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Potential operator of a recurrent strong Feller process in the strict sense and boundary value problem

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§ 0. Introduction.

The purpose of this paper is to define the potential operator for a recurrent strong Feller process in the strict sense on a compact metric space and apply it to the boundary value problem for elliptic differential operators of the second order.

G. A. Hunt [4] has mainly dealt with the potentials of transient Markov processes. Similar problems[1)] have been considered for recurrent Markov chains with discrete time parameter, by J. G. Kemeny-J. L. Snell [5] and for some classes of diffusion processes, by N. Ikeda [6] and S. Ito [9]. The fundamental idea of the latter works consists in excluding the infinite part of the Green operator $G_{0+}f(x)$ which may be divergent for recurrent Markov processes even for functions with compact carrier.

If a Markov process is a strong Feller process in the strict sense on a compact metric space, $\lim_{t\to\infty} T_t f(x)=\int f(x)m(dx)=mf$ converges (§ 1. Theorem 1.1).

Then, in § 2 we define the potential operator by

$$Kf(x)=\int_0^\infty (T_t f(x)-mf)dt\,.$$

If $mf=0$, $Kf(x)$ satisfies the Poisson equation.

In § 3, we consider the potential $R^\alpha f(x)=E_x\Bigl(\int_0^\infty e^{-\alpha t}f(x_t)d\varphi_t(w)\Bigr)$ corresponding to an additive functional $\varphi_t(w)$ and we investigate the finite part of this potential for $\alpha\to 0$.

We discuss in § 4 an application of the above results to the boundary value problem

$$Au=f, \qquad \text{on } D,$$

1) The author is informed recently that along the similar line, M. I. Freidlin [1] had some results. But, our paper deals with more general cases.

and

$$\frac{\partial}{\partial n}u=g, \qquad \text{on } \partial D,$$

where D is a compact domain in an N-dimensional manifold, ∂D is the boundary of D, A is a sufficiently smooth elliptic differential operator of the second order. The purely analytical approach for this problem is given in S. Ito [**9**].

We remark that the definition of the kernel of our potential is not a generalization of the kernel of the logarithmic potential, but is a generalization of the Neumann kernel. In fact, the kernel of the logarithmic potential is obtained by

$$\int_0^\infty (p(t, x, y)-p(t, x_0, y_0))dt=\frac{1}{\pi}\log\left|\frac{x_0-y_0}{x-y}\right|^{2)},$$

where $x \neq y$, $x_0 \neq y_0$, $p(t, x, y)=\frac{1}{(2\pi)}e^{-\frac{|x-y|^2}{2t}}$ and $x, y, x_0, y_0 \in R^2$.

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§1. Strong Feller processes in the strict sense.

Let $M=\{x_t, W, P_x, x\in S\}$ be a Markov process[3)], where S is a simply connected[4)] compact metric space and W is the space of continuous path functions on S.

We assume following conditions

(A.1.1) *$P(t, x, U)>0$, for any $x\in S$, $t>0$ and any non-null open set $U\in \mathscr{B}(S)$*[5)], *and $P(t, x, S)=1$.*

(A.1.2) *M is a strong Feller process in the strict sense, that is, each operator T_t $(t>0)$, defined by $T_tf(x)=E_x(f(x_t))=\int_S P(t, x, dy)f(y)$ maps any bounded set in $C(S)$*[6)] *into a compact set in $C(S)$.*

Condition (A.1.2) is equivalent to

(A.1.2′) *$P(t, x, \cdot)$, for each positive t, is a continuous function in x, taking values in measures on S, with respect to the norm of the total variation,*

and if (A.1.2′) is satisfied, then M is a strong Feller process. These were proved by Girsanov [**2**, Lemma 4.1].

2) $|\cdot|$ implies the distance in two-dimensional Euclidean space R^2.

3) For the definition of the Markov processes, vid. K. Ito [**7**].

4) We suppose it only for clarification of discussions.

5) $\mathscr{B}(\cdot)=$ the topological Borel field of $\cdot$.

6) $C(\cdot)=\{f; f$ is a continuous function on $\cdot\}$. The topology in $C(\cdot)$ is the one induced by the uniform norm.

If (A.1.1) holds and if M is a strong Feller process, then M is recurrent[7], that is, $P_x(\sigma_U < +\infty)=1$[8] for any $x \in S$, where U is any non-null open subset of S. For each fixed $t>0$, the family of measures $\{P(t, x, \cdot), x \in S\}$ on S are mutually absolutely continuous[9].

