{pmatrix} \quad (p=4t+3), \qquad U_4^p=\begin{pmatrix} 2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & 2(t+1) \end{pmatrix} \quad (p=8t+5),$$

$$U_3^p=\begin{pmatrix} 2 & 1 \\ 1 & -2t \end{pmatrix} \quad (p=4t+1)$$

(cf. Lemma 7). For a fixed p, any matrix A in $\hat{Q}^p$ is weakly equivalent to one of $\mathrm{diag.}(\pm U_i^p, \underbrace{V, \cdots, V}_{s})$ and $\mathrm{diag.}(\pm U_i^p, -\underbrace{V, \cdots, -V}_{s})$ for some non-negative integer s and $1 \leqq i \leqq 4$.

LEMMA 11. *If an indefinite symmetric matrix A in $\hat{Q}^n$ with $n=p_1 \cdots p_s$ $(s \geqq 1)$ is of rank $r \geqq 4s$, A is equivalent to a matrix diag. $(A_1, \cdots, A_s)$ where A_j are matrices in $\hat{Q}^{p_j}$ $(1 \leqq j \leqq s)$.*

(The condition for the rank of $A \in \hat{Q}^n$ can be improved if some p_i equals 2 or 3 mod 4.)

Obviously this implies the following.

THEOREM 12. *Any symmetric integral matrix A with determinant $\pm n = \pm p_1 \cdots p_s$ $(s \geqq 1)$, whose diagonal entries are all even, is weakly equivalent to diag. $(A_1, \cdots, A_{s+1})$ where A_j is one of $\pm U_i^{p_j}$ $(1 \leqq i \leqq 4)$ for $1 \leqq j \leqq s$, and A_{s+1} is a matrix $(V, \cdots, V)$ or $(-V, \cdots, -V)$.*

Furthermore, the number $N(n, I)$ of weak equivalence classes for fixed n, I is as follows:

$$N(1, I)=N(p, I)=1 \qquad \text{for a finite prime } p,$$

$$N(p_1 \cdots p_s, I)=2^{s-1} \qquad \text{for } s \geqq 1,\ p_1 \cdots p_s \equiv 1 \pmod 2,$$

$$N(p_1 \cdots p_s, I)=2^{s-2} \qquad \text{for } s \geqq 2,\ p_1 \cdots p_s \equiv 0 \pmod 2.$$

PROOF. The lemma clearly holds for $s=1$. We assume it for $s=t \geqq 1$ and prove it for $s=t+1$. Let $n=p_1 \cdots p_{t+1}$ and denote $p=p_{t+1}$, $q=p_1 \cdots p_t$, $p_1, \cdots, p_t$ being odd primes. We shall show that for any indefinite matrix A in $\hat{Q}^n$, there are matrices B, C in $\hat{Q}^p$ and $\hat{Q}^q$ respectively, such that A is equivalent to diag. (B, C). It suffices to show by Lemma 10 that the following conditions hold by choosing suitable matrices B and C:

(Cd) $\det A=(\det B)(\det C)$,

(Cr) $r(A)=r(B)+r(C)$,

(CI) $I(A)=I(B)+I(C)$,

(CH) i) $c_{p'}(B)=1$ for a finite prime p' not dividing $2p$,

$c_{p'}(C)=1$ for a finite prime p' not dividing $2q$,

ii) $c_2(A)=c_2(B)c_2(C)(-1,-1)_2(\det B, \det C)_2$,

iii) $c_p(A)=c_p(B)(\det B, \det C)_p=c_p(B)\left(\frac{\det C}{p}\right)$ for $p\neq 2$

where $\left(\frac{\det C}{p}\right)$ denotes Legendre's symbol (i. e. it is 1 or -1 according as $x^2\equiv \det C \pmod{p}$ has or has not a solution),

iv) $c_q(A)=\prod_{i=1}^{t} c_{p_i}(A)=c_q(C)\prod_{i=1}^{t}\left(\frac{\det B}{p_i}\right)$,

v) $c_\infty(A)=c_\infty(B)c_\infty(C)(-1,-1)_\infty(\det B, \det C)_\infty$.

