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Some remarks to the preceding paper of Tsukamoto

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We are going to supplement the preceding paper of Tsukamoto (referred as [T]) in the following two points. In the first place, he has considered exclusively the anti-hermitian forms over a quaternion *division* algebra. For applications, however, it is equally necessary to consider the case where the quaternion algebra splits over k. This case will be treated in N^0 **1** of this paper. In the second place, if G is the group of all automorphisms of an anti-hermitian space V over $\mathfrak{D}$ (division), it is known that V is anisotropic, if and only if G (viewed as a linear algebraic group over k) has no 'unipotent' element, and in particular in the case of local fields, if and only if G (viewed as a topological group with respect to the natural topology) is compact (cf. [T, Theorem 7]). We shall show in N^0 **2-4** that in the $\mathfrak{p}$-adic case (Case II in [T]) all the groups G corresponding to the anisotropic cases (listed in [T, Theorem 3]) come from certain division algebras over k. More precisely, it will be shown, by virtue of the well-known isomorphisms between classical groups, that such a group G is always isogeneous to a multiplicative group $\mathfrak{K}^{(1)}$ consisting of the elements of reduced norm 1 in a certain division algebra $\mathfrak{K}$ over k. The corresponding phenomena for other classical groups are well-known or easily reduced to the known case. Throughout the paper, the notation and the terminology in [T] will be used freely.

1. In this paragraph, we assume that $\mathfrak{D}$ is a splitting quaternion algebra over k and fix once for all an isomorphism $i: \mathfrak{D} \rightarrow M_2(k)$. It is clear that if $i(\xi)=\begin{pmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{pmatrix}$, we have

$$i(\bar{\xi})=\begin{pmatrix}\xi_{22} & -\xi_{12}\\ -\xi_{21} & \xi_{11}\end{pmatrix}=J{}^t i(\xi)J^{-1}, \qquad J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{1}$$

Let ε_{ij} $(i,j=1,2)$ denote the matrix units in $\mathfrak{D}$. Suppose that an n-dimensional vector space V over $\mathfrak{D}$ (i. e. a $\mathfrak{D}$-module with a basis consisting of n elements) is given. If we put

$$V'=V\varepsilon_{11}, \qquad V''=V\varepsilon_{22},$$

it is clear that V', V'' are $2n$-dimensional vector subspaces of V over k such that

$$V = V' + V'' \qquad \text{(direct sum)} \tag{2}$$

and that the mapping $\varphi: x' \in V' \to x'' = x'\varepsilon_{12} \in V''$ is a linear isomorphism over k from V' onto V''. Conversely, let V be a $4n$-dimensional vector space over k, V', V'' $2n$-dimensional vector subspaces of V over k such that (2) holds and let φ be a linear isomorphism over k from V' onto V''. Then, defining the (right) operations of $\mathfrak{D}$ on V by

$$x\varepsilon_{11} = x', \qquad x\varepsilon_{12} = \varphi(x'),$$
$$x\varepsilon_{21} = \varphi^{-1}(x''), \qquad x\varepsilon_{22} = x''$$

for $x = x' + x''$ with $x' \in V'$, $x'' \in V''$, one can verify immediately that V becomes an n-dimensional vector space over $\mathfrak{D}$. If $(x_1', \cdots, x_{2n}')$ is a basis of V' over k, $(x_1, \cdots, x_n)$ with $x_i = x_{2i-1}' + \varphi(x_{2i}')$ is a basis of V over $\mathfrak{D}$ and *vice versa*. A linear transformation ρ of V over k is $\mathfrak{D}$-linear, if and only if ρ leaves the decomposition (2) invariant and, denoting by ρ', ρ'' the restrictions of ρ on V', V'', respectively, we have $\rho'' \circ \varphi = \varphi \circ \rho'$. If $X = (\xi_{ij}) \in M_n(\mathfrak{D})$ is the matrix corresponding to a linear transformation ρ of V over $\mathfrak{D}$ in the basis $(x_1, \cdots, x_n)$, then the matrix corresponding to ρ' in the basis $(x_1', \cdots, x_{2n}')$ is given by $i(X) = (i(\xi_{ij}))$, which is a $2n \times 2n$ matrix obtained from X by replacing each element ξ_{ij} by $i(\xi_{ij})$. By definition, the reduced norm (from $M_n(\mathfrak{D})$ to k) $N(X)$ of $X \in M_n(\mathfrak{D})$ is equal to $\det(i(X))$.

