

Semisimple algebras over a commutative ring

Dedicated to Professor Y. Akizuki on his 60th birthday

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We intend in this paper to generalize the theory of semisimple algebras to the case where the domain of coefficients is a general commutative ring. In the classical case, a semisimple algebra A over a field K is characterized by one of the following three properties. 1) The direct sum of a finite number of simple algebras. 2) Vanishing of the radical. 3) Complete reducibility of every A -module. Each of these may be generalized in some manners to algebras over a commutative ring R . For instance, one may naturally call an algebra A over R to be simple if any two-sided ideal of A is of the form αA where α is an ideal of R . But, this definition seems to be too restrictive, since an extension of a Dedekind domain is not necessarily simple in this sense, even if it is unramified.

In this paper, we shall deal with the subject from the module-theoretical point of view 3). An algebra A over R will be called left semisimple if any extension of left A -modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \quad (\text{exact}),$$

with finitely generated M_3 , A -splits whenever it R -splits. In other words, A is left semisimple if every finitely generated left module is (A, R) -projective (Higman [9, 10], Hochschild [11]). We study in §1 some aspects of relative homological algebra for later use even in somewhat more generalized formulations than exactly needed, not pursuing, however, deeper results for its own sake.

§2 deals with some basic properties of semisimple algebras. As in the classical case, separable algebras form the most important class of semisimple algebras. The theory of separable algebras over a ring is successfully developed by Azumaya [3] and Auslander and Goldman [2]. It turns out that the relations of the separability with the semisimplicity in the classical case remain hold in our general case, except that the question of separability of central semisimple algebras is left open. In §3, we deal with the commuter relations of semisimple subalgebras of a central separable algebra.

If the base ring R is an integral domain, having K as its quotient field, then a semisimple algebra A over R is imbedded as an R -order in a semisimple algebra over K . Some results will be given in §4 of this case. For instance, applying the results of Auslander and Goldman [1], we prove that an R -projective semisimple order over an integrally closed Noetherian domain is a maximal order.

Throughout this paper, all rings have unit elements, and modules are unitary (unital). We say that an epimorphism (resp. monomorphism) $f: M \rightarrow N$ splits if $\text{Ker } f$ (resp. $\text{Im } f$) is a direct summand.

§ 1. From relative homological algebra.

1.1. To begin with, we recall the basic notions introduced by Hochschild [11]. Let A and B be rings, and ι a fixed unitary homomorphism $B \rightarrow A$. (For instance, B is a subring of A and ι the inclusion map.) A left A -module M is then viewed as a B -module *via* ι . A sequence of A -modules is (A, B) -exact if it is exact and splits as a sequence of B -modules. An A -module P is (A, B) -projective if the functor $\text{Hom}_A(P, _)$ maps every (A, B) -exact sequence into an exact sequence, or equivalently, if the natural epimorphism $\varepsilon: A \otimes_B P \rightarrow P$ defined by $\varepsilon(a \otimes x) = ax$ splits. An (A, B) -injective module Q is defined similarly by means of the functor $\text{Hom}_A(_, Q)$ or of the natural monomorphism $\varepsilon: Q \rightarrow \text{Hom}_B(A, Q)$ defined by $\varepsilon(x)(a) = ax$. Let M and N be A -modules. Then, an (A, B) -projective resolution of M is a left complex $\mathcal{P} = \{P_n\}$ of (A, B) -projective modules with an augmentation $\mathcal{P} \rightarrow M$, such that the sequence

$$\dots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is (A, B) -exact. An (A, B) -injective resolution of N is a right complex $\mathcal{Q} = \{Q^n\}$ of (A, B) -injective modules with an augmentation $N \rightarrow \mathcal{Q}$, such that the sequence

$$0 \longrightarrow N \longrightarrow Q^0 \longrightarrow \dots \longrightarrow Q^{n-1} \longrightarrow Q^n \longrightarrow \dots$$

is exact. Now one introduces two sequences of the relative functors $\text{Ext}_{A,B}^n$ and $\text{Tor}_n^{A,B}$, $n = 0, 1, 2, \dots$: $\text{Ext}_{A,B}^n(M, N)$ is described as $H^n(\text{Hom}_A(\mathcal{P}, N))$, as $H^n(\text{Hom}_A(M, \mathcal{Q}))$ or as $H^n(\text{Hom}_A(\mathcal{P}, \mathcal{Q}))$. Similarly, $\text{Tor}_n^{A,B}(L, M) = H_n(\mathcal{P}' \otimes_A N) = H_n(M \otimes_A \mathcal{P}) = H_n(\mathcal{P}' \otimes_A \mathcal{P})$, where \mathcal{P}' is an (A, B) -projective resolution of (a right A -module) L . Define relative dimensions as follows:

$$\dim_{A,B} M \leq n \text{ (resp. inj. dim}_{A,B} N \leq n) \text{ if } \text{Ext}_{A,B}^{n+1}(M, N) = 0$$

for every A -module N (resp. M).

$$\text{w. dim}_{A,B} M \leq n \text{ if } \text{Tor}_{n+1}^{A,B}(L, M) = 0 \text{ for every right } A\text{-module } L.$$

Further, set

$$(*) \quad \text{l. gl. dim}(A, B) = \sup \dim_{A,B} M = \sup \text{inj. dim}_{A,B} N,$$

$$(\dagger) \text{ w. gl. dim}(A, B) = \sup \text{ w. dim}_{A,B} M = \sup \text{ w. dim}_{A,B} L,$$

sup being taken over all left modules M, N and right modules L , respectively.

Let M be the direct limit of a system of A -modules M^ν , and $\mathcal{P}, \mathcal{P}^\nu$ be the standard (A, B) -projective resolutions of M, M^ν , respectively. \mathcal{P} is defined as follows: $P_0 = A \otimes_B M$, ε the natural epimorphism $P_0 \rightarrow M$. Assume we have defined P_i and $d_i: P_i \rightarrow P_{i-1}$, $0 \leq i \leq n$, then we put $P_{n+1} = A \otimes_B \text{Ker } d_n$, $d_{n+1} =$ the composition of $P_{n+1} \rightarrow \text{Ker } d_n$ with the inclusion $\text{Ker } d_n \rightarrow P_n$. Similarly for $\mathcal{P}^\nu = \{P_n^\nu\}$. Since the direct limit commutes with tensoring and preserves the exactness, we have $\mathcal{P} = \varinjlim \mathcal{P}^\nu$. It follows that $\text{Tor}_{A,B}$ commutes with the direct limit, and we have

PROPOSITION 1.1. *The equality (\dagger) holds when M (resp. L) ranges over finitely generated left (resp. right) A -modules.*

1.2. In this section we assume that B is left Noetherian and A is left B -finitely generated. Let M be a finitely generated left A -module. Then, all modules in the exact sequences

$$\begin{aligned} 0 \longrightarrow M_1 \longrightarrow A \otimes_B M \xrightarrow{\varepsilon} M \longrightarrow 0 \\ 0 \longrightarrow M \xrightarrow{\varepsilon} \text{Hom}_B(A, M) \longrightarrow M^1 \longrightarrow 0 \end{aligned}$$

are also finitely generated. Therefore M is (A, B) -projective (resp. injective) in the category of finitely generated A -modules if and only if it is (A, B) -projective (resp. injective) in the above sense. Further, the relative projective (resp. injective) dimension of a finitely generated module M in the category of finitely generated modules is identical with $\text{dim}_{A,B} M$ (resp. $\text{inj. dim}_{A,B} M$).

