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# A global representation of a fundamental set of solutions and a Stokes phenomenon for a system of linear ordinary differential equations

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## 1. Introduction.

Let us consider a system of linear ordinary differential equations of the first order

$$t\frac{d\vec{x}}{dt}=(A+tB)\vec{x} \tag{1.1}$$

where $\vec{x}$ is an $n$-vector, $t$ is a complex variable, $A$ and $B$ are $n$ by $n$ matrices with constant elements, such that $B$ is a diagonal matrix with mutually different diagonal elements, and that $A$ has no two eigenvalues which have integral difference.

The system has only two singular points in the entire complex plane—the one at the origin which is a regular singular point, and the other at the infinity, which is an irregular singular point. For any given eigenvalue $\rho$ of the matrix $A$, there is associated a solution of the form

$$\vec{x}_\rho(t)=t^\rho\sum_{m=0}^{\infty}\vec{g}_{\rho,m}t^m \tag{1.2}$$

where the power series converges in the whole open plane. Corresponding to $n$ different eigenvalues of $A$, there are $n$ such solutions which constitute a fundamental set of solutions in the open plane. On the other hand, there are $n$ formal solutions, corresponding to $n$ different eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $B$ of the form

$$\vec{\xi}_p(t)=e^{\lambda_p t}t^{a_{pp}}\sum_{s=0}^{\infty}\vec{h}_p(s)t^{-s} \qquad (p=1, 2, \cdots, n) \tag{1.3}$$

where $a_{pp}$ is the $p$-th diagonal element of the matrix $A$. It is known that these solutions represent actual solutions asymptotically in an arbitrarily given sectorial neighborhood of the point at infinity, provided that the central angle of the sector is sufficiently small. These $n$ actual solutions which are asymptotically represented by these $n$ formal solutions, also, constitute a fundamental set of solutions in this sectorial neighborhood.

A natural question arises. What is the relation between those two sets of fundamental solutions? Naturally, there must be a linear transformation $C$ which transforms the one set to the other set. However it was discovered by Stokes in 1857 that this transformation $C$ changes, in general, abruptly and discontinuously from sector to sector. Namely a formal solution expresses, asymptotically, different solutions in different sectorial neighborhoods. This fact is known by the name of "Stokes Phenomenon", and has been an object of extensive studies over a hundred years.

The object of this paper is to investigate this phenomenon from the inverse direction, without making any use of the classical theory of irregular singular points. We start from a solution of the form (1.2) and will obtain its asymptotic expansion as a linear combination of formal solutions (1.3) and an additional term which is estimated to be $O(t^h)$ for some positive real $h$. Since the existence of such linear combinations are evident from the classical theory, we laid our emphasis upon the method how to calculate the coefficients of the linear combination. We will carry out the calculation exactly for a sector $S$, which is maximally admissible in the sense that whenever we enlarge this sector the coefficients of the combination will, in general, necessarily change. Then we will show how to calculate these coefficients for another sector $S'$ from those for $S$.

An outline of our study can be described as follows. First we shall state in Theorem 1 the asymptotic behavior of an entire function defined by a Taylor series $\sum_{m=0}^{\infty} g_m t^m$, the coefficients of which can be expanded, for large values of $m$, in inverse factorial series of $m$. This theorem enables us to convert the differential system (1.1) into a system of linear difference equations satisfied by the coefficients $\vec{g}_{\rho,m}$ of the solution (1.2). For if we can expand the solution of the difference system, subject to the initial condition for $\vec{g}_{\rho,0}$, in inverse factorial series of $m$, then we will know the asymptotic behavior of the solution (1.2). Section 3 is concerned with function theoretic lemmas. Section 4 is devoted to the study of the difference system for $\vec{g}_{\rho,m}$. In the final section, our main results will be stated.

This conversion from a differential system to a difference system was originally proposed by W. B. Ford, and was used, although in somewhat incomplete manner, to the study of a second order linear differential equation. Turrittin in 1950 and Langer in 1955 treated the same problem for following equations respectively

$$\frac{d^n z}{ds^n} = s^p z \qquad (p: \text{ positive integer}),$$

$$\frac{d^3 y}{ds^3} = s\frac{dy}{ds} + \lambda y \qquad (\lambda; \text{ complex number}).$$

These equations can easily be transformed into the systems

$$t\frac{d\vec{z}}{dt}=\left\{\frac{p}{n+p}\begin{pmatrix}0 & & & \\ & 1 & & 0 \\ & & 2 & \\ & 0 & & \ddots \\ & & & & n-1\end{pmatrix}+t\begin{pmatrix}0 & 1 & 0\cdots 0\\ 0 & 0 & 1\cdots 0\\ \cdots & \cdots & \cdots\\ 0 & 0 & \cdots\cdots 1\\ 1 & 0 & \cdots\cdots 0\end{pmatrix}\right\}\vec{z},$$

$$t\frac{d\vec{y}}{dt}=\left\{\begin{pmatrix}0 & 0 & 0\\ 0 & \frac{1}{3} & 0\\ \frac{3}{2}\lambda & 0 & \frac{2}{3}\end{pmatrix}+t\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}\right\}\vec{y}$$

respectively. But in these cases, they could dispense with the theory of difference equations to derive inverse factorial series expansion of the coefficients, instead, simple iterations were sufficient for their purpose.

Although the system (1.1) treated here is of a very special form, it seems to be quite promising that our theory might work as a clue for the study of more general system. This possibility is suggested by the theorem of G. D. Birkhoff, which says, "every system of $n$ linear differential equations with an irregular singular point of rank unity at infinity is equivalent at infinity to a canonical system of the form (1.1)". Though in a certain case, the theorem is known to fail, it is very plausible that this theorem would hold under some additional restrictions. If so, our theory could cover a fairly wide class of systems with irregular singular point of rank unity.

Moreover, according to Birkhoff's theory, a system of equations with an irregular singular point of rank greater than unity at infinity is similarly reduced to a system of the form

$$t\frac{d\vec{x}}{dt}=(A_0+tA_1+\cdots+t^{\sigma}A_{\sigma})\vec{x} \qquad (\sigma\geqq 1).$$

A key to extend our theory to a system of this form lies in the reconstruction of the theory of the section 2, based on a single inhomogeneous equation

$$t\frac{dx}{dt}=(\mu_0+t\mu_1+\cdots+t^{\sigma}\mu_{\sigma})x-\frac{1}{\alpha}$$

while the present theory for (1.1) needed only the study of the equation

$$t\frac{dx}{dt}=(\mu+\lambda t)x-\frac{1}{\Gamma(-\mu)}.$$

We will use following notations in the sequel:

For a given non-zero complex number $\lambda$, a sectorial neighborhood of the point at infinity is defined to be

(1.4) $$S(\lambda)=\{t\,;\,|t|>K,\,|\arg\lambda t|\leqq\frac{3}{2}\pi-\varepsilon\}\,.$$

Throughout our discussions, the letter "$K$" stands for a positive constant which can be arbitrarily large, and "$\varepsilon$" for a positive constant which can be arbitrarily small.

A function $R_h(t, D)$ of a complex variable $t$, defined in $D$ and for a real number $h$ is a function for which

(1.5) $$R_h(t, D)=O(t^h)$$

holds as $t$ tends to the infinity in $D$.

An arrow on a letter means that the function or a constant denoted by that letter is an $n$ dimensional vector. A vector valued function is holomorphic in a domain, when each component of the vector is a holomorphic function in the domain. A vector is a constant vector when each component is a constant, and so on.

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## 2. An extension of the theory of Wright.

In this section we shall first establish the relation (2.1) for the sector $S(1)$. The relation was originary proved by Ford for $|\arg t|<\pi/2$, and independently by Wright for $|\arg t|\leqq\pi$. The extension to $|\arg t|\leqq\frac{3}{2}\pi-\varepsilon$ is essential, because our sector $S(1)$ is the sector of the maximal admissibility for the asymptotic expansion (2.1) (cf. Friedrichs [**1**]).

