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Some aspects of real-analytic manifolds and differentiable manifolds

By Kôji SHIGA

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Introduction

In 1958 C. B. Morrey [7] and H. Grauert [3] proved that any real-analytic manifold can be real-analytically imbedded in a Euclidean space by a regular and proper mapping. This, combined with the result of H. Whitney [12], shows that any differentiable manifold has a unique real-analytic structure, or in other words, every manifold has as many C^{ω}-structures as C^1-structures. We shall refer to this fact in speaking of the "constancy of differentiable structure" of manifolds.

Now, a fundamental tool in Whitney's work [12] was the approximation theorem, saying that any differentiable mapping f between two real-analytic manifolds M, N can be arbitrarily well approximated by a real-analytic mapping φ (cf. § 1 for an exact formulation). Actually Whitney [12] proved this under the condition that M and N are realized in a Euclidean space, but this condition can be removed owing to the result of Morrey-Grauert [7], [3]. From this it follows in particular that to any regular mapping f we can find a regular real-analytic approximation φ. Thus φ will be an analytic homeomorphism if f is a homeomorphism; i. e., the uniqueness of C^{ω}-structure compatible with a C^1-structure of a manifold—one half of the "constancy of differentiable structure"; another half being the existence of C^{ω}-structure—is a direct consequence of the approximation theorem.

In the present paper, we shall first state the generalized approximation theorem and some immediate consequences of it (§ 1). Now, corresponding to the case where f is injective, the approximation theorem has an application to fibre spaces; any differentiable fibre space $P=P(B, \pi)$ possesses a unique real-analytic structure as a fibre space when the projection π is proper (§ 2). On the other hand, in the case where f is injective, the approximation theorem combined with G. D. Mostow's theorem [8] concerning the equivariant imbeddings yields results related to transformation groups, one of which is formulated as follows: Let G be a compact Lie group acting on a compact real-analytic manifold M as a C^1-transformation group. Then G necessarily acts on M as a C^{ω}-transformation group (§3).

In §4, we deal with the classification problem of C^{ω}-fibre-bundles from the point of view of approximation theorem. The results obtained here are stated as follows. Any principal fibre bundle P of class C^0 has a real-analytic fibre bundle structure $\tilde{P}$, compatible with the given bundle structure. Moreover $\tilde{P}$ is uniquely determined. This essentially gives an answer of a problem raised by H. Cartan [2].

Finally, the author would like to express his gratitude to Professor S. Iyanaga for his constant encouragements. Also he thanks Professor T. Nagano for his valuable advices.

NOTATIONS: We assume manifolds to be paracompact, but not necessarily connected, while we assume for convenience' sake each connected component to have the same dimension. C^s-manifold $(1 \leqq s \leqq \infty, \omega)$ is, by definition, a topological manifold whose coordinate transition functions admit continuous derivatives of the order up to s if $s \leqq \infty$, and are real-analytic if $s = \omega$. $\{U_\iota; (x_\iota)\}_{\iota \in I}$ denotes a system of local coordinates on M: U_ι are coordinate neighborhoods with coordinates $(x_\iota) = (x_\iota^1, \cdots, x_\iota^m)$, where $m = \dim M$. It is useful to introduce a notation r^*, which we define by $r^* = r$ in case $1 \leqq r < \infty$, any positive integer in case $r = \infty, \omega$. Also we write conventionally $s < \omega$ for any $s = 1, 2, \cdots, \infty$. $P(B, F, G, \pi)$ denotes a fibre bundle with the base space B, the typical fibre F, the structural group G and the projection π.

1. Approximation theorem.

The following is well known [12]:

WHITNEY'S IMBEDDING THEOREM: *Any separable C^r-manifold M $(1 \leqq r \leqq \infty)$ can be imbedded in a Euclidean space E^k by a regular and proper C^r-mapping, and the imbedded manifold may be taken as a C^{ω}-manifold.*

The latter half of this statement obviously implies that any C^r-manifold has a C^{ω}-structure induced by the structure of the imbedded C^{ω}-manifold, which is compatible with the given C^r-structure. A similar result on C^{ω}-manifolds was obtained by C. B. Morrey [7] for compact case, and by Grauert [3] for general case:

MORREY-GRAUERT'S IMBEDDING THEOREM: *Any separable C^{ω}-manifold M can be imbedded in a Euclidean space E^k by a regular and proper C^{ω}-mapping.*

We wish to formulate the approximation theorem due essentially to H. Whitney [12]. Let M and N be C^s-manifolds with dimension m and n, respectively. $C^r(M, N)$ denotes the totality of C^r-mappings from M into N, where r satisfies $1 \leqq r \leqq s$. Let $f \in C^r(M, N)$. We need a precise description of C^s-mappings which approximate f. For this purpose we take local coordinate neighborhoods $\{U_\iota; (x_\iota)\}_{\iota \in I}$, $\{V_\iota\}$ on M and $\{W_\lambda\}_{\lambda \in \Lambda}$ on N such that $V_\iota \Subset U_\iota$

(this means that the closure of V_ι is contained in U_ι as a compact set). We assume that $\{U_\iota\}$, $\{V_\iota\}$ and $\{W_\lambda\}$ are all locally finite and that for any ι there is at least one $\lambda(\iota)$ such that $f(U_\iota)\Subset W_{\lambda(\iota)}$. Such coordinates obviously exist. Let $\mathcal{E}=\{\varepsilon_\iota\}_{\iota\in I}$ be a family of positive numbers indexed by $\iota\in I$. We shall call such $\mathcal{E}$ a *positive family*. Then we say that $\varphi\in C^s(M, N)$ gives an *$\mathcal{E}$-approximation to f up to order r^* or an $(\mathcal{E}, r^*)$-approximation for a given positive family $\mathcal{E}$* when the following conditions are satisfied:

(i) $\varphi(U_\iota)\Subset W_{\lambda(\iota)}$

(ii) $\| D_\iota^\nu f-D_\iota^\nu\varphi \|_{V_\iota}=\sup\limits_{1\leqq j\leqq n}\sup\limits_{x\in V_\iota}|D_\iota^\nu f_{\lambda(\iota)}^j(x)-D_\iota^\nu\varphi_{\lambda(\iota)}^j(x)|<\varepsilon_\iota$

for $\nu=0, 1, \cdots, r^*$. Here $(f_{\lambda(\iota)}^1, \cdots, f_{\lambda(\iota)}^n)$ and $(\varphi_{\lambda(\iota)}^1, \cdots, \varphi_{\lambda(\iota)}^n)$ denote coordinate components of f and φ on $W_{\lambda(\iota)}$, and the abbreviated notations $D_\iota^\nu f$ means 'in general'

$$D_\iota^\nu f(x)=\frac{\partial^\nu f}{(\partial x_\iota^1)^{\nu_1}\cdots(\partial x_\iota^m)^{\nu_m}},$$

where $\nu=\nu_1+\cdots+\nu_m$. ((ii) means that $\| D_\iota^\nu f-D_\iota^\nu\varphi \|_{V_\iota}$ is defined by the right hand side of (ii) for any kind of D_ι^ν, i.e., for any choice of $\nu_1, \cdots, \nu_m$ satisfying $\nu=\nu_1+\cdots+\nu_m$, and that $\| D_\iota^\nu f-D_\iota^\nu\varphi \|_{V_\iota}<\varepsilon_\iota$ is valid for every $\iota\in I$ and $\nu=0, 1, \cdots, r^*$). This definition of $(\mathcal{E}, r^*)$-approximation depends on the choice of local coordinates. But, as is easily seen, the property that for any $\mathcal{E}=\{\varepsilon_\iota\}$ f has $(\mathcal{E}, r^*)$-approximations each of which belongs to $C^s(M, N)$ does not depend on local coordinates used in the definition. In the sequel we deal only with such properties. Hence we shall no more refer explicitly to local coordinates.

We can now formulate the approximation theorem.

THEOREM A. *Let M and N be C^s-manifolds with $1\leqq s\leqq\omega$. Assume that a C^r-mapping f of M into N is given where r satisfies $1\leqq r\leqq s$. Then for any positive family $\mathcal{E}=\{\varepsilon_\iota\}_{\iota\in I}$ there exists a C^s-mapping φ of M into N giving an $(\mathcal{E}, r^*)$-approximation to f satisfying the "coincidence condition"*

$$D^\nu\varphi(x_l)=D^\nu f(x_l), \qquad \nu=0, 1, \cdots, r^*,$$

where $x_1, x_2, \cdots, x_l, \cdots$ is any sequence of points of M without accumulation points. In case where f is a proper mapping, φ can be taken as a proper mapping.

Actually H. Whitney [12] proved this theorem in case where M and N are imbedded manifolds. However the imbeddability of M and N imposes no restriction by virtue of Morrey-Grauert's theorem, which is applicable to each connected components of M and N. We remark that the above theorem holds true also in case $r=0$ (of course except the statement on regularity).

The following result is useful for our applications of Theorem A.

THEOREM B. *Let M and N be C^s-manifolds. Let f be a C^r-mapping f of M into N where r satisfies $1\leqq r\leqq s$. Assume that f has the one of the follow-*

ing properties:

(i) *f is C^r-immersion.*

(ii) *f is C^r-imbedding.*

(iii) *f is C^r-homeomorphism.*

Then, for a suitable choice of $\mathcal{E}_0$, any $(\mathcal{E}, r^*)$-approximation φ to f has the corresponding properties in the C^s-category, respectively, where $\mathcal{E}$ is so chosen that $0 < \mathcal{E} < \mathcal{E}_0$.

This theorem was found by H. Whitney [**12**]. A proof was given by J. Munkres [**9**]; to his proof we give a remark that the part of (iii) in the above theorem can be immediately deduced from (i) and (ii) when we consider the image of each connected component of M through φ.

COROLLARY 1. *M and N being as above, assume that we have a C^r-homeomorphism f of M onto N with $1 \leqq r \leqq s$. Then there exists a C^s-homeomorphism φ of M onto N which approximates f arbitrarily well up to order r^*.*

From this we can conclude the following

UNIQUENESS THEOREM. *Any C^r-manifold has a unique C^s-structure compatible with the given C^r-structure, where $1 \leqq r \leqq s$.*

Thus, in a particular case $s = \omega$, we have obtained three fundamental theorems for C^ω-manifold: (a) Morrey-Grauert's Imbedding Theorem, (b) Approximation Theorem, (c) Uniqueness Theorem. It should be noted that these are essentially equivalent. In fact, the implications (a)$\Rightarrow$(b)$\Rightarrow$(c) have been given above; (c)$\Rightarrow$(a) is seen as follows: Let M be a C^ω-manifold. Then M can be differentiably imbedded in E^k such that its image is a closed C^ω-submanifold M^*. We denote by $\tilde{M}$ the C^ω-manifold whose underlying space is M and whose C^ω-structure is induced from M^*. The validity (c) means $\tilde{M} = M$ as C^ω-manifolds. Hence $M \to M^*$ gives a C^ω-imbedding of M. This proves (a).

REMARK. B. Malgrange [**4**] has shown that (a), (b) and (c) are also equivalent to the following: Any C^ω-manifold admits a Riemannian metric of class C^ω.