Assume that $\text{dim}_{A,B} M \leq n$ for every finitely generated module M . Let N be a finitely generated module, and let

$$0 \longrightarrow N \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow \dots \longrightarrow Q^{n-1} \longrightarrow N' \longrightarrow 0$$

be an (A, B) -exact sequence such that Q^i ($i = 0, 1, \dots, n-1$) are (A, B) -injective and N' is finitely generated. Then we have, by assumption,

$$\text{Ext}_{A,B}^1(M, N') \cong \text{Ext}_{A,B}^{n+1}(M, N) = 0.$$

But, as is remarked above, N' is then (A, B) -injective, and we have $\text{inj. dim}_{A,B} N' \leq n$. Similarly, if $\text{inj. dim}_{A,B} N \leq n$ for every finitely generated module N , then $\text{dim}_{A,B} M \leq n$ for every finitely generated module M . Hence we have

$$\sup \text{proj. dim}_{A,B} M = \sup \text{inj. dim}_{A,B} N$$

in the category of finitely generated modules. We shall denote this common value by $\text{l. gl. dim}_f(A, B)$.

1.3. Let us consider the homomorphism

$$\sigma_0: \text{Hom}_B(N, Q) \otimes_A M \rightarrow \text{Hom}_B(\text{Hom}_A(M, N), Q) \quad ({}_A M, {}_A N_B, Q_B)$$

defined by

$$\sigma_0(g \otimes m)(f) = g(f(m)), \quad m \in M, \quad g \in \text{Hom}_B(N, Q), \quad f \in \text{Hom}_A(M, N).$$

If $M = A \otimes_B M_0$, σ_0 becomes

$$\text{Hom}_B(N, Q) \otimes_B M_0 \rightarrow \text{Hom}_B(\text{Hom}_B(M_0, N), Q).$$

This is an isomorphism if B is left Noetherian, M_0 is finitely generated and Q is injective (Cartan-Eilenberg [4, Prop. VI 5. 3]). By a direct sum argument we have

LEMMA. σ_0 is an isomorphism, if B is left Noetherian, M is both B -finitely generated and (A, B) -projective, and Q is B -injective.

Assume now A is left B -finitely generated, and let \mathcal{P} be the standard (A, B) -projective resolution of a finitely generated A -module M . Then every P_n is B -finitely generated, and we have by the above Lemma

$$\text{Hom}_B(N, Q) \otimes_A \mathcal{P} \cong \text{Hom}_B(\text{Hom}_A(\mathcal{P}, N), Q).$$

Passing to the homology,

$$\text{Tor}^{A,B}(\text{Hom}_B(N, Q), M) \cong \text{Hom}_B(\text{Ext}_{A,B}(M, N), Q).$$

Hence we have

PROPOSITION 1.2. *If B is left Noetherian and A is left B -finitely generated, then for a finitely generated left A -module M , we have*

$$\text{w. dim}_{A,B} M = \dim_{A,B} M.$$

Applying Proposition 1.1, we obtain

PROPOSITION 1.3. *Under the same assumptions as in Proposition 1.2, we have*

$$\text{w. gl. dim}(A, B) = \text{l. gl. dim}_f(A, B).$$

If B is both left and right Noetherian and A is both left and right B -finite, then we have

$$\text{l. gl. dim}_f(A, B) = \text{w. gl. dim}(A, B) = \text{r. gl. dim}_f(A, B).$$

We denote this common value by $\text{gl. dim}_f(A, B)$.

1.4. Let R be a commutative Noetherian ring, and A be an R -finitely generated algebra over R . (A ring A together with a unitary homomorphism of R into the center of A is called an algebra over R . The finite generation of A is in the module theoretical sense.) Let S be a multiplicative system in R , and R_S the quotient ring of R with respect to S . As usual A_S denotes the algebra $R_S \otimes_R A$ over R_S , and similarly for modules.

Let M be a finitely generated left A -module, and let $\mathcal{P} = \{P_n\}$ be an (A, R) -projective resolution of M such that each P_n is finitely generated. $\mathcal{P}_S = \{R_S \otimes P_n\}$ is then an (A_S, R_S) -projective resolution of $M_S = R_S \otimes M$. As

P_n is finitely generated and R_S is R -flat, we have for any left A -module N

$$H(\text{Hom}_{A_S}(\mathcal{P}_S, N_S)) \cong H(R_S \otimes \text{Hom}_A(\mathcal{P}, N)) \cong R_S \otimes H(\text{Hom}_A(\mathcal{P}, N)),$$

i. e.,

$$\text{Ext}_{A_S, R_S}(M_S, N_S) \cong \text{Ext}_{A, R}(M, N)_S.$$

Conversely, let M' be an A_S -module. Let M be an A -submodule of M' generated by a set of A_S -generators of M' , and define an epimorphism $f: R_S \otimes M \rightarrow M'$ by $f(\alpha' \otimes x) = \alpha'x$. Assume $f(\sum \alpha'_i \otimes x_i) = \sum \alpha'_i x_i = 0$. There exists $\xi \in S$ such that $\alpha'_i = \alpha_i/\xi$, $\alpha_i \in R$, for every i . Then we have $\sum \alpha'_i \otimes x_i = 1/\xi \otimes \sum \alpha_i x_i = 1/\xi \otimes \xi \sum \alpha'_i x_i = 0$. This shows that f is an isomorphism: $M_S \cong M'$. The standard reasoning then yields

PROPOSITION 1.4. *Let A be an R -finitely generated algebra over a Noetherian ring R . For a finitely generated A -module M , we have*

$$\dim_{A, R} M = \sup_{\mathfrak{m}} \dim_{A_{\mathfrak{m}}, R_{\mathfrak{m}}} M_{\mathfrak{m}},$$

and for the global dimensions

$$\text{gl. dim}_f(A, R) = \sup_{\mathfrak{m}} \text{gl. dim}_f(A_{\mathfrak{m}}, R_{\mathfrak{m}}),$$

where \mathfrak{m} ranges over all maximal ideals of R .