LEMMA 1. *For an arbitrarily given complex number $\beta$, we have*

(2.1) $$\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(m+\beta)}=t^{1-\beta}e^t+R_{-1}(t, S(1))\,.$$

PROOF. If $\beta=1$, from the definition of the exponential function, we have,

$$\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(m+1)}=e^t$$

If $\beta-1=m_0$ is a positive integer, we have

$$\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(m+\beta)}=\Big(\sum_{m=0}^{\infty}\frac{t^{m+m_0}}{\Gamma(m_0+m+1)}\Big)t^{-m_0}=t^{-m_0}\Big(\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(m+1)}-\sum_{m=0}^{m_0-1}\frac{t^m}{\Gamma(m+1)}\Big)$$

$$=t^{-m_0}e^t-\sum_{m=0}^{m_0-1}\frac{t^{m-m_0}}{\Gamma(m+1)}=t^{1-\beta}e^t+R_{-1}(t, S(1))\,.$$

If $\beta-1=-m_0$ is a negative integer, we have

$$\frac{1}{\Gamma(m+\beta)}=\frac{1}{\Gamma(1+m-m_0)}\equiv 0 \qquad (m=0,1,\cdots,m_0-1).$$

Hence, we have

$$\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(m+\beta)}=t^{m_0}\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(m+1)}=t^{1-\beta}e^t.$$

When $\beta-1$ is not an integer, we at once see the function $x(t)$ defined by a Taylor series

$$x(t)=\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(m+\beta)}$$

satisfies the following linear inhomogeneous differential equation:

$$\frac{dx}{dt}=\left(1+\frac{1-\beta}{t}\right)x+\frac{1}{\Gamma(\beta-1)\cdot t}. \tag{2.2}$$

By quadrature, the general solution with an arbitrary constant $c$ is shown to have the form

$$x(t)=e^t t^{1-\beta}\left(c-\int_t^{\infty}e^{-s}s^{\beta-2}\frac{1}{\Gamma(\beta-1)}ds\right). \tag{2.3}$$

Since, the identity $\int_0^{\infty}e^{-s}\cdot s^{\beta-2}ds=\Gamma(\beta-1)$ holds for non-integral value of $\beta-1$, and by the multiple valuedness of $t^{1-\beta}$, the only one possible choice of $c$, for which $x(t)$ has the desired series expansion, is $c=1$. This can be easily seen from

$$\begin{aligned}
x(t)&=t^{1-\beta}e^t\left[(c-1)+\int_0^t e^{-s}\cdot s^{\beta-2}\frac{1}{\Gamma(\beta-1)}ds\right]\\
&=(c-1)t^{1-\beta}e^t+e^t t^{1-\beta}\int_0^t(1-s+\cdots)\frac{s^{\beta-2}}{\Gamma(\beta-1)}ds\\
&=(c-1)t^{1-\beta}e^t+\frac{e^t t^{1-\beta}}{\Gamma(\beta-1)}\int_0^t(s^{\beta-2}-s^{\beta-1}+\cdots)ds\\
&=(c-1)t^{1-\beta}e^t+e^t\left[\frac{1}{\Gamma(\beta)}-\frac{t}{\beta\cdot\Gamma(\beta-1)}+\cdots\right].
\end{aligned}$$

Repeated applications of the integrations by parts to the second term on the right hand side of (2.3) will yield

$$x(t)=t^{1-\beta}e^t-\sum_{m=1}^{N}\frac{t^{-m}}{\Gamma(\beta-m)}+\frac{e^t t^{1-\beta+N}}{\Gamma(\beta-N-1)}\int_t^{\infty}e^{-s}s^{\beta-N-2}ds.$$

The last term on the right hand side is known to be $R_{\mathrm{Re}\beta-N}(t,S(1))$, for $N>\mathrm{Re}\beta$. This completes the proof.

The following lemma is the Lemma 5 of Wright's paper, so the proof is omitted.

LEMMA 2 (E. M. Wright). *If $\varphi(w)$ is holomorphic and bounded in the right half plane,*

$$\mathrm{Re}\, w \geqq h > 0 \qquad \left(h > \frac{3}{2} - \mathrm{Re}\, \beta\right), \tag{2.4}$$

*then we have*

$$\sum_{m=\sigma_0}^{\infty} \frac{\varphi(m)}{\Gamma(m+\beta)} t^m = F(t) + R_h(t, S(1)) \qquad (\sigma_0 = [h]+1), \tag{2.5}$$

$$F(t) = e^t R_{3/2-\mathrm{Re}\beta-\kappa}(t, S(1)) \tag{2.6}$$

*where* $\kappa$ *is equal to* $-\frac{1}{2}$ *or* 0 *according as* $|\mathrm{Re}\, t| \geqq 1$ *or not.*

THEOREM 1. *If* $g(w)$ *is a holomorphic function of* $w$ *which is bounded in the right half plane* (2.4), *and has an asymptotic expansion of the form*

$$g(w) \cong \sum_{s=0}^{\infty} \frac{A_s \lambda^w}{\Gamma(s+w+\beta)} \tag{2.7}$$

*where* $A_s$ *are constants independent of* $w$. *Then a function of a complex variable* $t$ *defined by a Taylor series*

$$f(t) = \sum_{m=\sigma_0}^{\infty} g(m) t^m \tag{2.8}$$

*has the form*

$$f(t) = (\lambda t)^{1-\beta} e^{\lambda t} A(t) + R_h(t, S(\lambda)) \tag{2.9}$$

*where* $A(t)$ *has the asymptotic expansion in* $S(\lambda)$ *of the form*

$$A(t) \cong \sum_{s=0}^{\infty} A_s (\lambda t)^{-s}. \tag{2.10}$$

PROOF. Since the series (2.8) converges absolutely for every finite value of $t$, we can change the order of summation, to have

$$\sum_{m=\sigma_0}^{\infty} g(m) t^m = \sum_{s=0}^{N} \left[ \sum_{m=\sigma_0}^{\infty} \frac{A_s (\lambda t)^m}{\Gamma(s+m+\beta)} \right] + \sum_{m=\sigma_0}^{\infty} \frac{\varphi_N(m)}{\Gamma(\beta+N+m+1)} (\lambda t)^m$$

where $\varphi_N(w)$ is a bounded function of $w$ in the right half plane (2.4). Applying Lemma 1 to the first $N$ terms, and Lemma 2 to the last term, we have

$$f(t) = \sum_{s=0}^{N} e^{\lambda t} (\lambda t)^{1-\beta-s} A_s + R_{\sigma_0-1}(t, S(\lambda)) + F(\lambda t) + R_h(t, S(\lambda))$$

where

$$F(\lambda t) = e^{\lambda t} R_{-N-\mathrm{Re}\beta}(t, S(\lambda)).$$

Thus we have

$$f(t) = \sum_{m=\sigma_0}^{\infty} g(m) t^m = e^{\lambda t} (\lambda t)^{1-\beta} \left[ \sum_{s=0}^{N} A_s (\lambda t)^{-s} + R_{-N-1}(t, S(\lambda)) \right] + R_h(t, S(\lambda)).$$

This proves the theorem.

## 3. An asymptotic formula.

The object of this section is to prove an asymptotic formula for an integral of Fourier-Mellin type. This formula is stated as a corollary to Lemma 4. Though we do not make any use of "an order of a function on its circle of convergence" originally proposed by J. Hadamard, Lemma 3 implies some essential property of the order.

Lemma 3. *Let $P(\xi)$ be a function of a complex variable $\xi$ given by a convergent power series expansion*

$$P(\xi)=\sum_{m=0}^{\infty} a_m(\xi-1)^m \tag{3.1}$$

*in its circle of convergence $|\xi-1|<1$. Suppose that the origin is the only possible singularity of $P(\xi)$ on $|\xi-1|=1$, and $P(\xi)$ satisfies the inequality*

$$|\xi\cdot P(\xi)|\leqq C \qquad (|\xi-1|\leqq 1) \tag{3.2}$$

*where $C$ is a certain positive constant. Then the series*

$$\sum_{m=0}^{\infty} a_m\xi^w(\xi-1)^m \tag{3.3}$$

*converges uniformly on the closed unit interval $0\leqq\xi\leqq 1$, for* $\operatorname{Re} w>1$.