THEOREM 1.1. *Assume* (A.1.1) *and* (A.1.2). *Then, there exists a unique probability measure* $m(\cdot)$ *on* S *such that*

$$|T_t f(x) - mf| \leqq Ke^{-ct} \|f\|_\infty{}^{10)}, \qquad (x \in S)$$

for any $f \in B(S)$[11] *and any* $t \geqq 0$, *where we put* $mf = \int_S f(y) m(dy)$ *and where* K *and* c *are constants independent of* $f(x)$, t *and* x.

PROOF[12]. We fix a positive number $t_0 > 0$. We put

$$\begin{aligned} Q(P) &= \frac{1}{2} \sup_{x,y \in S} \| P(t_0, x, \cdot) - P(t_0, y, \cdot) \| \\ &= \frac{1}{2} \sup_{x,y \in S} \sup_{f \in B_1(S)} \int_S (P(t_0, x, dz) - P(t_0, y, dz)) f(z) \end{aligned}$$

where $B_1(S) = \{f;\ f \in B(S),\ \|f\|_\infty \leqq 1\}$.

We prove that $Q(P) < 1$. If we assume the contrary, there are two sequences $x_n \in S$ and $y_n \in S$ with $\lim_{n \to \infty} \| P(t_0, x_n, \cdot) - P(t_0, y_n, \cdot) \| = 2$. Since S is compact, there are subsequences $x_{n'}$ and $y_{n'}$ with limits x_0 and y_0 in S respectively. By the continuity of $P(t_0, x, \cdot)$ in x with respect to the norm, we have

$$\| P(t_0, x_0, \cdot) - P(t_0, y_0, \cdot) \| = \lim_{n' \to \infty} \| P(t_0, x_{n'}, \cdot) - P(t_0, y_{n'}, \cdot) \| = 2\,.$$

By the Hahn decomposition of $P(t_0, x_0, \cdot) - P(t_0, y_0, \cdot)$, we have two mutually disjoint subsets S^+ and S^- for which we have

$$P(t_0, x_0, S^+) = P(t_0, y_0, S^-) = 1\,, \qquad P(t_0, x_0, S^-) = P(t_0, y_0, S^+) = 0\,.$$

This fact contradicts the mutual absolutely continuity of $P(t_0, x_0, \cdot)$ and $P(t_0, y_0, \cdot)$. Therefore, we have $Q(P) < 1$.

Hence, by the theorem in T. Ueno [**16**, 454-455], there exists a probability measure $m(\cdot)$ and positive constants c' and K' independent of x such that

$$\| P(nt_0, x, \cdot) - m(\cdot) \| \leqq K' e^{-c'n} \qquad \text{for any} \quad x \in S\,.$$

On the other hand, by means of the relation $T_{nt_0} T_s f(x) = T_s T_{nt_0} f(x)$ for

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7) E. g. vid. Nagasawa [**11**].

8) We denote by σ_U the first hitting time for U.

9) Vid. Hasminsky [**3**, p. 197].

10) $\|f\|_\infty = \sup_{x \in S} |f(x)|$.

11) $B(\cdot) = \{f;\ f$ is a bounded $\mathscr{B}(\cdot)$-measurable function on $\cdot\}$.

12) The half part of the proof of Theorem 1.1 is completely analogous to the Proposition 2.2 in T. eno [**15**], but we describe it for reader's convenience.

arbitrary $s>0$, we can see that $mT_sf=\int_S\int_S m(dy)P(s, y, dx)f(x)=\int_S m(dx)f(x)$, that is, $m(\cdot)$ is an invariant measure of T_t.

For any $t>0$, we put $n=\max_{kt_0\leqq t} k$ and $t-nt_0=s$. Then, we have

$$
\begin{aligned}
|T_tf(x)-mf| &= |T_{nt_0}T_sf(x)-mT_sf| \leqq \|T_sf\|_\infty\cdot\|P(nt_0, x, \cdot)-m(\cdot)\| \\
&\leqq \|f\|_\infty K'e^{-c'n} = \|f\|_\infty K'e^{-\left(\frac{c'}{t_0}\right)nt_0} \\
&= \|f\|_\infty K'e^{c'}e^{-\frac{c'}{t_0}(n+1)t_0} \qquad (1.1)\\
&\leqq \|f\|_\infty Ke^{-ct},
\end{aligned}
$$

where $K=K'e^{c'}$ and $c=\dfrac{c'}{t_0}$. From (1.1), we can see that $m(\cdot)$ is independent of t_0.

§2. The potential of the recurrent strong Feller processes in the strict sense.

Under the assumptions (A.1.1) and (A.1.2), we can define by Theorem 1.1.

$$Kf(x)=\int_0^\infty (T_tf(x)-mf)dt, \qquad \text{for} \quad f\in B(S). \tag{2.1}$$

We call K the potential operator of M.