§2. Semisimple algebras.

2.1. We shall call an algebra A over R *left semisimple* if $1. \text{gl. dim}_f(A, R) = 0$. This means that every finitely generated left A -module M is (A, R) -projective, namely, that there exists an A -homomorphism $\mu: M \rightarrow A \otimes_R M$ such that $\pi \circ \mu = 1_M$, where π is the canonical epimorphism $A \otimes_R M \rightarrow M$. The right semisimplicity is defined similarly. If A is left and right semisimple, we call A a *semisimple algebra*. If R is a semisimple ring in the classical sense, then the relative projectivity means the absolute projectivity, and a left (or right) semisimple algebra over R in our sense is nothing but a semisimple algebra in the classical sense.

If A is R -flat, and left semisimple over R , we have by Hochschild [12]

$$1. \text{gl. dim } A \leq \text{gl. dim } R.$$

It follows, in particular, that an R -flat left semisimple algebra over a hereditary ring R is left hereditary.

Concerning Noetherian rings, we have

THEOREM 2.1. *If R is Noetherian and A is an R -finitely generated algebra over R , then A is left semisimple over R if and only if it is right semisimple.*

This follows immediately from Proposition 1.3.

PROPOSITION 2.2. *Under the same assumptions as in Proposition 2.1, A is*

semisimple over R if and only if every finitely generated left (or right) A -module is (A, R) -injective.

This is clear from § 1.2.

2.2. As usual, an algebra A over R is considered as an $A^e = A \otimes_R A^\circ$ -module, where A° is an anti-isomorphic copy of A . Now A is called *separable* (Auslander-Goldman [2]) if A is A^e -projective, or equivalently, if the A^e -epimorphism $\varphi: A^e \rightarrow A$ defined by $a \otimes b^\circ \rightarrow ab$ splits, i. e., if there exists $\sum u_i \otimes v_i^\circ \in A^e$ satisfying $\sum u_i v_i = 1$ and such that

$$(*) \quad \sum a u_i \otimes v_i^\circ = \sum u_i \otimes (v_i a)^\circ \quad \text{for every } a \in A.$$

It is to be noted that A is separable if and only if A is (A^e, R) -projective, since the sequence $0 \rightarrow \text{Ker } \varphi \rightarrow A^e \rightarrow A \rightarrow 0$ splits in the category of R -modules.

PROPOSITION 2.3. *A separable algebra is semisimple.*

PROOF. For any left A -module M , we have a commutative diagram

$$\begin{array}{ccc} A^e \otimes_A M & \xrightarrow{\varphi \otimes 1} & A \otimes M \\ \wr \iota & & \wr \\ A \otimes_R M & \xrightarrow{\pi} & M \end{array}$$

where the isomorphism ι is defined by $\iota((a \otimes b^\circ) \otimes x) = a \otimes bx$. If ψ is an A^e -homomorphism $A \rightarrow A^e$, such that $\varphi \circ \psi = 1_A$, then $\psi \otimes 1$ gives rise to an A -homomorphism $\mu: M \rightarrow A \otimes_R M$ such that $\pi \circ \mu = 1_M$ (explicitly $\mu(x) = \sum u_i \otimes v_i x$). Hence A is left semisimple. Similarly A is right semisimple. q. e. d.

Let T be a two-sided A -module. Then $\text{Hom}_{A^e}(A, T)$ is naturally identified with $T^A = \{x \in T \mid ax = xa \text{ for every } a \in A\}$. If A is separable, then $\psi: A \rightarrow A^e$ yields $\text{Hom}_{A^e}(A^e, T) \rightarrow \text{Hom}_{A^e}(A, T)$, i. e., an R -contraction $t: T \rightarrow T^A$. Explicitly $t(x) = \sum u_i x v_i$. If \mathcal{Q} is a left (resp. right) operator domain of the left (resp. right) A -module T , then T^A is an \mathcal{Q} -submodule, and t becomes an \mathcal{Q} -homomorphism.

If, in particular, $T = \text{Hom}_R(M, N)$, where M and N are left A -modules, then we have the *mean* $t: \text{Hom}_R(M, N) \rightarrow \text{Hom}_A(M, N)$ defined by

$$t(g) = \sum u_i \circ g \circ v_i, \quad g \in \text{Hom}_R(M, N).$$

t is an R -homomorphism such that

- i) $t(g) = g$ for $g \in \text{Hom}_A(M, N)$,
- ii) $t(g \circ f) = t(g) \circ f$, $t(h \circ g) = h \circ t(g)$ for $f \in \text{Hom}_A(L, M)$,
 $h \in \text{Hom}_A(N, P)$.

For example, the A -homomorphism μ in the Proof of Proposition 2.3 is $t(\mu_0)$, where $\mu_0 \in \text{Hom}_R(M, A \otimes_R M)$ is defined by $\mu_0(x) = 1 \otimes x$. If \mathcal{Q} is an operator domain of the A -modules M and N , and g is an \mathcal{Q} -homomorphism, then $t(g)$ is also an \mathcal{Q} -homomorphism.

REMARK. The mean t may be interpreted as the composition

$$\mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_A(A \otimes_R M, N) \longrightarrow \mathrm{Hom}_A(M, N),$$

where the first arrow is the usual natural isomorphism, and the second is $\mathrm{Hom}(\mu, 1_N)$, μ being as above. In this way, the mean t can be defined for a finitely generated left A -module M and a left A -module N , when A is a left semisimple algebra over R . t satisfies then i) and the second identity of ii), but not necessarily the first of ii).

2.3. We now study the behavior of semisimple algebras in the tensor products and the coefficient ring extensions.

PROPOSITION 2.4. *If A is an R -finitely generated separable algebra over R and B is a left (resp. right) semisimple algebra over R' , where R' is a commutative algebra over R , then $A \otimes_R B$ is left (resp. right) semisimple over R' .*

PROOF. A finitely generated left $A \otimes B$ -module M is also finitely generated as a left B -module. By assumption, there is a B -homomorphism $\mu_1: M \rightarrow B \otimes_{R'} M$ satisfying $\pi_1 \circ \mu_1 = 1_M$, where π_1 is the natural epimorphism $B \otimes_{R'} M \rightarrow M$. Let ι be the B -homomorphism $B \otimes_{R'} M \rightarrow A \otimes_R B \otimes_{R'} M$ defined by $\iota(y) = 1 \otimes y$, $y \in B \otimes_{R'} M$. Set the mean $\mu = t(\iota \circ \mu_1)$ with respect to the separable algebra A over R . μ is then an $(A \otimes_R B)$ -homomorphism satisfying $\pi \circ \mu = 1_M$, where π is the natural epimorphism $(A \otimes_R B) \otimes_{R'} M \rightarrow M$.