Proof. For an arbitrarily small positive number $\varepsilon$, we define a function $Q(\xi)$ by the formula

$$Q(\xi)=\frac{(\xi-1)^{-\varepsilon-1}}{\Gamma(\varepsilon+1)}\int_1^{\xi}(\xi-\zeta)^{\varepsilon}P(\zeta)d\zeta\,. \tag{3.4}$$

Clearly $Q(\xi)$ is a holomorphic function of $\xi$ in $|\xi-1|\leqq 1$, except possibly for the origin, and has a power series expansion

$$Q(\xi)=\sum_{m=0}^{\infty} a_m\frac{\Gamma(m+1)}{\Gamma(m+\varepsilon+2)}(\xi-1)^m\,, \tag{3.5}$$

for we have, from the definition of the Beta function,

$$\int_1^{\xi}(\xi-\zeta)^{\varepsilon}(\zeta-1)^m d\zeta=(\xi-1)^{\varepsilon+m+1}\int_1^{\xi}\left(\frac{\zeta-1}{\xi-1}\right)^m\left(1-\frac{\zeta-1}{\xi-1}\right)^{\varepsilon}d\left(\frac{\zeta-1}{\xi-1}\right)$$

$$=(\xi-1)^{\varepsilon+m+1}\int_0^1 s^m(1-s)^{\varepsilon}ds=\frac{\Gamma(\varepsilon+1)\Gamma(m+1)}{\Gamma(\varepsilon+m+2)}(\xi-1)^{\varepsilon+m+1}\,.$$

We will show first that the following inequalities hold for real $\theta$:

$$|Q(1+e^{i\theta})|\leqq K, \tag{3.6}$$

$$|Q'(1+e^{i\theta})|\leqq K|\sin\theta|^{\frac{\varepsilon}{2}-1}+K. \tag{3.7}$$

Taking the path of integration on a straight line $\zeta=1+\sigma e^{i\theta}$ $(0\leqq\sigma\leqq 1)$, and noting the inequality

(3.8) $$|1+\sigma e^{i\theta}|\geqq 1-\sigma,$$

we see immediately the following sequence of inequalities:

$$|Q(1+e^{i\theta})|\leqq\frac{1}{\Gamma(\varepsilon+1)}\int_0^1(1-\sigma)^{\varepsilon}|P(1+\sigma e^{i\theta})|d\sigma$$

$$\leqq\frac{1}{\Gamma(\varepsilon+1)}\int_0^1(1-\sigma)^{\varepsilon-1}|(1+\sigma e^{i\theta})P(1+\sigma e^{i\theta})|d\sigma\leqq\frac{C}{\Gamma(1+\varepsilon)}\int_0^1(1-\sigma)^{\varepsilon-1}d\sigma.$$

The final term is less than a positive constant for positive $\varepsilon$.

To prove the second inequality (3.7), we remark

(3.9) $$|\sin\theta|\leqq|1+\sigma e^{i\theta}|$$

which can be seen easily if we consider the distance between the origin and the path of integration, $\zeta=1+\sigma e^{i\theta}$ $(0\leqq\sigma\leqq 1)$. Since $Q(\xi)$ is analytic on and inside the circle $|\xi-1|=1$, except for the origin, we have

$$Q'(\xi)=-(\varepsilon+1)\frac{(\xi-1)^{-\varepsilon-2}}{\Gamma(\varepsilon+1)}\int_1^{\xi}(\xi-\zeta)^{\varepsilon}P(\zeta)d\zeta+\frac{\varepsilon(\xi-1)^{-\varepsilon-1}}{\Gamma(\varepsilon+1)}\int_1^{\xi}(\xi-\zeta)^{\varepsilon-1}P(\zeta)d\zeta.$$

The first term on the right is finite on $|\xi-1|=1$, by (3.6). Accordingly we have only to prove the inequality

$$\left|\int_1^{\xi}(\xi-\zeta)^{\varepsilon-1}P(\zeta)d\zeta\right|\leqq K|\sin\theta|^{\frac{\varepsilon}{2}-1}.$$

Putting $\xi=1+e^{i\theta}$, $\zeta=1+\sigma e^{i\theta}$ $(0\leqq\sigma\leqq 1)$, as in the case of (3.6), we have

$$\left|\int_1^{\xi}(\xi-\zeta)^{\varepsilon-1}P(\zeta)d\zeta\right|\leqq\int_0^1(1-\sigma)^{\varepsilon-1}\cdot|P(1+\sigma e^{i\theta})|d\sigma$$

$$\leqq\int_0^1(1-\sigma)^{\frac{\varepsilon}{2}-1}\cdot(1-\sigma)^{\frac{\varepsilon}{2}}\frac{|(1+\sigma e^{i\theta})P(1+\sigma e^{i\theta})|}{|1+\sigma e^{i\theta}|}d\sigma$$

$$\leqq c\int_0^1(1-\sigma)^{\frac{\varepsilon}{2}-1}|1+\sigma e^{i\theta}|^{\frac{\varepsilon}{2}-1}d\sigma\leqq c|\sin\theta|^{\frac{\varepsilon}{2}-1}\int_0^1(1-\sigma)^{\frac{\varepsilon}{2}-1}d\sigma$$

by (3.8) and (3.9). Since we can choose $\varepsilon$ so small that $\frac{\varepsilon}{2}-1$ is negative, we have (3.7).

Using Cauchy's formula for (3.5), and integrating by parts, we have an identity

(3.10) $$ma_m\frac{\Gamma(\varepsilon+1)}{\Gamma(m+\varepsilon+2)}=\frac{1}{2\pi i}[-(\zeta-1)^{-m}Q(\zeta)]_{|\zeta-1|=1}+\frac{1}{2\pi i}\int_{|\xi-1|=1}(\zeta-1)^{-m}Q'(\zeta)d\zeta.$$

From this we can derive an estimate for the coefficients $a_m$ of $P(\xi)$. Since, by (3.6) and (3.7), the right hand side of this identity is bounded, we have

$$\lim_{m\to\infty}ma_m\frac{\Gamma(m+1)}{\Gamma(m+\delta+2)}=0,\quad\text{or}\quad\lim_{m\to\infty}\frac{\Gamma(m+1)\cdot\Gamma(\delta+1)}{\Gamma(m+\delta+1)}a_m=0$$

provided $\delta>\varepsilon$. This is equivalent to say that, for any positive $\varepsilon_0$, we can find a positive integer $m_0$, such that

$$|a_m| \leqq \frac{\Gamma(m+\delta+1)}{\Gamma(m+1)\Gamma(\delta+1)} \varepsilon_0 \qquad (m \geqq m_0) \tag{3.11}$$

holds.

Now back to the series (3.3) for $\xi$ in the interval [0.1], we have

$$\left|\sum_{m=m_0}^{\infty} a_m \xi^w (\xi-1)^m\right| \leqq \xi^{\mathrm{Re}w} \sum_{m=m_0}^{\infty} |a_m| \cdot (1-\xi)^m \leqq \xi^{\mathrm{Re}w} \cdot \varepsilon_0 \sum_{m=m_0}^{\infty} \frac{\Gamma(m+\delta+1)}{\Gamma(m+1)\Gamma(\delta+1)} (1-\xi)^m$$

$$\leqq \xi^{\mathrm{Re}w} \varepsilon_0 \cdot \sum_{m=0}^{\infty} \frac{\Gamma(m+\delta+1)}{\Gamma(m+1)\Gamma(\delta+1)} (1-\xi)^m = \xi^{\mathrm{Re}w} \varepsilon_0 (1-(1-\xi))^{-\delta-1} = \xi^{\mathrm{Re}w-\delta-1} \varepsilon_0 .$$

This proves our lemma, for $\varepsilon$, and accordingly $\delta$ can be chosen arbitrarily small.

