J. Math. Soc. Japan
Vol. 15, No. 4, 1963

On weak boundary components of a Riemann surface

Dedicated to Professor Y. Akizuki on his 60th birthday

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(Received March 15, 1963)
(Revised July 6, 1963)

Introduction

L. Sario [12] introduced the notion of weakness of an ideal boundary component of a Riemann surface[1] and properties of weak boundary components have been studied by his students. In this paper we use Jurchescu [3]'s modified definition as follows.

Let γ be an ideal boundary component of a Riemann surface R in Stoïlow's sense and $\{S_n\}$ be its defining system. Here we suppose that $\{S_n\}$ is defined by a canonical exhaustion $\{R_n\}$ of R, that is, S_n is a connected component of $R-R_n$. We put $\partial S_n=\gamma_n$ and $\partial R_n=\gamma_n\cup(\bigcup_i \beta_n^i)$, where each β_n^i is a connected component of ∂R_n. Let t_n be the harmonic function in R_n-R_0 which satisfies the following boundary conditions:

$$t_n=\begin{cases} d_n^\gamma(>0) \text{ and } \int dt_n^*=1 & \text{on } \gamma_n \\ d_n^i\,(d_n^\gamma>d_n^i>0) \text{ and } \int dt_n^*=0 & \text{on } \beta_n^i \\ 0 & \text{on } \partial R_0, \end{cases}$$

where t_n^* is the conjugate harmonic function of t_n. Then the Dirichlet integral $D(t_n)$ of t_n over R_n-R_0 equals d_n^γ and there always exists the limit of $D(t_n)$ and

$$\lim_{n\to\infty} D(t_n)=d^\gamma\ (\leqq\infty).$$

DEFINITION. γ is said to be weak when $d^\gamma=\infty$.
The property $d^\gamma=\infty$ does not depend on the choice of the exhaustion $\{R_n\}$.

If γ is not weak, in $R-R_0$, there exists the unique *extremal* harmonic function $t^\gamma(=\lim_{n\to\infty} t_n)$ with a finite Dirichlet integral. It has the minimal Dirichlet integral among those functions $\{t\}$ in $R-R_0$ which satisfy the following conditions:

1) In the case of a plane region Grötzsch first introduced the notion "vollkommenpunktförmig" which corresponds to "weak".

$$t=0 \quad \text{on} \quad \partial R_0$$

$$\int_{\gamma_n} dt^* = 1 \quad \text{and} \quad \int_{\beta_n^i} dt^* = 0,$$

and $D_{R-R_0}(t) = D_{R-R_0}(t^{\gamma}) + D_{R-R_0}(t-t^{\gamma})$.

In this paper we give three criterions for the weakness of γ (§ 1) and study some problems of classification of Riemann surfaces (§ 3). In the case of a bordered Riemann surface we give analogous criterions according to Jurchescu [4]'s modified definition (§ 2) and study the properties of subregions of a Riemann surface.

§ 1. Criterions of weakness.

Let $\{R_n\}$ be a canonical exhaustion of R and S_n be the component of $R-R_n$ which is a neighborhood of γ[2]. We put $\partial S_n = \gamma_n$ and consider Nevanlinna's function ω_{γ_n} with respect to γ_n, which we construct in the proof of Theorem 1. ω_γ means the limit of ω_{γ_n} when n tends to ∞ and γ_n to γ. Then we can find the following

THEOREM 1. *γ is weak if and only if $\omega_\gamma \equiv 0$.*

PROOF. If γ is not weak, there exists the non-constant extremal function $t^{\gamma} = \lim_{n\to\infty} t_n$ in $R-R_0$ such that $D_{R-R_0}(t^{\gamma}) = d^{\gamma} < \infty$.