As is remarked above, A is separable if A is (A^e, R) -projective. Hence we have

THEOREM 2.5. *An R -finitely generated algebra A over R is separable if and only if A^e is left (or right) semisimple over R .*

PROPOSITION 2.6. *Let A be an algebra over R , and R' be a commutative algebra over R . Then, the algebra $R' \otimes_R A$ over R' is left semisimple, if one of the following conditions is satisfied.*

- a) A is a separable algebra over R .
- b) A is left semisimple over R , and R' is an R -finitely generated separable algebra over R .

PROOF. The case a) is essentially a special case of Proposition 2.4, since, for $B = R'$, $\pi_1: B \otimes_{R'} M \rightarrow M$ is the identity. (In this case, $R' \otimes_R A$ is separable over R' (Auslander-Goldman [2, Cor. 1.6]).) Under the assumption b), $A \otimes R'$ is left semisimple over R by the same proposition, and *a fortiori* left semisimple over R' .

PROPOSITION 2.7. *If A is left semisimple over R , then $A/\mathfrak{a}A$ is left semisimple over R/\mathfrak{a} , where \mathfrak{a} is an ideal of R .*

PROOF. This is clear, since a finitely generated $A/\mathfrak{a}A$ -module M is finitely generated as an A -module, and the natural map $A/\mathfrak{a}A \otimes M \rightarrow M$ coincides with the natural map $A \otimes M \rightarrow M$.

PROPOSITION 2.8. *Let A be an algebra over R and B an algebra over R' , where R' is a commutative algebra over R . Assume that there exists a finitely generated left B -module N which has an R -direct summand N_0 isomorphic to R . Then, if $A \otimes_R B$ is left semisimple over R' , A is left semisimple over R .*

PROOF. Let M be a finitely generated left A -module and π be the natural epimorphism $A \otimes M \rightarrow M$. $M \otimes_R N$ is a finitely generated left $(A \otimes B)$ -module, and $\pi \otimes 1: A \otimes M \otimes N \rightarrow M \otimes N$ R' -splits. By the semisimplicity of $A \otimes B$, there exists an $A \otimes B$ -homomorphism $\mu': M \otimes N \rightarrow A \otimes M \otimes N$ such that $(\pi \otimes 1) \circ \mu' = 1_{M \otimes N}$. Now, let $\alpha: R \rightarrow N$ and $\beta: N \rightarrow R$ be R -homomorphisms such that $\beta \circ \alpha = 1_R$. Define $\mu: M \rightarrow A \otimes M$ by the commutativity of the diagram

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\mu'} & A \otimes M \otimes N \\ \uparrow 1 \otimes \alpha & & \downarrow 1 \otimes \beta \\ M = M \otimes R & \xrightarrow{\mu} & A \otimes M \otimes R = A \otimes M \end{array}$$

Then, μ is an A -homomorphism, and it is easy to see $\pi \circ \mu = 1_M$.

REMARK. If a B -module N has an R -direct summand N_0 isomorphic to R , then $\iota: R \rightarrow B$ is a monomorphism. But the converse is not true. Example: $R =$ ring of rational integers, $B =$ field of rational numbers.

COROLLARY 2.9. *Let R' be a commutative ring containing R , and assume that R is an R -direct summand of R' . Then, an algebra A over R is left semisimple, if $R' \otimes A$ is left semisimple over R' .*

Now we prove two criteria for separability.

PROPOSITION 2.10. *If R is a Noetherian ring such that R/\mathfrak{m} is perfect for every maximal ideal \mathfrak{m} , then an R -finitely generated semisimple algebra A over R is separable.*

PROOF. $A/\mathfrak{m}A$ is semisimple over R/\mathfrak{m} by Proposition 2.7, and therefore is separable by the perfectness of R/\mathfrak{m} . Our proposition now follows from Auslander-Goldman [2, Th. 4.7].

THEOREM 2.11. *An R -finitely generated algebra A over a Noetherian ring R is separable if and only if $S \otimes_R A$ is left (or right) semisimple over S for every R -finitely generated extension ring S of R .*

The 'only if' part is clear by Proposition 2.6. To prove the converse, we need

LEMMA. *Let R be a commutative ring, \mathfrak{m} a maximal ideal of R and $K = R/\mathfrak{m}$. For any extension field L of K of a finite degree, there exists an R -finitely generated extension ring S of R such that $S/\mathfrak{m}S \cong L$.*

PROOF. By way of the induction, we may assume $L = K(\alpha)$, a simple extension. Let $\varphi(X) = X^n + \dots \in K[X]$ be the minimal polynomial of α over K . Take $f(X) = X^n + \dots \in R[X]$ such that $\varphi(X) = f(X) \pmod{\mathfrak{m}[X]}$, and set

$S = R[X]/(f(X))$. Then we see immediately $S/mS \cong R[X]/(m[X] + (f(X))) \cong K[X]/(f(X)) \cong L$.

Proof of the 'if' part of Theorem 2.11. With the same notations as above, we have $L \otimes_K A/mA \cong S/mS \otimes_K A/mA \cong (S \otimes_R A)/m(S \otimes_R A)$. If $S \otimes_R A$ is left (say) semisimple over S , $L \otimes_K A/mA$ is semisimple over L by Proposition 2.7. If this is the case for every finite extension L of K , A/mA must be separable over $K = R/m$. If, besides, this is the case for every maximal ideal m of R , A must be separable over R by Auslander-Goldman [2, Th. 4.7].

2.4. Let S be a multiplicative system in R , and R_S be the ring of quotients of R with respect to S . For an algebra A over R , A_S denotes the algebra $R_S \otimes A$ over R_S .