In the following lemma we shall mean by a contour $C$ a contour which is composed of the three following parts; (i) the real axis from 0 to $1-\varepsilon$, (ii) a circle of radius $\varepsilon$ described in the negative direction about 1, (iii) the real axis from $1-\varepsilon$ to 0.

LEMMA 4. *Let $\psi(\zeta)$ be defined by a convergent power series at $\zeta=1$, and is holomorphic in a domain $D_\zeta$ defined by*

$$D_\zeta = \left\{\zeta \,;\, 0 \leqq |\zeta| < \infty, |\arg \zeta| < \frac{\pi}{\omega}\right\} \tag{3.12}$$

*where $\omega$ is a positive integer. If $\psi(1) \neq 0$ and $\gamma$ is not an integer, the integral*

$$I(w) = \frac{1}{2\pi i} \int_c \zeta^{w-1} (\zeta-1)^{\gamma-1} \psi(\zeta) d\zeta \tag{3.13}$$

*can be expanded in a uniformly convergent factorial series of $\frac{w}{\omega}$ in the right half plane*

$$\mathrm{Re}\, w \geqq \omega + \varepsilon . \tag{3.14}$$

PROOF. Make a change of variable

$$\zeta = \xi^{\frac{1}{\omega}} \tag{3.15}$$

and let $D_\xi$ be a domain corresponding to $D_\zeta$, then the circle $|\xi-1|=1$ is contained in $D_\xi$, and the only one possible singularity on this circle is the origin $\xi=0$. The integral (3.13) is transformed into the form

$$I(w) = \frac{1}{2\pi i \cdot \omega} \int_{C_\xi} \xi^{\frac{w}{\omega}} (\xi-1)^{\gamma-1} P(\xi) d\xi \tag{3.16}$$

where $C_\xi$ is the image of $C$ by (3.15). $P(\xi)$ has a power series expansion in $(\xi-1)$

$$P(\xi) = \xi^{-1} \left(\frac{\xi^{\frac{1}{\omega}}-1}{\xi-1}\right)^{\gamma-1} \psi(\xi^{\frac{1}{\omega}}) = \sum_{s=0}^{\infty} b_s (\xi-1)^s \tag{3.17}$$

where

$$b_0 = P(1) = \omega^{1-\gamma} \cdot \psi(1). \tag{3.18}$$

The series has a radius of convergence 1, for there is no singularity of this function in the circle $|\xi - 1| < 1$. Moreover, as $\xi$ approaches 0 in this circle, we have the inequality

$$|\xi \cdot P(\xi)| \leqq K. \tag{3.19}$$

Thus from the preceding lemma, the series $\xi^{\frac{w}{\omega}} P(\xi) = \sum_{s=0}^{\infty} b_s \xi^{\frac{w}{\omega}} (\xi - 1)^s$ converges uniformly for $0 \leqq \xi \leqq 1$. On the other hand, the series converges uniformly in and on the circle $|\xi - 1| \leqq r$ ($r < 1$), which is contained in the image of the circle $|\zeta - 1| = 1$ if $r$ is sufficiently small. Hence the convergence of $\xi^{\frac{w}{\omega}} P(\xi)$ is uniform on the contour $C_\xi$, and we can integrate (3.16) term by term

$$I(w) = \frac{1}{2\pi i \cdot \omega} \int_{C_\xi} \xi^{\frac{w}{\omega}} (\xi - 1)^{\gamma - 1} \sum_{s=0}^{\infty} b_s (\xi - 1)^s d\xi = \frac{1}{\omega} \sum_{s=0}^{\infty} b_s \cdot \frac{1}{2\pi i} \int_{C_\xi} \xi^{\frac{w}{\omega}} (\xi - 1)^{\gamma + s - 1} d\xi .$$

From the defining integral for Beta functions, we derive a formula

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_\xi} \xi^{\frac{w}{\omega}} (\xi - 1)^{\gamma + s - 1} d\xi &= \frac{e^{(\gamma + s - 1)\pi i}}{2\pi i} (1 - e^{-2\pi(\gamma + s - 1)i}) \cdot \int_0^1 \xi^{\frac{w}{\omega}} (1 - \xi)^{\gamma + s - 1} d\xi \\ &= \frac{\Gamma\left(\frac{w}{\omega} + 1\right)}{\Gamma(1 - \gamma - s) \cdot \Gamma\left(\frac{w}{\omega} + \gamma + s + 1\right)} . \end{aligned}$$

Using the above formula, we have the inverse factorial series expansion in the form

$$I(w) = \Gamma\left(\frac{w}{\omega} + 1\right) \cdot \sum_{s=0}^{\infty} \frac{b_s}{\omega} \frac{1}{\Gamma(1 - \gamma - s)} \cdot \frac{1}{\Gamma\left(\frac{w}{\omega} + \gamma + s + 1\right)} . \tag{3.20}$$

Once this series converges in an open half plane $\mathrm{Re}\, w > \omega$, then it is uniformly convergent in the closed right half plane (3.14). This completes the proof.

COROLLARY. *Let $v(\zeta)$ has a power series expansion*

$$v(\zeta) = \sum_{s=0}^{\infty} d_s (\zeta - 1)^s \tag{3.21}$$

*at $\zeta = 1$, and is holomorphic in $D_\zeta$. Then the function $g(w)$ of a complex variable $w$ defined by*

$$g(w) = \frac{1}{2\pi i \cdot \Gamma(w)} \int_C \zeta^{w-1} \cdot (\zeta - 1)^{\beta - 1} v(\zeta) d\zeta \tag{3.22}$$

*has an asymptotic expansion*

$$g(w) \cong \sum_{s=0}^{\infty} \frac{d_s}{\Gamma(1-\beta-s)\cdot\Gamma(w+\beta+s)} \tag{3.23}$$

*in the right half plane* (3.14), *for non-integral* $\beta$.

PROOF. This is an easy consequence of the Lemma 4. For if we integrate first $N$ terms of the integrand term by term, then the integral $g(w)$ takes the form

$$g(w) = \sum_{s=0}^{N-1} \frac{d_s}{\Gamma(1-\beta-s)\cdot\Gamma(w+\beta+s)} + \frac{1}{2\pi i\Gamma(w)} \int_C \zeta^{w-1}(\zeta-1)^{\beta+N-1}\psi(\zeta)d\zeta\,. \tag{3.24}$$

Applying the preceding lemma to the last term, we have a uniformly convergent series

$$\frac{1}{\omega} \frac{\Gamma\left(\frac{w}{\omega}+1\right)}{\Gamma(w)} \cdot \sum_{s=0}^{\infty} \frac{b_s}{\Gamma(1-\beta-N-s)\cdot\Gamma\left(\frac{w}{\omega}+\beta+N+s+1\right)}\,.$$

From the uniformity of the convergence, we can interchange the summation and limiting operation, and we have, by the Stirling formula

$$\lim_{w\to\infty} \Gamma(w+\beta+N)\cdot\frac{\Gamma\left(\frac{w}{\omega}+1\right)}{\Gamma(w)} \sum_{s=0}^{\infty} \frac{b_s}{\Gamma(1-\beta-N-s)\cdot\Gamma\left(\frac{w}{\omega}+\beta+N+s+1\right)}$$

$$= \omega^{\beta+N} b_0 \frac{1}{\Gamma(1-\beta-s)}\,.$$

This proves the asymptoticity in (3.14).

## 4. A fundamental set of solutions for a difference system.

Consider a system of linear difference equations

$$\{(\rho+w)E-A\}\vec{g}(w) = B\vec{g}(w-1) \tag{4.1}$$

with some preassigned initial condition

$$\vec{g}(0) = \vec{g}_{\rho,0}, \qquad (\rho E-A)\vec{g}_{\rho,0} = \vec{0}\,, \tag{4.2}$$

where $w$ is a complex variable, $\vec{g}(w)$ is an $n$-vector, and $\rho$ is an eigenvalue of the matrix $A$ which will be fixed throughout this section. Following assumptions are made on the matrices $A$ and $B$.

