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On ordinal diagrams

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G. Takeuti developed the theory of ordinal diagrams of order n (where n is a positive integer) in [**2**], and generalized it to the theory of ordinal diagrams constructed from well-ordered sets I, A, and S in [**3**]. It was necessary to consider S in order to prove the accessibility for Od(I, A, S) (the system of ordinal diagrams constructed from I, A and S) given in [**3**]. But S did not serve to extend the system of ordinal diagrams. In fact, if we denote Od(I, A, S) and O(I, A, S) with empty S by Od(I, A) and O(I, A) respectively, we can embed Od(I, A, S) (or O(I, A, S)) into Od($\{*\} \cup I, A \cup S$) (or O($\{*\} \cup I, A \cup S$)), where $*$ is distinct form any element of I, A and S; the notation $A \cup S$ means the well-ordered set obtained from A and S by keeping the orders in themselves and setting the elements of A before the elements of S. The embedding is defined as follows:

1. If $\alpha \in A$, then α^* is α.
2. If α is of the form (α_0, s), then α^* is $(*, \alpha_0{}^*, s)$.
3. If α is of the form (i, α_1, α_2), then α^* is $(i, \alpha_1{}^*, \alpha_2{}^*)$.
4. If α is of the form $\alpha_1 \sharp \alpha_2$, then α^* is $\alpha_1{}^* \sharp \alpha_2{}^*$.

Now we can simplify the proof of the accessibility of Od(I, A, S) in a similar way as in § 2 of [**2**], whether S is empty or not (cf. § 2 of this paper). In this paper, we shall construct a system Od(I), namely "the system of ordinal diagrams constructed from a well-ordered set I" (in § 1), and prove that the system is well-ordered for the given orderings in a similar way as in [**2**] (in § 2). Then we shall show that the present system is a generalization of previous systems. In fact, Od(I, A) is embedded into Od($I \cup A$) in § 3. By the way, we shall show that a formal theory of Od(I, A) can be formalized in the system developed in [**5**] and is consistent.

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§ 1. Ordinal diagrams constructed from *I*.

Let I be a well-ordered set with the order $<^*$ and o be the first element of I. In this section, we shall construct a kind of system of ordinal diagrams, called *ordinal diagrams constructed from* I and denoted by Od(I). Though the word o. d. is used in [2] and in [3] to denote an element of ordinal diagrams developed there, we use it instead of 'an element of Od(I)' for simplification throughout this and the next sections.

1. Od(I) is defined recursively as follows:
   1.1. If $i \in I$, then i is an o. d.
   1.2. If α and β are o. d.'s, then (α, β) is an o. d.
   1.3. If α and β are o. d.'s, then $\alpha \# \beta$ is an o. d.

2. An o. d. α is called a *c. o. d.* (connected ordinal diagram constructed from I), if and only if the operation used in the final step of construction of α is not $\#$.

3. Let α be an o. d. We define *components* of α recursively as follows:
   3.1. If α is a c. o. d., then α has exactly one component which is α itself.
   3.2. If α is an o. d. of the form $\alpha_1 \# \alpha_2$, then the components of α are the components of α_1 and of α_2.

4. Let α and β be o. d.'s. We define $\alpha = \beta$ recursively as follows:
   4.1. Let $\alpha \in I$. Then $\alpha = \beta$, if β is an element of I and equal to α in I.
   4.2. Let α be of the form (α_0, α_1). Then $\alpha = \beta$ if β is of the form (β_0, β_1) and $\alpha_0 = \beta_0$ and $\alpha_1 = \beta_1$.
   4.3. Let α have k components $\alpha_1, \cdots, \alpha_k$ $(k > 1)$. Then $\alpha = \beta$, if β has k-components, and $\beta_1, \cdots, \beta_k$ being these components, there exists a permutation $(m_1, \cdots, m_k)$ of $(1, \cdots, k)$ such that $\alpha_n = \beta_{m_n}$ for $n = 1, \cdots, k$.
   4.4. $\beta = \alpha$ if $\alpha = \beta$.