Now the definitions of an anti-hermitian form and the associated sesquilinear form given in [T, §1] are valid in our case also. Let H be an anti-hermitian form on V and Φ the associated anti-hermitian sesquilinear form. We can write

$$i(H(x)) = \begin{pmatrix} Q(x) & Q''(x) \\ Q'(x) & -Q(x) \end{pmatrix},$$
$$i(\Phi(x, y)) = \begin{pmatrix} B_1(x, y) & \frac{1}{2}B''(x, y) \\ \frac{1}{2}B'(x, y) & B_2(x, y) \end{pmatrix}. \tag{3}$$

Then it can easily be verified that Q, Q', Q'' are quadratic forms on V over k, that $B_1 - B_2$, B', B'' are symmetric bilinear forms on $V \times V$ associated with Q, Q', Q'', respectively, and that they satisfy the following relations

$$Q'(x) = Q'(x'), \qquad Q''(x) = Q''(x''),$$
$$Q''(x'\varepsilon_{12}) = -Q'(x'),$$
$$Q(x) = -\frac{1}{2}B'(x', x''\varepsilon_{21}), \tag{4}$$
$$B_1(x, y) = -B_2(y, x) = -\frac{1}{2}B'(y', x''\varepsilon_{21})$$

for any $x=x'+x'', y=y'+y''$ with $x', y' \in V', x'', y'' \in V''$. Thus H is uniquely determined by any one of Q, Q', Q''. Conversely, suppose that a quadratic form Q' on V' over k is given. Then, defining Q, Q', Q'', B_1, B_2 by (4) and H, Φ by (3), one can verify immediately that H becomes an anti-hermitian form on V over $\mathfrak{D}$ with the associated sesquilinear form Φ. Thus there exists a one-to-one correspondence between the anti-hermitian forms H on V over $\mathfrak{D}$ and the quadratic forms Q' on V' over k. If we denote again by H, Q' the matrices $(\Phi(x_i, x_j)) \in M_n(\mathfrak{D})$, $\left(\frac{1}{2}B'(x_i', x_j')\right) \in M_{2n}(k)$ corresponding to H, Q', respectively, we have from (3), (4)

$$Q' = (-J\ i(\Phi(x_i, x_j))) = -(J \otimes 1_n) \cdot i(H)^{1)} . \tag{5}$$

It follows also that a linear transformation ρ of V over k is an automorphism of the anti-hermitian space V over $\mathfrak{D}$, if and only if it satisfies the following conditions. Namely, ρ leaves the decomposition (2) invariant and, denoting by ρ', ρ'' the restrictions of ρ on V', V'', respectively, ρ' is an orthogonal transformation of V' with respect to Q' and $\rho'' \circ \varphi = \varphi \circ \rho'$. Thus *the group* G (resp. G^+) *of all automorphisms* (resp. *automorphisms of reduced norm* 1) *of the anti-hermitian space* V (*with* H) *is isomorphic to the orthogonal group* (resp. *the special orthogonal group*) *of the corresponding quadratic space* V' (*with* Q').

2. Now we return to the case where $\mathfrak{D}$ is a division algebra and restrict ourselves to Case II. Our purpose here is to show that the group $G=G_n$ of the anisotropic space of dimension $n=1,2,3$ in [T, Theorem 3] is isogeneous to $\mathfrak{K}^{(1)}$ with a suitable division algebra $\mathfrak{K}$.