(B.1) *$B$ is a diagonal matrix with non-zero diagonal elements $\lambda_1, \lambda_2, \cdots, \lambda_n$.*

(B.2) *There is a finite positive integer $\omega$ such that*

$$|\arg\lambda_j - \arg\lambda_k| \geqq \frac{\pi}{\omega} \qquad (j\neq k,\ j,k=1,2,\cdots,n)\,. \tag{4.3}$$

(A.1) *There is no integral value of $w$ for which*

(4.4) $$\det[(\rho+w)E-A]=0\,.$$

(A.2) *Quantities* $\beta_p$ *defined by*

(4.5) $$\beta_p=\rho-a_{pp}+1 \qquad (p=1,2,\cdots,n)$$

*with the p-th diagonal element* $a_{pp}$ *of* $A$, *are not integers.*

THEOREM 2. *With conditions* (B.1), (B.2), (A.1) *and* (A.2), *there is a set of solutions* $\vec{g}_1(w),\vec{g}_2(w),\cdots,\vec{g}_n(w)$ *of* (4.1), *which is linearly independent except for those values of* $w$ *which are congruent to a finite set of complex numbers, and each* $\vec{g}_p(w)$ *admits an asymptotic expansion of the form*

(4.6) $$\vec{g}_p(w)\cong\lambda_p^w\sum_{s=0}^{\infty}\frac{1}{\Gamma(w+\beta_p+s)}\vec{A}_p(s)$$

*in the right half plane*

(4.7) $$\mathrm{Re}\,w\geqq\omega+\varepsilon$$

*where* $\vec{A}_p(s)$ *are constant vectors independent of* $w$. *Specifically,*

(4.8) $$\vec{A}_p(0)=\lambda_p^{\beta_p-1}\cdot\frac{1}{\Gamma(1-\beta_p)}\vec{1}_p$$

*where* $\vec{1}_p$ *is a vector with all its elements zero except the p-th.*

*Moreover, there is a uniformly convergent expression of* $\vec{g}_p(w)$, *in the form*

(4.9) $$\vec{g}_p(w)=\lambda_p^w\sum_{s=0}^{\infty}\frac{1}{\Gamma\left(\frac{w}{\omega}+\beta_p+s\right)\cdot\Gamma(1-\beta_p+s)}\vec{b}_p(s)$$

*which is also valid in* (4.7).

PROOF. Consider a system of linear differential equations

(4.10) $$(uE-B)\frac{d\vec{v}}{du}=(\rho E-A)\vec{v}\,.$$

This equation has regular singular points at $u=\lambda_p$ $(p=1,2,\cdots,n)$ with characteristic exponents $0,0,\cdots,\beta_p-1,\cdots,0$, and there are no other singularities in the finite $u$-plane. Define a function $\vec{v}_p(u)$ as the solution of this system which corresponds to the exponent $\beta_p-1$ at $u=\lambda_p$. By the assumption (A.2), $\vec{v}_p(u)$ has a convergent power series expansion of the form

(4.11) $$\vec{v}_p(u)=(u-\lambda_p)^{\beta_p-1}\cdot\sum_{s=0}^{\infty}(u-\lambda_p)^s\vec{d}_p(s)$$

with constant vectors $\vec{d}_p(s)$. We remark that we can always define $\vec{v}_p(u)$ so that the condition

(4.12) $$\vec{d}_p(0)=\vec{1}_p$$

is satisfied.

Then, as can easily be verified, the integral expression

$$\vec{g}_p(w) = \frac{1}{2\pi i \cdot \Gamma(w)} \int_{l_p} u^{w-1} \vec{v}_p(u) du \tag{4.13}$$

is a solution of (4.1), provided that the increment of $u^{w-1}(uE-B)\vec{v}_p(u)$ along the path of integration is zero, namely,

$$[u^{w-1}(uE-B)\vec{v}_p(u)]_{l_p} = \vec{0}. \tag{4.14}$$

The function $(u-\lambda_p)^{1-\beta_p}\vec{v}_p(u) \equiv \sum_{s=0}^{\infty}(u-\lambda_p)^s \vec{d}_p(s)$ is a holomorphic function of $u$ in a simply connected domain which does not contain any singularity $u=\lambda_j$ except for $u=\lambda_p$. So by the hypotheses (B.1) and (B.2), such a domain can be so chosen that it contains the angular domain

$$D_u = \left\{u\,;0 \leqq |u| < \infty, |\arg u - \arg \lambda_p| < \frac{\pi}{\omega}\right\}. \tag{4.15}$$

By the change of variable $\zeta = \dfrac{u}{\lambda_p}$, this domain $D_u$ is transformed into the domain $D_\zeta$ defined by (3.12). We shall define the path of integration $l_p$ as the inverse image of the contour $C$ defined in Lemma 4, by this transformation. Then we see that the condition (4.14) is automatically satisfied for $w$ in the right half plane $\mathrm{Re}\, w \geqq 1+\varepsilon$.

If we write $\vec{V}_p(\zeta)$ for $\vec{v}_p(u) = \vec{v}_p(\lambda_p\zeta)$, then $\vec{V}_p(\zeta)$ is holomorphic in $D_\zeta$, and has a convergent power series expansion in $(\zeta-1)$ in any circle around $\zeta=1$ contained in $D_\zeta$:

$$\vec{V}_p(\zeta) = \lambda_p^{\beta_p-1} \cdot (\zeta-1)^{\beta_p-1} \cdot \sum_{s=0}^{\infty} \lambda_p^s (\zeta-1)^s \vec{d}_p(s). \tag{4.11$'$}$$

Applying the corollary of Lemma 4, with $\beta$ replaced by $\beta_p$, we have the expansion

$$\vec{g}_p(w) \cong \lambda_p^{w+\beta_p-1} \cdot \sum_{s=0}^{\infty} \frac{\lambda_p^s}{\Gamma(1-\beta_p-s) \cdot \Gamma(w+\beta_p+s)} \vec{d}_p(s) \tag{4.16}$$

which is valid in (4.7).

Convergent expression is obtained in the form (4.9) from Lemma 4 (cf. (3.20)). So it only remains to prove the linear independence of the solutions. By taking the first terms of the expansion (4.16) for each $p$, we have an asymptotic expansion of Casorati determinant $G(w) = \det(\vec{g}_1(w), \vec{g}_2(w), \cdots, \vec{g}_n(w))$ for large positive value of $\mathrm{Re}\, w$ in a form

$$G(w) = \prod_{p=1}^{n} \lambda_p^{\beta_p+w-1} \cdot \frac{1}{\Gamma(1-\beta_p) \cdot \Gamma(w+\beta_p)} \cdot \det[\vec{1}_1, \vec{1}_2, \cdots, \vec{1}_n] \cdot \left(1+O\left(\frac{1}{w}\right)\right).$$

Since $\lambda_p \neq 0$ by (B.1), and the expression $\dfrac{1}{\Gamma(1-\beta_p) \cdot \Gamma(w+\beta_p)}$ vanishes only for those values of $w$ such that

$$w+\beta_p=\alpha:\ \text{negative integer},$$

$G(w)\neq 0$ if $\operatorname{Re} w$ is large. As $G(w)$ satisfies the difference equation

$$\det[(\rho+w)E-A]G(w)=\det\cdot B\cdot G(w-1),$$

it happens that

$$G(w)\neq 0,\quad\text{and}\quad G(w-1)=G(w-2)=\cdots=0$$

only when $w$ is equal to one of the values for which

$$\det[(\rho+w)E-A]=0.$$

Therefore the linear independence breaks down only for $w=\alpha-1, \alpha-2, \cdots$, where $\alpha$ denotes any one of the roots of $\det[(\rho+w)E-A]=0$. This completes the proof.