Since $\int m(dx)P(t, x, E)=m(E)$ for any $E\in\mathscr{B}(S)$, $m(E)=0$ implies $P(t, x, E)=0$ for all x except for a set of $m(\cdot)$-measure zero. But, since $P(t, x, E)$ is continuous in x, $P(t, x, E)=0$ for all $x\in S$. Therefore, $P(t, x, \cdot)$ is absolutely continuous with respect to $m(\cdot)$. Denoting by $p(t, x, y)$ the density of $P(t, x, \cdot)$ in the sense of Radon-Nykodium with respect to $m(\cdot)$, we have

$$p(t+s, x, y)=\int_S p(t, x, z)m(dz)p(s, z, y), \tag{2.2}$$

$$\int_S m(dx)p(t, x, y)=1, \tag{2.3}$$

for each $x\in S$, $t>0$, $s>0$, and for almost all y with respect to $m(\cdot)$.

Furthermore, we have

$$\int_0^\infty dt\int_S |p(t, x, y)-1||f(y)|m(dy)\leqq\int_0^\infty \|P(t, x, \cdot)-m(\cdot)\|dt\cdot\|f\|_\infty<+\infty.$$

By Fubini's Theorem, we have

$$\int_0^\infty |p(t, x, y)-1|dt<+\infty,$$

for all y except for a subset N of $m(\cdot)$-measure zero. Therefore, if we put, for each $x\in S$, $K(x, y)=\int_0^\infty(p(t, x, y)-1)dt$ for $y\in S-N$ and $K(x, y)=+\infty$ for $y\in N$, then we can write

$$Kf(x)=\int_S K(x, y)f(y)m(dy)\,.$$

By Theorem 1.1 and the property (A.1.2), we can easily prove that $Kf(x)\in C(S)$ for any $f\in B(S)$.

Thus, we have

THEOREM 2.1. *Assume that* (A.1.1) *and* (A.1.2) *hold. Then,* $Kf(x)=\int_0^\infty(T_tf(x)-mf)dt$ *converges absolutely and* $Kf(x)\in C(S)$ *for any* $f\in B(S)$. *Also,* $G_\alpha f(x)-mf/\alpha=\int_0^\infty e^{-\alpha t}(T_tf(x)-mf)dt$ *is convergent to* $Kf(x)$ *as* α *tends to zero.*

THEOREM 2.2. *Under the same assumptions as Theorem* 2.1, *we have* $Kf\in D(\mathcal{G}_I)$. *Furthermore, if* $mf=0$,

$$\mathcal{G}_IKf(x)=-f(x)\,,\qquad x\in S\,, \tag{2.4}$$

holds for any $f\in B(S)$, *where* $\mathcal{G}_I$ *is the generator in the sense of K. Ito* [7]. *The same statement holds if we replace* $\mathcal{G}_I$ *by the Hille-Yosida generator* A *and* $B(S)$ *by* $C(S)$.

PROOF OF THEOREM 2.2. By the resolvent equation, we have

$$(G_\beta f(x)-mf/\beta)-(G_\alpha f(x)-mf/\alpha)=(\alpha-\beta)G_\beta\{G_\alpha f(x)-mf/\alpha\}\,. \tag{2.5}$$

Let α tend to zero in (2.5). Then, we have $G_\beta f(x)-mf/\beta-Kf(x)=-\beta G_\beta Kf(x)$. Thus, we have

$$Kf(x)=G_\beta(f(x)+\beta Kf(x)-mf)\,, \tag{2.6}$$

since $G_\alpha 1=1/\alpha$.

Hence, by the definition of $D(\mathcal{G}_I)$, we have $Kf(x)\in D(\mathcal{G}_I)$. If $mf=0$, also by (2.6), we have (2.4). Accordingly, we have the first part of the theorem.

For the proof of the second part, we have only to remark

$$T_tKf(x)-Kf(x)=-\int_0^t T_sf(x)ds+mf\cdot t\,,$$

which follows from (2.1) by Dynkin's formula.

If we replace mf/α in (2.5) by $mG_\alpha f$ $(=mf/\alpha)$, we have

$$(G_\beta f(x)-mf/\beta)-(G_\alpha f(x)-mf/\alpha)=(\alpha-\beta)G_\beta\{G_\alpha f(x)-mG_\alpha f\}\,. \tag{2.7}$$

Letting β tend to zero in (2.7), we have

$$Kf(x)-G_\alpha f(x)+mf/\alpha=\alpha KG_\alpha f(x)\,,$$

which implies $G_\alpha f(x)=mf/\alpha+K(f-\alpha G_\alpha f)(x)$. Since Kf is not a non-zero constant for any $f\in B(S)$, we have $G_\alpha f(x)\in \text{Range}(K)=\{g;\ Kf=g,\ f\in B(S)\}$ if and only if $mf=0$. Hence, we have

THEOREM 2.3. *Range*(K) *coincides with the set*

$$\{g\,;\ G_\alpha f=g\,,\ mf=0\,,\ f\in B(S)\}\,.$$

§ 3. The potential corresponding to an additive functional.