We construct the harmonic function u_n of $R_n - R_0$ as follows

$$u_n = \begin{cases} d_n & \text{on } \gamma_n \text{ and } \int_{\gamma_n} du_n^* = 1 \\ 0 & \text{on } \partial(R_n - R_0) - \gamma_n . \end{cases}$$

Then $D_{R_n-R_0}(u_n) = \int_{\gamma_n} u_n du_n^* = d_n$, and for $n > m$

$$0 \leqq D_{R_m-R_0}(u_n - u_m) = D(u_n) - 2\int_{\partial(R_m-R_0)} u_m du_n^* + D(u_m)$$
$$= D(u_n) - D(u_m),$$

so $d_n = D_{R_n-R_0}(u_n) > D_{R_m-R_0}(u_n) \geqq D_{R_m-R_0}(u_m) = d_m$. Therefore $\{d_n\}$ is monotone increasing, and its limit d is finite, because

$$0 \leqq D_{R_n-R_0}(t^{\gamma} - u_n) = D_{R_n-R_0}(t^{\gamma}) - D_{R_n-R_0}(u_n) \quad \text{and} \quad D_{R-R_0}(t^{\gamma}) < \infty .$$

The sequence $\{u_n\}$ is uniformly bounded ($0 \leqq u_n \leqq d$), so it converges uniformly on every compact set. We put $v_n = u_n/d_n$, then

$$v_n = \begin{cases} 1 & \text{on } \gamma_n \\ 0 & \text{on } \partial(R_n - R_0) - \gamma_n , \end{cases}$$

2) By a "Neighborhood of γ" we mean an end of R which belongs to a defining system of γ.

and $v=\lim_{n\to\infty} v_n=\lim u_n/\lim d_n$ converges uniformly on every compact set.

Here we construct Nevanlinna's function ω_{γ_n} as follows and compare it with v. Let S_n be an end of R, whose relative boundary is γ_n, and for $m>n$, let $R'_m=(R-R_0-S_n)\cap R_m$. In R'_m we consider the following harmonic function

$$\omega_{nm}=\begin{cases} 1 & \text{on } \gamma_n \\ 0 & \text{on } \partial R'_m-\gamma_n. \end{cases}$$

$\omega_{nm}\geqq v_m$ in R_n-R_0, so $\omega_n=\lim_{R'_m\to(R-R_0-S_n)}\omega_{nm}\geqq\lim_{m\to\infty} v_m=v$ on $R-R_0-S_n$, and $\omega_{\gamma_n}=\lim_{n\to\infty}\omega_n\geqq\lim_{m\to\infty} v_m=v$. Therefore, on putting $\omega_\gamma=\lim_{n\to\infty}\omega_{\gamma_n}$ we have

$$\sup_{R-R_0}\omega_\gamma>\sup_{R-R_0} v>0 \quad\text{and}\quad \omega_\gamma=0 \quad\text{on}\quad \partial R_0.$$

Consequently ω_γ is not a constant.

Conversely, we suppose that γ is weak. We consider the following harmonic function v'_{nm} in R'_m.

$$v'_{nm}=\begin{cases} k_{nm}\,(>0) & \text{on } \gamma_n \text{ and } \int_{\gamma_n} dv'^{*}_{nm}=1 \\ l_{nm}\ (0<l_{nm}<k_{nm}) & \text{on } \partial R_n-\partial R_0-\gamma_n \text{ and } \int_{(\partial R_n-\partial R_0-\gamma_n)} dv'^{*}_{nm}=0 \\ 0 & \text{on } \partial R_0. \end{cases}$$

If we put $v_{nm}=v'_{nm}/k_{nm}$, $\{v_{nm}\}$ is uniformly bounded for m, so we can choose a convergent subsequence which we denote by $\{v_{nm}\}$ again. Since $k_{nm}=D_{R'_m}(v'_{nm})$ is monotone increasing for m, we get

$$v'_n=\lim_{m\to\infty} v'_{nm}=\lim_{m\to\infty} k_{nm}\lim_{m\to\infty} v_{nm}=k_n v_n,$$

while, for the extremal function t_n, $D_{R_n-R_0}(t_n)\leqq D_{R-S_n-R_0}(v'_n)=k_n$. If γ is weak $\lim k_n=\infty$. Hence

$$D_{R-S_n-R_0}(v_n)=\lim_{m\to\infty} D_{R'_m}(v_m)=\lim_{m\to\infty} D_{R'_m}(v_{nm})=\lim_{m\to\infty} 1/k_{nm}=1/k_n$$

tends to zero, if γ is weak. But Nevanlinna's function satisfies the inequality $v_n\geqq\omega_{\gamma_n}$, and for all $m>n$, $v_n\geqq\omega_{\gamma_n}>\omega_{\gamma_m}$ on $R-S_n-R_0$. Therefore, if $\lim_{n\to\infty} v_n=0$, we have $\omega_\gamma=0$. q.e.d.