As is remarked in § 1.4, an A_S -module M' is isomorphic to $M_S = R_S \otimes M$, where M is an A -module, and if M' is finitely generated then M may be chosen to be finitely generated. Since M_S is (A_S, R_S) -projective if M is (A, R) -projective, A_S is left semisimple over R_S if A is left semisimple over R . In particular, we have

PROPOSITION 2.12. *Let \mathfrak{p} be a prime ideal of R . If A is left semisimple over R , then $A_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes A$ is left semisimple over $A_{\mathfrak{p}}$.*

PROPOSITION 2.13. *If A is left semisimple over an integral domain R , then the rational hull $A_K = K \otimes A$ is semisimple over the quotient field K of R .*

Finally, we have by Proposition 1.4

THEOREM 2.14. *An R -finitely generated algebra A over a Noetherian ring R is semisimple over R , if and only if $A_{\mathfrak{m}}$ is semisimple over $R_{\mathfrak{m}}$ for every maximal ideal \mathfrak{m} of R .*

§ 3. Semisimple subalgebras of a central separable algebra.

3.1. Let A be an algebra over R , and B a subalgebra of A . A is considered as a left $B \otimes A^\circ$ -module in the natural way, where A° is an inverse-isomorphic copy of A corresponding to the right multiplications. We denote by ϕ the $B \otimes A^\circ$ -epimorphism $B \otimes A^\circ \rightarrow A$ defined by $\phi(b \otimes a^\circ) = ba$. For an $B \otimes A^\circ$ -module (i.e., a left B - and right A -module) T , $\text{Hom}_{B \otimes A^\circ}(A, T)$ is isomorphic to the submodule $T^B = \{x \in T \mid bx = xb \text{ for every } b \in B\}$ of T , and is identified to it.

Now assume that A is $B \otimes A^\circ$ -projective. Then, the sequence $0 \rightarrow \text{Ker } \phi \rightarrow B \otimes A^\circ \rightarrow A \rightarrow 0$ splits, and we have a contraction

$$t: T (= \text{Hom}_{B \otimes A^\circ}(B \otimes A^\circ, T)) \rightarrow T^B (= \text{Hom}_{B \otimes A^\circ}(A, T)).$$

If, in particular, we take $T = A$, then $T^B =$ the commuter of B in A , and we have

PROPOSITION 3.1. *If A is left $B \otimes A^\circ$ -projective, where B is a subalgebra of*

A , then the commuter C of B in A is a direct summand of A regarded as a left C -module.

If $B=A$, this proposition reduces to Auslander-Goldman [2, Prop. 1.2]. Similarly as there, it follows immediately

COROLLARY 3.2. For a right ideal r of C , we have $rA \cap C = r$.

3.2. Let A be a central separable algebra over R , where central means $R =$ the center of A . By Auslander-Goldman [2, Th. 2.1], A is a finitely generated R -projective module, the homomorphism

$$\tau_A : \text{Hom}_{A^e}(A, A^e) \otimes A \rightarrow A^e$$

defined by $\tau_A(f \otimes x) = f(x)$ is surjective, and the homomorphism

$$\eta : A \otimes A^\circ \rightarrow \text{Hom}_R(A, A)$$

defined by $\eta(a \otimes b^\circ)(x) = (a \otimes b^\circ)(x) = axb$ is an isomorphism.

Let B be a subalgebra of A , and C be the commuter of B . As in [2, § 2], we consider the following commutative diagram

$$\begin{array}{ccc} \text{Hom}_{B \otimes A^\circ}(A, B \otimes A^\circ) \otimes_C A & \xrightarrow{\zeta \otimes 1} & \text{Hom}_C(A, C) \otimes_C A \\ \downarrow \tau_B & & \downarrow \lambda \\ B \otimes A^\circ & \xrightarrow{\eta} & \text{Hom}_C(A, A) \end{array}$$

where A is considered as a left $B \otimes A^\circ$ - and left C -module, and τ_B, ζ, λ are defined by $\tau_B(g \otimes x) = g(x)$, $\zeta(g)(a) = \eta(g(1))(a) = g(1)(a) (\in C)$, and $\lambda(\alpha \otimes a)(x) = \alpha(x)a$, respectively.

PROPOSITION 3.3. Let B be a subalgebra of a central separable algebra A over R , and C its commuter. Assume further that B is a left B -direct summand of A . Then, $\eta : B \otimes A^\circ \rightarrow \text{Hom}_C(A, A)$ is an isomorphism, A is left C -projective, and B is the commuter of C .

PROOF. Let γ be a left B -projection $A \rightarrow B$. If $f_i \in \text{Hom}_{A^e}(A, A^e)$ and $a_i \in A$ are such that $\tau_A(\sum f_i \otimes a_i) = \sum f_i(a_i) = 1 \otimes 1^\circ$, then $g_i = (\gamma \otimes 1) \circ f_i \in \text{Hom}_{B \otimes A^\circ}(A, B \otimes A^\circ)$ satisfy $\sum g_i(a_i) = (\gamma \otimes 1) \sum f_i(a_i) = 1 \otimes 1^\circ$. Hence the homomorphism τ_B is surjective, and the first two assertions of Proposition 3.3 are immediate consequences of Auslander-Goldman [1, Th. A.2]. To see the last assertion, let a be in the commuter of C . Then $\eta(a \otimes 1^\circ)$ is in $\text{Hom}_C(A, A) = \eta(B \otimes A^\circ)$. Hence $a \otimes 1 \in B \otimes A^\circ$. As R is a direct summand of A° , we have $a \in B$, as desired.

REMARK. If a separable algebra A is left C -projective, where C is a subalgebra of A , then A is left $C \otimes A^\circ$ -projective. For, let π be the $C \otimes A^\circ$ -homomorphism $C \otimes A^\circ \rightarrow A$ defined by $c \otimes a^\circ \rightarrow ca$. If A is C -projective, there exists a C -homomorphism $\mu : A \rightarrow C \otimes A^\circ$ such that $\pi \circ \mu =$ identity. Take the mean $\mu^* = t(\mu)$ of μ with respect to the right multiplications of A , explicitly

$\mu^*(x) = \sum(1 \otimes v_i^\circ)\mu(xu_i)$, where u_i and v_i are as in § 2.2. μ^* is a $C \otimes A^\circ$ -homomorphism $A \rightarrow C \otimes A^\circ$, such that $\pi \circ \mu^* = \text{identity}$, and A is $C \otimes A^\circ$ -projective.