For a given matrix A, we shall first choose a suitable matrix B of rank $r(B)\leqq 4$ which satisfies the conditions (CH) i) for $p=2$, and i), iii) for $p\neq 2$. Next, we shall show that there exists a matrix C whose determinant, rank, signature and Hasse's symbols satisfy these conditions.

There are three cases: $n\equiv 1, 2$ and 3 mod 4. For $n\equiv 1$ mod 4, we have $p\equiv q \pmod 4$ so we choose B, for instance, with determinant p as follows:

$p\equiv 1 \pmod 4$: $B=\text{diag.}(U_3^p, U)$ for $c_p(B)\left(\frac{2}{p}\right)=1$,

$B=U_4^p$ for $c_p(B)\left(\frac{2}{p}\right)=-1$,

$p\equiv 3 \pmod 4$: $B=U_2$ for $c_p(B)=1$,

$B=-U_2$ for $c_p(B)=-1$,

where $c_p(B)=c_p(A)\left(\frac{\det C}{p}\right)$ by (CH) iii). For $p\equiv 3 \pmod 4$, $c_p(B)$ can be 1 or -1 freely, by the relation $\left(\frac{-q}{p}\right)=-\left(\frac{q}{p}\right)$. It is easy to see that there are suitable matrices for $n\equiv 2$ or 3 mod 4.

Now $\det B$, $r(B)$ and $I(B)$ determine $c_\infty(B)$ so we have relations in $\det C$, $I(C)$, $r(C)$ and $c_{p'}(C)$ for all primes p'. Thus it suffices to show by Lemmas 6, 10 that the following conditions hold:

(a) $c_p(C)=1$ for a finite prime p not dividing $2q$.

(b) $\prod_p c_p(C)=1$ for all primes p.

(c) $(r(C)-I(C))/2\equiv c_\infty(C)+\{1+(-1, \det C)_\infty\}/2 \pmod 4$.

(d) $(-1)^{r(C)/2}\equiv \det C \pmod 4$.

We can examine all of these using the conditions (Cd), (Cr), (CI), (CH), and relations $\sum_{i=1}^{t}(p_i-1)/2\equiv(q-1)/2 \pmod 2$, $\sum_{i=1}^{t}(p_i^2-1)/8\equiv(q^2-1)/8 \pmod 2$.

§ 4. Classification of manifolds.

In this section we shall restrict k to 2, 3, 4, 5 and 7. (For other values of k, we can discuss analogously but it cannot be decided whether a prime p and the invariant $\bar{\lambda}$ characterize the diffeomorphism class or not. In other words if we put $I_k=2^{2k+1}(2^{2k-1}-1)a_k\cdot b_k$, there exist at most b_k manifolds with distinct differentiable structures having the same homotopy type and the same invariant $\bar{\lambda}$, and we have $b_9=43867$ (Adams [1]).)

Let $\partial\mathcal{H}'_n(2k)$ denote the set of boundaries, whose $(2k-1)$-th homology groups are cyclic of order $n=p_1\cdots p_s$ $(s\geqq 0)$, of parallelizable handlebodies in $\mathcal{H}'(2k)$. Notice that $\partial\mathcal{H}'_n(2k)=\partial\mathcal{H}_n(2k)$ for $k=3$.

Let M be a manifold in $\partial\mathcal{H}'_n(2k)$. By S. Smale ([13], Th. 6.1) there is a non-degenerate C^∞-function with just four critical points and non-trivial type numbers $M_0=M_{2k-1}=M_{2k}=M_{4k-1}=1$. This and the fact that M is a π-manifold imply that $M-\operatorname{Int}D$ has the same homotopy type as $S^{2k-1}\underset{f}{\cup}D^{2k}$ where D is a $(4k-1)$-cell imbedded in M and $f:\partial D^{2k}\to S^{2k-1}$ is an attaching map of degree n. So the homotopy type of such manifolds is uniquely determined by k and n.