5. Let α be an o. d. The *rank* of α means the sum of the number of (,) and $\#$ in α.

6. Let α, β and ξ be o. d.'s. We define the relations $\beta \subset_\xi \alpha$ (to read: β is a *ξ-section* of α) and $\beta <_\xi \alpha$, $\beta <_\infty \alpha$ and '*index* of α' simultaneously as follows:
   6.1. If $\alpha, \beta \in I$, then $\beta <_\xi \alpha$ and $\beta <_\infty \alpha$ means $\beta <^* \alpha$.
   6.2. Let one (or both) of α and β be not a c. o. d., and the components of α and β be $\alpha_1, \cdots, \alpha_h$ and $\beta_1, \cdots, \beta_k$ respectively. $\beta <_\xi \alpha$ holds if one of the following conditions is satisfied:
   6.2.1. There exists an α_m $(1 \leqq m \leqq h)$ such that $\beta_n <_\xi \alpha_m$ holds for every n $(1 \leqq n \leqq k)$.
   6.2.2. $h > 1$, $k = 1$ and $\beta_1 = \alpha_m$ for some m $(1 \leqq m \leqq h)$.
   6.2.3. $h > 1$, $k > 1$ and there exist an α_m $(1 \leqq m \leqq h)$ and a β_n $(1 \leqq n \leqq k)$ such that $\alpha_m = \beta_n$ and

$$\beta_1 \# \cdots \# \beta_{n-1} \# \beta_{n+1} \# \cdots \# \beta_k <_\xi \alpha_1 \# \cdots \# \alpha_{m-1} \# \alpha_{m+1} \# \cdots \# \alpha_h .$$

$\beta <_\infty \alpha$ holds if one of 6.2.1-6.2.3 with ∞ in place of ξ is fulfilled.

6.3. If $\alpha \in I$, then $\beta \subset_{\xi} \alpha$ never holds.

6.4. Let α be of the form (α_0, α_1).

6.4.1. If $\xi <_o \alpha_0$, then $\beta \subset_{\xi} \alpha$ if and only if $\beta \subset_{\xi} \alpha_1$.

6.4.2. If $\xi = \alpha_0$, then $\beta \subset_{\xi} \alpha$ if and only if β is α_1.

6.4.3. If $\alpha_0 <_o \xi$, then $\beta \subset_{\xi} \alpha$ never holds.

6.5. Let α be of the form $\alpha_1 \# \alpha_2$. Then $\beta \subset_{\xi} \alpha$ if and only if either $\beta \subset_{\xi} \alpha_1$ or $\beta \subset_{\xi} \alpha_2$ holds.

6.6. ξ is called an *index* of α, if α has a ξ-section.

In the following we shall simply say 'ξ is less (or greater) than η' and 'ξ is the minimum (or maximum)' in place of 'ξ is less (or greater) than η in the sense of $<_o$' and 'ξ is the minimum (or maximum) in the sense of $<_o$', respectively.

6.7. Let α and β be c. o. d's. If there exists an index η of α and/or β such that $\xi <_o \eta$, then ξ^+ is defined to be the minimum of such indices; otherwise, ξ^+ is defined to be ∞. Then $\beta <_{\xi} \alpha$, if and only if one of the following conditions is fulfilled:

6.7.1. There exists a ξ-section α_0 of α such that $\beta \leqq_{\xi} \alpha_0$.

6.7.2. $\beta_0 <_{\xi} \alpha$ for every ξ-section β_0 of β and $\beta <_{\xi^+} \alpha$.

6.8. Let α and β be c. o. d,'s of the form (α_0, α_1) and (β_0, β_1) respectively. $\beta <_{\infty} \alpha$ if and only if one of the following conditions is fulfilled:

6.8.1. $\beta_0 <_o \alpha_0$.

6.8.2. $\beta_0 = \alpha_0$ and $\beta_1 <_{\alpha_0} \alpha_1$.

6.9. Let $\alpha \in I$ and β be a c. o. d. of the form (β_0, β_1). $\alpha <_{\infty} \beta$ if $\alpha \leqq_o \beta_0$, $\beta <_{\infty} \alpha$ if $\beta_0 <_o \alpha$.

Under these definitions the following propositions are easily proved.

