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Intuitionistic analysis and Gödel's interpretation

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§ 0. Introduction

Gödel defined, in his paper [2], a quantifier-free system with primitive recursive functionals of any finite type, named T, and a transformation of each formula in Heyting's arithmetic (abbr.: HA) to a certain formula which was called by Kreisel [4] *Gödel's interpretaion.* Gödel's interpretation of any formula has the form

$$\exists\varphi_1 \cdots \exists\varphi_m \forall\psi_1 \cdots \forall\psi_n A(\varphi_1, \cdots, \varphi_m; \psi_1, \cdots, \psi_n),$$

where $\varphi_1, \cdots, \varphi_m, \psi_1, \cdots, \psi_n$ are variables for functionals of higher type, in general, and $A(\varphi_1, \cdots, \varphi_m; \psi_1, \cdots, \psi_n)$ is a quantifier-free formula. It was proved by Gödel [2] that if the original formula is provable in HA, then there exist primitive recursive functionals $\varphi_1, \cdots, \varphi_m$ such that the formula

$$A(\varphi_1, \cdots, \varphi_m; \alpha_1, \cdots, \alpha_n)$$

with free variables $\alpha_1, \cdots, \alpha_n$ is provable in T.

Kreisel [5] introduced further a system T_1^c, obtained as follows. Let T_1 be the system obtained from T by adding the intuitionistic predicate calculus, including the quantifications for functionals of any finite type, but excluding the axiom of choice, and adding the mathematical induction for all formulas of the extended notation. We obtain T_1^c by adding to T_1 the axiom schema

$$\forall\varphi\exists\psi\mathfrak{A}(\varphi, \psi) \supset \exists\chi\forall\varphi\mathfrak{A}(\varphi, \chi(\varphi))$$

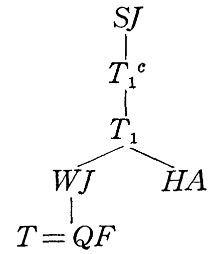
representing a special case of the axiom of choice. Kreisel proved the theorem which states that if a formula $\mathfrak{A} \supset \mathfrak{B}$ is provable in T_1^c, and if the Gödel's interpretations of $\mathfrak{A}$ and $\mathfrak{B}$ have the quantifier-free formulas

$$A(\varphi_1, \cdots; \psi_1, \cdots) \quad \text{and} \quad B(\varphi_1', \cdots; \psi_1', \cdots)$$

as their kernels, respectively, then there exist primitive recursive functionals $\varphi_1, \cdots, \varphi_1', \cdots$ such that $B(\varphi_1', \cdots; \alpha_1', \cdots)$ is provable in T from $A(\varphi_1, \cdots; \alpha_1, \cdots)$, where $\alpha_1, \cdots, \alpha_1', \cdots$ mean free variables for functionals.

In this paper, we shall introduce two new Gentzen-type sequential intuitionistic calculi SJ (strong system) and WJ (weak system). The above-mentioned system T will be renamed as QF (quantifier-free system) in the following.

The following diagram will show the situations of those systems:



SJ is a real extension of $T_1{}^c$, because $\neg\forall \boldsymbol{x} \neg A(\boldsymbol{x}) \rightarrow \exists \boldsymbol{x} A(\boldsymbol{x})$ is not $T_1{}^c$-provable for all quantifier-free $A(\boldsymbol{x})$, but is SJ-provable for all such $A(\boldsymbol{x})$ (cf. [**5**]).

We shall define *Gödel's interpretation* $\mathfrak{S}^G$ and *quantifier-free reductions* for each sequence $\mathfrak{S}$ in SJ and WJ, appropriately. Then we show that the following four conditions are equivalent to one another:

1) $\mathfrak{S}$ is provable in SJ;
2) $\mathfrak{S}^G$ is provable in SJ;
3) $\mathfrak{S}^G$ is provable in WJ;
4) There exists a quantifier-free reduction of $\mathfrak{S}$ which is provable in QF.

§ 1. Systems

1.0. Fundamental system

1.0.1. Type

As *types* we use only those given by the following 1) and 2).

1) 0 is a type.
2) If σ and τ are types, then $\sigma\langle\tau\rangle$ is a type.

We shall use the abbreviated notation $\sigma\langle\tau_1, \tau_2, \cdots, \tau_n\rangle$ to represent the type $\sigma\langle\tau_1\rangle\langle\tau_2\rangle\cdots\langle\tau_n\rangle$.

1.0.2. DEFINITION of 'term' (inductive definition)

1) 0 is a term of type 0.
2) A free variable of type τ is a term of type τ.

(Informally, each variable of type τ represents an arbitrary functional of type τ. Especially, each variable of type 0 means a non-negative integer.)

3) If t is a term of type 0, then t' (which means informally $t+1$) is a term of type 0.

4) If $f(\alpha)$ is a term of type σ and α is a free variable of type τ, then $\lambda\varphi f(\varphi)$ is a term of type $\sigma\langle\tau\rangle$, where φ is an arbitrary bound variable of type τ not contained in $f(\alpha)$, and $f(\varphi)$ means the result of substituting φ for α throughout $f(\alpha)$.

5) If s is a term of type $\sigma\langle\tau\rangle$ and t is a term of type τ, then $s\langle t\rangle$ is a term of type σ.