It is to be remarked that, according to the hypothesis (A.1), linear independence is preserved for all integral values of $w$.

COROLLARY. *There is associated a set of $n$ constants $c_1, c_2, \cdots, c_n$ for any preassigned initial condition* (4.2) *for the system* (4.1), *such that the solution $\vec{g}(m)$ of the initial value problem is given by*

$$\vec{g}(m)=\sum_{p=1}^{n} c_p\vec{g}_p(m) \tag{4.17}$$

*which is valid in* (4.7).

PROOF. Since the existence of the inverse $[(\rho+w)E-A]^{-1}$ is assured for integral value of $w$ by (A.1), we can determine $\vec{g}(m_0)$ $(m_0\geqq\omega+\varepsilon)$ from $\vec{g}(0)$ by the finite number of iterations

$$\vec{g}(m)=[(\rho+m)E-A]^{-1}\cdot B\vec{g}(m-1).$$

Then a system of $n$ algebraic equations of the first order

$$\vec{g}(m_0)=\sum_{p=1}^{n} c_p\vec{g}_p(m_0) \tag{4.18}$$

for $n$ unknowns $c_1, c_2, \cdots, c_n$ has a unique solution.

It is to be noted that the asymptotic expansion (4.6) is an asymptotic expansion of $\vec{g}_p(w)$ in a neighborhood of the point $w=\infty$, and we can determine the constants $c_1, c_2, \cdots, c_n$ explicitly with the help of the convergent expansion (4.9). That is to say we have solved explicitly the Stokes Phenomenon for the difference system (4.1).

REMARK. We shall indicate some relaxation of the condition (A.2).

(i) When $\beta_p-1$ is zero, we take that path $l_p$ of the integration as the straight line joining the origin to the point $u=\lambda_p$. The condition (4.14) is satisfied, and we can apply the Lemma 4 to have a convergent expansion, and its corollary to have the asymptotic expansion, respectively. We have slightly different expression

$$\vec{g}_p(w)\begin{cases} \cong \dfrac{1}{2\pi i}\displaystyle\sum_{s=0}^{\infty}(-\lambda_p)^s\cdot\Gamma(s+1)\cdot\dfrac{1}{\Gamma(w+s+1)}\vec{d}_p(s)\,, \\ =\dfrac{1}{2\pi\omega i}\displaystyle\sum_{s=0}^{\infty}(-\lambda_p)^s\cdot\Gamma(s+1)\cdot\dfrac{1}{\Gamma\left(\dfrac{w}{\omega}+s+1\right)}\vec{b}_p(s)\,. \end{cases}$$

(ii) When $\beta_p-1$ is a positive integer, the system (4.10) has a logarithmic singularity at $u=\lambda_p$, corresponding to the exponent $\beta_p-1$. In this case the expansion (4.11) takes the form

$$\vec{v}_p(u)=(u-\lambda_p)^{\beta_p-1}[\log(u-\lambda_p)]\sum_{s=0}^{\infty}\vec{d}_p(s)(u-\lambda_p)^s+(u-\lambda_p)^{\beta_p-1}\cdot\vec{v}_p^*(u)$$

where $\vec{v}_p^*(u)$ is a holomorphic function of $u$ in $D_u$. We will take the same path $l_p$ as in the proof of the theorem, then since $(u-\lambda_p)^{\beta_p-1}\vec{v}_p^*(u)$ is holomorphic in $D_u$ for a positive integer $\beta_p-1$, we have by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{l_p}u^{w-1}(u-\lambda_p)^{\beta_p-1}\vec{v}_p^*(u)du=0\,.$$

Similarly, for a holomorphic function $\varphi(u)$ in $D_u$

$$\begin{aligned}&\frac{1}{2\pi i}\int_{l_p}u^{w-1}[\log(u-\lambda_p)]\cdot\varphi(u)du\\ &=\frac{1}{2\pi i}\int_0^{\lambda_p}u^{w-1}\{[\log(u-\lambda_p)]\varphi(u)-[\log(u-\lambda_p)-2\pi i]\varphi(u)\}du\\ &=\int_0^{\lambda_p}u^{w-1}\cdot\varphi(u)du\,.\end{aligned}$$

Thus we have

$$\begin{aligned}\vec{g}_p(w)&=\frac{1}{2\pi i\Gamma(w)}\int_{l_p}u^{w-1}\cdot\vec{v}_p(u)du=\frac{1}{\Gamma(w)}\int_0^{\lambda_p}(u-\lambda_p)^{\beta_p-1}u^{w-1}\sum_{s=0}^{\infty}(u-\lambda_p)^s\vec{d}_p(s)du\\ &\cong\lambda_p^{w+\beta_p-1}\sum_{s=0}^{\infty}e^{\pi i(\beta_p-1+s)}\cdot\frac{\Gamma(\beta_p+s)}{\Gamma(w+\beta_p+s)}\lambda_p^s\vec{d}_p(s)\end{aligned}$$

in (4.7). The convergent expression can be obtained similarly in the same domain.

The case when $\beta_p-1$ is a negative integer is not solved yet. As it will be clarified later, the essential hypothesis for the matrix $B$ is that it has mutually different eigenvalues. The condition (A.1) means that $A$ has no two eigenvalues which are congruent modulo integer. So (A.2) is the only one assumption which is not invariant under a linear transformation.

## 5. Main theorems.

Now we can state the theorem about the asymptotic behavior of the function $\vec{x}_\rho(t)$ defined by (1.2) for an eigenvalue $\rho$ of the matrix $A$.

THEOREM 3. *With conditions* (B.1), (B.2), (A.1) *and* (A.2) *imposed upon the matrices A and B,* $\vec{x}_\rho(t)$ *has an expression*

$$\vec{x}_\rho(t)=\sum_{p=1}^{n} c_p \vec{\xi}_p^{(\rho)}(t) \tag{5.1}$$

*where* $\vec{\xi}_p^{(\rho)}(t)$ *is a vector admitting an asymptotic expansion*

$$\vec{\xi}_p^{(\rho)}(t)\cong e^{\lambda_p t}t^{a_{pp}}\sum_{s=0}^{\infty}\frac{\vec{d}_p(s)}{\Gamma(1-\beta_p-s)}t^{-s}+R_h(t,S(\lambda_p)) \tag{5.2}$$

*as t tends to infinity in an angular domain S defined by*

$$S=\bigcap_{p=1}^{n} S(\lambda_p) \tag{5.3}$$

*and h is a certain positive number satisfying the inequality*

$$h>\max\left\{\omega,\frac{1}{2}+\mathrm{Re}\,a_{11},\frac{1}{2}+\mathrm{Re}\,a_{22},\cdots,\frac{1}{2}+\mathrm{Re}\,a_{nn}\right\}.$$

PROOF. Since the coefficient $g_{\rho,m}$ in the series (1.2) satisfies the system (4.1) with the initial condition (4.2), applying the corollary of Theorem 2 we have an asymptotic expansion (see (4.16) and (4.17))

$$\vec{g}_{\rho,m}\cong\sum_{p=1}^{n}c_p\sum_{s=0}^{\infty}\frac{\lambda_p^m}{\Gamma(m+\beta_p+s)}\vec{A}_p(s)\quad\left(\vec{A}_p(s)=\lambda_p^{\beta_p+s-1}\cdot\frac{\vec{d}_p(s)}{\Gamma(1-\beta_p-s)}\right). \tag{5.4}$$

By Theorem 1, each expression in the linear combination $t^\rho\sum_{m=0}^{\infty}\vec{g}_{\rho,m}t^m = \sum_{p=1}^{m}c_p(\sum_{m=0}^{\infty}\vec{g}_p(m)t^m)\cdot t^\rho$ can be expanded asymptotically into the form

$$\begin{aligned}t^\rho\sum_{m=0}^{\infty}\vec{g}_p(m)t^m &\cong e^{\lambda_p t}t^{1-\beta_p+\rho}\cdot\lambda_p^{1-\beta_p}\cdot\sum_{s=0}^{\infty}\vec{A}_p(s)(\lambda_p t)^{-s}+R_h(t,S(\lambda_p))\\ &\cong e^{\lambda_p t}t^{a_{pp}}\sum_{s=0}^{\infty}\frac{1}{\Gamma(1-\beta_p-s)}\vec{d}_p(s)t^{-s}+R_h(t,S(\lambda_p))\end{aligned}$$

as $t$ tends to infinity in the angular domain $S(\lambda_p)$.