In this section, we assume (A.1.1) and (A.1.2). For the purpose of more delicate discussions, upon $p(t, x, y)$, given in § 1 and a regular positive Borel measure $\mu(\cdot)$ concentrated on a non-null closed subset B of S, we impose the following conditions:

C) *$p(t, x, y)$ is $\mathscr{B}([0,\infty])\otimes\mathscr{B}(S)\otimes\mathscr{B}(S)$-measurable with respect to (t, x, y), where $t\in[0,\infty)$, x and $y\in S$. Let $g_\alpha(x, y)=\int_0^\infty e^{-\alpha t}p(t, x, y)dt$ (not possibly finite). Then, $R^\alpha f(x)=\int_B g_\alpha(x, y)f(y)\mu(dy)$, for any $f\in B(S)$ and $\int_0^t ds\int_B p(s, x, y)$ $f(y)\mu(dy)$, for any $f\in B(S)$ and any $t>0$, are absolutely convergent.* (2.2) *and* (2.3) *are satisfied for almost all y with respect to the measure $\mu(\cdot)$.*

We remark that the last assumption in C) is satisfied if

$$\text{(A.3.1)}\qquad \begin{cases} p(t, x, \cdot)\in C(S)\,, \\ \int_S m(dy)p(t, y, \cdot)\in C(S)\,. \end{cases}$$

Let us consider the finite part of $R^\alpha f(x)$ for $\alpha\to 0$. If we fix arbitrary $s>0$, we have

$$\begin{aligned} R^\alpha f(x) &= \int_0^s e^{-\alpha t}dt\int_B f(y)p(t, x, y)\mu(dy)+\int_s^\infty e^{-\alpha t}dt\int_B f(y)p(t, x, y)\mu(dy) \\ &= \int_0^s e^{-\alpha t}dt\int_B f(y)p(t, x, y)\mu(dy)+e^{-\alpha s}\int_0^\infty e^{-\alpha t}dt\int_B f(y)p(t+s, x, y)\mu(dy) \\ &= \int_0^s e^{-\alpha t}dt\int_B f(y)p(t, x, y)\mu(dy)+e^{-\alpha s}\int_0^\infty e^{-\alpha t}dt\int_S p(t, x, y)g(y)m(dy)\,, \end{aligned}$$

where $g(y)=\int_B p(s, y, z)f(z)\mu(dz)$.

Furthermore, we have

$$\begin{aligned} R^\alpha f(x)-\frac{1}{\alpha}\int_S g(y)m(dy) &= \int_0^s e^{-\alpha t}dt\int_B f(y)p(t, x, y)\mu(dy) \\ &\quad +e^{-\alpha s}\int_0^\infty e^{-\alpha t}\Big(\int_S p(t, x, y)g(y)m(dy)-\int_S g(y)m(dy)\Big)dt \\ &\quad +\frac{1}{\alpha}\int_S g(y)m(dy)(e^{-\alpha s}-1)\,. \end{aligned}$$

If we assume for any $f\in B(S)$ and for any $s>0$, (A.3.2) $$\sup_{x\in S}\int p(s, x, z)|f(z)|\mu(dz) < +\infty$$

which is satisfied if

(A.3.2′) $$\int_B p(s, \cdot, y)f(y)\mu(dy) \in C(S), \quad \text{for any } f\in B(S),$$

then, we have by Theorem 2.1,

$$\lim_{\alpha\to 0}\left(R^{\alpha}f(x) - \frac{1}{\alpha}\int g(y)m(dy)\right) = \int_0^s dt\int_B p(t, x, y)f(y)\mu(dy)$$
$$+\int K(x, y)g(y)m(dy) - s\cdot\int g(y)m(dy).$$

Moreover, if we assume

(A.3.3) $$\int_0^s dt\int_B f(y)p(t, x, y)\mu(dy) \in C(S), \quad \text{for any } f\in B(S)$$

then, $\lim_{\alpha\to 0}\left(R^{\alpha}f(x) - \frac{1}{\alpha}\int g(y)m(dy)\right) = \tilde{K}^{\mu}f(x)$ exists and $\tilde{K}^{\mu}f(x)\in C(S)$.

