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On local class field theory*

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The purpose of this paper is to show how the basic theorems of local class field theory can be deduced without recourse to the somewhat cumbersome computations of indices of norm class groups. Basic facts employed will be the properties of unramified (cyclic) extensions of complete fields assuming that the residue class fields with respect to the given valuation have for each positive integer exactly one extension of that degree[1].

The classical inequalities are seen to be Chevalley's famous Théorème 0[2], that is, existence of a cocycle in the second cohomology whose order equals the degree of the normal extension in question, and a lemma, implicit in the work of Kummer on explicit laws of reciprocity,[3] which states that the second cohomology group is trivial if the residue class field is algebraically closed. Throughout the paper constant use is made of the properties of Hochschild's transfer mapping[4]. The theory of the norm residue is obtained as a simple corollary to the isomorphism theorem for abelian extensions which is formulated by means of the homomorphism first used successfully by Akizuki and Nakayama[5]. Thus, the approach presented here does not make any distinction between the "characteristic unequal and equal cases" as required heretofore, if the theory of algebras as such is to be excluded.

Suppose that K/F is a normal extension of degree n with the Galois group $\mathcal{K}=\{\sigma, \tau, \rho, \cdots\}$. In the sequel the second cohomology group $H^2(\mathcal{K}, K^*)$[6] and homomorphisms related to it will be of primary importance. This cohomology group may be defined[7] as the factor group of 2-cocycles (factor sets) $f(\sigma, \tau) \in K^*$ (which form a group $Z^2(\mathcal{K}, K^*)$) satisfying the n^3 relations

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1) As to the significance of this assumption as a necessary condition for the validity of local class field theory see [**14**], [**17**] and [**19**]. The numbers in [...] refer to the bibliography at the end of the paper.

2) See [**3**, p. 142]; and [**9**, p. 341].

3) See [8]; and [**20**, p. 508].

4) See [**9**].

5) See [**2**], [**9**] and [**15**].

6) K^* denotes the multiplicative group of non-zero elements in K. See [**9**] for the definitions of cocycle, coboundary and cohomology group.

7) For definitions and proofs of algebraic properties see [**9**].

$$f(\tau, \rho)^{\sigma} f(\sigma, \tau\rho) = f(\sigma\tau, \rho) f(\sigma, \tau)^{8)}$$

modulo the group of coboundaries (transformation sets) $B^2(\mathcal{K}, K^*) = \{f(\tau)^{\sigma} f(\sigma) f(\sigma\tau)^{-1}\}$.[9]

Immediate consequences of these defining relations and definitions are[10]:

(i) $H^2(\mathcal{K}, K^*)$ is an abelian group of exponent n,

(ii) the element $\prod_{\tau \in \mathcal{K}} f(\tau, \sigma) = g(\sigma)$ lies in F[11];

(iii) the mapping $\sigma \to J_c(\sigma) = g(\sigma) N(K^*)$ is a homomorphism of the Galois group $\mathcal{K}$ into the factor group $F^*/N(K^*)$. Here $g(\sigma)$ denotes the product, according to (ii), for a representative $f(\tau, \sigma)$ of the cohomology class of $c \in H^2(\mathcal{K}, K^*)$, N denotes the norm taken from K to F.

Suppose now that $\mathcal{H}$ is a subgroup of $\mathcal{K}$ having the field S/F for its invariant field elements. Then[12], given a couple $c = c(\sigma, \tau)$ of $H^2(\mathcal{K}, K^*)$, the restriction of the arguments σ, τ to elements of $\mathcal{H}$ determines an element $P_{\mathcal{H}}[c(\sigma, \tau)]$ of $H^2(\mathcal{H}, K^*)$. The resulting mapping $c(\sigma, \tau) \to P_{\mathcal{H}}[c(\sigma, \tau)]$ is a homomorphism $P_{\mathcal{H}}$ of $H^2(\mathcal{K}, K^*)$ into $H^2(\mathcal{H}, K^*)$.

If $\mathcal{H}$ is a normal subgroup of K, the kernel of $P_{\mathcal{H}}$ can readily be determined, using the fact that the first cohomology group of K/F is trivial[13]. For this purpose a homomorphism $\Lambda_{S,K}$ from $H^2(\mathcal{K}/\mathcal{H}, S^*)$ into $H^2(\mathcal{K}, K^*)$ is used. This mapping is determined by the equations $c(\sigma, \tau) = \Lambda_{S,K}[c(\sigma^*, \tau^*)]$ where σ^*, τ^* are the images of σ, τ in $\mathcal{K}$ in the Galois group $\mathcal{K}/\mathcal{H}$ of S/F. It turns out that $\Lambda_{S,K}$ actually is an isomorphism of $H^2(\mathcal{K}/\mathcal{H}, S^*)$ into $H^2(\mathcal{K}, K^*)$.

Then the kernel of the homomorphism $P_{\mathcal{H}}$, $\mathcal{H}$ normal in $\mathcal{K}$, is found to be $\Lambda_{S,K}[H^2(\mathcal{K}/\mathcal{H}, S^*)]$.

Suppose now that L/F is another normal extension with Galois group $\mathcal{L}$. Denote by M the union of K and L in an algebraic closure of F. Assume that L belongs to the subgroup $\mathcal{H}$ of the Galois group of M/F. Then a cohomology class f in $H^2(\mathcal{K}, K^*)$ is said to be split by L/F, provided the "lift". $\Lambda_{K,M}(f)$ lies in the kernel of $P_{\mathcal{H}}$[14]. Here the symbol f is simultaneously used for a cocycle representing the given cohomology class. Thus, if f is split by L/F, then $\Lambda_{K,M}(f) = \Lambda_{L,M}(f')$ for some cocycle f' for L/F. The cocycle f' shall

8) Exponentiation by σ signifies taking the conjugate for the automorphism $\sigma \in \mathcal{K}$.