6) If s is a term of type τ and t is a term of type $\tau\langle 0, \tau\rangle$, then $\rho[s, t]$ is a term of type $\tau\langle 0\rangle$.

(The intended sense of $\rho[s, t]$ is the functional φ defined as

$$\varphi(0) = s, \qquad \varphi(x') = t(x, \varphi(x)),$$

where x represents all non-negative integers.)

7) The only terms are those given by 1)-6).

We shall use the abbreviated notation $s\langle t_1, t_2, \cdots, t_n\rangle$ to represent the term $s\langle t_1\rangle\langle t_2\rangle \cdots \langle t_n\rangle$,

By *primitive recursive functional* (abbr.: prf) we mean a term containing no free variables. Each prf represents a primitive recursive functional in Gödel's sense, informally. Conversely, for any primitive recursive functional in Gödel's sense, there exists at least one prf such that the latter represents the former.

1.0.3. Formula

Our *prime formulas* are of the form $s = t$, where s and t are arbitrary terms of the same type. The *formulas* are constructed, as usual, by the propositional connectives $\wedge, \vee, \neg, \supset$ and by quantifiers $\forall\varphi$ and $\exists\varphi$ for every type, starting from the prime formulas. We use $\mathfrak{A} \sim \mathfrak{B}$ as the abbreviation of $(\mathfrak{A} \supset \mathfrak{B}) \wedge (\mathfrak{B} \supset \mathfrak{A})$.

1.0.4. Sequence

We shall use any *sequence* $\Gamma \rightarrow \Delta$, provided that

at most one formula of Δ *contains quantifiers*,

where Γ and Δ mean the finite series (possibly void) of formulas in our sense, respectively.

1.0.5. Schemata for 'beginning sequence'

(1) $$\rightarrow t = t$$

(2) $$s = t \rightarrow t = s$$

(3) $$r = s, s = t \rightarrow r = t$$

(4) $$s = t \rightarrow s' = t'$$

(5) $$s_1 = s_2, t_1 = t_2 \rightarrow s_1\langle t_1\rangle = s_2\langle t_2\rangle$$

(6) $$\rightarrow (\lambda\varphi f(\varphi))\langle t\rangle = f(t)$$

(7) $$\rightarrow \rho[s, t]\langle 0\rangle = s$$

(8) $$\rightarrow \rho[s, t]\langle r'\rangle = t\langle r, \rho[s, t]\langle r\rangle\rangle$$

(9) $$t' = 0 \rightarrow$$

(10) $$s' = t' \rightarrow s = t$$

In those schemata, $r, s, s_1, s_2, t, t_1, t_2$ mean arbitrary terms, especially s and

t in (4), (9), (10) and r in (8) are of type 0, and $f(t)$ means the result of substituting t for φ throughout $f(\varphi)$.

1.0.6. Inference

All schemata of inferences in Gentzen's LK with the restriction that, in applications of them, as their *upper sequences and lower sequences only those defined in* **1.0.4.** are admitted. (Cf. Gentzen [**1**]. See also Takeuti [**6**], for the names of inferences.)

The following schemata of inferences are added:

$$\forall\varphi\begin{cases}\text{left:} & \dfrac{\mathfrak{F}(t), \Gamma \to \Delta}{\forall\varphi\mathfrak{F}(\varphi), \Gamma \to \Delta} \\ \text{right:} & \dfrac{\Gamma \to \Delta, \mathfrak{F}(\alpha)}{\Gamma \to \Delta, \forall\varphi\mathfrak{F}(\varphi)}\end{cases} \qquad \exists\varphi\begin{cases}\text{right:} & \dfrac{\Gamma \to \Delta, \mathfrak{F}(t)}{\Gamma \to \Delta, \exists\varphi\mathfrak{F}(\varphi)} \\ \text{left:} & \dfrac{\mathfrak{F}(\alpha), \Gamma \to \Delta}{\exists\varphi\mathfrak{F}(\varphi), \Gamma \to \Delta}\end{cases},$$

where φ is an arbitrary bound variable and t and α are an arbitrary term and an arbitrary free variable of the same type as φ, respectively, with the restriction that α does not appear in the lower sequences.

1.0.7. Proof-figure

We use the proof-figure in tree form whose uppermost sequences are beginning sequences defined in **1.0.5.** We use the terminology 'provable' as usual.

1.1. System SJ

This system is obtained by adding the inferences represented by the following schemata to the above fundamental system:

1) a. c. (axiom of choice):

$$\frac{\Gamma \to \Delta, \forall\varphi\exists\psi\mathfrak{F}(\varphi, \psi)}{\Gamma \to \Delta, \exists\chi\forall\varphi\mathfrak{F}(\varphi, \chi(\varphi))};$$

2) $\exists\supset$:

$$\frac{\Gamma \to \Delta, \mathfrak{A} \supset \exists\varphi\mathfrak{F}(\varphi)}{\Gamma \to \Delta, \exists\varphi(\mathfrak{A} \supset \mathfrak{F}(\varphi))},$$

where $\mathfrak{A}$ is a universal prenex formula;

3) $\exists\neg$:

$$\frac{\Gamma \to \Delta, \neg\forall\varphi\mathfrak{A}(\varphi)}{\Gamma \to \Delta, \exists\varphi\neg\mathfrak{A}(\varphi)},$$

where $\mathfrak{A}$ is a universal prenex formula;