First it is trivial that for $V=V(c)$ ($c \nsim 1$) we have

$$G_1 \cong k(\sqrt{c})^{(1)} . \tag{6}$$

Before we enter the considerations on G_2, G_3, we make some preliminary observations. Let $(1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2)$ be a basis of $\mathfrak{D}$ over k such that $\varepsilon_1^2=c_1$, $\varepsilon_2^2=c_2$ with $c_1, c_2 \in k^*$ and $\varepsilon_1\varepsilon_2=-\varepsilon_2\varepsilon_1$. Put

$$K_1 = k(\sqrt{c_1}), \qquad K = k(\sqrt{c_1}, \sqrt{c_2}) .$$

Then, identifying K_1 with the quadratic subfield $k(\varepsilon_1)$ in $\mathfrak{D}$, we may write $\mathfrak{D}=K_1+\varepsilon_2 K_1$. This expression gives the following representation i of $\mathfrak{D}$ into $M_2(K_1)$:

$$i(\xi) = \begin{pmatrix} \xi_0+\xi_1\sqrt{c_1} & c_2(\xi_2+\xi_3\sqrt{c_1}) \\ \xi_2-\xi_3\sqrt{c_1} & \xi_0-\xi_1\sqrt{c_1} \end{pmatrix} \tag{7}$$

for $\xi=\xi_0+\varepsilon_1\xi_1+\varepsilon_2\xi_2+\varepsilon_1\varepsilon_2\xi_3 \in \mathfrak{D}$ with $\xi_i \in k$. The image $i(\mathfrak{D})$ is formed of all the matrices $Y \in M_2(K_1)$ such that

1) 1_n denotes the identity matrix of degree n.

$$\begin{pmatrix}0 & c_2\\ 1 & 0\end{pmatrix}\bar{Y}\begin{pmatrix}0 & c_2\\ 1 & 0\end{pmatrix}^{-1}=Y,$$

i.e. the matrices Y commuting with the following semilinear transformation of $K_1^2=\left\{y=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \mid \eta_1, \eta_2 \in K_1\right\}$:

$$y \to \begin{pmatrix}0 & c_2\\ 1 & 0\end{pmatrix}\bar{y}.$$

Now let K' be any field containing K_1 and let $\mathfrak{D}^{K'}$ denote the algebra over K' obtained from $\mathfrak{D}$ by the scalar extension K'/k. Then $\mathfrak{D}^{K'}$ is a splitting quaternion algebra over K' and the natural extension of i gives an isomorphism $\mathfrak{D}^{K'} \to M_2(K')$. Call further $G_n^{+K'}$ the group formed of all $X \in M_n(\mathfrak{D}^{K'})$ such that ${}^t\bar{X}HX=H$, $N(X)=1$. Then, from what we have stated in N^0 **1**, the restriction on $G_n^{+K'}$ of the isomorphism i:

$$M_n(\mathfrak{D}^{K'}) \ni X=(\xi_{ij}) \to i(X)=(i(\xi_{ij})) \in M_{2n}(K')$$

gives the following isomorphism:

$$G_n^{+K'} \cong O_{2n}^{+}(K', Q'), \tag{8}$$

Q' being given by (5). In view of the fact that $O_{2n}^{+}(K', Q')$ is an irreducible algebraic group, $G_n^{+K'}$ may be regarded as the algebraic group obtained from G_n^{+} by the scalar extension K'/k.

Moreover, it is known that, for $X \in G_n$, the condition $N(X)=1$ is automatically satisfied, so that (on considering only k-rational points) we have $G_n=G_n^{+}$ [2].

3. *The case* $\dim V=3$, $\delta(V)=1$. We choose the basis of $\mathfrak{D}$ in such a way that the condition $c_1c_2 \not\sim 1$ is satisfied, in addition to the usual conditions $c_1 \not\sim 1$, $c_2 \notin N(k(\sqrt{c_1})^*)$. (This is possible, since we are in Case II.) Then we may assume that $V=V(c_1, c_2, c_1c_2)$, i.e. that V has an orthogonal basis (x_1, x_2, x_3) such that

$$\begin{gathered} H(x_1)=\varepsilon_1, \quad H(x_2)=\varepsilon_2, \quad H(x_3)=\gamma=\varepsilon_1\gamma_1+\varepsilon_2\gamma_2+\varepsilon_1\varepsilon_2\gamma_3 \\ \text{with } \gamma^2=c_1\gamma_1^2+c_2\gamma_2^2-c_1c_2\gamma_3^2=c_1c_2. \end{gathered} \tag{9}$$