Now we shall study the number of differentiable manifolds of such homotopy type. In § 1, it was proved that $\varphi':\mathcal{H}'(4k, r, 2k)\to Q(r)$ is bijective. If a matrix A in $\hat{Q}^n(r)$ is equivalent to a matrix diag.(A_1, A_2), then W is diffeomorphic to W_1+W_2, where W, W_1 and W_2 are corresponding to $\pi_1(A)$, $\pi_1(A_1)$ and $\pi_1(A_2)$ under φ' respectively. The sum W_1+W_2 of two compact oriented differentiable n-manifolds with boundaries will mean the compact oriented differentiable n-manifold with boundary obtained from the disjoint union of W_1 and W_2 by $f_1(x)$ with $f_2(x)$ $(x\in D^{n-1})$, where $f_1:D^{n-1}\to\partial W_1$ (resp. $f_2:D^{n-1}\to\partial W_2$) is an orientation-preserving (resp. orientation-reversing) imbedding of $(n-1)$-disk D^{n-1}. $\partial(W_1+W_2)$ coincides with the connected-sum $\partial W_1\#\partial W_2$ of their boundaries (cf. [12]). Weakly equivalent matrices A and diag.(A, U) determine manifolds W and W' (i.e. $\varphi'(W)=\pi_1(A)$ and $\varphi'(W')=\pi_1(\text{diag.}(A, U))$) having the same boundary, strictly speaking we can obtain W from W' by performing a surgery Killing homotopies corresponding to the matrix U without modifying its boundary (cf. [8], [9]). Thus any two matrices in a weak equivalence class $\boldsymbol{A}$ determine the unique manifold M (we shall denote it by $\psi(\boldsymbol{A})$) and $\bar{\lambda}(M)$ equals the signature of $\boldsymbol{A}$ mod $2^{2k+1}(2^{2k-1}-1)a_k$, an invariant for $\boldsymbol{A}$ (cf. § 3). We have thus the correspondence $\psi:\boldsymbol{Q}^n\to\partial\mathcal{H}'_n(2k)$.

U_1, U_2^p, U_3^p and U_4^p for a fixed prime p, are matrices of reduced types of signatures 1, 2, 0 and 4 and determinants 2, p, $-p$ and p respectively (cf. § 3). Let W_0 be the handlebody corresponding to the matrix V, and M_0 the boundary of W_0. Likewise let W_1, W_i^p $(i=2,3,4)$ correspond to the matrices U_1, U_i^p respectively and let M_1, M_i^p be their respective boundaries. Notice that M_0 is the generator of $\Theta^{4k-1}(\partial\pi)$ (cf. [8]) and M_1 is the total space of the tangent sphere bundle over S^{2k} (cf. § 1, Remark). By Theorem 12, any symmetric integral matrix A with determinant $\pm n=\pm p_1\cdots p_s$ $(s\geqq 1)$, whose diagonal entries are all even, is weakly equivalent to $\operatorname{diag.}(A_1,\cdots,A_{s+1})$ where A_j is one of $\pm U_i^{p_j}$ $(1\leqq i\leqq 4)$ for $1\leqq j\leqq s$, and A_{s+1} is a matrix $\operatorname{diag.}(V,\cdots,V)$ or $\operatorname{diag.}(-V,\cdots,-V)$. If A_{s+1} is equivalent to $\operatorname{diag.}(\underbrace{V,\cdots,V}_{m})$ (resp. $\operatorname{diag.}(\underbrace{-V,\cdots,-V}_{m})$) $(m\geqq 0)$, we have $I(A_{s+1})=8m$ (resp. $I(A_{s+1})=-8m$) and $\pi(\hat{\varphi}'^{-1}(A_{s+1}))=\underbrace{W_0+\cdots+W_0}_{m}$ (resp. $\underbrace{(-W_0)+\cdots+(-W_0)}_{m}$). So two matrices A and $\operatorname{diag.}(A_1,\cdots,A_{s+1})$ determine the same manifold $M_1\#\cdots\# M_s\# M_0\#\cdots\# M_0$ (m-fold connected sum of M_0) where $M_j=\psi(\boldsymbol{A}_j)$ is $M_i^{p_j}$ or $-M_i^{p_j}$ according to $A_j=U_i^{p_j}$ or $-U_i^{p_j}$ for a suitable i $(1\leqq j\leqq s)$.