4) s-ind.[1] (strong induction):

$$\frac{\Gamma_1 \to \Delta_1, \mathfrak{F}(0, \mathfrak{b}) \quad \mathfrak{F}(a, \mathfrak{r}\langle a, \mathfrak{b}\rangle), \Gamma_2 \to \Delta_2, \mathfrak{F}(a', \mathfrak{b})}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, \mathfrak{F}(s, \mathfrak{t})},$$

---

1) Our form of the inference of mathematical induction is equivalent to the usual one. We take the present form according to Kreisel [**4**]. And see also an appendix.

where a is a free variable of type 0 and $\mathfrak{b}$ is a series of free variables, respectively, and a and $\mathfrak{b}$ do not occur in the lower sequence, s is an arbitrary term of type 0 and $\mathfrak{t}$ is a series of arbitrary terms of appropriate types. $\mathfrak{r}$ is a series of arbitrary terms of appropriate types, containing neither a nor $\mathfrak{b}$.

1.2. System *WJ*

This system is obtained by adding the inferences named 'w-ind. (weak induction)' to the fundamental system. A w-ind. is an s-ind. containing no quantifiers.

1.3. System *QF*

This system is obtained from *WJ* by the following restriction:

The formulas of *QF* are all quantifier-free.

The consistency of *QF* can be reduced to that of the calculation procedure of primitive recursive functionals in Gödel's sense [2].

§ 2. Gödel's interpretation

2.1. Gödel's interpretation of formula (cf. Gödel [2]).

To each formula of *SJ* (or of *WJ*) we associate, as follows, a formula of the form

$$\exists\varphi_1 \cdots \exists\varphi_m \forall\psi_1 \cdots \forall\psi_n A(\varphi_1, \cdots, \varphi_m, \psi_1, \cdots, \psi_n)\,;\ m, n \geqq 0$$

or, for short, $\exists\mathfrak{x}\forall\mathfrak{y}A(\mathfrak{x}, \mathfrak{y})$, where $A(\mathfrak{x}, \mathfrak{y})$ is quantifier-free. We call it *Gödel's interpretation* of the original formula $\mathfrak{A}$ and write it as $\mathfrak{A}^G$.

1) If $\mathfrak{A}$ is quantifier-free, then $\mathfrak{A}^G$ is $\mathfrak{A}$.

2) If $\mathfrak{A}^G$ is $\exists\mathfrak{x}\forall\mathfrak{y}A(\mathfrak{x}, \mathfrak{y})$ and $A(\mathfrak{x}, \mathfrak{y})$ is quantifier-free, then $(\neg\mathfrak{A})^G$ is $\exists\mathfrak{z}\forall\mathfrak{x}\neg A(\mathfrak{x}, \mathfrak{z}\langle\mathfrak{x}\rangle)$.

3) If $\mathfrak{A}^G$ is $\exists\mathfrak{x}\forall\mathfrak{y}A(\mathfrak{x}, \mathfrak{y})$, $\mathfrak{B}^G$ is $\exists\mathfrak{x}'\forall\mathfrak{y}'B(\mathfrak{x}', \mathfrak{y}')$, and $A(\mathfrak{x}, \mathfrak{y})$ and $B(\mathfrak{x}', \mathfrak{y}')$ are quantifier-free, then $(\mathfrak{A}\wedge\mathfrak{B})^G$, $(\mathfrak{A}\vee\mathfrak{B})^G$ and $(\mathfrak{A}\supset\mathfrak{B})^G$ are

$$\exists\mathfrak{x}\exists\mathfrak{x}'\forall\mathfrak{y}\forall\mathfrak{y}'(A(\mathfrak{x}, \mathfrak{y}) \wedge B(\mathfrak{x}', \mathfrak{y}'))\,,$$

$$\exists x\exists\mathfrak{x}\exists\mathfrak{x}'\forall\mathfrak{y}\forall\mathfrak{y}'((x=0 \wedge A(\mathfrak{x}, \mathfrak{y})) \vee (x \neq 0 \wedge B(\mathfrak{x}', \mathfrak{y}')))$$

and

$$\exists\mathfrak{u}\exists\mathfrak{z}\forall\mathfrak{x}\forall\mathfrak{y}'(A(\mathfrak{x}, \mathfrak{z}\langle\mathfrak{x}, \mathfrak{y}'\rangle) \supset B(\mathfrak{u}\langle\mathfrak{x}\rangle, \mathfrak{y}'))\,,$$

respectively.

4) If $\mathfrak{F}(\alpha)^G$ is $\exists\mathfrak{x}\forall\mathfrak{y}F(\alpha, \mathfrak{x}, \mathfrak{y})$ and $F(\alpha, \mathfrak{x}, \mathfrak{y})$ is quantifier-free, then $(\forall\varphi\mathfrak{F}(\varphi))^G$ and $(\exists\varphi\mathfrak{F}(\varphi))^G$ are

$$\exists\mathfrak{z}\forall\varphi\forall\mathfrak{y}F(\varphi, \mathfrak{z}\langle\varphi\rangle, \mathfrak{y}) \quad \text{and} \quad \exists\varphi\exists\mathfrak{x}\forall\mathfrak{y}F(\varphi, \mathfrak{x}, \mathfrak{y})\,,$$

respectively.