Then we have from (5), (7)

$$-Q'=\begin{pmatrix} 0 & \sqrt{c_1} & & & & \\ \sqrt{c_1} & 0 & & & & \\ & & -1 & 0 & & \\ & & 0 & c_2 & & \\ & & & & -\gamma_2+\gamma_3\sqrt{c_1} & \gamma_1\sqrt{c_1} \\ & & & & \gamma_1\sqrt{c_1} & c_2(\gamma_2+\gamma_3\sqrt{c_1}) \end{pmatrix},$$

2) See [**3**, p. 197, Lemme 1].

which can also be written in the form $-Q'={}^tPQ_0P$ in $K=k(\sqrt{c_1},\sqrt{c_2})$ with

$$Q_0=\frac{1}{2}\begin{pmatrix}0&1&&&&\\1&0&&&&\\&&0&-1&&\\&&-1&0&&\\&&&&0&1\\&&&&1&0\end{pmatrix},$$

$$P=\begin{pmatrix}1&0&&&&\\0&2\sqrt{c_1}&&&&\\&&1&\sqrt{c_2}&&\\&&1&-\sqrt{c_2}&&\\&&&&-\gamma_2+\gamma_3\sqrt{c_1}&(\gamma_1+\sqrt{c_2})\sqrt{c_1}\\&&&&1&\dfrac{(-\gamma_1+\sqrt{c_2})\sqrt{c_1}}{\gamma_2-\gamma_3\sqrt{c_1}}\end{pmatrix}.$$

Hence from (8) we get an isomorphism

(10) $$G_3^{+K}\cong O_6^+(K,Q_0),$$

given by

$$G_3^{+K}\ni X\to Y=Pi(X)P^{-1}\in O_6^+(K,Q_0).$$

On the other hand, by a canonical isomorphism between classical groups ([4], [6]), we have the isomorphism

(11) $$O_6^+(K,Q_0)\cong\tilde{L}/\tilde{L}_0,$$

where

$$\tilde{L}=\{(\lambda,U)\mid\lambda\in K^*,\ U\in GL_4(K),\ \det U=\lambda^2\},$$

$$\tilde{L}_0=\{(\lambda^2,\lambda 1_4)\mid\lambda\in K^*\}\cong K^*,$$

the mapping from $\tilde{L}$ onto $O_6^+(K,Q_0)$ being given by

$$\tilde{L}\ni(\lambda,U)\to Y=\lambda^{-1}U^{(2)}\in O_6^+(K,Q_0),$$

where $U^{(2)}$ denotes the representation of U by the bivectors, indexed as $(\xi_{12},\xi_{34},\xi_{13},\xi_{24},\xi_{14},\xi_{23})$. $\tilde{L}/\tilde{L}_0$ is clearly a group isogeneous to the special linear group $SL_4(K)$. Combining the two isomorphisms (10), (11), we get an isomorphism f from $\tilde{L}/\tilde{L}_0$ onto G_3^{+K} given by

(12) $$f(\lambda,U)=i^{-1}(\lambda^{-1}P^{-1}U^{(2)}P).$$

Now we have to determine the subgroup of $\tilde{L}/\tilde{L}_0$ corresponding to G_3 itself under the isomorphism (12). Call σ, τ the Galois automorphisms of K/k such that $\sqrt{c_1}^\sigma=-\sqrt{c_1}$, $\sqrt{c_2}^\sigma=\sqrt{c_2}$, $\sqrt{c_1}^\tau=\sqrt{c_1}$, $\sqrt{c_2}^\tau=-\sqrt{c_2}$. Then, for any element ξ in $\mathfrak{D}^K$, we have

$$i(\xi^\sigma)=\begin{pmatrix}0 & c_2\\ 1 & 0\end{pmatrix} i(\xi)^\sigma \begin{pmatrix}0 & c_2\\ 1 & 0\end{pmatrix}^{-1}, \qquad i(\xi^\tau)=i(\xi)^\tau .$$