Since $\int m(dy)p(t, y, x) = 1$ for μ-almost all x, we have $\int_S m(dy)g(y) = \int_B\int_S m(dy)p(t, y, x)f(x)\mu(dx) = \int_B f(x)\mu(dx)$. Therefore, we see that $\lim_{\alpha\to 0}\Big(R^{\alpha}f(x) - \frac{1}{\alpha}\int g(y)m(dy)\Big) = \lim_{\alpha\to 0}\left(R^{\alpha}f(x) - \frac{1}{\alpha}\int f(x)\mu(dx)\right)$ is independent of s.

Summing up the above discussions, we have

THEOREM 3.1. *Under the conditions* (A.1.1), (A.1.2), C), (A.3.2) *and* (A.3.3), $R^{\alpha}f(x) - \frac{1}{\alpha}\int_B f(y)\mu(dy) = \int_0^{\infty} e^{-\alpha t}dt\int_B (p(t, x, y) - 1)f(y)\mu(dy)$ *converges to* $\tilde{K}^{\mu}f(x) \in C(S)$ *as* α *tends to zero for any* $f\in B(S)$.

THEOREM 3.2. *Assume that* (A.1.1) *and* (A.1.2) *hold. For* $p(t, x, y)$ *and a regular, positive, Borel measure* $\mu(\cdot)$, *we assume that* $R^{\alpha}f(x)\in C(S)$ *for any* $f\in B(S)$ *and that* (A.3.1), (A.3.2′) *and* (A.3.3) *hold. Then the result of Theorem* 3.1 *follows.*

We suppose that there exists a α-th order continuous additive[13)] functional $\varphi_t^{\alpha, f}(w)$ such that $E_x(\varphi_{\infty}^{\alpha, f}(w)) = R^{\alpha}f(x)$. If we put $\varphi_t(w) = \int_0^t e^{\alpha s}d\varphi_s^{\alpha, 1}(w)$ we have $E_x(\varphi_t(w)) = \int_0^t ds\int_B p(s, x, b)\mu(db)$, by the definition of $\varphi_t^{\alpha, 1}(w)$. We assume that for σ_B (the first hitting time for B)

(A.3.4) $$P_x(\varphi_{\sigma_B}(w) = 0) = 1$$

and for any $t > 0$

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13) For the definition of the α-th order continuous additive functional, vid. [**13**].

(A.3.5) $$P_x(\varphi_{\sigma_B+t}(w)>0)=1\,, \qquad \text{for any} \quad x\in S\,.$$

Then, by use of the argument in Theorem 4.1 of Nagasawa-Sato [**12**], we can prove that

$$R^\alpha f(x)=E_x\Big(\int_0^\infty e^{-\alpha t}f(x_t)d\varphi_t(w)\Big)\,.$$

Now, putting $\tau_t(w)=\inf\{s\,;\,\varphi_s(w)>t\}$ if $\varphi_\infty(w)>t$ and $=+\infty$ if $\varphi_\infty(w)\leqq t$, we have

(3.1) $$R^\alpha f(x)=E_x\Big(\int_0^\infty e^{-\alpha\tau_t(w)}f(x_{\tau_t(w)}(w))dt\Big)\,.$$

We put

(3.2) $$E_x(e^{-\alpha\tau_t(w)}f(x_{\tau_t(w)}(w)))=\tilde{T}_t^{(\alpha)}f(x)\,,$$

and

$$E_x(f(x_{\tau_t(w)}(w)))=\tilde{T}_tf(x)\,.$$

Then, we can see that $\tilde{T}_t^{(\alpha)}$ and $\tilde{T}_t$ are semi-groups on $B(S)$, by the definition of $\tau_t(w)$. Furthermore, we have $\lim_{\alpha\to 0}\tilde{T}_t^{(\alpha)}f(x)=\tilde{T}_tf(x)$ for any $f\in C(S)$ and $\lim_{t\to 0}\tilde{T}_tf(x)=f(x)$ for $f\in C(S)$ and $x\in B$.

THEOREM 3.3. *Assume the same conditions as Theorem* 3.2. *Moreover, assume that* (A.3.4) *and* (A.3.5) *hold. Let A be the weak infinitesimal generator of the semi-group $\tilde{T}_t$ on B. Then, if $\int_B f(z)\mu(dz)=0$, it holds that*

(3.3) $$A\tilde{K}^\mu f(x)=-f(x)\,, \qquad \textit{for} \quad x\in B \textit{ and } f\in C(S)\,,$$

(3.4) $$h_B\tilde{K}^\mu f(x)=\tilde{K}^\mu f(x)\,, \qquad \textit{for} \quad x\in S \textit{ and } f\in B(S)\,,$$

where $h_Bf(x)=E_x(f(x_{\sigma_B}))$.