Especially, formulas expressed as $\mathfrak{A}\vee\mathfrak{B}$ in 3) should *not* be quantifier-free.

THEOREM 1. *Gödel's interpretation of a formula is SJ-equivalent to the original formula, i.e. for each formula* $\mathfrak{D}$ *the formula*

$$\mathfrak{D}^G \sim \mathfrak{D}$$

is provable in SJ.

The proof of this theorem will be carried out by the mathematical induction on the number of the logical symbols in $\mathfrak{D}$.

0) The case where $\mathfrak{D}$ is quantifier-free. The theorem holds evidently.

1) The case where the outermost logical symbol of $\mathfrak{D}$ is $\supset$. Let $\mathfrak{D}$ be $\mathfrak{A} \supset \mathfrak{B}$ and let $\mathfrak{A}^G$ and $\mathfrak{B}^G$ be $\exists \mathfrak{x} \forall \mathfrak{y} A(\mathfrak{x}, \mathfrak{y})$ and $\exists \mathfrak{x}' \forall \mathfrak{y}' B(\mathfrak{x}', \mathfrak{y}')$, respectively. We can see that $(\mathfrak{A} \supset \mathfrak{B})^G$ is *SJ*-equivalent to $\mathfrak{A}^G \supset \mathfrak{B}^G$ by the *SJ*-equivalence of the following formulas, which is almost clear from the fact that $A(\mathfrak{x}, \mathfrak{y})$ and $B(\mathfrak{x}', \mathfrak{y}')$ are quantifier-free:

$$\exists \mathfrak{x} \forall \mathfrak{y} A(\mathfrak{x}, \mathfrak{y}) \supset \exists \mathfrak{x}' \forall \mathfrak{y}' B(\mathfrak{x}', \mathfrak{y}'),$$
$$\forall \mathfrak{x} (\forall \mathfrak{y} A(\mathfrak{x}, \mathfrak{y}) \supset \exists \mathfrak{x}' \forall \mathfrak{y}' B(\mathfrak{x}', \mathfrak{y}')),$$
$$\forall \mathfrak{x} \exists \mathfrak{x}' (\forall \mathfrak{y} A(\mathfrak{x}, \mathfrak{y}) \supset \forall \mathfrak{y}' B(\mathfrak{x}', \mathfrak{y}')),$$
$$\forall \mathfrak{x} \exists \mathfrak{x}' \forall \mathfrak{y}' (\forall \mathfrak{y} A(\mathfrak{x}, \mathfrak{y}) \supset B(\mathfrak{x}', \mathfrak{y}')),$$
$$\forall \mathfrak{x} \exists \mathfrak{x}' \forall \mathfrak{y}' \exists \mathfrak{y} (A(\mathfrak{x}, \mathfrak{y}) \supset B(\mathfrak{x}', \mathfrak{y}')),$$
$$\forall \mathfrak{x} \exists \mathfrak{x}' \exists \mathfrak{v} \forall \mathfrak{y}' (A(\mathfrak{x}, \mathfrak{v}\langle \mathfrak{y}' \rangle) \supset B(\mathfrak{x}', \mathfrak{y}')),$$
$$\exists \mathfrak{z} \exists \mathfrak{u} \forall \mathfrak{x} \forall \mathfrak{y}' (A(\mathfrak{x}, \mathfrak{u}\langle \mathfrak{x}, \mathfrak{y}' \rangle) \supset B(\mathfrak{z}\langle \mathfrak{x} \rangle, \mathfrak{y}')).$$

By the induction hypothesis, $\mathfrak{A}^G \supset \mathfrak{B}^G$ is *SJ*-equivalent to $\mathfrak{A} \supset \mathfrak{B}$, so is the formula $(\mathfrak{A} \supset \mathfrak{B})^G$.

2) The case where the outermost logical symbol of $\mathfrak{D}$ is $\neg$. The treatment is similar to 1).

3) The other cases. The treatments are almost clear.

2.2. Gödel's interpretation of sequence

Gödel's interpretation of a sequence

$$\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_m \rightarrow \mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_n \qquad (m, n \geqq 0)$$

is defined to be the sequence

$$\mathfrak{A}_1^G, \mathfrak{A}_2^G, \cdots, \mathfrak{A}_m^G \rightarrow \mathfrak{B}_1^G, \mathfrak{B}_2^G, \cdots, \mathfrak{B}_n^G.$$

§ 3. Main theorems

DEFINITION. Let

$$\exists \mathfrak{x}_1 \forall \mathfrak{y}_1 A_1(\mathfrak{x}_1, \mathfrak{y}_1), \cdots, \exists \mathfrak{x}_m \forall \mathfrak{y}_m A(\mathfrak{x}_m, \mathfrak{y}_m) \rightarrow \Theta, \exists \mathfrak{u} \forall \mathfrak{v} B(\mathfrak{u}, \mathfrak{v}), \Lambda$$