Referring to the coincidence condition of Theorem A, we can easily deduce

COROLLARY 2 (H. Cartan). *Let M be a C^ω-manifold and $x_1, x_2, \cdots, x_l, \cdots$ be a sequence of points of M without accumulation points. Let r be any positive integer. Then there is a C^ω-function on M such that its partial derivatives up to order r take any assigned values at $x_1, x_2, \cdots, x_l, \cdots$.*

H. Cartan [**2**] proved this result as a consequence of his theorem that any C^ω-manifold necessarily becomes a 'real Stein manifold'. Now, let M be a *connected* C^ω-manifold. Observe that the group composed of all diffeomorphisms on M transitively operates on M. Hence for any points $x_0, y_0 \in M$ we can find a diffeomorphism f such that $f(x_0) = y_0$. Approximate f by a C^ω-homeomorphism φ satisfying $\varphi(x_0) = f(x_0)$. Then we have

COROLLARY 3. *The group composed of all C^{ω}-homeomorphisms on M transitively operates on M.*

2. Fibre spaces.

DEFINITION 1. A C^s-manifold $P=P(B,\pi)$ is called a *C^s-fibre space* if the base space B is a C^s-manifold and the projection $\pi: P\to B$ is a regular onto C^s-mapping.

Clearly the total space of a C^s-fibre bundle gives an example of a C^s-fibre space. For any C^s-fibre space P, put $F(b)$ for $\pi^{-1}(b)$; $F(b)$ is a closed C^s-submanifold of P for each $b\in B$ and is referred to as the *fibre* over b. It is evident that $P=\bigcup_{b\in B} F(b)$. Since π is a regular mapping of P onto B, for any $p_0\in F(b_0)$ there is a coordinate neighborhood $U(p_0)$ in P so that local coordinates on $U(p_0)$ are given by $(y^1\circ\pi, \cdots, y^n\circ\pi, x^1, \cdots, x^m)$, where $y^1, \cdots, y^n$ denote local coordinates on $\pi(U(p_0))$. Accordingly, $F(b_1)\cap U(p_0)$ is characterized by the equations

$$y^1\circ\pi(p)=y^1(b_1), \cdots, y^n\circ\pi(p)=y^n(b_1),$$

where b_1 is in $\pi(U(p_0))$. In this sense we may well say that $F(b)$ are closed submanifolds depending on b as parameters of class C^s, in terms of local coordinates on P.

Let $P=P(B,\pi)$ be a C^s-fibre space. Regard P simply as a C^s-manifold and imbed P in a Euclidean space E^k by a regular and proper C^s-mapping. We identify P with the imbedded manifold. Each fibre $F(b)$, $b\in B$, is a closed C^s-submanifold of P and thus of E^k, whence to each point $p\in F(b)$ we can attach the normal plane $\beta(p)$ to $F(b)$ passing through p, the dimension of which is denoted by l. $G(k,l)$ denotes the Grassmann manifold consisting of all l-dimensional linear spaces passing through the origin of E^k, with known C^{ω}-structure. Now $\beta(p)$ naturally gives rise to a mapping $\bar{\beta}$ from P into $G(k,l)$: $\bar{\beta}(p)$ is obtained by the parallel displacement of $\beta(p)$ to the plane passing through the origin. $\bar{\beta}(p)$ is a C^{s-1}-mapping in p. By making use of Theorem A, we can find a mapping $\bar{\alpha}$ of P into $G(k,l)$ such that $\bar{\alpha}(p)$ is a C^s-mapping in p and that $\bar{\alpha}$ approximates $\bar{\beta}$ so well that each $\alpha(p)$ (plane passing through p parallel to $\bar{\alpha}(p)$) is independent of the tangent plane to $F(b)$ at p. Express $\alpha(p)$ in terms of local coordinates and apply the implicit function theorem. Then it is easily verified that $\alpha(p)$ has the following properties [**12**; p. 667].

(i) Tubular neighborhood $T_\rho(F(b))$ along the fibre $F(b)$ is filled up by $\alpha(p)$ in one-to-one way where

$$T_\rho(F(b))=\{q\mid \|q-p\|<\rho(p),\ p\in F(b)\},$$

$\|\cdot\|$ being a Euclidean distance in E^k, and $\rho(p)$ is a positive continuous function on P chosen suitably.

(ii) Let $p \in F(b)$ and $q \in \alpha(p)$. The assignment $q \to p$ gives a C^s-mapping of $T_\rho(F(b))$ onto $F(b)$, depending on b as C^s-mapping. We denote this mapping by $\tau(q\,; b)$.

Henceforth, a family of planes $\alpha(p)$ and the projection $\tau(q\,; b)$ are fixed. $\alpha(p)$ is called to be a *quasi-normal plane* to $F(b)$ at p. Now consider the case where the projection π is proper. Then for any $b \in B$, $F(b) = \pi^{-1}(b)$ is compact, and so we can find a neighborhood $V(b)$ such that $\pi^{-1}(V(b))$ is C^s-homeomorphic to $F(b) \times V(b)$; in fact, this C^s-homeomorphism can be given through the mapping $\tau(q\,; b)$ on a tubular neighborhood $T_\rho(F(b)) \cap P$ of $F(b)$.

Let P and P_1 be C^s-fibre spaces and suppose that we have a C^s-mapping f of P into P_1. If f sends each fibre $F(b)$ of P into a fibre $F_1(b_1)$ of P_1, we call f a *fibre-preserving map.* If f is C^s-homeomorphic and fibre-preserving, we say that P and P_1 are *C^s-equivalent* to each other. A fundamental theorem is now stated as follows.

THEOREM 2. *Let $P = P(B, \pi)$ be a C^s-fibre space. Assume that π is proper. Then there exists a C^ω-fibre space $P_1 = P_1(B_1, \pi_1)$ which is C^s-equivalent to P.*

Note that if P_1 is such a fibre space, the underlying C^s-structure of P_1 is essentially the same to that of P. Moreover the proof below shows that the base space B_1 may be simply regarded as B when both C^s-structures are concerned.