PROPOSITION 1. *$=$ is an equivalence relation between o. d.'s.*

PROPOSITION 2. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be o. d.'s. $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$ imply $\alpha_1 \# \alpha_2 = \beta_1 \# \beta_2$.*

PROPOSITION 3. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be o. d.'s and γ be an o. d. or ∞. Then $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$ and $\alpha_1 <_{\gamma} \alpha_2$ imply $\beta_1 <_{\gamma} \beta_2$.*

PROPOSITION 4. *Each of the relations $<_{\xi}$, where ξ is an o. d. or ∞, defines a linear order between o. d.'s.*

PROPOSITION 5. *Let α and β be o. d.'s. Then $\beta <_{\xi} (\alpha, \beta)$ for every γ such that $\gamma \leqq_o \alpha$.*

§ 2. Accessibility of Od(I).

Let S be a system with a linear order $<$. An element s of S is called 'accessible in S (or accessible for $<$)', if the subsystem of S consisting of elements, which are not greater than s in the sense of $<$, is well-ordered. S is called accessible, if the whole system is well-ordered by $<$.

1. Let α and β be o.d.'s. We define a relation $\beta \lessdot \alpha$ (to read; β is a *value* of α) as follows:

1.1. If $\alpha \in I$, then α has no value, that is, $\beta \lessdot \alpha$ never holds.

1.2. Let α be not a c.o.d. and have components $\alpha_1, \cdots, \alpha_k$. Then $\beta \lessdot \alpha$, if $\beta \lessdot \alpha_m$ for some m $(1 \leqq m \leqq k)$.

1.3. Let α be of the form (α_0, α_1). Then $\beta \lessdot \alpha$, if β is α_0 or $\beta \lessdot \alpha_0$ or $\beta \lessdot \alpha_1$.

2. Let α and β be o.d.'s. β is called a $(\xi_1, \cdots, \xi_n)$-*section* of α, if the following conditions are fulfilled:

2.1. $\xi_1 \leqq_o \xi_2 \leqq_o \cdots \leqq_o \xi_n$.

2.2. There exists a series of o.d.'s $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$ such that α_k is the maximal component of a ξ_k-section of α_{k-1} in the sense of $<_{\xi_k}$ for every k $(k = 1, 2, \cdots, n)$.

3. Let ξ be an o.d. $\xi \# o$ is called the *successor* of ξ and sometimes denoted as ξ'. (It is clearly seen that no o.d. lies between ξ and $\xi \# o$ for $<_\eta$ where η is an o.d. or ∞). An o.d. ξ is called a *l.o.d.* (limit ordinal diagram constructed from I), if every component of ξ is different from o.

4. Let α be an o.d. and ξ be an o.d. accessible for $<_o$. We define 'α is a ξ-fan' and 'α is ξ-accessible' by transfinite induction on ξ for $<_o$ as follows:

4.1. An o.d., every value of which is accessible for $<_o$, is an o-fan.

4.2. α is ξ-accessible, if and only if α is a ξ-fan and accessible for $<_o$ in the system of ξ-fans.

4.3. α is $\xi \# o$-fan, if and only if α is a ξ-fan and every ξ-section of α is ξ-accessible.

4.4. Let ξ be a l.o.d. α is a ξ-fan, if and only if α is an η-fan for every η satisfying $\eta <_o \xi$.

Let α be an o.d. α is called an ∞-fan, if α is a ξ-fan for every o.d. ξ accessible for $<_o$, and is called to be ∞-accessible, if α is an ∞-fan and accessible for $<_\infty$ in the system of ∞-fans.

The following propositions are easily proved.

PROPOSITION 1. *Let α and ξ be o.d.'s. If every o.d. less than α in the sense of $<_\xi$ is accessible for $<_\xi$, then α is accessible for $<_\xi$.*

PROPOSITION 2. *Let α and ξ be o.d.'s. If α is accessible for $<_\xi$, then every o.d. less than α in the sense of $<_\xi$ is accessible for $<_\xi$.*

PROPOSITION 3. *Let $\alpha_1, \cdots, \alpha_n$ and ξ be o.d.'s. If $\alpha_1, \cdots, \alpha_n$ are accessible for $<_\xi$, then $\alpha_1 \# \cdots \# \alpha_n$ is accessible for $<_\xi$.*

These propositions remain correct, if we replace 'o.d. ξ', 'o.d.'s $\alpha, \alpha_1, \cdots, \alpha_n$' and 'accessible for $<_\xi$' by 'o.d. ξ accessible for $<_o$', 'ξ-fans $\alpha, \alpha_1, \cdots, \alpha_n$' and '$\xi$-accessible', respectively. We refer to thus replaced propositions as Propositions 1*-3*.