be Gödel's interpretation of an arbitrary sequence $\mathfrak{S}$ of *SJ*, and Θ and Λ be the series of some quantifier-free formulas (the formula $\exists \mathfrak{u} \forall \mathfrak{v} B(\mathfrak{u}, \mathfrak{v})$ may also be quantifier-free). Let $\mathfrak{s}_1, \cdots, \mathfrak{s}_m$ and $\mathfrak{t}$ be $m+1$ finite series of arbitrary terms, respectively, of appropriate types which do not contain any free variable except ones occurring in $\mathfrak{S}, \mathfrak{f}_1, \cdots, \mathfrak{f}_m$ and $\mathfrak{g}$ be the series of free variables, respectively, of appropriate types, not contained in $\mathfrak{S}$ and $\mathfrak{f}$ be the series of all free variables belonging to any one of $\mathfrak{f}_1, \cdots, \mathfrak{f}_m$. Then we call any sequence of the form

$$(1) \qquad A_1(\mathfrak{f}_1, \mathfrak{s}_1\langle \mathfrak{f}, \mathfrak{g}\rangle), \cdots, A_m(\mathfrak{f}_m, \mathfrak{s}_m\langle \mathfrak{f}, \mathfrak{g}\rangle) \to \Theta, B(\mathfrak{t}\langle \mathfrak{f}\rangle, \mathfrak{g}), \Lambda$$

a *quantifier-free reduction* of $\mathfrak{S}$.

THEOREM 2. *If a sequence $\mathfrak{S}$ is SJ-provable, then there exists at least one QF-provable quantifier-free reduction of $\mathfrak{S}$.*

COROLLARY. *Let the quantifier-free reduction in Theorem 2 be of the form (1). If $\mathfrak{S}$ does not contain free variables, then $\mathfrak{s}_1, \cdots, \mathfrak{s}_m$ and $\mathfrak{t}$ are the series of* prfs, *respectively.*

PROOF OF THEOREM 2. Mathematical induction on the number of inferences in the proof-figure of $\mathfrak{S}$[2].

1. When $\mathfrak{S}$ is a beginning sequence, the quantifier-free reduction of $\mathfrak{S}$ is $\mathfrak{S}$ itself, and it is QF-provable, i.e. it is also a beginning sequence of QF.

2. Induction steps:

Case 1: $\mathfrak{S}$ is the lower sequence of a contraction:

$$\frac{\mathfrak{D}, \mathfrak{D}, \Gamma \to \Delta}{\mathfrak{D}, \Gamma \to \Delta}.$$

Let $\mathfrak{D}^G$ be $\exists\mathfrak{x}\forall\mathfrak{y}D(\mathfrak{x}, \mathfrak{y})$ and $\mathfrak{f}$ and $\mathfrak{h}$ be series of free variables, respectively. By the induction hypothesis, a quantifier-free reduction of the upper sequence,

$$(2) \qquad D(\mathfrak{f}, \mathfrak{s}_1\langle \mathfrak{f}, \mathfrak{h}\rangle), D(\mathfrak{f}, \mathfrak{s}_2\langle \mathfrak{f}, \mathfrak{h}\rangle), \Gamma^* \to \Delta^*,$$

is QF-provable. We can show that there is a series of terms $\mathfrak{r}$ such that the sequences

$$(3) \qquad \neg D(\mathfrak{f}, \mathfrak{s}_1\langle \mathfrak{f}, \mathfrak{h}\rangle) \to D(\mathfrak{f}, \mathfrak{r}\langle \mathfrak{f}, \mathfrak{h}\rangle) \sim D(\mathfrak{f}, \mathfrak{s}_1\langle \mathfrak{f}, \mathfrak{h}\rangle),$$

and

$$(4) \qquad D(\mathfrak{f}, \mathfrak{s}_1\langle \mathfrak{f}, \mathfrak{h}\rangle) \to D(\mathfrak{f}, \mathfrak{r}\langle \mathfrak{f}, \mathfrak{h}\rangle) \sim D(\mathfrak{f}, \mathfrak{s}_2\langle \mathfrak{f}, \mathfrak{h}\rangle)$$

are both QF-provable. From (2), (3) and (4) we can show that the sequence

$$D(\mathfrak{f}, \mathfrak{r}\langle \mathfrak{f}, \mathfrak{h}\rangle), \Gamma^* \to \Delta^*,$$

which is a quantifier-free reduction of $\mathfrak{S}$, is QF-provable.

Case 2: $\mathfrak{S}$ is the lower sequence of a cut:

$$\frac{\Gamma_1 \to \Delta_1, \mathfrak{D} \quad \mathfrak{D}, \Gamma_2 \to \Delta_2}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2}.$$

By the induction hypothesis, quantifier-free reductions of the upper sequences,

2) It is sufficient to prove the theorem for the cases in which the terms of $\mathfrak{s}_1, \cdots, \mathfrak{s}_m$ and $\mathfrak{t}$ may contain some free variables not occurring in $\mathfrak{S}$.