PROOF. P and B possess a unique C^ω-structure compatible with the given C^s-structures, respectively. Hence we may assume without loss of generality P and B are C^ω-manifolds. Since π is a regular and proper C^s-mapping of P onto B, it follows from Theorem A that there exists a regular and proper C^ω-mapping π_1 of P into B, approximating π very well up to order s^*.

We wish to show that π_1 is an onto-mapping. Without loss of generality, we may assume that P (and so B) is connected. Since π_1 is regular, $\pi_1(P)$ is an open subset of B. On the other hand, $\pi_1(P)$ is closed. In fact, assume that $b_n \in \pi_1(P)$, $b_n \to b_0$. π being proper, we have $\pi^{-1}(V(b_0)) \cong F(b_0) \times V(b_0)$ for a properly chosen $V(b_0)$. Hence $\{\pi_1^{-1}(b_n)\}$ are assumed to be contained in a compact subset of P. Then it is easily seen that for some $p_0 \in P$ we have $\pi_1(p_0) = b_0$, which shows the desired result. Consequently, $\pi_1(P)$ is open and closed, whence we find that $\pi_1(P) = B$.

It follows that P has a C^ω-structure as a fibre space, having the base space B and the projection π_1. Put $P_1 = P(B, \pi_1)$ for this C^ω-fibre space. We have to prove that P_1 is C^s-equivalent to P if π_1 is suitably chosen. As remarked above, we see that $\pi^{-1}(V(b))$ is C^s-homeomorphic to $F(b) \times V(b)$, for some neighborhood $V(b)$ of V. We take π_1 so that π_1 approximates π well enough to satisfy $\pi(F_1(b)) \subset V(b)$, whence we obtain a mapping $\tau_b : F_1(b) \to F(b)$ through the projection $\tau(\ ; b)$. Expressing the local triviality $\pi^{-1}(V(b)) \cong F(b) \times V(b)$ explicitly, we can take local coordinates (x, y) around p $(\in F(b))$, such that x denote fibre coordinates of p and y denote base-space coordinates of p. Then $\pi^j(x, y) = y^j$ $(j = 1, 2, \cdots, n)$ and thus $\dfrac{\partial \pi^j}{\partial x^i} = 0$, $\dfrac{\partial \pi^j}{\partial y^{j'}} = \delta^j_{j'}$ $(i = 1, 2, \cdots, m)$. Observe that the values of $\dfrac{\partial \pi_1^j}{\partial x^i}$ and $\dfrac{\partial \pi_1^j}{\partial y^{j'}}$ are approximately equal to the corresponding values of π, since π_1 approximates π up to order 1. It follows that in each point (x, y) of $F_1(b)$ the tangent vectors to $F_1(b)$ are transversal to the vectors $\dfrac{\partial}{\partial y^j}$ $(j = 1, 2, \cdots, n)$. Thus τ_b induces a C^s-homeomorphism from a neighborhood $V(x, y) \cap F_1(b)$ into $F(b)$. This correspondence at each point of $F_1(b)$ together gives the C^s-mapping τ_b of $F_1(b)$ into $F(b)$ which is a local homeomorphism. This shows in particular that the image of $F_1(b)$ is an open subset of $F(b)$. On the other hand, $F_1(b)$ is compact since π_1 is proper. From this it is easily verified that τ_b gives a covering map of $F_1(b)$ onto $F(b)$. Now, for a properly chosen neighborhood $V(b)$ of b, we have $\pi_1^{-1}(V(b)) \cong F_1(b) \times V(b)$ and $\pi^{-1}(V(b)) \cong F(b) \times V(b)$. Moreover, for $c \in V(b)$, τ_c depends on c as C^s-parameters. Hence each covering map τ_c of $F_1(c)$ onto $F(c)$ naturally induces a covering map of $F_1(b) \times V(b)$ onto $F(b) \times V(b)$.

Accordingly, all τ_b, $b \in B$, together give rise to a covering map Ψ of P_1 onto P, i.e., Ψ is defined to be $\Psi \mid F_1(b) = \tau_b$ $(b \in B)$. Since P_1 is homeomorphic to P as topological space, we can conclude that Ψ is a trivial covering map, in other words, Ψ is a C^s-homeomorphism. From the construction of Ψ, this in turn implies that Ψ is fibre-preserving. Hence the fibre spaces P and P_1 are C^s-equivalent, which is the desired result.

Next we wish to generalize the approximation theorem to the case of fibre-preserving maps. The following Proposition 1 is essential for that purpose. Suppose that we have a C^ω-function defined on a set $W \times W'$, say $\Phi(X, Y)$ $(X \in W,\ Y \in W')$, where W, W' are open sets in E^l, $E^{l'}$ respectively. Letting U be an open set in E^k, take f and g such that $f \in C^s(U, W)$ and $g \in C^s(U, W')$. We are concerned with C^ω-approximations to the function $\Phi(f, g)$. We obtain

PROPOSITION 1. *Let F be an $(\mathcal{E}_1, s^*)$-approximation to f and G be an $(\mathcal{E}_2, s^*)$-approximation to g, where F and G are C^ω-mappings from U into open sets W and W' respectively. Let $\mathcal{E}$ be any given positive family. Then, by a suitable choice of $\mathcal{E}_1$ and $\mathcal{E}_2$, $\Phi(F, G)$ gives an $(\mathcal{E}, s^*)$-approximation to $\Phi(f, g)$. $\Phi(F, G)$ is clearly a C^ω-function on U.*