Therefore, the subgroup of $O_6^+(K, Q_0)$ corresponding to G_3 under the isomorphism (10) is formed of all $Y \in O_6^+(K, Q_0)$ such that

$$C_\sigma Y^\sigma C_\sigma^{-1} = C_\tau Y^\tau C_\tau^{-1} = Y,$$

where

$$C_\sigma = P\begin{pmatrix} 0 & c_2 & & & & \\ 1 & 0 & & & & \\ & & 0 & c_2 & & \\ & & 1 & 0 & & \\ & & & & 0 & c_2 \\ & & & & 1 & 0 \end{pmatrix} P^{-\sigma}$$

$$= \begin{pmatrix} 0 & -c_2/2\sqrt{c_1} & & & & \\ 2\sqrt{c_1} & 0 & & & & \\ & & \sqrt{c_2} & 0 & & \\ & & 0 & -\sqrt{c_2} & & \\ & & & & 0 & \sqrt{c_1}(\gamma_1+\sqrt{c_2}) \\ & & & & -c_2/\sqrt{c_1}(\gamma_1+\sqrt{c_2}) & 0 \end{pmatrix},$$

$$C_\tau = PP^{-\tau} = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & -\gamma_2+\gamma_3\sqrt{c_1} \\ & & & & (-\gamma_2+\gamma_3\sqrt{c_1})^{-1} & 0 \end{pmatrix}.$$

Moreover, according to the general principle yielding the isomorphism (11), the semilinear transformations $y \rightarrow C_\sigma y^\sigma$, $y \rightarrow C_\tau y^\tau$ of K^6 should also come from certain semilinear transformations of K^4. In fact, we can write

$$C_\sigma = -\frac{2\sqrt{c_2}}{\gamma_1+\sqrt{c_2}} D_\sigma^{(2)}, \qquad C_\tau = \frac{1}{\gamma_2-\gamma_3\sqrt{c_1}} D_\tau^{(2)},$$

where

$$D_\sigma = \begin{pmatrix} & & -\frac{\gamma_1+\sqrt{c_2}}{2} & 0 \\ & & 0 & \frac{\sqrt{c_2}}{2\sqrt{c_1}} \\ 1 & 0 & & \\ 0 & \frac{\sqrt{c_1}}{\sqrt{c_2}}(\gamma_1+\sqrt{c_2}) & & \end{pmatrix},$$

$$D_\tau=\begin{pmatrix} 0 & -\gamma_2+\gamma_3\sqrt{c_1} & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & \gamma_2-\gamma_3\sqrt{c_1} & 0 \end{pmatrix}.$$

Thus we see that the semilinear transformations $x\to D_\sigma x^\sigma$, $x\to D_\tau x^\tau$ of K^4 give rise, up to scalar factors, to the above semilinear transformations of K^6. We also get the relations

$$f(\lambda, U)^\sigma=f(\lambda^\sigma, D_\sigma U^\sigma D_\sigma^{-1}),\quad f(\lambda, U)^\tau=f(\lambda^\tau, D_\tau U^\tau D_\tau^{-1}) \tag{13}$$

for all $(\lambda, U)\in\tilde{L}$.

Now, if one considers the subgroup of $GL_4(K)$ formed of all the elements U in $GL_4(K)$ commuting with the semilinear transformations $x\to D_\sigma x^\sigma$, $x\to D_\tau x^\tau$, i.e. such that $D_\sigma U^\sigma D_\sigma^{-1}=D_\tau U^\tau D_\tau^{-1}=U$, it turns out that this group consists of the matrices of the following form