(5) $$\Gamma_1^* \to \Delta_1^*, D(\mathfrak{s}_1\langle\mathfrak{h}\rangle, \mathfrak{g})$$

and

(6) $$D(\mathfrak{f}, \mathfrak{s}_2\langle\mathfrak{f}, \mathfrak{h}'\rangle), \Gamma_2^* \to \Delta_2^*,$$

are QF-provable. As $\mathfrak{s}_1\langle\mathfrak{h}\rangle$ and $\mathfrak{f}$, and $\mathfrak{s}_2\langle\mathfrak{f}, \mathfrak{h}'\rangle$ and $\mathfrak{g}$ are of the same types, respectively, we can substitute $\mathfrak{s}_1\langle\mathfrak{h}\rangle$ for $\mathfrak{f}$ and $\mathfrak{s}_2\langle\mathfrak{f}, \mathfrak{h}'\rangle$ for $\mathfrak{g}$. The resulting sequences

(5′) $$\Gamma_1^{**} \to \Delta_1^{**}, D(\mathfrak{s}_1\langle\mathfrak{q}\rangle, \mathfrak{s}_2\langle\mathfrak{s}_1\langle\mathfrak{h}\rangle, \mathfrak{h}'\rangle)$$

and

(6′) $$D(\mathfrak{s}_1\langle\mathfrak{h}\rangle, \mathfrak{s}_2\langle\mathfrak{s}_1\langle\mathfrak{q}\rangle, \mathfrak{h}'\rangle), \Gamma_2^{**} \to \Delta_2^{**}$$

are QF-provable. By (5′) and (6′), we obtain a quantifier-free reduction of $\mathfrak{S}$, which is QF-provable.

Case 3: $\mathfrak{S}$ is the lower sequence of an $\forall\varphi$, left:

$$\frac{\mathfrak{F}(t), \Gamma \to \Delta}{\forall\varphi\mathfrak{F}(\varphi), \Gamma \to \Delta}.$$

Let $\mathfrak{F}(\alpha)^G$ be $\exists\mathfrak{x}\forall\mathfrak{y}\mathfrak{F}(\alpha, \mathfrak{x}, \mathfrak{y})$, where α is a free variable of the same type as t. By the induction hypothesis a quantifier-free reduction of the upper sequence,

(7) $$F(t, \mathfrak{f}, \mathfrak{s}\langle\mathfrak{f}, \mathfrak{h}\rangle), \Gamma^* \to \Delta^*,$$

is QF-provable. Then the sequence

(7′) $$F(t, \mathfrak{g}\langle t\rangle, \mathfrak{s}\langle\mathfrak{g}\langle t\rangle, \mathfrak{h}\rangle), \Gamma^{**} \to \Delta^{**},$$

is QF-provable, where $\mathfrak{g}$ is an arbitrary series of free variables not contained in (7), of appropriate types. From (7′) we have a QF-provabble quantifier-free reduction of $\mathfrak{S}$,

$$F(\lambda\mathfrak{u}\lambda\mathfrak{v}t\langle\mathfrak{g}, \mathfrak{h}\rangle, \mathfrak{g}\langle\lambda\mathfrak{u}\lambda\mathfrak{v}t\langle\mathfrak{g}, \mathfrak{h}\rangle\rangle, \mathfrak{s}\langle\mathfrak{g}\langle\lambda\mathfrak{u}\lambda\mathfrak{v}t\langle\mathfrak{g}, \mathfrak{h}\rangle\rangle, \mathfrak{h}\rangle), \Gamma^{**} \to \Delta^{**}.$$

Case 4: $\mathfrak{S}$ is the lower sequence of an $\forall\varphi$, right:

$$\frac{\Gamma \to \Delta, \mathfrak{F}(\alpha)}{\Gamma \to \Delta, \forall\varphi\mathfrak{F}(\varphi)}.$$

By the induction hypothesis a quantifier-free reduction of the upper sequence,

(8) $$\Gamma^* \to \Delta, F(\alpha, \mathfrak{s}(\alpha)\langle\mathfrak{h}\rangle, \mathfrak{g}),$$

is QF-provable. Write $\lambda\mathfrak{u}\varphi(\mathfrak{s}(\varphi)\langle\mathfrak{u}\rangle)$ as $\mathfrak{s}_0$ (if $\mathfrak{s}$ expresses the series of terms $s_1, \cdots, s_k$, then we use the abbreviated notation $\lambda\mathfrak{u}\varphi(\mathfrak{s}(\varphi)\langle\mathfrak{u}\rangle)$ for the series of terms $\lambda\mathfrak{u}\lambda\varphi(s_1(\varphi)\langle\mathfrak{u}\rangle), \cdots, \lambda\mathfrak{u}\lambda\varphi(s_k(\varphi)\langle\mathfrak{u}\rangle)$). Then the quantifier-free reduction of $\mathfrak{S}$,

$$\Gamma^* \to \Delta, F(\alpha, \mathfrak{s}_0\langle\mathfrak{h}, \alpha\rangle, \mathfrak{g}),$$

is QF-provable.

Case 5: $\mathfrak{S}$ is the lower sequence of an a.c., an $\exists\supset$ or an $\exists\neg$.

As Gödel's interpretations of the upper sequences and of the lower sequences in these cases have just the same forms, respectively, we can take the same terms with those assumed for the upper sequences.