3.3. Concerning semisimplicity, we have

PROPOSITION 3.4. *Let B be a subalgebra of A , and C the commuter of B . If $B \otimes A^\circ$ is right semisimple and A is left $B \otimes A^\circ$ -projective as well as left C -projective, then C is left semisimple.*

PROOF. For a finitely generated left C -module M , set $(B \otimes A^\circ)^B \otimes_C M$, where $(B \otimes A^\circ)^B = \text{Hom}_{B \otimes A^\circ}(A, B \otimes A^\circ)$ is a right $B \otimes A^\circ$ - and right C -module. As A is left $B \otimes A^\circ$ -projective, $(B \otimes A^\circ)^B$ is a right direct summand of $B \otimes A^\circ$, and $(B \otimes A^\circ)^B \otimes_C M$ is finitely generated as a right $B \otimes A^\circ$ -module. Furthermore, tensoring with the left $B \otimes A^\circ$ - and C -module A , we have C -isomorphisms $((B \otimes A^\circ)^B \otimes_C M) \otimes_{B \otimes A^\circ} A \cong ((B \otimes A^\circ)^B \otimes_{B \otimes A^\circ} A) \otimes_C M \cong C \otimes_C M \cong M$ (cf. Auslander-Goldman [1, Prop. A.4]). As $B \otimes A^\circ$ is right semisimple, $(B \otimes A^\circ)^B \otimes_C M$ is a direct summand of $((B \otimes A^\circ)^B \otimes_C M) \otimes (B \otimes A^\circ)$. Tensoring with A over $B \otimes A^\circ$, we see that M is a C -direct summand of $((B \otimes A^\circ)^B \otimes_C M) \otimes A$. But, the latter module is (C, R) -projective, since A is left C -projective. Hence M is also (C, R) -projective, and C is left semisimple.

Let A be a central separable algebra over R . Let \mathcal{L} (resp. \mathcal{R}) denote the set of subalgebras B of A such that A is left (resp. right) B -projective and B is a left (resp. right) B -direct summand of A . Further, let \mathcal{S}_l (resp. \mathcal{S}_r) be the set of left (resp. right) semisimple subalgebras B which are R -direct summands of A (or, equivalently, left (resp. right) B -direct summands of A). If $B \in \mathcal{S}_l$, then A is left (B, R) -projective as well as R -projective, hence A is left B -projective. It follows that $\mathcal{S}_l \subset \mathcal{L}$. Similarly, we have $\mathcal{S}_r \subset \mathcal{R}$. If, finally, we denote by \mathcal{S} the set of (both left and right) semisimple subalgebras which are R -direct summands of A , then it is clear that $\mathcal{S} = \mathcal{S}_l \cap \mathcal{S}_r = (\mathcal{S}_l \cap \mathcal{R}) \cap (\mathcal{S}_r \cap \mathcal{L})$. With these notations, we summarize the above discussions in the following

THEOREM 3.5. *The formation of the commuter $B \rightarrow V(B)$ in the set of subalgebras of a central separable algebra A over R provides a one-to-one correspondence of \mathcal{L} to itself (resp. \mathcal{R} to itself) such that $V^2 = \text{identity}$. In $\mathcal{L} \cap \mathcal{R}$, V yields a one-to-one correspondence of $\mathcal{S}_l \cap \mathcal{R}$ to $\mathcal{S}_r \cap \mathcal{L}$, and, in particular, of \mathcal{S} to itself.*

REMARK. If B is a left semisimple subalgebra of an R -finite separable algebra A , and if B is a $B \otimes B^\circ$ -direct summand of A , then B is also separable. Indeed, since $B \otimes A^\circ$ is left semisimple, $B \otimes A^\circ$ -epimorphism $B \otimes A^\circ \rightarrow A$ splits, and there exist $b_i \in B$ and $a_i \in A$ such that $\sum b_i a_i = 1$ and $\sum b b_i \otimes a_i^\circ = \sum b_i \otimes (a_i b)^\circ$ for every $b \in B$. Now, let γ be a $B \otimes B^\circ$ -projection $A \rightarrow B$, and put $a'_i = \gamma(a_i) \in B$. Then, we have $\sum b_i a'_i = 1$ and $\sum b b_i \otimes a'_i = \sum b_i \otimes (a'_i b)^\circ$ for every $b \in B$. This shows that B is separable.

§ 4. Semisimple orders.

4.1. Let R be an integral domain, and K its quotient field. If A is an R -finitely generated, torsion free left semisimple algebra over R , then the rational hull $\mathfrak{A} = A_K = A \otimes_R K$ is a semisimple algebra of a finite rank over K , and A is canonically imbedded in \mathfrak{A} as an R -order (Deuring [6], Auslander-Goldman [1]).

In this section 4.1, we assume that R is a Prüfer domain. This means that every finitely generated torsion free R -module is R -projective (Cartan-Eilenberg [4]). Now, let M be a finitely generated A -module, and consider the \mathfrak{A} -space $M_K = M \otimes_R K$. If $M = M_1 \oplus M_2$, then obviously $M_K = M_{1K} \oplus M_{2K}$. Conversely, there holds

PROPOSITION 4.1. *Let M be a finitely generated R -torsion free A -module. For any \mathfrak{A} -subspace U of M_K , there exists a direct sum decomposition $M = M_1 \oplus M_2$ such that $M_{1K} = U$.*

PROOF. Let $M_1 = M \cap U$, M being canonically imbedded in M_K . Then M/M_1 is R -finitely generated and torsion free, whence R -projective. Since A is R -semisimple, it is A -projective. Hence we have a direct sum decomposition $M = M_1 \oplus M_2$. It is clear that $M_{1K} = U$.

Applying this to $M = A$, we have

COROLLARY 4.2. *For any left ideal \mathfrak{I} of \mathfrak{A} , there exists a direct sum decomposition $A = I \oplus J$, such that $I_K = \mathfrak{I}$.*

This means that if e is an idempotent in \mathfrak{A} , then there exists an idempotent e' in $A \cap \mathfrak{A}e$ such that $\lambda e = ae'$ for some $\lambda \neq 0$ in R and a in A . It follows immediately $e'e = e'$, and $ee' = e$. If e is central, we have $e = e' \in A$. Thus, we obtain

THEOREM 4.3. *A contains all central idempotents of \mathfrak{A} . If $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$, be the decomposition of \mathfrak{A} into the direct sum of simple algebras, then $A = A_1 \oplus \cdots \oplus A_r$, where $A_i = A \cap \mathfrak{A}_i$, $\mathfrak{A}_i = A_{iK}$, and A_i are left semisimple over R ($i = 1, \dots, r$).*

The study of left semisimple algebra A over a Prüfer domain is therefore reduced to the case where \mathfrak{A} is simple. We shall call A in this case a *left simple algebra* over R , though this nomination seems somewhat unsuitable.

REMARK. Proposition 4.1 is valid also under the assumptions that R is a Dedekind domain, A is an R -finitely generated, torsionfree hereditary algebra over R , and M is a finitely generated A -projective module (Auslander-Goldman [1, Prop. 2.8]).