Now we restrict n to p (a prime) and $2p$ (twice an odd prime). For these n, we have $N(n,I)=1$ for a fixed signature I (cf. Theorem 12). Since M_0 is the generator of $\Theta^{4k-1}(\partial\pi)$, matrices A and B of signature $I(A)\equiv I(B)$ (mod I_k) in $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\hat{\boldsymbol{Q}}^n$ determine the same manifold $\varphi(\boldsymbol{A})=\varphi(\boldsymbol{B})=M$ with $\bar{\lambda}(M)\equiv I(A)$ (mod I_k). Conversely by Lemma 5 and $N(n,I)=1$, for $M,M'\in\partial\mathcal{H}'_n(2k)$ $\bar{\lambda}(M)=\bar{\lambda}(M')$ implies $\hat{\varphi}(F)\underset{w}{\sim}\hat{\varphi}(F')$ by choosing suitable W, W' and presentations F, F'.

Thus we have

THEOREM 13. *Let M, M' be two manifolds in $\partial\mathcal{H}'_n(2k)$ for $n=p$ (a prime) or $2p$ (twice an odd prime) and $k=2,3,4,5$ or 7. M is diffeomorphic to M' (we shall denote it by $M=M'$) if and only if $\bar{\lambda}(M)$ equals $\bar{\lambda}(M')$. Furthermore we have the following:*

Case $n=p=2$. $\bar{\lambda}(M)=\pm1+8s$ for a certain integer $0\leqq s<I_k$. $M=M_1\# M_0\#\cdots\# M_0$ (s-fold connected sum of M_0) if $\bar{\lambda}(M)=1+8s$, $M=(-M_1)\# M_0\#\cdots\# M_0$ (s-fold connected sum of M_0) if $\bar{\lambda}(M)=-1+8s$.

Case $n=p\equiv 3$ (mod 4). $\bar{\lambda}(M)=\pm2+8s$, for $0\leqq s<I_k$. $M=M_2^p\# M_0\#\cdots\# M_0$ (s-fold connected sum of M_0) if $\bar{\lambda}(M)=2+8s$ and $M=(-M_2^p)\# M_0\#\cdots\# M_0$ (s-fold connected sum of M_0) if $\bar{\lambda}(M)=-2+8s$.

Case $n=p\equiv 1$ (mod 4). $\bar{\lambda}(M)=8s$ or $4+8s$ $(0\leqq s<I_k)$. $M=M_3^p\# M_0\#\cdots\# M_0$ (s-fold connected sum of M_0) if $\bar{\lambda}(M)=8s$ and $M=M_4^p\# M_0\#\cdots\# M_0$ (s-fold connected sum of M_0) if $\bar{\lambda}(M)=4+8s$.
Furthermore we have $M_3^p=-M_3^p$ and $M_4^p=(-M_4^p)\# M_0$.

Case $n=2p$ (p*: an odd prime*). $\bar{\lambda}(M)\equiv 1 \pmod 2$.

(i) $p\equiv 1 \pmod 4$. $\bar{\lambda}(M)=\pm 1+8s$ *or* $\pm 3+8s$ *for* $0\leqq s<I_k$. $M=M_1\# M_3^p \# M_0\# \cdots M_0$ (*resp.* $(-M_1)\# M_3^p\# M_0\# \cdots \# M_0$) (*s-fold connected sum of* M_0) *if* $\bar{\lambda}(M)=1+8s$ (*resp.* $-1+8s$) *and* $M=(-M_1)\# M_4^p\# M_0\# \cdots \# M_0$ (*resp.* $M_1\# M_4^p \# M_0\# \cdots \# M_0$) (*s-fold connected sum of* M_0) *if* $\bar{\lambda}(M)=3+8s$ (*resp.* $5+8s$).

