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On the field of definition of Borel subgroups of semi-simple algebraic groups

Dedicated to Professor Y. Akizuki for his 60th birthday

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Let k be a perfect field and let G be a connected semi-simple algebraic group defined over k. It is known that G has a maximal torus T defined over k (Rosenlicht [2]). Fixing once for all such a torus T, denote by $\boldsymbol{B}$ the set of all Borel subgroups of G containing T. Our purpose is to prove the following

THEOREM. *Every group in $\boldsymbol{B}$ is defined over k if and only if T is trivial over k. When that is so, all groups in $\boldsymbol{B}$ are conjugate by k-rational points of the normalizer of T.*

For some purpose the following trivial restatement is useful.

COROLLARY. *Let K/k be an extension such that K is perfect. Then, every group in $\boldsymbol{B}$ is defined over K if and only if T is split by K. When that is so, all groups in $\boldsymbol{B}$ are conjugate by K-rational points of the normalizer of T.*

PROOF OF THEOREM. We begin with arranging the basic notions in Séminaire Chevally [1] from the Galois theoretical view point.

Denote by N the normalizer of T and by W the Weyl group N/T of T. Let $\bar{k}$ be the algebraic closure of k and $\mathfrak{g}=\mathfrak{g}(\bar{k}/k)$ be the Galois group of $\bar{k}/k$. Since every coset of W contains a $\bar{k}$-rational point, one can define the action of $\mathfrak{g}$ on W by

$$w^{\sigma}=s^{\sigma} \bmod T, \quad \text{where} \quad w=s \bmod T \quad \text{and} \quad s\in N_{\bar{k}}.^{*}$$

The group $\mathfrak{g}$ acts on the character module $\hat{T}$ since every character is $\bar{k}$-rational. Furthermore, W acts on $\hat{T}$ by

$$(w\chi)(t)=\chi(s^{-1}ts), \quad \text{where} \quad w=s \bmod T, \quad s\in N.$$

One verifies easily that

$$(w\chi)^{\sigma}=w^{\sigma}\chi^{\sigma} \qquad \text{for} \quad \sigma\in\mathfrak{g},\ w\in W,\ \chi\in\hat{T}.$$

In other words, $\hat{T}$ has a $(\mathfrak{g}, W)$-module structure. By linearity, this structure is trivially extended to the vector space $\hat{T}^{\boldsymbol{Q}}=\boldsymbol{Q}\otimes\hat{T}$.

* For an algebraic set A we denote by A_K the subset of K-rational points.

Let K/k be a finite Galois splitting field for T. The action of $\mathfrak{g}$ on $\hat{T}$ is essentially that of the finite group $\mathfrak{g}(K/k)$, the Galois group of K/k. Denoting by (ξ, η) a usual noncanonical inner product on $\hat{T}^{\boldsymbol{Q}}$, put

$$\langle \xi, \eta \rangle = \sum_{\substack{w \in W \\ \sigma \in \mathfrak{g}(K/k)}} ((w\xi)^{\sigma}, (w\eta)^{\sigma}).$$

It is clear that $\langle \xi, \eta \rangle$ is a positive definite inner product on $\hat{T}^{\boldsymbol{Q}}$ which is $(\mathfrak{g}, W)$-invariant in the sense that

$$\langle w\xi, w\eta \rangle = \langle \xi^{\sigma}, \eta^{\sigma} \rangle = \langle \xi, \eta \rangle \qquad \text{for} \quad w \in W,\ \sigma \in \mathfrak{g}.$$