$$\begin{pmatrix} \zeta_0 & -(\gamma_2-\gamma_3\sqrt{c_1})\zeta_1^\tau & -\frac{\gamma_1+\sqrt{c_2}}{2}\zeta_2^\sigma & \frac{\sqrt{c_2}}{2\sqrt{c_1}}\zeta_3^{\sigma\tau} \\ \zeta_1 & \zeta_0^\tau & \frac{\sqrt{c_2}}{2\sqrt{c_1}}\zeta_3^\sigma & -\frac{\gamma_1-\sqrt{c_2}}{2(\gamma_2-\gamma_3\sqrt{c_1})}\zeta_2^{\sigma\tau} \\ \zeta_2 & -\zeta_3^\tau & \zeta_0^\sigma & -\frac{\sqrt{c_2}(\gamma_2+\gamma_3\sqrt{c_1})}{\sqrt{c_1}(\gamma_1+\sqrt{c_2})}\zeta_1^{\sigma\tau} \\ \zeta_3 & (\gamma_2-\gamma_3\sqrt{c_1})\zeta_2^\tau & \frac{\sqrt{c_1}}{\sqrt{c_2}}(\gamma_1+\sqrt{c_2})\zeta_1^\sigma & \zeta_0^{\sigma\tau} \end{pmatrix}. \tag{14}$$

This is nothing other than a representation in K of an element $\zeta=\zeta_0+\omega_1\zeta_1+\omega_2\zeta_2+\omega_3\zeta_3$ in an algebra $\tilde{\mathfrak{D}}$ of dimension 16 over k defined as follows:

$$\tilde{\mathfrak{D}}=K+\omega_1K+\omega_2K+\omega_3K, \tag{15}$$

$$\text{with}\begin{cases} \omega_1^2=-\gamma_2+\gamma_3\sqrt{c_1}, \quad \omega_2^2=-\frac{1}{2}(\gamma_1+\sqrt{c_2}), \\ \omega_2\omega_1=\omega_1\omega_2\frac{\sqrt{c_2}(\gamma_2-\gamma_3\sqrt{c_1})}{\sqrt{c_1}(\gamma_1+\sqrt{c_2})}=\omega_3(\gamma_2-\gamma_3\sqrt{c_1}), \\ \omega_1^{-1}\eta\omega_1=\eta^\tau, \quad \omega_2^{-1}\eta\omega_2=\eta^\sigma \quad \text{for } \eta\in K. \end{cases}$$

Therefore we have $\tilde{\mathfrak{D}}^K=M_4(K)$, and, if we put

$$L=\{(\lambda,\zeta)\mid\lambda\in k^*,\ \zeta\in\tilde{\mathfrak{D}}^*,\ \tilde{n}(\zeta)=\lambda^2\}, \tag{16}$$

$\tilde{n}$ denoting the reduced norm from $\tilde{\mathfrak{D}}$ to k, it is easy to see that $\tilde{L}$ may be regarded as the algebraic group obtained from L by the scalar extension K/k. As the rational homomorphism f (defined over K) commutes with the Galois automorphisms of K/k operating on G_3^{+K} and on $\tilde{L}$, by (13), f is in fact defined over k, and thus maps the set of k-rational points of $\tilde{L}/\tilde{L}_0$ onto G_3. Since $\tilde{L}_0 \cong K^*$, it follows from the Theorem 90 of Hilbert that any k-rational coset modulo $\tilde{L}_0$ contains a k-rational representative. Therefore, putting

$$L_0 = \{(\lambda^2, \lambda) \mid \lambda \in k^*\} \cong k^*,$$

we finally conclude that f induces the isomorphism

$$G_3 \cong L/L_0, \tag{17}$$

which is a rational isomorphism defined over k. The fact that $\tilde{\mathfrak{D}}$ is a division algebra follows either directly or from the fact that $\tilde{\mathfrak{D}}^*$ contains no unipotent element. L/L_0 is clearly isogeneous to $\tilde{\mathfrak{D}}^{(1)} = \{\zeta \in \tilde{\mathfrak{D}} \mid \tilde{n}(\zeta) = 1\}$.