(ii) $p\equiv 3 \pmod 4$. $\bar{\lambda}(M)=\pm 1+8s$ *or* $\pm 3+8s$ *for* $0\leqq s<I_k$. $M=(-M_1) \# M_2^p\# M_0\# \cdots \# M_0$ (*resp.* $M=M_1\#(-M_2^p)\# M_0\# \cdots \# M_0$) (*s-fold connected sum of* M_0) *for* $\bar{\lambda}(M)=1+8s$ (*resp.* $\bar{\lambda}(M)=-1+8s$), $M=M_1\# M_2^p\# M_0\# \cdots \# M_0$ (*resp.* $M=(-M_1)\#(-M_2^p)\# M_0\# \cdots \# M_0$) (*s-fold connected sum of* M_0) *for* $\bar{\lambda}(M)=3+8s$ (*resp.* $\bar{\lambda}(M)=-3+8s$).

THEOREM 14. *Let* M *be a manifold in* $\partial\mathcal{H}_n'(2k)$ *for* $n=p_1\cdots p_s$ ($s\geqq 1$) *having distinct prime factors* ($k=2, 3, 4, 5$ *and* 7). M *can be obtained by forming connected sums of some of the standard manifolds:* M_0, M_1 *for* $p_j=2$, $M_2^{p_j}$ *for* $p_j\equiv 3 \pmod 4$, $M_3^{p_j}$, $M_4^{p_j}$ *for* $p_j\equiv 1 \pmod 4$ *and manifolds with the reversed orientation.*

REMARK. We cannot decide whether the representation of M by the connected sum operation of some standard manifolds in Theorem 14 is unique or not, except in the case of Theorem 13.

COROLLARY 15. *There exist precisely 1984 distinct* 4*-connected closed oriented differentiable* 11*-manifolds whose fifth homology groups are cyclic of order* $n=p$ (p*: a prime*) (*resp.* $n=2p$, p*: an odd prime*). *There exist precisely* 56 *distinct* 2*-connected closed differentiable* π*-manifolds of dimension* 7 *whose third homology groups are cyclic of order* $n=p$ (p*: a prime*) (*resp.* $n=2p$, p*: an odd prime*). *They all have the same homotopy type and the invariant* $\bar{\lambda}$ *characterizes these manifolds. There is only one topological manifold for* $p=2$ *or* $p\equiv 3 \pmod 4$ *and there are at most two for* $p\equiv 1 \pmod 4$.

COROLLARY 16. *There exist precisely* 16256 (*resp.* 523264, 67100672) *distinct* 6*-connected* (*resp.* 8*-connected,* 12*-connected*) *closed oriented differentiable* 15*-manifolds* (*resp.* 19*-manifolds,* 27*-manifolds*) *which bound* π*-manifolds and whose first non-trivial homology groups are cyclic of a prime order or twice an odd prime order.*

§ 5. 3-sphere bundles over the 4-sphere.

In this section we shall compute the invariant $\bar{\lambda}$ for total spaces of 3-sphere bundles over the 4-sphere. First we shall recall some results about them (cf. Tamura [**15**]).

Let $\rho, \sigma: S^3\to SO(4)$ be maps defined by

$$\rho(u)v=uvu^{-1}, \qquad \sigma(u)v=uv,$$

where u and v denote quaternions with norm 1. The homotopy classes $\{\rho\}$ and $\{\sigma\}$ are generators of $\pi_3(SO(4)) \cong Z+Z$. Let

$$\mathfrak{B}_{m,n} = \{B_{m,n}, \pi_{m,n}, S^4, S^3\}$$

be the S^3-bundle over S^4 with the characteristic map $m\{\rho\}+n\{\sigma\}$. Moreover let

$$\bar{\mathfrak{B}}_{m,n} = \{\bar{B}_{m,n}, \bar{\pi}_{m,n}, S^4, D^4\}$$

be the 4-cell bundle over S^4 associated with $\mathfrak{B}_{m,n}$. $B_{m,n}$ and $\bar{B}_{m,n}$ have differentiable structures naturally defined by bundle structures. Thus $B_{m,n}$ is a closed 7-manifold and $\bar{B}_{m,n}$ is a compact 8-manifold with the boundary $\partial \bar{B}_{m,n} = B_{m,n}$.