The above constructed ω_γ has a relation with the harmonic measure of Royden's harmonic boundary [**5**]. Each $\gamma_n=\partial S_n$ divides R into two parts, one of which is S_n and the other is $R-S_n$. Let R^* be Royden [**11**]'s compactification of $R-R_0$, Δ be the harmonic boundary of it and Δ_n be the harmonic boundary of the ideal boundary of S_n, that is, $\Delta_n=\bar{S}_n\cap\Delta$ where $\bar{S}_n$ is the closure of S_n in R^*. We define the harmonic boundary component γ_Δ

which corresponds to γ by $\gamma_{\Delta}=\cap\Delta_n$. Both the harmonic measure Ω_{Δ_n} of Δ_n and Nevanlinna's function ω_{γ_n} are bounded harmonic functions with finite Dirichlet integral (we denote it by *HBD*) in $R-R_0$. Since

$$\omega_{\gamma_n}=\begin{cases}0 & \text{on } \Delta-\Delta_n\\ 1 & \text{on } \Delta_n,\end{cases}$$

$\omega_{\gamma_n}=\Omega_{\Delta_n}$ on Δ, and $\omega_{\gamma_n}=\Omega_{\Delta_n}$ on R^*. $\{\omega_{\gamma_n}\}$ decreases monotone when n increases. $\{\Omega_{\Delta_n}\}$ also decreases by the way of construction of Ω_{Δ_n} and by the fact $\Delta_n\supset\Delta_{n+1}$ [5]. Then $\Omega_{\gamma_\Delta}=\lim\Omega_{\Delta_n}$ is the harmonic measure of γ_Δ, and $\Omega_{\gamma_\Delta}=\omega_\gamma$ in $R-R_0$ because $\Omega_{\Delta_n}=\omega_{\gamma_n}$ in $R-R_0$. Therefore, $\Omega_{\gamma_\Delta}=\omega_\gamma$ on R^* and by Theorem 1 we get the following

THEOREM 2. *γ is weak if and only if $\Omega_{\gamma_\Delta}\equiv 0$.*

This theorem gives an example of a Riemann surface each of its harmonic boundary components has a vanishing harmonic measure, while R^*-R has a positive harmonic measure.

Namely, let S be a compact N_{SD}-set in the extended plane W and have a positive capacity, then the Riemann surface $W-S$ is the example, because every boundary component of $W-S$ is weak [3].

THEOREM 3. *Let u be an arbitrary HD-function defined in a neighborhood of γ. γ is weak if and only if we can find for u a sequence of dividing cycles γ_n which tends to γ and is such that $\lim_{n\to\infty}\int_{\gamma_n}du^*=0$.*

PROOF. Let $w=u+iu^*$ and c be a dividing cycle, then

$$\left|\int_c du^*\right|\leqq\int_c|du^*|.$$

If γ is weak, the perimeter of γ with respect to any point of R is zero [3], so the extremal length $\lambda_{\{c\}}$ of the family $\{c\}$ of dividing cycles which separate γ from a parametric disk of a point of R is zero. Therefore $\inf_{\{c\}}\left|\int du^*\right|=0$, and we can find a sequence $\{\gamma_n\}\subset\{c\}$ for which $\lim_{n\to\infty}\int_{\gamma_n}du^*=0$.

If γ is not weak, there exists Jurchescu [3]'s extremal function $u\in HD$ in $R-R_0$ and for which $\int_c du^*=1$, so $\inf_{\{c\}}\int du^*=1$. Therefore, there is no sequence such as in Theorem 3. q.e.d.

§ 2. Subregions.

We consider a subregion R of a Riemann surface together with its relative boundary ∂R and put $X=R\cup\partial R$. An ideal boundary component γ of this subregion is defined by a family $\{S_n\}$ of non-compact region S_n of X which satisfies the following condition: $S_n'\supset S_{n+1}'$, $\cap S_{n+1}'=\phi$. (S_n' is the closure of S_n

in X.) Compact exhaustion $\{X_n\}$ of X consists of compact regions X_n on X. Here we may suppose that S_n is a connected component of $X-X_n$.