PROPOSITION 4. *Let ξ be an o.d. accessible for $<_o$. If α is $\xi\#o$-accessible, then α is ξ-accessible.*

PROOF. α is a ξ-fan by the definition. We may assume that every ξ'-fan β satisfying $\beta <_{\xi'} \alpha$ is ξ-accessible. We shall prove that every ξ-fan β such that $\beta <_\xi \alpha$ is ξ'-fan and ξ-accessible by induction on the rank of β. Let β be a ξ-fan such that $\beta <_\xi \alpha$. If β has a ξ-section β_0, β_0 is a ξ-fan and $\beta_0 <_\xi \alpha$. Then β_0 is ξ-accessible by the hypothesis of induction. We see that β is a ξ'-fan, whether β has a ξ-section or not. Then one of the following conditions holds:

(1) $\beta <_{\xi'} \alpha$.

(2) There exists a ξ-section α_0 of α such that $\beta \leqq_\xi \alpha_0$.

In the former case, β is ξ-accessible by our assumption. In the latter case, since α_0 is ξ-accessible, ξ-accessibility of β follows from Proposition 1*, q.e.d.

PROPOSITION 5. *Let ξ be a l.o.d. accessible for $<_o$, and the following condition* (C) *be satisfied:*

(C) *For any η, ζ such that $\eta <_o \zeta <_o \xi$, every ζ-accessible ξ-fan is η-accessible. Then 'α is ξ-accessible' implies 'α is η-accessible' for every η less than ξ.*

PROOF. Let the condition (C) be astisfied and α be ξ-accessible. Let ξ_0 be the successor of the greatest index less than ξ. We have only to prove that α is η-accessible for every η such that $\xi_0 \leqq_o \eta \leqq_o \xi$. We shall prove this by transfinite induction for $<_\xi$ on α. We may assume that every ξ-fan such that $\beta <_\xi \alpha$ is ζ-accessible for every ζ less than ξ. For the proof we define an auxiliary notion 'γ is the n-th η-branch of β with respect to ζ_0 and ζ_1' recursively as follows:

5.1. If $\zeta_0 \leqq_o \eta <_o \zeta_1$ and $\gamma \subset_\eta \beta, \gamma$ is the 1 st η-branch of β with respect to ζ_0 and ζ_1.

5.2. Let $\gamma \subset_\eta \delta$ and δ be the n-th ζ-branch of β with respect to ζ_0 and ζ_1. If $\zeta_0 \leqq_o \eta <_o \zeta$, then γ is the n-th η-branch of β. If $\zeta \leqq_o \eta <_o \zeta_1$ then γ is the $n+1$-st η-branch of β with respect to ζ_0 and ζ_1.

Let η satisfy $\xi_0 \leqq_o \eta <_o \xi$, and β be an η-fan and $\beta <_\eta \alpha$. We shall prove that β is a ξ-fan and ζ-accessible of every ζ such that $\xi_0 \leqq_o \zeta <_o \xi$ by induction on the number of branches of β with respect to ξ_0 and ξ. Let β_0 be an arbitrary ζ_0-branch of β ($\xi_0 \leqq_o \zeta_0 <_o \xi$). Using the hypothesis of induction, we see that β_0 is a ξ-fan. $\beta_0 <_\xi \alpha$ holds by means of $\beta <_\eta \alpha$. Then β_0 is ζ_0-accessible by the hypothesis of transfinite induction for $<_\xi$. Thus we may consider β as a ξ-fan. $\beta <_\xi \alpha$ holds by means of $\beta <_\eta \alpha$. Then β is ζ-accessible for every ζ less than ξ by the hypothesis of transfinite induction. From this our proposition follows by Proposition 1*. q.e.d.