PROOF. From the definition of $\mathcal{E}$-approximation, it follows immediately that the proposition is reduced to the case where U and U' have compact closures in the domains of f and g, respectively. It is then clear that $\Phi(F, G)$ gives a sufficiently good approximation to $\Phi(f, g)$ in order 0. Set $\Psi(x) = \Phi(f(x), g(x))$ and $\Psi_1(x) = \Phi(F(x), G(x))$. Differentiating them, we obtain

$$\frac{\partial \Psi}{\partial x^i} = \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial \Phi}{\partial X^\alpha}\Big|_{X=f(x)} + \sum_\beta \frac{\partial g^\beta}{\partial x^i} \frac{\partial \Phi}{\partial Y^\beta}\Big|_{Y=g(x)},$$

$$\frac{\partial \Psi_1}{\partial x^i} = \sum_\alpha \frac{\partial F^\alpha}{\partial x^i} \frac{\partial \Phi}{\partial X^\alpha}\Big|_{X=F(x)} + \sum_\beta \frac{\partial G^\beta}{\partial x^i} \frac{\partial \Phi}{\partial Y^\beta}\Big|_{Y=G(x)}.$$

Since $\Phi(X, Y)$ is a C^ω-function, all partial derivatives of Φ satisfy the Lipschitz condition on any compact set $(\subset W \times W')$. From this it follows that $\frac{\partial \Phi}{\partial X^\alpha}\Big|_{X=F(x)}$ and $\frac{\partial \Phi}{\partial Y^\beta}\Big|_{Y=G(x)}$ have an upper bound K, independent of F and G. Therefore we get

$$\left|\frac{\partial \Psi}{\partial x^i} - \frac{\partial \Psi_1}{\partial x^i}\right| \leqq K \sum_\alpha \left|\frac{\partial f^\alpha}{\partial x^i} - \frac{\partial F^\alpha}{\partial x^i}\right| + K \sum_\beta \left|\frac{\partial g^\beta}{\partial x^i} - \frac{\partial G^\beta}{\partial x^i}\right|$$

whence $\Psi_1 = \Phi(F, G)$ gives an $\mathcal{E}$-approximation of $\Psi = \Phi(f, g)$ up to order 1 when $\mathcal{E}_1$ and $\mathcal{E}_2$ are taken small enough. The proof proceeds in the same way in case of higher order approximations.

THEOREM 2. *Let $P_1(B_1, \pi_1)$ and $P(B, \pi)$ be C^s-fibre spaces where π is proper, and suppose r satisfies $1 \leqq r \leqq s$. Assume that a fibre-preserving C^r-map f of P_1 into P is given. Then for any positive family $\mathcal{E} = \{\varepsilon_\iota\}$ we have a fibre-preserving C^s-map φ of P_1 into P giving an $(\mathcal{E}, r^*)$-approximation to f.*

PROOF. By Theorem 1 we can take local coordinates on P_1 and P such that P_1 and P have structures of C^ω-fibre spaces with respect to these coordinates. Hence we may assume that P_1 and P are C^ω-fibre spaces. Take a C^ω-Riemannian metric on P. Since each fibre $F(b)$ $(b \in B)$ is a closed C^ω-submanifold of P, it is possible to find a tubular neighborhood $T(F(b))$ such that $T(F(b))$ is simply covered by normal geodesics ν, passing through each point p of $F(b)$. Thus we have a parametrization of $T(F(b))$, in terms of fibre coordinates (x) of p, the direction α to ν and the length s of ν. For $q \in T(F(b))$, denote this parametrization by

$$q = L(x, \alpha, s; b);$$

as is well known, L is a C^ω-function in $(x, \alpha, s; b)$. Set $x = \mu(q; b)$. Then by the implicit function theorem $\mu(q; b)$ is a C^ω-function in $(q; b)$.

The given C^r-mapping f naturally induces a C^r-mapping g of B_1 into B. Choose a C^ω-mapping Ψ of B_1 into B approximating g up to order r^* closely enough to satisfy $f(p_1) \in T(\Psi(\pi_1(p_1)))$. This is possible since π is proper by the assumption. Next take a C^ω-approximation Φ to f up to order r^* such that $\Phi(p_1) \in T(\Psi(\pi_1(p_1)))$. Now we define the C^ω-mapping φ of P_1 into P by setting

$$\varphi(p_1)=\mu(\Phi(p_1)\,;\,\Psi(\pi_1(p_1)))\,.$$

φ is obviously a fibre-preserving map. On the other hand, we have

$$f(p_1)=\mu(f(p_1)\,;\,g(\pi_1(p_1)))\,.$$

Hence Proposition 1 immediately yields the desired result.

Theorem 2, combined with Corollary 1 to Theorems A, B, gives

THEOREM 3. *Let r and s satisfy $1\leqq r\leqq s$. If C^s-fibre spaces P and P_1 having the proper projections are C^r-equivalent to each other through a C^r-mapping f, then they are C^s-equivalent to each other through a C^s-mapping φ. φ is taken arbitrarily near to f up to order r^*.*

3. Transformation groups.

Let G be a Lie group acting on a C^s-manifold M as a transformation group. We say that G is a *C^s-transformation group* on M $(1\leqq s\leqq\omega)$ when, for each fixed $g\in G$, gx gives a C^s-mapping of M onto M. The following is known [**6**; p. 213]: If G is a C^s-transformation group on M, then the mapping defined by $(g, x)\rightarrow gx$ gives a C^s-mapping of $G\times M$ onto M. G_p denotes the stabilizer of a point $p\in M$: $G_p=\{g\,|\,gp=p\}$. The orbits G_p and G_q are called *equivalent* if $G_q=gG_pg^{-1}$ for some $g\in G$. If M is a compact C^s-manifold and G a compact Lie group acting on M as C^s-transformation group, then G has only a finite number of inequivalent orbits.

The following theorem concerns a classical problem on transformation groups treated by Bochner-Montgomery [**1**].