Since, to $n$ different eigenvalues $\lambda_p$ $(p=1,2,\cdots,n)$ of $B$, correspond $n$ different sectors $S(\lambda_p)$ $(p=1,2,\cdots,n)$, the asymptotic expansion of the function (5.1) is valid in their intersection $S$. This completes the proof.

It is easy to verify that the leading term of the asymptotic expansion of $\xi_p^{(\rho)}(t)$ coincides with the formal solutions of (1.1) at infinity, which corresponds to the eigenvalue $\lambda_p$. More exactly, we have the following lemma.

LEMMA 5. *The formal series*

$$e^{\lambda_p t}t^{a_{pp}}\cdot\sum_{s=0}^{\infty}\frac{\vec{d}_p(s)}{\Gamma(1-\beta_p-s)}t^{-s} \tag{5.5}$$

*is a formal solution of the system* (1.1).

PROOF. Substituting the formal solution (1.3) into the system (1.1), we have a system of difference equations for $\vec{h}_p(s)$

$$(5.6)\qquad (\lambda_p-B)\vec{h}_p(s+1)=(A-a_{pp}+s)\vec{h}_p(s),\quad \vec{h}_p(0)=\vec{1}_p\,.$$

On the other hand, since $\vec{d}_p(s)$ is the coefficient of the power series expansion (4.11), we have a system of difference equations for it, by substituting the expansion (4.11) into (4.10),

$$(5.7)\qquad (\lambda_p-B)(\beta_p+s)\vec{d}_p(s+1)=(a_{pp}-s-A)\vec{d}_p(s)\,.$$

The substitution $\vec{h}_p^*(s)=\dfrac{\vec{d}_p(s)}{\Gamma(1-\beta_p-s)}$ yields a same system as (5.6) for $\vec{h}_p^*(s)$, and the initial condition $\vec{h}_p^*(0)=\dfrac{1}{\Gamma(1-\beta_p)}\vec{1}_p$ which is different from that for $\vec{h}_p(s)$ by only a constant multiplier. This completes the proof.

To verify the Theorem 3, and Lemma 5 for cases where some of the quantities $\beta_p-1$ are zero or positive integers, we can only modify the asymptotic expansion of $\vec{\xi}_p^{(p)}(t)$. For in these cases we have, by the remark in § 4,

$$\vec{g}_p(w)\cong\lambda_p^{w+\beta_p-1}\cdot\sum_{s=0}^{\infty}\Gamma(\beta_p+s)\cdot\frac{(-\lambda_p)^s}{\Gamma(w+\beta_p+s)}\vec{d}_p(s)\,.$$

Accordingly,

$$\vec{\xi}_p^{(p)}(t)\cong e^{\lambda_p t}t^{a_{pp}}\cdot\sum_{s=0}^{\infty}(-1)^s\Gamma(\beta_p+s)\vec{d}_p(s)t^{-s}+R_h(t,S(\lambda_p))$$

where $\vec{d}_p(s)$ satisfies the same difference equations as (5.7). The substitution $\vec{h}_p^*(s)=(-1)^s\Gamma(\beta_p+s)\vec{d}_p(s)$ yields the same system (5.6).

So, in the following discussions, we will replace the condition (A.2) by a milder condition:

(A.2)* $\beta_p-1$ are not negative integers.

We shall denote by a matrix $X(t)$ the matrix whose $k$-th column is a vector $x_{\rho_k}(t)$ of the form (1.2) corresponding to the $k$-th eigenvalue $\rho_k$ of $A$. Thus $X(t)$ is a fundamental set of solutions at the origin. And let $\Xi(t)$ be a matrix which has the formal solution $\vec{\xi}_p(t)$ defined by (1.3) as the $p$-th column vector. There are infinitely many determinations of the arguments of the eigenvalues $\lambda_p$, which are different by integral multiples of $2\pi$. Let us denote by $S(\theta_1,\theta_2,\cdots,\theta_n)$ the angular domain

$$S=\bigcap_{p=1}^{n}S(\lambda_p)$$

which corresponds to the set of determinations

$$(5.8)\qquad \arg\lambda_p=\theta_p\qquad(p=1,2,\cdots,n)\,.$$

Then we can summarize our discussion in the form:

THEOREM 4. *With conditions* (B.1), (B.2) *imposed upon the matrix $B$, and conditions* (A.1), (A.2)* *for every eigenvalue $\rho$ of $A$ imposed upon the matrix $A$, there is a unique $n$ by $n$ matrix $C(\theta_1, \theta_2, \cdots, \theta_n)$ corresponding to each sector $S(\theta_1, \theta_2, \cdots, \theta_n)$ such that the asymptotic expansion*

$$X(t) \cong \Xi(t)C(\theta_1, \theta_2, \cdots, \theta_n)+R_h(t, S(\theta_1, \theta_2, \cdots, \theta_n)) \tag{5.9}$$

*holds as $t$ tends to infinity in the sector $S(\theta_1, \theta_2, \cdots, \theta_n)$.*

The column $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ of the matrix $C(\theta_1, \theta_2, \cdots, \theta_n)$ which corresponds to the solution $\vec{x}_\rho(t)$, or to the eigenvalue $\rho$ of $A$, is determined from the equations

$$\vec{g}(m_0)=\sum_{p=1}^{n} c_p \vec{g}_p(m_0)\,.$$

If we move from the sector $S(\theta_1, \cdots, \theta_p, \cdots, \theta_n)$ to the sector $S(\theta_1, \cdots, \theta_p+2q_p\pi, \cdots, \theta_n)$ ($q_p$: integer), which is supposed to be non-empty, this change reflects in the vector $\vec{g}_p(m_0)$ as follows: From the defining equation (4.13), we have

$$\vec{g}(m_0)=\frac{1}{2\pi i\cdot\Gamma(m_0)}\int_{l_p} u^{w-1}(u-\lambda_p)^{\beta_p-1}\vec{\varphi}_p(u)du$$

where $\vec{\varphi}_p(u)$ is holomorphic in $D_u$, which is defined by (4.15). Corresponding to the change, $\lambda_p \to \lambda_p\cdot\exp(2q_p\pi i)$, we change the variable of integration from $u$ to $u\cdot\exp(2\pi q_p i)$. Thus for an integer $m_0$, the change which takes place is

$$\begin{aligned}\vec{g}_p(m_0) &\to \frac{1}{2\pi i\cdot\Gamma(m_0)}\int_{l_p} u^{m_0-1}\exp[2q_p(\beta_p-1)\pi i](u-\lambda_p)^{\beta_{p1}-}\vec{\varphi}_p(u)du \\ &=\exp[2q_p(\beta_p-1)\pi i]\vec{g}_p(m_0)\,.\end{aligned}$$

Accordingly, we have

$$c_p \to c_p\exp[-2q_p(\beta_p-1)\pi i]=c_p\exp[2q_p(a_{pp}-\rho)\pi i]\,.$$

Thus we have an important connection formula:

THEOREM 5. *Let $c_{pk}(\theta_1, \theta_2, \cdots, \theta_n)$ be the $(p, k)$ element of the matrix $C(\theta_1, \theta_2, \cdots, \theta_n)$, then we have the relation*

$$\begin{aligned}&c_{pk}(\theta_1+2q_1\pi, \theta_2+2q_2\pi, \cdots, \theta_n+2q_n\pi) \\ &\qquad =c_{pk}(\theta_1, \theta_2, \cdots, \theta_n)\cdot\exp[2q_k(a_{pp}-\rho_k)\pi i]\,.\end{aligned} \tag{5.10}$$