An injective homomorphism x, defined over $\bar{k}$, of the additive group G_a of the universal domain into G is called a one parameter group of G. By a root of G with respect to T we mean a character $\alpha \in \hat{T}$ for which holds the relation

$$t x_{\alpha}(\lambda) t^{-1} = x_{\alpha}(\alpha(t)\lambda), \quad t \in T,\ \lambda \in G_a$$

for a suitable one parameter group x_{α}. The totality of roots with respect to T will be denoted by Δ. It is easy to verify that α^{σ} belongs to the one parameter group $x_{\alpha^{\sigma}} = x_{\alpha}^{\sigma}$ and $w\alpha$ belongs to the one parameter group $x_{w\alpha}(\lambda) = s x_{\alpha}(\lambda) s^{-1}$, with $w = s \bmod T$. Thus, $\mathfrak{g}$ and W induce permutations on Δ. As a group of linear transformations on the vector space $\hat{T}^{\boldsymbol{Q}}$, W is generated by the symmetries w_{α} with respect to $\alpha \in \Delta$. By using the inner product $\langle \xi, \eta \rangle$, w_{α} is expressed as

$$w_{\alpha}\xi = \xi - \frac{2\langle \alpha, \xi \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad \xi \in \hat{T}^{\boldsymbol{Q}}.$$

In view of the $(\mathfrak{g}, W)$-invariance of the $\langle \xi, \eta \rangle$ on the $(\mathfrak{g}, W)$-space $\hat{T}^{\boldsymbol{Q}}$, one verifies easily that

(1) $$w_{\alpha}^{\sigma} = w_{\alpha^{\sigma}} \qquad \text{for} \quad \sigma \in \mathfrak{g},\ \alpha \in \Delta.$$

Let H_{α} be the hyperplane composed of $\xi \in \hat{T}^{\boldsymbol{Q}}$ such that $\langle \xi, \alpha \rangle = 0$. Any maximal convex set of the complement of $\bigcup_{\alpha \in \Delta} H_{\alpha}$ in $\hat{T}^{\boldsymbol{Q}}$ is called a chamber. Each chamber C is characterized by a $\{\pm 1\}$-valued function $\varepsilon(\alpha)$ with $\varepsilon(-\alpha) = -\varepsilon(\alpha)$ in such a way that

$$C = \{\xi \in \hat{T}^{\boldsymbol{Q}};\ \varepsilon(\alpha)\langle \xi, \alpha \rangle > 0 \quad \text{for all } \alpha \in \Delta\}.$$

Since C^{σ} is characterized by the function $\varepsilon_{\sigma}(\alpha) = \varepsilon(\alpha^{\sigma^{-1}})$, the Galois group $\mathfrak{g}$ permutes chambers. On the other hand, it is well known that the Weyl group W permutes chambers simply and transitively.

Now, let B be a Borel subgroup of G containing T: $B \in \boldsymbol{B}$. There is at least one B which is defined over $\bar{k}$. Since any other $B_1 \in \boldsymbol{B}$ is written as $B_1 = sBs^{-1}$ with $s \in N_{\bar{k}}$, one sees that every group in $\boldsymbol{B}$ is defined over $\bar{k}$. It is fundamental that W permutes groups in $\boldsymbol{B}$ simply and transitively by

$$wB = sBs^{-1}, \quad w = s \mod T$$

(Chevalley [1, Exposé n°. 9, § 3]). The Galois group $\mathfrak{g}$ also permutes these groups in an obvious way. For a group B in $\boldsymbol{B}$, put

$$\Delta_B = \{\alpha \in \Delta ;\ \mathrm{Im}\ x_\alpha \subset B\} .$$

One can easily verify that $w\Delta_B = \Delta_{wB}$ and $\Delta_B^\sigma = \Delta_{B^\sigma}$. Since Δ_B satisfies the condition for "positive roots", the set

$$C_B = \{\xi \in \hat{T}^{\boldsymbol{Q}} ;\ \langle \xi, \alpha \rangle > 0 \quad \text{for all } \alpha \in \Delta_B\}$$

becomes a chamber (Chevalley [1, Exposé n°. 14, § 4]). By virtue of the $(\mathfrak{g}, W)$-invariance of $\langle \xi, \eta \rangle$, one sees that