Now, let γ_n be a portion of $\partial X_n \cap R$ which separates γ from X_0, and put $\partial X_n \cap R = \gamma_n \cup (\bigcup_i \beta_n^i)$.

DEFINITION 1. A harmonic function u in $X_n - X_0$ is said to be *admissible* when it satisfies the following conditions:

$$u=0 \quad \text{on} \quad \partial X_0, \quad \int_{\gamma_n} du^* = 1, \quad \int_{\beta_n^i} du^* = 0$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial X \cap (X_n - X_0).$$

LEMMA (Jurchescu [4]). *There exists the harmonic function among admissible functions of $X_n - X_0$ such that*

1. $u_n = const. = \begin{cases} d_n^\gamma & on \quad \gamma_n \\ d_n^i & on \quad \beta_n^i \end{cases}$ [3)]
2. $d_n = D(u_n) = \min D(u)$, *where* min *is taken over the class of admissible functions of $X_n - X_0$.*

DEFINITION 2. γ is said to be parabolic when $\lim_{n\to\infty} D(u_n) = \infty$.

In the case of subregions we modify Nevanlinna's function ω'_{γ_n} in $X_n - X_0$ by the condition $\frac{\partial \omega'_{\gamma_n}}{\partial n} = 0$ on $\partial R \cap (X_n - X_0)$, and get the following theorem similar to Theorem 1.

THEOREM 4. *γ is parabolic if and only if $\omega'_\gamma \equiv 0$.*

A harmonic boundary component $\gamma_\varDelta$ is defined by $\gamma_\varDelta = \bigcap_n \varDelta_n$, where $\varDelta_n = \bar{S}_n \cap \varDelta$ and $\varDelta$ is the harmonic boundary of $X - X_0$ and $\bar{S}_n$ is the closure of S_n in $(X-X_0)^*$.

Denoting the harmonic measure of $\gamma_\varDelta$ by $\Omega'_{\gamma_\varDelta}$, we get the following

THEOREM 5. *If γ is parabolic, then $\Omega'_{\gamma_\varDelta} \equiv 0$.*

PROOF. $\gamma_\varDelta = \bigcap_n \varDelta_n$ is compact because $\varDelta_n = \bar{S}_n \cap \varDelta$ is compact. Let $\Omega'_{\gamma_\varDelta}$ be the harmonic measure of $\gamma_\varDelta$, then

$$\Omega'_{\gamma_\varDelta} = \begin{cases} 1 & \text{on} \quad \gamma_\varDelta \\ 0 & \text{on} \quad \varDelta - \gamma_\varDelta . \end{cases}$$

And modified Nevanlinna's function ω'_{γ_n} satisfies the following boundary condition:

$$\omega'_{\gamma_n} = \begin{cases} 1 & \text{on} \quad \gamma_\varDelta \\ \geqq 0 & \text{on} \quad \varDelta - \gamma_\varDelta . \end{cases}$$

3) $d_n^\gamma = D(u_n)$ is monotone increasing with n [4]

Therefore, $\omega'_{\gamma_n}-\Omega'_{\gamma_\Delta}\geqq 0$ on the harmonic boundary of $X-S_n-X_0$, and by Nakai's theorem [6] $\omega'_{\gamma_n}-\Omega'_{\gamma_\Delta}\geqq 0$ on $X-S_n-X_0$. This inequality holds for all γ_n, so we can conclude $\omega'_\gamma \geqq \Omega'_{\gamma_\Delta}$. Consequently, if γ is parabolic $\Omega'_{\gamma_\Delta}\equiv 0$.

§ 3. An application to the classification of Riemann surfaces and of subregions.

As a direct consequence of Theorem 3, we can enunciate, *if $R\in O_{HD}-O_G$, R has only one non-weak ideal boundary component*; if $R\in O_{HD}-O_G$, R has the unique harmonic boundary point (which corresponds to an HD-indivisible set in Constantinescu-Cornea [1]'s sense) which is contained in one ideal boundary component. Then the harmonic measure of the ideal boundary of R measured in R-(compact region) is an HD-function which does not satisfy the condition of Theorem 3.

In the case of a subregion, *every subregion of class $NO_{HD}-M_0$ has only one non-parabolic ideal boundary component,* where NO_{HD} denotes a class of subregions on which there exist no non-constant HD-functions whose normal derivatives on the relative boundary are zero, and M_0 a class of subregions whose doubles belong to O_G [4].