Non-trivial homology groups of $B_{m,n}$ are as follows:

$$H_0(B_{m,n}) \cong H_7(B_{m,n}) \cong H_4(B_{m,0}) \cong Z, \qquad H_3(B_{m,n}) \cong Z/nZ.$$

$\bar{B}_{m,n}$ has the homotopy type of S^4.

The first Pontrjagin class of $\bar{B}_{m,n}$ (resp. $B_{m,n}$) is given by

$$p_1(\bar{B}_{m,n}) = \pm 2(2m+n)\alpha \text{ (resp. } p_1(B_{m,n}) = \pm 4m\alpha')$$

where α is a generator of $H^4(\bar{B}_{m,n}) \cong Z$ (resp. α' is a generator of $H^4(B_{m,n}) \cong Z/nZ$). Let $a \in H_4(\bar{B}_{m,n})$ be the dual of α. We choose the orientation of $\bar{B}_{m,n}$ in such a way that $\langle a, a\rangle$ is positive and the orientation of $B_{m,n}$ to be compatible with that of $\bar{B}_{m,n}$.

$\mathfrak{B}_{-1,2}$ is the tangent sphere bundle of S^4 and $B_{-1,2}$ bounds a parallelizable 8-manifold $\bar{B}_{-1,2}$. $\bar{B}_{-1,2}$ is of signature 1 and diffeomorphic to W_1 as stated above (cf. §§ 1, 4). $B_{pm,p}$ with odd primes p, are π-manifolds for arbitrary integers m (cf. Tamura [**17**], Lemma 2).

Let us compute the invariant $\bar{\lambda}$ of $B_{pm,p}$. Suppose that $B_{pm,p}$ bounds a compact parallelizable 3-connected oriented differentiable 8-manifold W. Let V be the closed 2-connected oriented differentiable 8-manifold obtained from the disjoint union of $\bar{B}_{pm,p}$ and $-W$ by identifying their common boundary $B_{pm,p}$. We have $I(V) = I(\bar{B}_{pm,p}) - I(W) = 1 - I(W)$, $p_1^2(V)[V] = 2^2p^3(2m+1)^2$ by $i^{*-1}(\alpha \cup \alpha)[V] = p$ where $i: \bar{B}_{pm,p} \to V$ is the natural inclusion map. Thus the index theorem $I(V) = \frac{1}{45}(7p_2(V) - p_1^2(V))[V]$ and the integrality of $\hat{A}$-genus $\hat{A}(V) = \frac{1}{2^7 \cdot 45}(-4p_2(V) + 7p_1^2(V))[V]$ imply $I(W) \equiv 1 - p^3(2m+1)^2 \bmod 2^5 \cdot 7$.

Thus we obtain

THEOREM 17. *The residue classes* $\bar{\lambda}(B_{pm,p}) \bmod 2^5 \cdot 7$ *for an arbitrary* m *and odd primes* p *are as follows:*

$$\bar{\lambda}(B_{pm,p}) \equiv 1 - p^3 + 4m(m+1) \pmod{2^5 \cdot 7} \quad \textit{if} \quad p \equiv 3, 19, 27 \pmod{28},$$
$$1 - p^3 - 52m(m+1) \pmod{2^5 \cdot 7} \quad \textit{if} \quad p \equiv 5, 13, 17 \pmod{28},$$

$$1-p^3-28m(m+1) \pmod{2^5\cdot 7} \quad \text{if} \quad p\equiv 7 \pmod{28},$$
$$1-p^3-4m(m+1) \pmod{2^5\cdot 7} \quad \text{if} \quad p\equiv 1, 9, 25 \pmod{28},$$
$$1-p^3+52m(m+1) \pmod{2^5\cdot 7} \quad \text{if} \quad p\equiv 11, 15, 23 \pmod{28},$$
$$1-p^3+28m(m+1) \pmod{2^5\cdot 7} \quad \text{if} \quad p\equiv 21 \pmod{28}.$$