By Propositions 4 and 5, we see easily

PROPOSITION 6. *Let ξ be an o.d. accessible for $<_o$ and the condition* (C) *hold. Then for every η less than ξ, 'α is ξ-accessible' implies 'α is η-accessible'.*

PROPOSITION 7. *The condition* (C) *holds for an arbitrary o.d. ξ accessible for $<_o$.*

PROOF. We prove this by transfinite induction on ξ. Suppose now the proposition holds for every ξ_0 less than ξ. If ξ is a *l.* o. d., our assertion is clear by the definition of ξ-fan. If $\xi=\zeta_0\#o$, our assertion holds for ζ less than ζ_0 by the hypothesis of induction and for $\zeta=\zeta_0$ by Proposition 6.

From Propositions 6 and 7 follows

PROPOSITION 8. *Let ξ be an o.d. accessible for $<_o$, α be ξ-accessible and $\eta<_o\xi$. Then α is η-accessible.*

From Proposition 8 follows

PROPOSITION 9. *For any o.d.'s η,ζ accessible for $<_o$ and $\eta<_o\zeta$ every ζ-accessible ∞-fan is η-accessible.*

PROPOSITION 10. *If α is ∞-accessible, then α is ξ-accessible for every o.d. ξ accessible for $<_o$.*

PROOF. Following the proof of Proposition 5, we can prove this by the help of Proposition 9.

By transfinite induction over *I*, we have

PROPOSITION 11. *Every ∞-fan is ∞-accessible.*

From Propositions 10 and 11, we see easily

PROPOSITION 12. *Every ∞-fan is ξ-accessible for every ξ accessible for $<_o$.*

PROPOSITION 13. *Every o-fan is ξ-accessible where ξ is an arbitrary o.d. accessible for $<_o$ or ξ is ∞.*

We see easily the following proposition.

PROPOSITION 14. *Let α and β be c.o.d.'s and ξ an o.d. If $\alpha<_\xi\beta$, then $\alpha<_\infty\beta$ or there exists a $(\xi_1,\cdots,\xi_n)$-section β_0 of β such that $\xi\leqq_o\xi_1$ and $\alpha\leqq_\infty\beta_0$.*

Then we have

PROPOSITION 15. *Every value of an o.d. α is less than α.*

PROPOSITION 16. *Let α be an o.d. and not an o-fan. Then there exists an o-fan β such that $\beta<_o\alpha$ and β is not accessible for $<_o$.*

PROOF. We prove this by induction on the rank of α. By the hypothesis of the proposition, there exists a value α_0 of α not accessible for $<_o$. We have $\alpha_0<_o\alpha$ by Proposition 15. If α_0 is an o-fan, we can take α_0 as β. If α_0 is not an o-fan, there exists an o-fan β such that $\beta<_o\alpha_0$ and β is not accessible for $<_o$ by the hypothesis of induction. Then β has the required property.

q. e. d.

PROPOSITION 17. *Every o-fan is accessible for $<_o$.*

PROOF. We prove this by transfinite induction for $<_o$ on the system of o-fans (cf. Proposition 13). Let α be an o-fan. We may assume that every

o-fan β less than α is accessible for $<_o$. Under this hypothesis and Proposition 16, we see easily that, if $\gamma <_o \alpha$ then γ is an o-fan. Then we have the proposition by Proposition 1.

PROPOSITION 18. *Every o.d. is an o-fan.*

PROPOSITION 19. *Every o.d. is accessible for* $<_o$.

THEOREM. *Every o.d. is accessible for* $<_\xi$, *where* ξ *is an arbitrary o.d. or* ∞.

PROOF. It follows from Propositions 18, 19 and 13.

§ 3. Relations between Od(I, A) and Od(I).

In this section we shall show that Od(I, I) is embedded into Od(J), where J is a union of two sets isomorphic to I.