The sector $S(\theta_1, \theta_2, \cdots, \theta_n)$ has usually its central angle at least $\pi$, if we restrict ourselves for the principal values for the arguments of $\lambda_p, p=1, 2, \cdots, n$. For example, if the angles $\theta_1, \theta_2, \cdots, \theta_n$ are ordered in such an order as

$$0 \leqq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi\,,$$

we have, by assumption (B.2),

$$\theta_1+2\pi-\theta_n \geqq \frac{\pi}{\omega}.$$

From the definition, the central angle of $S(\theta_1, \theta_2, \cdots, \theta_n)$ is given by

$$-\left(\frac{3}{2}\pi-\varepsilon\right)-\theta_1 \leqq \arg t \leqq \frac{3}{2}\pi-\varepsilon-\theta_n$$

and hence

$$\left(\frac{3}{2}\pi-\theta_n-\varepsilon\right)-\left(-\frac{3}{2}\pi+\varepsilon-\theta_1\right)=\pi+(\theta_1+2\pi-\theta_n)\geqq \pi+\frac{\pi}{\omega}-2\varepsilon.$$

In a special case, when the inequality $\theta_n-\theta_1<\pi$ holds, we have

$$\left(\frac{3}{2}\pi-\theta_n-\varepsilon\right)-\left(-\frac{3}{2}\pi+\varepsilon-\theta_1\right)=3\pi-(\theta_n-\theta_1)-2\varepsilon>2\pi-2\varepsilon.$$

That is to say, the sector $S(\theta_1, \theta_2, \cdots, \theta_n)$ covers an entire angle $2\pi$ of the plane.

In the classical theory of irregular singular points, the domain of asymptotic validity for a formal solution $\vec{\xi}_p(t)$ of the form (1.3) is determined by the quantities $\arg(\lambda_p-\lambda_k), p, k=1, 2, \cdots, n$. Because the classical theory has its foundation on the integral equations, where the relative magnitude of the quantities $\exp(\lambda_p t), p=1, 2, \cdots, n$ to each other plays a fundamental role. Since we have an additional term $O(t^h)$, our asymptoticity is concerned with the relative magnitude of $\exp(\lambda_p t)$ to this expression. Thus our domain of validity is determined by the arguments of the eigenvalues of $B$, $\arg\lambda_p$, $p=1, 2, \cdots, n$. In a way, our domain of validity is more natural than that of the classical theory, because we deduced it from the corresponding single inhomogeneous equation (2.2).

The conditions (B.1) and (B.2) are not essential for our discussion. For, if $B$ has different eigenvalues, its diagonalization is always possible, and by the transformation

$$\vec{x}(t) \to e^{\lambda t}\vec{x}(t)$$

we can always find a complex number $\lambda$ such that the conditions (B.1) and (B.2) are satisfied for the new set of eigenvalues $(\lambda_1-\lambda, \lambda_2-\lambda, \cdots, \lambda_n-\lambda)$. We note that in this case the additional term has the form $O(e^{\lambda t}t^h)$ and the sector $S$ depends on the choice of $\lambda$.

We shall remark the relation between our theory and that of Hopf-Knobloch. The latter theory has its foundation on the following formula based on the Cauchy's theorem

$$\vec{g}(m_0)=\frac{1}{2\pi i}\oint \vec{x}_\rho(t)t^{-\rho-m_0-1}dt.$$

The evaluation of this integral is done on a sufficiently large circle so that the integrand $\vec{x}_\rho(t)t^{-\rho-m_0-1}$ can be replaced by its asymptotic expansion. If, as is shown by Knobloch, the formal term by term integration is admitted for each term in the linear combination,

$$\vec{g}(m_0)=\frac{1}{2\pi i}\oint[\sum_{p=1}^{n} c_p\vec{\xi}_p(t)]t^{-\rho-m_0-1}dt=\sum_{p=1}^{n} c_p\cdot\frac{1}{2\pi i}\oint \vec{\xi}_p(t)t^{-\rho-m_0-1}dt\,.$$

The resulting series is

$$\frac{1}{2\pi i}\oint e^{\lambda_p t}t^{\alpha_{pp}-\rho-m_0-1}\cdot\sum_{s=0}^{\infty}\frac{\vec{d}_p(s)}{\Gamma(1-\beta_p-s)}t^{-s}ds=\sum_{s=0}^{\infty}\lambda_p^{m_0+\beta_p+s}\cdot\frac{\vec{d}_p(s)}{\Gamma(1-\beta_p-s)\cdot\Gamma(m_0+s+\beta_p)}.$$

The formal series on the right is exactly the asymptotic expansion of $g(m_0)$. Thus our theory gives another foundation for the formal procedure of Hopf. In the theory of Knobloch, the above formal series has a meaning only when the function defined by

$$\sum_{s=0}^{\infty}\frac{\vec{d}_p(s)}{\Gamma(1-\beta_p-s)\cdot\Gamma(m_0+s+\beta_p)}z^{m_0+\beta_p+s}\qquad (|z|:\ \text{sufficiently small})$$

is continued analytically to the point $\lambda_p$. However he did not give the method how to carry out the continuation procedure explicitly. Our theory furnishes the method of the calculation, and assures the asymptoticity with respect to $m_0$.

We have reasons to believe that *the additional term $O(t^h)$ is identically zero*. We shall give a proof of this fact for a system of two equations with $B=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and with any $A$ for which conditions (A.1) and (A.2) are satisfied.

Let us define sectors $S_-$ and $S_+$ by

$$\begin{cases} S_-=\left\{t\,;\,|t|\geqq K,\,-\frac{\pi}{2}+\varepsilon\leqq\arg t\leqq\frac{\pi}{2}\right\}, \\ S_+=\left\{t\,;\,|t|\geqq K,\,\frac{\pi}{2}\leqq\arg t\leqq\frac{3}{2}\pi-\varepsilon\right\}. \end{cases}$$

Let $\vec{\xi}_+(t)$ and $\vec{\xi}_-(t)$ be the formal solutions corresponding to the eigenvalues 1 and $-1$ respectively, then we know in the classical theory that there are *uniquely* determined solutions $\vec{x}_+(t)$ and $\vec{x}_-(t)$ which have the asymptotic expansion

$$\text{(5.11)}\qquad \begin{cases} \vec{x}_-(t)\cong\vec{\xi}_-(t) & (t\in S_-), \\ \vec{x}_+(t)\cong\vec{\xi}_+(t) & (t\in S_+). \end{cases}$$

These solutions constitute a fundamental set of solutions in the combined sector $S=S_+\cup S_-$, and they maintain the same asymptotic expansions respectively in $S$. Let $\vec{x}_p(t)$ be a solution defined by (1.2). Then by Theorem 3 there is a set of constants $c_+$ and $c_-$ such that

$$\vec{x}_\rho(t) \cong c_-\vec{\xi}_-(t)+c_+\vec{\xi}_+(t)+R_h(t, S) \qquad (t \in S).$$

The solution $\vec{x}(t) \equiv \vec{x}_\rho(t)-(c_-\vec{x}_-(t)+c_+\vec{x}_+(t))$ has an asymptotic expansion

$$\vec{x}(t)=R_h(t, S) \cong \begin{cases} c_-^*\vec{\xi}_-(t) & (t \in S_-), \\ c_+^*\vec{\xi}_+(t) & (t \in S_+), \end{cases}$$

but since $\vec{x}_+(t)$ and $\vec{x}_-(t)$ constitute a fundamental set in $S$, and since they are the *unique* solutions for which (5.11) hold, we have

$$\vec{x}(t)=c_-^*\vec{x}_-(t)=c_+^*\vec{x}_+(t) \qquad (t \in S).$$

This is possible if and only if $c_-^*=c_+^*=0$ by the linear independence of $\vec{x}_-(t)$ and $x_+(t)$.

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