THEOREM 4. *Let G be a compact Lie group acting on a C^s-manifold M as a C^1-transformation group. Then G necessarily acts on M as a C^s-transformation graup if one of the following conditions is valid:*

(i) *G acts faithfully on M and has only a finite number of inequivalent orbits.*

(ii) *M is a compact manifold.*

To prove this, we use the following result due to G. D. Mostow [8]: When either (i) or (ii) is the case, there exist a regular C^1-homeomorphism f of M into a Euclidean space E^k and an isomorphism θ of G into the unitary group on E^k such that f is *equivariant* with respect to θ, i. e., $f(gp)=\theta(g)f(p)$ $(g\in G, p\in M)$.

PROOF. Assume that f and θ have the same meanings as above. Approximate f by a regular C^s-mapping φ of M into E^k up to order 1. We may assume

(1) $\sup_{g\in G}\|\varphi(gp)-f(gp)\|_{V_\iota}<\varepsilon_\iota$

(2) $\sup_{g \in G} \| D_\iota \varphi(gp) - D_\iota f(gp) \|_{V_\iota} < \varepsilon_\iota$,
since G is compact. Here $\|\cdot\|$ means the Euclidean distance on E^k, while a positive family $\mathcal{E} = \{\varepsilon_\iota\}$ will be determined immediately later. Now consider

$$\psi(p) = \int_G \theta(g)^{-1} \varphi(gp) dg\,,$$

the integral being taken with respect to the Haar measure on G, normalized as $\int_G dg = 1$. Then it is easily verified that $\psi(gp) = \theta(g)\psi(p)$ and that ψ gives a C^s-mapping of M into E^k. Moreover, since f is equivariant, we obtain

$$\psi(p) - f(p) = \int_G \theta(g)^{-1}(\varphi(gp) - f(gp)) dg\,.$$

If $\mathcal{E} = \{\varepsilon_\iota\}$ is taken sufficiently small, the right hand side becomes arbitrarily small on each V_ι since $\theta(g)$ is in the unitary group and (1) holds. The same is valid for the derivatives in view of (2). Thus, referring to Theorem B we can conclude that ψ gives rise to a C^s-equivariant imbedding of M so that $\psi(M)$ is a regular C^s-submanifold of E^k. Being a linear group, $\theta(g)$ obviously operates on $\psi(M)$ as a C^s-transformation group. This, together with the equivariance of ψ, yields the fact that $\psi(p) \to \psi(gp)$ gives a C^s-mapping in p for each $g \in G$, and so $x \to gx$ is a C^s-mapping on M because ψ is a C^s-homeomorphism. This completes the proof of Theorem 4.

4. Fibre bundles.

PROPOSITION 2. *Let $P = P(B, F, G, \pi)$ be a C^ω-fibre-bundle. Assume that P possesses a C^0-cross-section s over B. Then there exists a C^ω-cross-section σ over B which approximates s arbitrarily well.*

PROOF. It is well known that, under the assumption, we have a C^∞-cross-section $\tilde{s}$ approximating s arbitrarily well. Now $\tilde{s}$ satisfies $\pi \circ \tilde{s} =$ identity, and so $\tilde{s}(B)$ is a regular submanifold of P. Regarding $\tilde{s}$ as a mapping of B into P, apply Theorem A to $\tilde{s}$. Then we get a C^ω-approximation $\tilde{\sigma}$ to $\tilde{s}$ up to order 1. Choosing $\tilde{\sigma}$ properly, we can assume that $\tilde{\sigma}$ satisfies the following two conditions:

(i) $\tilde{\sigma}$ is a C^ω-imbedding of B into P.

(ii) For each $b \in B$, $\sigma(x)$ is contained in a tubular neighborhood $T(F_x)$ where $F_x = \pi^{-1}(x)$.

Let τ_x be a C^ω-projection of $T(F_x)$ onto F_x and set $\sigma(x) = \tau_x \circ \tilde{\sigma}(x)$. Then from (i) and (ii) it follows that $\sigma(x)$ is a C^ω-cross-section over B, approximating $s(x)$ arbitrarily well. This completes the proof.

Corresponding to the unique existence of C^ω-structure on any differentiable manifold, the similar problem on the fibre-bundle level can be described as follows: Does any C^r-fibre-bundle admit a unique C^s-bundle-structure, compatible with the given C^r-bundle-structure, where $1 \leqq r \leqq s \leqq \omega$? In the following, we wish to give an affirmative answer to this problem. More precisely, the conclusion of Theorem 5 below shows that the classification problem of C^ω-fibre-bundles is essentially equivalent to that of C^0-fibre-bundles, which gives a solution of H. Cartan's problem [**2**, p. 90].

First, we state the following theorem for later use.

THEOREM C. *Let M and N be C^s-manifolds, and f, g be C^s-mappings from M into N. Assume that f and g are C^0-homotopic. Then f and g are C^s-homotopic.*

This theorem is essentially due to Whitney [**11**]. However, for the sake of completeness, we shall give a proof to it in the appendix according to the lines of Whitney's idea.

THEOREM 5. *Let $P = P(B, F, G, \pi)$ be a C^0-fibre-bundle, where B and F are C^s-manifolds. Assume that G operates on F as C^s-transformation group. Then P has a unique C^s-fibre-bundle structure $\tilde{P}$ which is compatible with the given structure as C^0-fibre-bundle, where $1 \leqq s \leqq \omega$.*

PROOF. Referring to Proposition 2, we can easily verify that it suffices to prove the theorem in the case where G is a compact Lie group and P is a principal fibre bundle $P(B, G, \pi)$, and so we assume it in the sequel. Now we shall devide the proof in two parts:

(I) Existence of $\tilde{P}$. P is induced from a C^0-mapping of B in a higher-dimensional classifying space B of G. Alternatively, there is a higher-dimensional universal fibre-bundle $Q = Q(B^*, G, \pi^*)$ and a C^0-mapping f of B into B^* such that $P = f^*Q$. It is known that Q can be chosen so that Q is a C^ω-fibre-bundle. Since B admits the unique C^ω-structure, we can apply Theorem A to the C^0-mapping f. Then it follows that there is a C^s-mapping φ approximating f well. Set $\tilde{P} = \varphi^*Q$. P is obviously a C^s-fibre-bundle over B. Moreover, since f and φ lie near, they are C^0-homotopic. Thus P and $\tilde{P}$ are C^0-homotopic each other, and a fortiori compatible as C^0-bundles.