1. Let I be well-ordered, $<$ be the well-ordering of I, and the first element of I be denoted by o.

We define $\tilde{I}$ to be a set consisting of all the i and $\tilde{i}$ where $i \in I$. $\tilde{<}$ is a well-ordering of $\tilde{I}$, which is defined as follows:

1.1. If $i<j$, then $i \tilde{<} j$.

1.2. If $i \in I$ and $j \in I$, then $i \tilde{<} \tilde{j}$.

1.3. If $i<j$, then $\tilde{i} \tilde{<} \tilde{j}$.

2. In the following some notations (e.g. $\#$, ∞) are used in both Od(I, I) and Od($\tilde{I}$).

Let α be an element of Od(I, I). α^* is defined recursively as follows:

2.1. If $\alpha \in I$, then α^* is $\tilde{\alpha}$.

2.2. If α is of the form (i, α_0, α_1), then α^* is $(\alpha_0^*, (i, \alpha_1^*))$.

2.3. If α is of the form $\alpha_1 \# \alpha_2$, then α^* is $\alpha_1^* \# \alpha_2^*$.

We see easily the following propositions.

PROPOSITION 1. *If* α *is an element of* Od(I, I), *then* α^* *is an element of* Od($\tilde{I}$).

PROPOSITION 2. *Let* α *and* β *be elements of* Od(I, I), $\alpha^* = \beta^*$ *if and only if* $\alpha = \beta$

PROPOSITION 3. *If* i *and* α *belong to* I *and* Od(I, I) *respectively, then* $i <_\xi \alpha^*$ *where* ξ *is an arbitrary element of* Od($\tilde{I}$) *or* ∞.

PROOF. We prove this by induction on the rank of α. If $\alpha \in I$, then it is clear by 1.2. If α is of the form (j, α_1, α_2) then α^* is $(\alpha_1^*, (j, \alpha_2^*))$. By the hypothesis of induction $i <_o \alpha_1^*$, whence follows $i <_\infty \alpha^*$. Then $i <_\xi \alpha^*$ for every $\xi \geqq_o \alpha_1^*$. Since α^* contains no ξ-section such that $j <_o \xi <_o \alpha_1^*$, this implies $i <_\xi \alpha^*$ for $j <_o \xi <_o \alpha_1^*$. Since $i <_j \alpha_2^*$ holds by the hypothesis of induction, $i <_j \alpha^*$ holds. From this we see easily the proposition.

PROPOSITION 4. *Let* α *and* β *be elements of* Od(I, I) *and* $i \in I$. β^* *is an* i*-section of* α^*, *if and only if* β *is an* i*-section of* α.

PROOF. We see easily the proposition by induction on the rank of α and Proposition 3.

PROPOSITION 5. *Let α and β be elements of* Od(I, I). *If $\alpha <_i \beta$, then $\alpha^* <_i \beta^*$ where $i \in I$ or i is ∞.*

PROOF. We shall prove this by double induction on the sum of ranks of α and β and the number of indices greater than i in α and/or β.

First we shall prove the case $i = \infty$. We have only to prove $\alpha <_\infty \beta$ implies $\alpha^* <_\infty \beta^*$ under the following hypothesis of induction:

(H1) Let γ and δ be any elements of Od(I, I), and the sum of the ranks of γ, δ be less than the sum of the ranks of α and β. Then $\gamma <_j \delta$ implies $\gamma^* <_j \delta^*$ where $j \in I$ or j is ∞.

To show this we separate the cases according to the forms of α and β. Since other cases are easily treated, we treat here only the case that α and β are of the form (i, α_0, α_1) and (j, β_0, β_1) respectively. If $\alpha_0 <_o \beta_0$, then $\alpha_0^* <_o \beta_0^*$ by (H1), which implies $\alpha^* <_\infty \beta^*$. If $\alpha_0 = \beta_0$, then we have only to prove $(i, \alpha_1^*) <_{\alpha_0^*} (j, \beta_1^*)$ (by Proposition 2), which follows from $(i, \alpha_1^*) <_\infty (j, \beta_1^*)$ (by Proposition 3). $(i, \alpha_1^*) <_\infty (j, \beta_1^*)$ follows from $i < j$, or $i = j$ and $\alpha_1^* <_i \beta_1^*$ according as $i < j$, or $i = j$ and $\alpha_1 <_i \beta_1$.

Then we prove that $\alpha <_i \beta$ implies $\alpha^* <_i \beta^*$ for $i \in I$ under (H1) and the following hypothesis of induction:

(H2) $\alpha <_j \beta$ implies $\alpha^* <_j \beta^*$ for every j such that the number of indices greater than j in α and/or β is less than the number of indices greater than i in α and/or β.