4.2. We still assume that R is a Prüfer domain, and A is an R -finitely generated torsionfree left semisimple algebra over R . As is mentioned in the Proof of Proposition 4.1, a finitely generated torsionfree A -module is A -projective. In particular, all finitely generated left ideals of A are A -projective.

In other words, A is a left semihereditary ring.

REMARK. We introduced in [8] the notions of torsionfree modules and torsion modules over a general ring. A projective module is always torsionfree. In the present case, conversely, every finitely generated A -torsionfree module, being R -torsionfree, is A -projective. We do not know whether all left semihereditary rings enjoy this last property.

If A is simple and commutative, then A is an integral domain, because a simple commutative algebra over a field is a field. Hence A is also a Prüfer domain. Similarly, if R is a Dedekind domain, an R -finitely generated torsionfree simple commutative algebra over R is also a Dedekind domain. We shall call a Dedekind domain S containing R *weakly unramified* over R if S is R -finitely generated and every prime ideal \mathfrak{p} of R decomposes in S in the form $\mathfrak{p}S = \mathfrak{P}_1 \cdots \mathfrak{P}_g$, where \mathfrak{P}_i are different primes in S .

PROPOSITION 4.4. *Let R be a Dedekind domain and S a commutative algebra over R which is R -finitely generated and torsionfree. Then S is semisimple over R if and only if S is a direct sum of a finite number of weakly unramified Dedekind domains over R .*

PROOF. By the facts mentioned above, it suffices to prove that, on assuming S to be a Dedekind domain, S is semisimple if and only if it is weakly unramified. If S is semisimple, then $S/\mathfrak{p}S$ is semisimple over R/\mathfrak{p} . Hence there are prime ideals $\mathfrak{P}_1, \dots, \mathfrak{P}_g$ of S such that $\mathfrak{p}S = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_g = \mathfrak{P}_1 \cdots \mathfrak{P}_g$. Conversely, we assume that S is weakly unramified. By the elementary divisor theory over S , a finitely generated module over S is a direct sum of a projective module and (a finite number of) torsion modules of type S/\mathfrak{P}^e (\mathfrak{P} a prime in S) (cf. e. g. Chevalley [5]). We have therefore to prove that S/\mathfrak{P}^e is (S, R) -projective. Let $\mathfrak{p} = \mathfrak{P} \cap R$, R being canonically imbedded in S , and put $\mathfrak{p}S = \mathfrak{P}\mathfrak{Q}$. Then we have $(\mathfrak{P}, \mathfrak{Q}) = 1$ by assumption. Let $a \in S$ be such that $a \in \mathfrak{Q}^e$, $a \notin \mathfrak{P}$. There exists $b \in S$ such that $ab \equiv 1 \pmod{\mathfrak{P}^e}$. If we denote the residue class of $b \pmod{\mathfrak{P}^e}$ by \bar{b} , we have $\mathfrak{P}^e(a \otimes \bar{b}) = 0$ in $S \otimes S/\mathfrak{P}^e$. Indeed, if $x \in \mathfrak{P}^e$ then $xa \in \mathfrak{P}^e\mathfrak{Q}^e = \mathfrak{p}^eS$; hence $xa = \sum s_i y_i$, $s_i \in S$, $y_i \in \mathfrak{p}^e$, and we have $x(a \otimes \bar{b}) = \sum s_i \otimes y_i \bar{b} = \sum s_i \otimes \overline{y_i b} = 0$. We can therefore define an S -homomorphism $\mu: S/\mathfrak{P}^e \rightarrow S \otimes S/\mathfrak{P}^e$ by $1 \rightarrow a \otimes \bar{b}$. Clearly the composition of μ with the natural homomorphism $S \otimes S/\mathfrak{P}^e \rightarrow S/\mathfrak{P}^e$ is the identity. Hence S/\mathfrak{P}^e is (S, R) -projective.

THEOREM 4.5. *Let A be a finitely generated torsionfree simple algebra over a Dedekind domain R . Then the center C of A is a simple (i. e., weakly unramified) Dedekind domain over R .*

PROOF. We first prove that C is a Dedekind domain. Let $\mathfrak{A} = A_K$, K being the quotient field of R and Z the center of \mathfrak{A} . Z is an extension field of finite degree over K , and $C = Z \cap A$. It suffices therefore to show that all those

elements of Z which are integral over R are contained in A (cf. e. g. Zariski-Samuel [13, p. 281]). Since $A = \bigcap A_{\mathfrak{p}}$, \mathfrak{p} running over all prime ideals of R , and since $A_{\mathfrak{p}}$ is simple over $R_{\mathfrak{p}}$, our problem may be reduced to the \mathfrak{p} -local cases. Thus, we assume that R is a discrete valuation ring of rank 1 with the maximal ideal $\mathfrak{p} = (\pi)$. Assume that an element $z = \pi^{-e}a$ ($e > 0$, $a \in A$, $a \notin \pi A$) in Z is integral over R . Then, there exists a certain number $f \geq 0$ such that $\pi^f z^k \in A$, for all $k = 1, 2, \dots$. Taking k large enough, we have $a^k \in \pi A$. But, as $A/\pi A$ is semisimple over $R/(\pi)$, $A/\pi A$ contains no central nilpotent element other than 0. Thus, we have $a \in \pi A$, a contradiction.

Finally, $C/\mathfrak{p}C$ is a central subalgebra of the semisimple algebra $A/\mathfrak{p}A$ over R/\mathfrak{p} , so that it is also semisimple. It follows that \mathfrak{p} decomposes in the form $\mathfrak{p}C = \mathfrak{P}_1 \cdots \mathfrak{P}_g$, and C is weakly unramified over R .

4.3. We now pass to the problem of maximality of semisimple orders, and we begin with the local study. So, let \mathfrak{o} be a local ring with the maximal ideal \mathfrak{m} .

LEMMA. *Let A be an \mathfrak{o} -finitely generated semisimple algebra over \mathfrak{o} . Then, the radical of A is $\mathfrak{m}A$, and A has only a finite number of maximal two-sided ideals.*

PROOF. If a maximal left ideal \mathfrak{l} of A does not contain $\mathfrak{m}A$, then $A = \mathfrak{m}A + \mathfrak{l}$. As A is \mathfrak{o} -finite, we would have $A = \mathfrak{l}$. Hence $\mathfrak{l} \supset \mathfrak{m}A$. On the other hand, as $A/\mathfrak{m}A$ is a semisimple algebra over the field $\mathfrak{o}/\mathfrak{m}$, there exist maximal ideals \mathfrak{M}_i , $i = 1, \dots, g$, such that $\mathfrak{m}A = \bigcap \mathfrak{M}_i$. It follows that $\mathfrak{m}A$ coincides with the radical of A . Furthermore, any maximal two-sided ideal \mathfrak{M} of A contains $\mathfrak{m}A$, and therefore is identical with one of \mathfrak{M}_i 's.