PROOF. Since $R^\alpha f(x)\in C(S)$, $R^\alpha f(x)$ $(f\geqq 0)$ is the regular excessive function in the sense of Shur [**14**] and Meyer [**10**]. Therefore, there exists a continuous additive functional $\varphi_t(w)$ such that $E_x(\varphi_t(w))=\int_0^t ds\int_B p(s,x,b)\mu(db)$. By (3.1) and (3.2), we have $R^\alpha f(x)=\int_0^\infty \tilde{T}_t^{(\alpha)}f(x)dt$. We put $\tilde{K}_\alpha^\mu f(x)=R^\alpha f(x)-\frac{1}{\alpha}\int f(z)\mu(dz)$. Then, we have

$$\begin{aligned}\tilde{K}_\alpha^\mu f(x)&=\int_0^\infty\Big(\tilde{T}_s^{(\alpha)}f(x)-e^{-\alpha s}\cdot\int f(z)\mu(dz)\Big)ds\\&=\int_0^t\tilde{T}_s^{(\alpha)}f(x)ds-\int_0^t e^{-\alpha s}\Big(\int f(z)\mu(dz)\Big)ds\\&\quad+\int_t^\infty\Big(\tilde{T}_s^{(\alpha)}f(x)-e^{-\alpha s}\cdot\int f(z)\mu(dz)\Big)ds\\&=\int_0^t\Big(\tilde{T}_s^{(\alpha)}f(x)-e^{-\alpha s}\cdot\int f(z)\mu(dz)\Big)ds\end{aligned}$$

$$+\tilde{T}_t^{(\alpha)}\Big(\int_0^\infty\Big(\tilde{T}_s^{(\alpha)}f(x)-e^{-\alpha s}\cdot\int f(z)\mu(dz)\Big)ds\Big)$$

$$+(\tilde{T}_t^{(\alpha)}1-e^{-\alpha t})\cdot\int_0^\infty e^{-\alpha s}ds\cdot\int_B f(z)\mu(dz)\,.$$

Since $\lim_{\alpha\to 0}\tilde{K}_\alpha^\mu f(x)=\tilde{K}^\mu f(x)$ and $\lim_{\alpha\to 0}\tilde{T}_t^{(\alpha)}f(x)=\tilde{T}_t f(x)$ boundedly on S, if $\int_B f(z)\mu(dz)=0$, we have

$$\tilde{K}^\mu f(x)=\int_0^t \tilde{T}_s f(x)ds+\tilde{T}_t\tilde{K}^\mu f(x)\,. \tag{3.6}$$

As $\tilde{T}_t f(x)$ is right-continuous in $t\in[0,\infty)$ and $\lim_{t\to 0}\tilde{T}_t f(x)=f(x)$ for $x\in B$, we have (3.3) by (3.6).

Since $\varphi_{\sigma_B}(w)=0$, we have

$$E_x(e^{-\alpha\sigma_B}\tilde{K}_\alpha^\mu f(x_{\sigma_B}))=\tilde{K}_\alpha^\mu f(x)\,,\qquad \text{for all}\quad x\in S\,,$$

from which we have (3.4).

REMARK. If $E_x(\tau_t(w))<+\infty$, we have, by (3.5),

$$\tilde{K}^\mu f(x)=\int_0^t \tilde{T}_s f(x)ds+\tilde{T}_t\tilde{K}^\mu f(x)-E_x(\tau_t(w))\cdot\int_B f(z)\mu(dz)\,,$$

for any $x\in S$.

§ 4. Boundary value problem for elliptic differential operators of the second order.

Let D be a connected domain with compact closure $\bar{D}$ in an N-dimensional orientable manifold of class C^∞ and the boundary ∂D consists of a finite number of $N-1$ dimensional hypersurface of class C^3. Let $\{x_t, W, \mathcal{B}, P_x\}$ be a reflecting A-diffusion[14] on D with transition probability $U(t,x,y)dy$[15]. $U(t,x,y)$ is the unique fundamental solution[16] for the initial value problem of the equation

$$\frac{\partial}{\partial t}u(t,x)=Au(t,x)$$

with the boundary condition

$$\frac{\partial}{\partial n}u(t,x)=0\,.$$

A is a second order elliptic differential operator

$$Au(x)=\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\Big(a^{ij}(x)\sqrt{a(x)}\frac{\partial u(x)}{\partial x^j}\Big)+b^i(x)\frac{\partial u(x)}{\partial x^i}\ ^{17)},$$

14) For the definition of a reflecting A-diffusion, vid. [**13**].
15) We denote the local coordinate of the point x as $(x^1,\cdots,x^N)$.
16) Its construction is given by S. Ito [8].
17) Here, we used the summation convention in differential geometry.

where $a^{ij}(x)$ and $b^i(x)$ are contravariant tensors on $\bar{D}$ of class C^3, $a^{ij}(x)$ is symmetric and strictly positive definite for each $x \in \bar{D}$ and $a(x) = \det(a^{ij}(x))^{-1}$. The operator $\frac{\partial}{\partial n}$ is defined as

$$\frac{\partial u(x)}{\partial n} = \frac{1}{\sqrt{a^{NN}(x)}} a^{Ni}(x) \frac{\partial u(x)}{\partial x^i}, \qquad x \in \partial D,$$

when in a neighborhood of x.