Case 6: $\mathfrak{S}$ is the lower sequence of an s-ind.:

$$\frac{\Gamma_1 \rightarrow \varDelta_1, \mathfrak{F}(0, \mathfrak{b}) \quad \mathfrak{F}(a, \mathfrak{r}\langle a, \mathfrak{b}\rangle), \Gamma_2 \rightarrow \varDelta_2, \mathfrak{F}(a', \mathfrak{b})}{\Gamma_1, \Gamma_2 \rightarrow \varDelta_1, \varDelta_2, \mathfrak{F}(s, \mathfrak{t})}.$$

Let $\mathfrak{F}(a, \mathfrak{b})^G$ be $\exists\mathfrak{x}\forall\mathfrak{y}F(a, \mathfrak{b}, \mathfrak{x}, \mathfrak{y})$. By the induction hypothesis, quantifier-free reductions of the upper sequences

$$\Gamma_1^* \rightarrow \varDelta_1^*, F(0, \mathfrak{b}, \mathfrak{s}_1(\mathfrak{b})\langle\mathfrak{h}\rangle, \mathfrak{g}) \tag{9}$$

and

$$F(a, \mathfrak{r}\langle a, \mathfrak{b}\rangle, \mathfrak{f}, \mathfrak{s}_2(a, \mathfrak{b})\langle\mathfrak{f}, \mathfrak{h}_1, \mathfrak{g}\rangle), \Gamma_2^* \rightarrow \varDelta_2^*, F(a', \mathfrak{b}, \mathfrak{s}_3(a, \mathfrak{b})\langle\mathfrak{f}, \mathfrak{h}_1\rangle, \mathfrak{g}) \tag{10}$$

are QF-provable. We write the series of terms

$$\rho[\lambda\mathfrak{u}\mathfrak{v}\mathfrak{w}(\mathfrak{s}_1(\mathfrak{u})\langle\mathfrak{v}\rangle), \lambda x\mathfrak{z}\mathfrak{u}\mathfrak{v}\mathfrak{w}(\mathfrak{s}_3(x, \mathfrak{u})\langle\mathfrak{z}\langle\mathfrak{r}\langle x, \mathfrak{u}\rangle, \mathfrak{v}, \mathfrak{w}\rangle, \mathfrak{w}\rangle)]$$

as $\mathfrak{t}_0$,

$$\lambda x\mathfrak{u}\mathfrak{y}(\mathfrak{t}_0\langle x, \mathfrak{u}, \mathfrak{h}, \mathfrak{h}_1\rangle)$$

as $\tilde{\mathfrak{t}}_0$,

$$\lambda x\mathfrak{u}\mathfrak{y}(\mathfrak{r}\langle x, \mathfrak{u}\rangle)$$

as $\mathfrak{r}_0$,

$$\lambda x\mathfrak{u}\mathfrak{v}\mathfrak{w}\mathfrak{y}(\mathfrak{s}_2(x, \mathfrak{u})\langle\mathfrak{t}^0\langle x, \mathfrak{r}\langle x, \mathfrak{u}\rangle, \mathfrak{v}, \mathfrak{w}\rangle, \mathfrak{w}, \mathfrak{y}\rangle)$$

as $\mathfrak{s}_0$, and

$$\lambda x\mathfrak{u}\mathfrak{y}(\mathfrak{s}_0\langle x, \mathfrak{u}, \mathfrak{h}, \mathfrak{h}_1, \mathfrak{y}\rangle)$$

as $\tilde{\mathfrak{s}}_0$. Substitute $\tilde{\mathfrak{t}}_0\langle a, \mathfrak{r}\langle a, \mathfrak{b}\rangle, \mathfrak{g}\rangle$ for $\mathfrak{f}$ of the sequence in (10). Then we see from (9) and (10) that the sequences

$$\Gamma_1^* \rightarrow \varDelta_1^*, F(0, \mathfrak{b}, \tilde{\mathfrak{t}}_0\langle 0, \mathfrak{b}, \mathfrak{g}\rangle, \mathfrak{g}) \tag{11}$$

and

$$F(a, \mathfrak{r}_0\langle a, \mathfrak{b}, \mathfrak{g}\rangle, \tilde{\mathfrak{t}}_0\langle a, \mathfrak{r}_0\langle a, \mathfrak{b}, \mathfrak{g}\rangle, \tilde{\mathfrak{s}}_0\langle a, \mathfrak{b}, \mathfrak{g}\rangle\rangle, \tilde{\mathfrak{s}}_0\langle a, \mathfrak{b}, \mathfrak{g}\rangle), \Gamma_2^{**} \rightarrow \varDelta_2^{**}, F(a', \mathfrak{b}, \tilde{\mathfrak{t}}_0\langle a', \mathfrak{b}, \mathfrak{g}\rangle, \mathfrak{g}) \tag{12}$$

are QF-provable. Taking (11) and (12) as the upper sequences of a w-ind., the sequence

$$\Gamma_1^*, \Gamma_2^{**} \rightarrow \varDelta_1^*, \varDelta_2^{**}, F(s, \mathfrak{t}, \tilde{\mathfrak{t}}_0\langle s, \mathfrak{t}, \mathfrak{g}\rangle, \mathfrak{g})$$

is QF-provable, from which we have a quantifier-free reduction of $\mathfrak{S}$,

$$\Gamma_1^*, \tilde{\Gamma}_2 \rightarrow \varDelta_1^*, \tilde{\varDelta}_2, F(s, \mathfrak{t}, \mathfrak{t}_0\langle s, \mathfrak{t}, \mathfrak{h}, \mathfrak{h}_1\rangle, \mathfrak{g}\rangle,$$

which is QF-provable.

The other cases are treated similarly.

Thus Theorem 2 is proved.

THEOREM 3. *SJ is consistent.*

PROOF. If the empty sequence $\rightarrow$ were SJ-provable, then it should be QF-provable by Theorem 2, contradicting the consistency of QF (cf. **1.3.**).