(II) Uniqueness of P. We first prove that for any C^s-fibre-bundle $P = P(B, G, \pi)$ there is a C^s-map φ from B into B^* such that $P = \varphi^*Q$. As is well-known, this is essentially equivalent to show the existence of a bundle map ϕ of P into Q. Now if $s < \omega$, this result is known. Hence, in particular, when we simply regard P as C^0-fibre bundle P^0, P^0 admits a bundle map ϕ^0 of P^0 into Q.

We assume that G operates on P from the right and on Q from the left. Consider the associated fibre bundle E of P defined by $E = P \times_G Q$; E consists of the equivalence classes $(x, y) \sim (xg, g^{-1}y)$, $(x \in P, y \in Q, g \in G)$, which is known to possess the C^s-fibre-bundle structure over B having the typical fibre

Q. Let $\sigma(p)$ $(p \in B)$ be a cross-section of E. To each p $\sigma(p)$ assigns an equivalence class $(x(p), y(p)) \sim (x(p)g, g^{-1}y(p))$ $\pi \circ x(p) = p$. It follows immediately that the correspondence $x(p) \rightarrow y(p)$ gives rise to a well defined bundle map $\tilde{\sigma}(p)$. Conversely, any bundle map $\tilde{\sigma}$ induces a cross-section of σ[1].

Thus ψ^0 gives a C^0-cross-section $\bar{\psi}^0$ of E over B. Apply Proposition 2 to this situation. Then we can find a C^s-cross-section $\bar{\psi}$ of E, which in turn gives a C^s-bundle map ψ of P into Q. This however is the desired result.

Now the uniqueness can be proved as follows: Assume that there are two C^s-fibre-bundle structures $\tilde{P}_1$, $\tilde{P}_2$, both of which have a compatible C^0-fibre-bundle structure $P = P(B, G)$ in common. It follows from the above that we can find C^s-bundle maps ψ_i of P into Q $(i = 1, 2)$. Since $\tilde{P}_1$ and $\tilde{P}_2$ have the same C^0-structure, ψ_1 and ψ_2 necessarily becomes C^0-homotopic as bundle maps. In particular, the induced base maps φ_1 of ψ_1 and φ_2 of ψ_2 are C^0-homotopic. Hence, referring to Theorem C, we know that φ_1 and φ_2 are C^s-homotopic. Thus $\tilde{P}_1 = \varphi_1^* Q$ and $\tilde{P}_2 = \varphi_2^* Q$ are C^s-homotopic, which implies $\tilde{P}_1$ and $\tilde{P}_2$ are C^s-equivalent as fibre bundles. This completes the proof.

Appendix.

In this appendix, we give a proof of Theorem C stated in § 4. Let $M_1 = M \times [0, 1]$, and consider a double $\tilde{M}$ of M_1 [**9**; p. 52]; note that $M \subset \tilde{M}$ and $\tilde{M}$ has a C^s-structure naturally induced from M. Then f and g given in Theorem C are regarded as C^s-mappings of $M \times 0$ into N, and $M \times 1$ into N, respectively. The assumption that f and g are C^0-homotopic implies that they are extendable over $\tilde{M}$ as C^0-mappings. Accordingly, Theorem C is a special case of the following extension theorem, and a proof will be given in this theorem:

THEOREM. *Let M and N be C^s-manifolds and let L be a closed C^s-submanifold of M. Suppose that we have a C^r-mapping f from M into N such that $f(p)|L$ gives a C^s-mapping of L into N, where r satisfies $0 \leqq r \leqq s$. Then, for any positive family $\mathcal{E}$, there exists a C^s-mapping φ from M into N having the following properties:*

(i) *φ gives an $\mathcal{E}$-approximation to f in order* 0.

(ii) *$\varphi(p)|L = f(p)|L$.*

In case $r = \infty$, $s = \omega$, we have a supplementary result as follows: When f gives a C^ω-mapping from a neighborhood $V(L)$ of L into N, the condition (i) *can be replaced by*

(iii) *φ gives an $\mathcal{E}$-approximation to f up to any finite order.*

In order to prove this, first we need to generalize Theorem A in case of

1) Professor J. Milnor kindly suggested this fact to the author.

C^{ω}-manifolds. Here we use the same notations as in §1; in particular, the local coordinate neighborhoods $\{U_\iota\}$, $\{V_\iota\}$ on M and $\{W_\lambda\}$ on N are taken fixed whenever $f\in C^r(M, N)$ is given. We take a metric $\rho(x, y)$ on M, such that $\rho(x, y)\to\infty$ whenever y tends towards the outside of any compact set. Let x_0 be any fixed point of M. For any given V_ι, we can attach an integer $[V_\iota]$ as follows: $[V_\iota]$ is the least integer such that V_ι is contained in the ball with center x_0 and radius r. While H. Whitney [**11**] proved the following generalization of Theorem A in a special case where M and N are Euclidean spaces, the result can be immediately extended to any C^{ω}-manifolds.