(4.1) ∂D is represented as $x^N = 0$, and D as $x^N > 0$.

dx and $d\tilde{x}$ are the Riemannian volume and surface elements respectively, that is, $dx = \sqrt{a(x)}dx^1 \cdots dx^N$ and in case of (4.1), $d\tilde{x} = \sqrt{a(x)}\sqrt{a^{NN}(x)}dx^1 \cdots dx^{N-1}$.

THEOREM 4.1. *The reflecting A-diffusion satisfies* (A.1.1) *and* (A.1.2).

PROOF. From the properties of the fundamental solution, shown by S. Ito [8, p. 83], we can prove that (A.1.1).

For the verification of (A.1.2), we have only to show that any family of functions: $\{\varphi_\alpha(x); T_t f_\alpha(x) = \varphi_\alpha(x), \alpha \in \Lambda, \|f_\alpha\|_\infty \leqq 1, f_\alpha \in C(S)\}$ are equicontinuous and uniformly bounded. But, $\{\varphi_\alpha, \alpha \in \Lambda\}$ are uniformly bounded since $\|\varphi_\alpha\|_\infty \leqq 1$.

On the other hand, $T_t f(x) \in C^1(\bar{D})$ and

$$\left|\frac{\partial}{\partial x^i} T_t f(x)\right| \leqq \int_{\bar{D}} \left|\frac{\partial U(t,x,y)}{\partial x^i}\right| |f(y)| dy \leqq \|f\|_\infty \int_D \left|\frac{\partial U(t,x,y)}{\partial x^i}\right| dy \leqq M(t)\|f\|_\infty,$$

by the result in S. Ito [8], where $M(t)$ is a constant independent of x. Then, by the mean value theorem, we have for an appropriate ξ,

$$T_t f(x) - T_t f(y) = \sum_{i=1}^N (x_i - y_i) \left(\frac{\partial T_t f(x)}{\partial x^i}\right)_{x=\xi}.$$

Hence,

$$|T_t f(x) - T_t f(y)| \leqq \max_{1 \leqq i \leqq N} |x_i - y_i| \cdot M(t),$$

from which follows the equicontinuity of $\{\varphi_\alpha, \alpha \in \Lambda\}$, completing the proof of the theorem.

Now, we remark that the transition probability function $P(t, x, \cdot)$ of the reflecting A-diffusion is written as follows

$$P(t, x, \cdot) = \int_{\cdot} U(t,x,y) \frac{1}{k(y)} k(y) dy = \int_{\cdot} p(t,x,y) m(dy), \qquad \text{for } \cdot \in \mathscr{B}(\bar{D}),$$

where $p(t,x,y) = U(t,x,y)\frac{1}{k(y)}$, $k(y)dy = m(dy)$, $k(y) \in C^1(\bar{D})$ and $k(y)dy$ is the invariant in the sense of § 1[18]. By means of the properties of the fundamental

18) This is shown in [11].

solution, given in [8, §4], there exists $R^{\alpha}f(x)=\int_0^{\infty}e^{-\alpha t}dt\int_{\partial D}U(t, x, y)f(y)d\tilde{y}$ $=\int_0^{\infty}e^{-\alpha t}dt\int_{\partial D}p(t, x, y)f(y)k(y)d\tilde{y}$. If we put $k(y)d\tilde{y}=\mu(dy)$, we can write $R^{\alpha}f(x)=\int_0^{\infty}e^{-\alpha t}dt\int_{\partial D}p(t, x, y)f(y)\mu(dy)=\int_{\partial D}g_{\alpha}(x, y)f(y)\mu(dy)$, where $g_{\alpha}(x, y)$ $=\int_0^{\infty}e^{-\alpha t}p(t, x, y)dt$. For such $\mu(\cdot)$ and the $p(t, x, y)$, we can see that conditions C), (A.3.1), (A.3.2′), (A.3.3) are satisfied and that $R^{\alpha}f(x)\in C(\bar{D})$, by use of the result of [8]. Since $R^{\alpha}f(x)\in C(\bar{D})$ and path functions are continuous, we can define a continuous additive functional $\varphi_t(w)$ such that $E_x(\varphi_t(w))=\int_0^t ds$ $\int_{\partial D}p(s, x, y)\mu(dy)$. Then, $\varphi_{\sigma_{\partial D}}(w)=0$ and $P_x(\varphi_{\sigma_{\partial D}+t}(w)>0)=1$ for $t>0$, are shown in Theorem 1 of [13].