(2) $$wC_B = C_{wB}, \quad w \in W,$$

(3) $$C_B^\sigma = C_{B^\sigma}, \quad \sigma \in \mathfrak{g}.$$

Since the set $\{wC_B, w \in W\}$ forms a partition of the complement of $\bigcup_{\alpha \in \Delta} H_\alpha$ in $\hat{T}^{\boldsymbol{Q}}$, (2) implies that the set $\{C_B, B \in \boldsymbol{B}\}$ forms the same partition. Hence, from (3), one gets

(4) $$C_B^\sigma = C_B \quad \Leftrightarrow \quad B^\sigma = B .$$

We are now ready to prove our theorem. Suppose first that T is trivial over k. In terms of characters, this means that $\xi^\sigma = \xi$ for all $\sigma \in \mathfrak{g}$, $\xi \in \hat{T}^{\boldsymbol{Q}}$. Since the chambers C_B are subsets of $\hat{T}^{\boldsymbol{Q}}$, $C_B^\sigma = C_B$ for all $\sigma \in \mathfrak{g}$, $B \in \boldsymbol{B}$. Hence, by (4), $B^\sigma = B$ for all $\sigma \in \mathfrak{g}$, $B \in \boldsymbol{B}$, i.e., every $B \in \boldsymbol{B}$ is defined over k. Conversely, suppose that every $B \in \boldsymbol{B}$ is defined over k. Again by (4) $C_B^\sigma = C_B$ for all $\sigma \in \mathfrak{g}$, $B \in \boldsymbol{B}$, and hence every chamber is invariant under $\mathfrak{g}$. From (2), (3), $wC_B = (wC_B)^\sigma = (C_{wB})^\sigma = C_{(wB)^\sigma} = C_{w^\sigma B^\sigma} = w^\sigma C_{B^\sigma} = w^\sigma C_B^\sigma = w^\sigma C_B$, and so $w^\sigma = w$ for all $\sigma \in \mathfrak{g}$, $w \in W$. Hence, by (1), $w_\alpha = w_{\alpha^\sigma}$ for all $\alpha \in \Delta$, $\sigma \in \mathfrak{g}$. Thus α and α^σ are colinear and, since both are roots, one must have $\alpha^\sigma = \pm\alpha$. Suppose that $\alpha^\sigma = -\alpha$ and take $B \in \boldsymbol{B}$ such that $\alpha \in \Delta_B$. Then $-\alpha = \alpha^\sigma \in \Delta_B^\sigma = \Delta_{B^\sigma} = \Delta_B$, a contradiction. Hence every $\alpha \in \Delta$ is invariant by $\mathfrak{g}$. Since roots generate $\hat{T}^{\boldsymbol{Q}}$, the $\mathfrak{g}$-module $\hat{T}$ is trivial, i.e., T is trivial over k. Finally, suppose that T is trivial over k. Take any $B, B_1 \in \boldsymbol{B}$. There is an $s \in N_{\bar{k}}$ such that $B_1 = sBs^{-1}$. Since B, B_1 are defined over k by what we have proved, one has $B_1 = s^\sigma B s^{-\sigma}$ for $\sigma \in \mathfrak{g}$, and hence $s^{-1}s^\sigma \in$ (normalizer of $B) \cap N_{\bar{k}} = B \cap N_{\bar{k}} = T_{\bar{k}}$ (Chevalley [1, Exposé n°. 9, § 3]). As $(s^{-1}s^\sigma)$ is a cocycle of $\mathfrak{g}$ in T_k and T is trivial over k, one can find, by Hilbert's Theorem 90, a point $t \in T_{\bar{k}}$ such that $s^{-1}s^\sigma = t^{-1}t^\sigma$. Hence $B_1 = uBu^{-1}$ with $u = st^{-1} \in N_{\bar{k}}$, Q. E. D.

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References

[1] C. Chevalley, Classification des groupes de Lie algébriques, Paris, 1958.
[2] M. Rosenlicht, Some rationality questions on algebraic groups, Ann. Mat. Pura Appl., **43** (1957), 25-50.