THEOREM A′. *Let M and N be C^{ω}-manifolds. Assume that a C^{∞}-mapping f of M into N is given. Then for any positive family $\mathcal{E}=\{\varepsilon_\iota\}_{\iota\in I}$, there exists a C^{ω}-mapping φ of M into N such that*

$$\| D_\iota^\nu f-D_\iota^\nu\varphi \|_{V_\iota}<\varepsilon_\iota$$

where $\nu=0, 1, 2, \cdots, [V_\iota]$; φ satisfies the coincidence condition

$$D^\nu\varphi(x_l)=D^\nu f(x_l), \qquad \nu=0, 1, \cdots, [V_\iota^{(l)}],$$

where $x_1, x_2, \cdots, x_l, \cdots$ is any sequence of points of M such that $x_l\in V_\iota^{(l)}$.

PROOF OF THEOREM. We only prove the theorem in case $s=\omega$. If $s\neq\omega$, the similar arguments as in (I) below will establish the theorem.

(I) Imbed M and N into a Euclidean space E^k by regular and proper C^{ω}-mappings, and identify them with the imbedded manifolds. We take and fix a C^{ω}-Riemannian metric on M and consider a tubular neighborhood $T(L)$ of L in M. Let π be a C^{ω}-projection of $T(L)$ onto L obtained in a canonical way. For any given $p\in T(L)$, we denote by $s(p)$ the length from p to $\pi(p)$ along the normal geodesic passing through p. Letting $\xi(p)$ be a small positive C^{ω}-function on L, set

$$T(L;\xi)=\{p\,|\,p\in T(L),\ s(p)<\xi(p)\}.$$

We may assume that $T(L;\xi)\Subset T(L)$. A C^{ω}-mapping $\tilde{f}$ from $T(L;\xi)$ into N is defined to be $f\circ\pi$. Let $g(p)$ be a C^{∞}-mapping from M into N which approximates $f(p)$ well in order 0. Take a C^{∞}-function $\Lambda(t)$ defined on the real axis such that $\Lambda(t)=0$ for $t\leqq 1/2$, $\Lambda(t)=1$ for $t\geqq 1$, and that

$$\left|\Lambda(t)-2\left(t-\frac{1}{2}\right)\right|<\delta \qquad \text{for } \frac{1}{2}\leqq t\leqq 1.$$

Now put

$$\tilde{F}(p)=\begin{cases}\tilde{f}(p), & \text{if } p\in T(L;\xi/2)\\ \Lambda\left(\dfrac{3}{2}-\dfrac{s(p)}{\xi(p)}\right)\tilde{f}(p)+\Lambda\left(\dfrac{s(p)}{\xi(p)}\right)g(p), & \text{if } p\in T(L;\xi)-T(L;\xi/2)\\ g(p), & \text{if } p\notin T(L;\xi).\end{cases}$$

$\hat{F}(p)$ gives a C^∞-mapping from M into a tubular neighborhood $T(N)$ of N in E^k if $\xi(p)$ and δ are taken sufficiently small. Let $\tilde{\pi}$ be the C^ω-projection of $T(N)$ onto N and set $F(p)=\tilde{\pi}\circ\hat{F}(p)$. Then F is a C^∞-mapping from M into N such that

(a) $F(p)|L=f(p)|L$.

(b) $F(p)$ is a C^ω-mapping of $T(L;\xi/2)$ into N.

(c) $F(p)$ approximates f well in order 0 whenever $\xi(p)$ and δ are chosen small.

We replace f by F in the statement of the theorem and in what follows we prove (ii) and (iii) of the theorem under the above conditions (a) and (b) of F. If we know that F satisfies (ii) and (iii), then f necessarily satisfies (i) and (ii) by virtue of (a) and (c); this however shows the validity of Theorem.

(II) Now we take a metric ρ on $M-L$ such that $\rho(x,y)\to\infty$ whenever y tends towards the outside of any compact set of $M-L$. We regard F as a mapping of $C^\infty(M-L,N)$ and apply Theorem A′ to this situation. Then we can get $\psi\in C^\omega(M-L,N)$ giving an $\mathcal{E}$-approximation to F up to any finite order in the sense of Theorem A′. Consequently, taking $\mathcal{E}$ properly, we can assume that each partial derivative of $\psi(p)$ converges rapidly to that of $F(p)$ whenever p tends to a point of L. Since L is a closed submanifold, we may assume that $\{L\cap V_\iota\}$ forms an open covering of local coordinate neighborhoods on L, which is clearly locally finite. Let $x_0\in L$ and suppose that V_{ι_0} contains x_0. Then we can easily verify that

$$\lim_{x\to x_0} D^\nu_{\iota_0}\psi(x)=D^\nu_{\iota_0}F(x_0)\,,\qquad \nu=0,1,2,\cdots. \tag{1}$$

Put

$$\varphi(p)=\begin{cases}\psi(p)\,, & \text{for } p\in M-L\,,\\ F(p)\,, & \text{for } p\in L\,.\end{cases}$$

Then φ is a desired mapping. In fact, it is clear that $\varphi(p)|L=f(p)|L$ by (a), and that φ gives a C^ω-mapping from $M-N$ into N; on the other hand, F has the property (b) and φ satisfies (1) at each point x_0 of L, which implies that φ gives a C^ω-mapping of M into N. Finally we note that φ approximates F well up to any finite order. This completes the proof of Theorem.

COROLLARY. *Let M be a C^ω-manifold and L a closed C^ω-submanifold of M. Then any C^ω-function φ on L can be extended to a C^ω-function ψ on M in such a manner that ψ becomes arbitrarily small outside any neighborhood of L.*

Department of Mathematics
Tokyo Institute of Technology

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Added in proof: According to a brief resumé reported in Sûgaku, Vol. 15, No. 4, we have known that Professor H. Cartan delivered a lecture on approximation of differentiable mappings at Kyoto University (Nov. 15, 1963), the subjects of which are closely related to those of this paper; in particular, he also proved Theorem 5.