This proposition is due to the following facts: If $G\in NO_{HD}-M_0$ the ideal boundary of the double $\hat{G}$ has two symmetric harmonic boundary points (symmetric HD-indivisible sets) or $\hat{G}$ belongs to $O_{HD}-O_G$ [5], and, since each harmonic boundary point is contained in one ideal boundary component, G has only one non-parabolic component. *The same proposition holds for $NO_{HB}-M_0$* (cf. [2]).

In the light of this proposition, we consider a metrical criterion when R has infinite genus. Let $d\rho$ be a conformal metric on R, and $\Gamma_\rho=\{P;\ P\in R,\ d(P_0, P)=\rho\}$ be the geodesic circle about $P_0\in R$ with radius $d(P_0, P)=\rho$. We divide Γ_ρ into dividing cycles Γ_i, that is, $\Gamma_\rho=\bigcup_i \Gamma_i$, and let $L_i(\rho)$ be the length of Γ_i measured by $d\rho$. Putting $L(\rho)=\max_i L_i(\rho)$, according to Royden, if

$$\int^{\rho_\infty}\frac{d\rho}{L(\rho)}=\infty\,,\quad (\rho_\infty=d(P_0,\ \text{ideal boundary of } R))$$

then R belongs to O_{FD} [11]. While, for the family $\{\Gamma_\gamma\}$ of dividing cycles $\Gamma_\gamma\subset\Gamma_\rho$ which separate γ and P_0, Savage [13] showed that, if

$$\int^{\rho_\infty}\frac{d\rho}{L_\gamma(\rho)}=\infty\,,\quad (L_\gamma(\rho)=\text{length of } \Gamma_\gamma),$$

then γ is weak. But if

$$\int^{\rho_\infty}\frac{d\rho}{L(\rho)}=\infty\,,\quad \text{then}\quad \int^{\rho_\infty}\frac{d\rho}{L_\gamma(\rho)}=\infty$$

for all ideal boundary components γ, that is, all components are weak. However, if $R \in O_{HD} - O_G$, R has a non-weak boundary component γ, and

$$\int^{\rho\infty} \frac{d\rho}{L_\gamma(\rho)} < \infty .$$

Therefore,

$$\int^{\rho\infty} \frac{d\rho}{L(\rho)} < \infty \quad \text{for} \quad R \in O_{HD} - O_G .$$

Consequently we get the following

THEOREM 6. *If* $R \in O_{HD}$ *and* $\int^{\rho\infty} \frac{d\rho}{L(\rho)} = \infty$ *for a conformal metric* $d\rho$, *then* $R \in O_G$. *In other words, if* $R \in O_{HD} - O_G$, $\int^{\rho\infty} \frac{d\rho}{L(\rho)} < \infty$ *for all conformal metrics.*

REMARK. The converse of Savage's theorem is true, that is, *if* γ *is weak, there exists a conformal metric for which* $\int \frac{d\rho}{L_\gamma(\rho)} = \infty$. If γ is weak, there exists a canonical exhaustion $\{R_n\}$ for which $\sum_n \mu_n^\gamma = \infty$, where μ_n^γ is modulus of a component F_n^γ of $R_n - R_{n-1}$ which is contained in S_n [**9**]. We construct Noshiro's graph [**8**] of R with respect to $\{R_n\}$. By the piecewise conformal mapping of R into the graph, euclidian metric of the graph induces a conformal metric $d\rho = \rho(z)|dz|$ on R. On the other hand, the extremal harmonic function which gives modulus of F_n^γ induces a conformal metric dl on F_n^γ. The metrics $d\rho$ and dl are homothetic on F_n^γ, that is, geodesic lines of each metric coincide. Therefore,

$$\mu_n^\gamma \leqq \int_{(F_n^\gamma)} \frac{d\rho}{L_\gamma(\rho)} \quad \text{and} \quad \sum \mu_n^\gamma \leqq \int_{(R)} \frac{d\rho}{L_\gamma(\rho)} .$$

Therefore, if γ is weak $\int \frac{d\rho}{L_\gamma(\rho)} = \infty$ for $d\rho = \rho(z)|dz|$.

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