Now, we assume that \mathfrak{o} is a local domain, and the maximal ideal \mathfrak{m} is principal $\mathfrak{m} = t\mathfrak{o}$. We shall consider the \mathfrak{m} -adic completion \hat{A} of a semisimple \mathfrak{o} -order A . Assume that the semisimple algebra $A/\mathfrak{m}A$ over $\mathfrak{o}/\mathfrak{m}$ has g simple components, and let \mathfrak{M}_i , $i = 1, \dots, g$, be maximal two-sided ideals of A such that

$$\bigcap \mathfrak{M}_i = \mathfrak{m}A, \quad \mathfrak{M}_i + \bigcap_{j \neq i} \mathfrak{M}_j = A, \quad i = 1, \dots, g.$$

Further, let the identity of the simple component $\bigcap_{j \neq i} \mathfrak{M}_j/\mathfrak{m}A$ be the residue class of e_i modulo $\mathfrak{m}A$ ($e_i \in A$). We have

$$e_i \in \mathfrak{M}_j (j \neq i), \quad e_i \mathfrak{M}_i \subset \mathfrak{m}A, \quad e_i e_j \equiv \delta_{ij} e_i \pmod{\mathfrak{m}A}.$$

We shall show that

$$A/\mathfrak{m}^n A \cong A/\mathfrak{M}_1^n \oplus \cdots \oplus A/\mathfrak{M}_g^n, \quad n = 1, 2, \dots$$

by induction on n . For $n = 1$, this is clear. Let $n > 1$.

i) $\bigcap \mathfrak{M}_i^n = \mathfrak{m}^n A$. By the induction-hypothesis, $a \in \bigcap \mathfrak{M}_i^n (\subset \bigcap \mathfrak{M}_i^{n-1})$ may be written as $a = t^{n-1}b$, $b \in A$. Then, $t^{n-1}be_i \in \mathfrak{M}_i^n e_i \subset t\mathfrak{M}_i^{n-1}$. It follows that

$t^{n-2}be_i \in \mathfrak{M}_i^{n-1}$. Since $b \equiv be_i \pmod{\mathfrak{M}_i}$, we have $t^{n-2}b \equiv t^{n-2}be_i \pmod{\mathfrak{M}_i m^{n-2}}$ and *a fortiori* $t^{n-2}b \equiv t^{n-2}be_i \equiv 0 \pmod{\mathfrak{M}_i^{n-1}}$. i being arbitrary, we have $t^{n-2}b \in \bigcap \mathfrak{M}_i^{n-1} = t^{n-1}A$. Hence, we have $b \in tA$, so that $a \in t^n A$.

ii) We must solve $x \equiv a_i \pmod{\mathfrak{M}_i^n}$, $i = 1, \dots, g$, for arbitrarily given a_i 's. If $\mathfrak{M}_i^n + \bigcap_{j \neq i} \mathfrak{M}_j^n = A$ for every i , let $1 = u_i + v_i$, $u_i \in \mathfrak{M}_i^n$, $v_i \in \bigcap_{j \neq i} \mathfrak{M}_j^n$. Then $x = \sum v_i a_i$ will give a solution. Suppose that $\mathfrak{M}_i^n + \bigcap_{j \neq i} \mathfrak{M}_j^n$ is not identical with A for some i . Then, it is contained in some maximal two-sided ideal of A , say \mathfrak{M}_k . If $k \neq i$, then we would have $e_k^n \in \mathfrak{M}_i^n \subset \mathfrak{M}_k$, which is impossible. If $k = i$, then $e_i^n \in \bigcap_{j \neq i} \mathfrak{M}_j^n \subset \mathfrak{M}_i$, which is also impossible. Hence we must have $\mathfrak{M}_i^n + \bigcap_{j \neq i} \mathfrak{M}_j^n = A$ for all i .

As the diagram

$$\begin{array}{ccccccc} A/m^n A & \cong & A/\mathfrak{M}_1^n & \oplus & \dots & \oplus & A/\mathfrak{M}_g^n \\ \uparrow & & \uparrow & & & & \uparrow \\ A/m^{n+1} A & \cong & A/\mathfrak{M}_1^{n+1} & \oplus & \dots & \oplus & A/\mathfrak{M}_g^{n+1} \end{array}$$

is commutative, where the vertical maps are natural homomorphisms, we may pass to the completion, and we have

$$\varprojlim A/m^n A \cong \varprojlim A/\mathfrak{M}_1^n \oplus \dots \oplus \varprojlim A/\mathfrak{M}_g^n.$$

Thus, we have a direct sum decomposition of the m -adic completion of A :

$$(*) \quad \hat{A} \cong A_1 \oplus \dots \oplus A_g.$$

Using this, we shall prove

THEOREM 4.6. *Let R be an integrally closed Noetherian domain, K its quotient field, and \mathfrak{A} a central simple algebra over K . Then, an R -order A in \mathfrak{A} which is R -projective and semisimple over R is a maximal order.*

PROOF. By Auslander-Goldman [1, Th. 1.5] and § 2.4, we may localize the problem by minimal primes of R . So we assume that R is a discrete valuation ring \mathfrak{o} of rank 1, \mathfrak{A} is a central simple algebra over K , the quotient field of \mathfrak{o} , and A is an semisimple \mathfrak{o} -order in \mathfrak{A} . Let $\hat{\mathfrak{o}}, \hat{K}, \hat{A}$, be the m -adic completions of \mathfrak{o}, K, A , respectively. Then, we have by (*)

$$\mathfrak{A} \otimes_K \hat{K} = A \otimes_{\mathfrak{o}} \hat{K} = \hat{A} \otimes_{\hat{\mathfrak{o}}} \hat{K} = A_1 \otimes_{\hat{\mathfrak{o}}} \hat{K} \oplus \dots \oplus A_g \otimes_{\hat{\mathfrak{o}}} \hat{K}.$$

But, as \mathfrak{A} is central simple, g must be 1. This shows that mA is the unique maximal two-sided ideal of A . Since A is hereditary, it follows from Auslander-Goldman [1, Th. 2.3] that A is maximal, as desired.

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