As is easily seen, we have $\bar{\lambda}(B_{pm,p})\equiv -2, 2, 0$ or $4 \pmod 8$ for $p\equiv 3, 7, 1$ or $5 \pmod 8$.

COROLLARY 18. *$B_{pm,p}$ for arbitarary m and a fixed odd prime p, are homeomorphic to each other and $B_{pm,p}$ is diffeomorphic to $B_{pm',p}$ as oriented manifolds if and only if*

$$m(m+1)\equiv m'(m'+1) \pmod 8 \quad \text{for} \quad p\equiv 7, 21 \pmod{28},$$
$$m(m+1)\equiv m'(m'+1) \pmod{56} \quad \text{for} \quad p\not\equiv 7, 21 \pmod{28}.$$

REMARK. Tamura proved in his paper [16] that $B_{pm,p}$ for arbitrary m and a fixed odd prime p are homeomorphic to each other.

Japan Women's University

References

[1] J.F. Adams, Applications of the Grothendieck-Atiyah-Hirzebruch functor $K(X)$, Colloquium on Algebraic Topology, Mat. Inst. Aarhus Univ., 1962, 104-113.
[2] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces III, Amer. J. Math. **82** (1960), 491-504.
[3] M. Eichler, Quadratische Formen und orthogonale Gruppen, Berlin, 1952.
[4] M. Eichler, Die Ähnlichkeitsklassen indefiniter Gitter, Math. Z., **55** (1952), 216-252.
[5] F. Hirzebruch, Neue topologische Methoden in der algebraischen Geometrie, Berlin, 1956.
[6] B.W. Jones, The arithmetic theory of quadratic forms, New York, 1950.
[7] J. Milnor, On simply connected 4-manifolds, Symp. Intern. de Topologia Algebraica, Mexico, 1958, 122-128.
[8] J. Milnor, Differentiable manifolds which are homotopy spheres, (mimeographed), Princeton Univ., 1959.
[9] J. Milnor, A procedure for Killing homotopy groups of differentiable manifolds, Proc. of Symp. Differential Geometry, 1961, 39-55.
[10] J. Milnor and M. Kervaire, Bernoulli numbers, homotopy groups and a theorem of Rohlin, Proc. of Intern. Congr. of Math., Edinburgh, 1958, 454-458.
[11] S. Smale, Generalized Poincaré's conjecture in dimensions greater than four, Ann. of Math., **74** (1961), 391-406.
[12] S. Smale, On the structure of 5-manifolds, Ann. of Math., **75** (1962), 38-46.
[13] S. Smale, On the structure of manifolds, Amer. J. Math., **84** (1962), 387-399.
[14] N.E. Steenrod, The topology of fibre bundles, Princeton, 1951.
[15] I. Tamura, On Pontrjagin classes and homotopy types of manifolds, J. Math. Soc. Japan, **9** (1957), 250-262.
[16] I. Tamura, Homeomorphy classification of total spaces of sphere bundles over spheres, J. Math. Soc. Japan, **10** (1958), 29-43.
[17] I. Tamura, Differentiable 7-manifolds with a certain homotopy type, J. Math. Soc. Japan, **14** (1962), 292-299.
[18] H. Toda, Composition methods in homotopy groups of spheres, Ann. of Math. Studies, 49, Princeton, 1962.
[19] C.T.C. Wall, Classification of $(n-1)$-connected $2n$-manifolds, Ann. of Math., **75** (1962), 163-189.