REMARK. In the case $\dim V = 3$, $\delta(V) \not\sim 1$, we can choose the basis of $\mathfrak{D}$ and V in such a way that

$$H = \begin{pmatrix} \varepsilon_1 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}. \tag{18}$$

Then, proceeding quite similarly as above, we conclude that under the isomorphism (12) we have

$$G_3 \cong L/L_0, \tag{19}$$

where

$$L = \{(\lambda, U) \mid \lambda \in k^*,\ U \in GL_4(K_1),\ {}^t\bar{U}DU = \lambda D,\ \det U = \lambda^2\}$$

$$\text{with}\quad D = \begin{pmatrix} 1 & & & \\ & -c_2 & & \\ & & 0 & \frac{c_2}{\sqrt{c_1}} \\ & & -\frac{c_2}{\sqrt{c_1}} & 0 \end{pmatrix},$$

$$L_0 = \{(\lambda^2, \lambda 1_4) \mid \lambda \in k^*\}.$$

4. *The case* $\dim V = 2$, $\delta(V) \not\sim 1$. Taking suitable basis of $\mathfrak{D}$ and V, we may assume, in the notation of N^0 3, that $\delta(V) \sim c_1$, $V = V(c_2, c_1c_2)$ and

$$H = \begin{pmatrix} \varepsilon_2 & 0 \\ 0 & \gamma \end{pmatrix}. \tag{20}$$

Hence G_2 can be identified with the subgroup of G_3 for $V(c_1, c_2, c_1c_2)$ consisting of those elements X which leave x_1 fixed, i.e. of the form

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & * & \end{pmatrix}.$$

Therefore, under the isomorphism (17), we have $f(\lambda, \zeta) \in G_2$ for $(\lambda, \zeta) \in L$, if and only if the matrix (14) corresponding to ζ is of the form

$$U = \begin{pmatrix} \overset{2}{U_1} & \overset{2}{0} \\ 0 & U_2 \end{pmatrix}, \qquad \det U_1 = \det U_2 = \lambda,$$

i.e. if and only if

$$\zeta = \zeta_0 + \omega_1 \zeta_1 \in \mathfrak{D}_1 = K + \omega_1 K,$$

$$\zeta_0^{1+\tau} + (\gamma_2 - \gamma_3 \sqrt{c_1}) \zeta_1^{1+\tau} = \lambda.$$

Here it is clear that $\mathfrak{D}_1$ is a quaternion division algebra over K_1 and that, denoting by n_1 the reduced norm from $\mathfrak{D}_1$ to K_1, we have

$$n_1(\zeta) = \zeta_0^{1+\tau} + (\gamma_2 - \gamma_3 \sqrt{c_1}) \zeta_1^{1+\tau}.$$

Thus we obtain the isomorphism

$$G_2 \cong L/k^*, \qquad L = \{\eta \in \mathfrak{D}_1^* \mid n_1(\eta) \in k^*\}. \tag{21}$$

L/k^* is clearly isogeneous to $\mathfrak{D}_1^{(1)} = \{\eta \in \mathfrak{D}_1 \mid n_1(\eta) = 1\}$.

REMARK. In the case $\dim V = 2$, $\delta(V) \sim 1$, we may assume that

$$H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{22}$$

Then it follows either directly or by a similar argument as above starting from the case $\dim V = 3$, $\delta(V) \nsim 1$, that we have

$$G_2 \cong L/L_0, \qquad L = \{(\xi, Y) \mid \xi \in \mathfrak{D}^*,\ Y \in GL_2(k),\ n(\xi) = \det Y\}, \qquad L_0 = \{(\lambda, \lambda 1_2) \mid \lambda \in k^*\}. \tag{23}$$

L/L_0 is clearly isogeneous to $\mathfrak{D}^{(1)} \times SL_2(k)$.

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Bibliography

[1] N. Bourbaki, Eléments de mathématique, Livre II Algèbre, Chap. 9, Formes sesquilinéaires et formes quadratiques. Hermann, 1959.

[2] J. Dieudonné, Les extensions quadratiques des corps non commutatifs et leurs applications. Acta Math., t. 87 (1952), 175-242.

[3] J. Dieudonné, Sur les groupes unitaires quaternioniques à deux et à trois variables. Bull. des Sci. Math., t. 77 (1953), 195-213.

[4] J. Dieudonné, La géométrie des groupes classiqes. Springer, 1955.

[5] T. Tsukamoto, On the local theory of quaternionic anti-hermitian forms. This Journal, 387-400.

[6] B. L. van der Waerden, Gruppen von linearen Transformation. Springer, 1935.