If there exists an i-section β_0 of β such that $\alpha \leqq_i \beta_0$, then β_0^* is an i-section of β^* and $\alpha^* \leqq_i \beta_0^*$ by Proposition 4 and (H1). Let $\alpha_0 <_i \beta$ for every i-section α_0 of α and $\alpha <_j \beta$ where j is defined as follows: If there exists an index of α and/or β greater than i, then j is defined to be the minimum of such indices; othewise, j is defined to be ∞. Then $\alpha_0^* <_i \beta^*$ for every i-section α_0^* of α^* and $\alpha^* <_j \beta^*$ by Proposition 4 and (H2). From this follows $\alpha^* <_i \beta^*$ by Proposition 4.

From these propositions follows

THEOREM 1. Od(I, I) *is embedded into* Od$(\tilde{I})$.

We define a subsystem O(I) of Od(I) recursively as follows:

3.1. If $i \in I$ then $i \in$ O(I).

3.2. If $i \in I$ and $\alpha \in$ O(I), then $(i, \alpha) \in$ O(I).

3.3. If $\alpha \in$ O(I) and $\beta \in$ O(I), then $\alpha \# \beta \in$ O(I).

Then we have

COROLLARY 1. O(I, I) *is embedded into* O$(\tilde{I})$.

Let I and A be well-ordered. We have the following theorem in the same way as above.

THEOREM 2. *If I and A have no element in common,* Od(I, A) *is embedded into* Od$(I \cup A)$.

COROLLARY 2. *If I and A have no element in common,* O(I, A) *is embedded into* O$(I \cup A)$.

§ 4. On a formal theory of Od(I, A).

In [5], G. Takeuti proved the consistency of a logical system. We shall consider the following slight modification of this system: Let $I(a)$, $A(a)$, $a <^* b$ and $a \overset{..}{<} b$ be primitive recursive predicates, and $<^*$ and $\overset{..}{<}$ well-orderings of I and A, where I and A are $\{a \mid I(a)\}$ and $\{a \mid A(a)\}$ respectively.

1. Every beginning sequence is of the form $D \to D$ or of the form $a = b$, $F(a) \to F(b)$ or a 'mathematische Grundsequénz' in Gentzen [1], or one of the following forms:

$$I(a), A_m(a, b) \to G_m(a, b, \{x, y\}(A_m(x, y) \wedge x <^* a));$$
$$I(a), G_m(a, b, \{x, y\}(A_m(x, y) \wedge x <^* a)) \to A_m(a, b);$$
$$A(a), B_n(a, b) \to H_n(a, b, \{x, y\}(B_n(x, y) \wedge x \overset{..}{<} a));$$
$$A(a), H_n(a, b, \{x, y\}(B_n(x, y) \wedge x \overset{..}{<} a)) \to B_n(a, b);$$

where $m, n = 0, 1, 2, \cdots$, $A_0, A_1, \cdots$, $B_0, B_1, \cdots$ are symbols for predicate and G_m and H_n are arbitrary formulas satisfying the following conditions:

(a) $G_m(a, b, \alpha)$ and $H_n(a, b, \alpha)$ do not contain $A_m, A_{m+1}, A_{m+2}, \cdots, B_0, B_1, B_2, \cdots$ and $B_n, B_{n+1}, B_{n+2}, \cdots$ respectively.

(b) If $G_m(a, b, \alpha)$ or $H_n(a, b, \alpha)$ contains a formula of the form $\forall \varphi F(\varphi)$, then $F(\beta)$ contains no bound f-variable.

2. The following inference 'induction' is added:

$$\frac{F(a),\ \Gamma \to \Delta,\ F(a+1)}{F(0),\ \Gamma \to \Delta,\ F(t)}$$

where a is contained in none of $F(0)$, Γ and Δ, and t is an arbitrary term.

3. The inference $\forall$ left on f-variable

$$\frac{F(V),\ \Gamma \to \Delta}{\forall \varphi F(\varphi),\ \Gamma \to \Delta}$$

is restricted by the condition that $F(\beta)$ contains no bound f-variable.