THEOREM 4. *If one of the quantifier-free reductions of* $\mathfrak{S}$ *is QF-provable, then* $\mathfrak{S}^G$ *is WJ-provable.*

PROOF. Note that if a quantifier-free reduction of $\mathfrak{S}$ is QF-provable, then it is WJ-provable and also that $\mathfrak{f}_1, \cdots, \mathfrak{f}_m$ and $\mathfrak{g}$ in Theorem 2 are different from those free variables which appear in $\mathfrak{S}$. Then the proof goes quite easily.

THEOREM 5. *If* $\mathfrak{S}^G$ *is WJ-provable, then* $\mathfrak{S}$ *is SJ-provable.*

This is obvious by Theorem 1.

THEOREM 6. *If* $\mathfrak{S}^G$ *is WJ-provable, then there exists a quantifier-free reduction of* $\mathfrak{S}$ *which is QF-provable.*

This theorem is obvious by Theorems 5 and 2, but it is also proved directly using the following Theorem:

Any proof-figure of WJ can be transformed to a normal form in which any cut inference has a quantifier-free formula as the cut-formula.

Now, by this theorem, a proof-figure of $\mathfrak{S}^G$ can be transformed to a normal form, in which all inferences of propositional calculus used concern solely with the quantifier-free formulas. By these arguments, the proof of Theorem 6 can be carried out by the mathematical induction on the number of inferences occurring in the normal proof-figure of $\mathfrak{S}^G$.

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Appendix

PROOF of the proposition that a w-ind. is equivalent in QF to an ordinary induction,

$$\frac{\Gamma_1 \rightarrow \Delta_1, \mathfrak{G}(0) \quad \mathfrak{G}(a), \Gamma_2 \rightarrow \Delta_2, \mathfrak{G}(a')}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, \mathfrak{G}(s)},$$

where the free variable a, of type 0, does not occur in the lower sequence and s is an arbitrary term of type 0. The same holds in WJ (and also in SJ).

It is clear that an ordinary induction is a special w-ind. So we will prove the converse. Consider a w-ind. of the form

$$\frac{\Gamma_1 \rightarrow \Delta_1, \mathfrak{F}(0, \mathfrak{b}), \quad F(a, \mathfrak{r}\langle a, \mathfrak{b}\rangle), \Gamma_2 \rightarrow \Delta_2, \mathfrak{F}(a', \mathfrak{b})}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, \mathfrak{F}(s, \mathfrak{t})},$$

and assume that the upper sequences

(1) $$\Gamma_1 \rightarrow \Delta_1, \mathfrak{F}(0, \mathfrak{b})$$

and

(2) $$\mathfrak{F}(a, \mathfrak{r}\langle a, \mathfrak{b}\rangle), \Gamma_2 \rightarrow \Delta_2, \mathfrak{F}(a', \mathfrak{b})$$

are QF-provable. Write the term

$$\lambda y(\rho[t, \lambda x \lambda \mathfrak{u} \mathfrak{r}\langle s \dot{-} x', \mathfrak{u}\rangle]\langle s \dot{-} y\rangle)$$

as $\mathfrak{p}$, where $s - y$ means $s-y$ if $s \geqq y$ and 0 if $s < y$, informally (cf. [3]). Then the sequences obtained from (1) and (2) by substituting $\mathfrak{p}\langle 0\rangle$ and $\mathfrak{p}\langle a'\rangle$ for $\mathfrak{b}$, respectively, i. e.,

(3) $$\Gamma_1 \rightarrow \Delta_1, \mathfrak{F}(0, \mathfrak{p}\langle 0\rangle)$$

and

(4) $$\mathfrak{F}(a, \mathfrak{r}\langle a, \mathfrak{p}\langle a'\rangle\rangle), \Gamma_2 \rightarrow \Delta_2, \mathfrak{F}(a', \mathfrak{p}\langle a'\rangle)$$

are QF-provable. By (4) and the QF-provable sequence

$$a' \leqq s \rightarrow \mathfrak{p}\langle a\rangle = \mathfrak{r}\langle a, \mathfrak{p}\langle a'\rangle\rangle$$

we see that the sequence

(5) $$a' \leqq s, \mathfrak{F}(a, \mathfrak{p}\langle a\rangle), \Gamma_2 \rightarrow \Delta_2, \mathfrak{F}(a', \mathfrak{p}\langle a'\rangle)$$

is QF-provable. By (3) and (5) we have that the sequences

(6) $$\Gamma_1 \rightarrow \Delta_1, 0 \leqq s \supset \mathfrak{F}(0, \mathfrak{p}\langle 0\rangle)$$

and

(7) $$a \leqq s \supset \mathfrak{F}(a, \mathfrak{p}\langle a\rangle), \Gamma_2 \rightarrow \Delta_2, a' \leqq s \supset \mathfrak{F}(a', \mathfrak{p}\langle a'\rangle)$$

are QF-provable, respectively. Taking the sequences (6) and (7) as the upper sequences, we have that, by an ordinary induction, the sequence

(8) $$\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, s \leqq s \supset \mathfrak{F}(s, \mathfrak{p}\langle s\rangle)$$

is QF-provable. Then by (8) and the QF-provable sequence

$$\rightarrow \mathfrak{p}\langle s\rangle = \mathfrak{t},$$

the lower sequence of the w-ind. above,

$$\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, \mathfrak{F}(s, \mathfrak{t})$$

is QF-provable.

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