Thus, we can apply the theorems in §2 and §3 to the reflecting A-diffusion. Especially, by Theorem 2.2, there exists

$$Kf(x)=\int_0^{\infty}(T_tf(x)-mf)dt\,, \qquad \text{for any} \quad f\in B(\bar{D})\,.$$

By Dynkin's formula, we have, for any $t>0$,

$$Kf(x)=\int_0^t T_sf(x)ds-mf\cdot t+T_tKf(x)\,. \tag{4.2}$$

For $R^{\alpha}g(x)$, by Theorem 3.2, there exists

$$\begin{aligned}\tilde{K}g(x)&=\lim_{\alpha\to 0}\Big(R^{\alpha}g(x)-\frac{1}{\alpha}\int_{\partial D}g(y)\mu(dy)\Big)\\ &=\lim_{\alpha\to 0}\int_0^{\infty}e^{-\alpha t}dt\int_{\partial D}(p(t, x, y)-1)g(y)k(y)d\tilde{y}\,, \qquad \text{for any} \quad g\in B(\partial D)\,.\end{aligned}$$

Thus,

$$\begin{aligned}\tilde{K}g(x)&=\lim_{\alpha\to 0}\int_0^t e^{-\alpha s}ds\int_{\partial D}(p(s, x, y)-1)g(y)k(y)d\tilde{y}\\ &\quad+\lim_{\alpha\to 0}\int_t^{\infty}e^{-\alpha s}ds\int_{\partial D}(p(s, x, y)-1)g(y)k(y)d\tilde{y}\\ &=\int_0^t ds\int_{\partial D}U(s, x, y)g(y)d\tilde{y}-t\cdot\int_{\partial D}g(y)k(y)d\tilde{y}\\ &\quad+\lim_{\alpha\to 0}e^{-\alpha t}T_t\Big(\int_0^{\infty}e^{-\alpha s}ds\int_{\partial D}(p(s, x, y)-1)g(y)k(y)dy\Big)\,.\end{aligned}$$

Therefore, we have

$$T_t\tilde{K}g(x)-\tilde{K}g(x)=-\int_0^t ds\int_{\partial D}U(s, x, y)g(y)d\tilde{y}+t\cdot\int_{\partial D}g(y)\mu(dy)\,. \tag{4.3}$$

If we put $\tilde{K}g(x)-Kf(x)=u_0(x)$, by (4.2) and (4.3), we have

(4.4)
$$u_0(x)=T_t u_0(x)-\int_0^t ds\int_D U(s,x,y)f(y)dy$$
$$+\int_0^t ds\int_{\partial D} U(s,x,y)g(y)d\tilde{y}+t\cdot(\mu g-mf),$$

where we put $\mu g=\int_{\partial D} g(y)\mu(dy)$.

Now, we can prove

THEOREM 4.2. *Assume that $f(x)$ is Hölder continuous and bounded on D, that $g(x)$ is Hölder continuous on ∂D and that $\mu g=mf$.*

Then, $u_0(x)$ is the solution of the differential equation

$$Au=f, \quad \text{for} \quad x\in D,$$

with boundary condition

$$\frac{\partial}{\partial n}u=g, \quad \text{for} \quad x\in\partial D.$$

PROOF. If $\mu g=mf$, from (4.3), we have

$$u_0(x)=\int_D U(t,x,y)u_0(y)dy-\int_0^t ds\int_D U(s,x,y)f(y)dy$$
$$+\int_0^t ds\int_{\partial D} U(s,x,y)g(y)d\tilde{y}.$$

When f and g satisfy the conditions stated above, by Theorem 1 VI)* in S. Ito [8], u_0 satisfies the differential equation

$$\frac{\partial}{\partial t}u_0(x)=Au_0(x)-f(x), \quad \text{for} \quad x\in D,$$

and

$$\frac{\partial}{\partial n}u_0(x)=g(x), \quad \text{for} \quad x\in\partial D.$$

But, $\frac{\partial}{\partial t}u_0(x)=0$, therefore, we have

$$Au_0(x)=f(x), \quad x\in D,$$

and

$$\frac{\partial}{\partial n}u_0(x)=g(x), \quad x\in\partial D,$$

which proves the theorem.

REMARK. Theorem 4.2 was obtained by S. Ito [**9**] in the case Laplace-Beltrami operator. We are informed the result for general cases has been proved in his forthcoming paper.

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