Then we have the following

THEOREM. *This system is consistent.*

PROOF. Let J be $I \cup A$, $\prec$ be a well-ordering of J defined as follows:

1. If $i <^* j$, then $i \prec j$.
2. If $i \in I$ and $a \in A$, then $i \prec a$.
3. If $a \overset{..}{<} b$, then $a \prec b$.

Then the proof is performed as in [5] considering J as I.

We see easily from the proof of § 2, that the proof for accessibility of $\mathrm{Od}(I, A)$ can be given in a similar way as in §2 of [2]. We can develop a formal theory of $\mathrm{Od}(I, A)$ in a subsystem of the above system such that $m=0,1$ and $n=0$. It is noticed that for the consistency-proof for this subsystem, we have only to use $\{\infty\} \cup J_0 \cup J_1$ instead of J_∞. We shall not give an exact treatment of the formal theory here, but show how to develop it. First we give all the necessary concepts concerning the construction of $\mathrm{Od}(I, A)$ as the mathematische Grundsequenzen in the same way as in [4]. Let $I(a), A(a)$, $a <^* b, a \stackrel{..}{<} b$, $O(a)$, $<(i, a, b)$, $\subset(i, a, b)$ and $\lessdot(a, b)$ be the formal counterparts of '$a \in I$', '$a \in A$', 'a is less than b in I', 'a is less than b in A', '$a \in \mathrm{Od}(I, A)$', '$a <_i b$', '$a \subset_i b$' and '$a \lessdot b$', respectively. We use further the following abbreviations:

$J^*(a)$ for $\forall\varphi(\forall x(I(x) \wedge \forall y(y <^* x \vdash \varphi[y]) \vdash \varphi[x]) \vdash \varphi[a])$;

$D^*(a, \alpha)$ for $\forall x(x <^* a \vdash \alpha[x]) \vdash J^*(a)$;

$\ddot{J}(a)$ for $\forall\varphi(\forall x(A(x) \wedge \forall y(y \stackrel{..}{<} x \vdash \varphi[y]) \vdash \varphi[x]) \vdash \varphi[a])$;

$\ddot{D}(a, \alpha)$ for $\forall x(x \stackrel{..}{<} a \vdash \alpha[x]) \vdash \ddot{J}(a)$;

$A(i, \alpha, a)$ for $\forall\varphi(\forall x(\alpha[x] \wedge \forall y(\alpha[y] \wedge <(i; y, x) \vdash \varphi[y]) \vdash \varphi[x]) \vdash \varphi[a])$;

$A(i, a)$ for $A(i, \{x\} O(x), a)$;

$\tilde{O}(a)$ for $O(a) \wedge \forall x(\lessdot(x, a) \vdash A(1, x))$, where 1 stands for the formal counterpart of the first element of I;

$B(i, a, \alpha)$ for

$I(i) \wedge \tilde{O}(a) \wedge \forall x(x <^* i \vdash \alpha[x, a] \wedge \forall y(\subset(x; y, a) \vdash A(x, \{u\} \alpha[x, u], y)))$;

$\tilde{I}(i)$ for $I(i) \wedge i=0$, where 0 stands for the formal counterpart of ∞.

Then the following sequences are also used as beginning sequences of our system:

1.1. $I(i), C^*(i) \rightarrow D^*(i, \{x\}(C^*(x) \wedge x <^* i))$.

1.2. $I(i), D^*(i, \{x\}(C^*(x) \wedge x <^* i)) \rightarrow C^*(i)$.

1.3. $A(a), \ddot{C}(a) \rightarrow \ddot{D}(a, \{x\}(\ddot{C}(x) \wedge x \stackrel{..}{<} a))$.

1.4. $A(a), \ddot{D}(a, \{x\}(\ddot{C}(x) \wedge x \stackrel{..}{<} a)) \rightarrow \ddot{C}(a)$.

1.5. $I(i), F(i, a) \rightarrow B(i, a, \{x, y\}(F(x, y) \wedge x <^* i))$.

1.6. $I(i), B(i, a, \{x, y\}(F(x, y) \wedge x <^* i)) \rightarrow F(i, a)$.

We can prove that the sequence $O(a), \tilde{I}(i) \rightarrow A(i, a)$ is provable in our system. This is done similarly as in [4], using the above proof of accessibility.

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