9) See [**9**] for general coboundary operations.

10) Statements (i) to (iii) are obtained by multiplying the defining relations over the arguments ρ, τ and σ, each varying over all elements of $\mathcal{K}$.

11) See [**15**, p. 85], Theorem 1. Also note that equivalent cocycles give rise to products $g(\ldots)$ which belong to the same coset of F^* modulo the norm group $N(K^*)$.

12) For the sequel, see, e. g., [**9**].

13) Loc. cit. [**9**], "Noether's equations."

14) This is an apparent simplification of Hochschild's definition, loc. cit, p. 336; here the burden of proof is placed on the statement that all algebraic closures of F are isomorphic over F.

be called the "transfer" $T_{K,L}(f)$ of f to L/F. Passage to the cohomology class of f' yields that $T_{K,L}$ actually is an isomorphism of the subgroup of cohomology classes of $H^2(\mathcal{K}, K^*)$ which are split by L, denoted by $H^2(\mathcal{K}, K^*)_L$, into $H^2(\mathcal{L}, L^*)$. As a matter of fact $T_{K,L}[H^2(\mathcal{K}, K^*)_L] = H^2(\mathcal{L}, L^*)_K$; thus $T_{L,K}$ is the inverse mapping of $T_{K,L}$.

Suppose now that the field F is complete with respect to a discrete valuation V[15] of rank one. Assume that the associated residue class field admits for every positive integer n exactly one cyclic extension of degree n. Let U_i, $i=1,2,\cdots$, be an arbitrary ordering of the unramified extensions of F. Then the union $k=\bigcup_i U_i$ is a relatively complete field whose *residue class field is algebraically closed*. The completion $\bar{k}$ of k has the same value group as the fields F, U_i, and k.

THEOREM 1. *If K/k is a normal finite extension with the Galois group $\mathcal{K}$, then $H^2(\mathcal{K}, K^*)=1$.*

For the proof first note that it suffices to establish that

(A) $$H^2(\mathcal{Z}, Z^*)=1 \quad \text{for cyclic extensions } Z/H$$

of prime degree over a finite extension H of k.

Note that abelian groups and p-groups $\mathcal{M}$ (with corresponding field M) contain normal subgroups $\mathcal{M}_i$, $1 \subsetneqq \mathcal{M}_1 \subsetneqq \cdots \subsetneqq \mathcal{M}_i \subsetneqq \cdots \subsetneqq \mathcal{M}$ such that $\mathcal{M}_{i+1}/\mathcal{M}_i$ has prime index p_i in $\mathcal{M}/\mathcal{M}_i$. Consider therefore the subfield M_1/H which corresponds to $\mathcal{M}_1$. The kernel of the restriction $P_{\mathcal{M}_1}$ of $H^2(\mathcal{M}, M^*)$ equals $\Lambda_{M_1,M}[H^2(\mathcal{M}/\mathcal{M}_1, M_1^*)]$. Furthermore, this restriction $P_{\mathcal{M}_1}[H^2(\mathcal{M}, M^*)]$ is a subgroup of $H^*(\mathcal{M}_1, M^*)$ which is the identity if (A) is established. Consequently $H^2(\mathcal{M}, M^*)$ coincides with $\Lambda_{M_1,M}[H^2(\mathcal{M}/\mathcal{M}_1, M_1^*)]$. Hence, induction on the subgroups $\mathcal{M}_i$ implies $H^2(\mathcal{M}, M^*)=1$.

This argument can be applied to a normal extension K/k with the ramification field K_0/k. By the Hilbert theory[16] the subgroup K_0 of K belonging to K_0 is a p-group (p being the common characteristic of the residue class fields of k, U_i and F) with the cyclic factor group K/K_0.

Then $H^2(\mathcal{K}/\mathcal{K}_0, K_0^*)=1$ and also $H^2(\mathcal{K}_0, K^*)=1$. Reasoning as before, using the restriction mapping $P_{\mathcal{K}_0}$ and (A), it follows that $H^2(\mathcal{K}, K^*)=1$.

Now it remains to prove assumption (A). For this purpose observe that the completion $\bar{Z}$ of a finite separable extension Z/H has degree $[Z:H]$ over the completion $\bar{H}$[17]. The field $Z=H(z)$ is equal to the union $HH_0(z)$ where

15) For the elementary properties of complete and relatively complete fields see [**18**].

16) See [**18**, p. 8].

17) See [**18**, p. 195 et sequ.] for arguments dealing with the approximation of extensions of an infinite algebraic extension.

H_0 is a finite extension of F in a chain of approximating fields $F \subset H_0 \subset \cdots \subset H_i \subset \cdots \subset H$ with $\bigcup H_i = H$. Let N, T, respectively, denote the norm and trace for $H_0(z)/H_0$ then $N(1+xk) = 1 + T(k)x + \cdots = 1 + y = \varepsilon$, where $k \in H_0(z)$ has non-zero trace and $x \in H_0$ has a solution for given y, according to the Lagrange inversion formula[18], provided $V_{H_0}(y) \geqq \mu$, for sufficiently large positive μ. This means that every unit $\varepsilon \equiv 1\,(P_{H_0}^{\mu})$, where P_{H_0} denotes the prime ideal of H_0, is the norm of a unit in $H_0(z)$. Note that the same value μ can be used for $H_i(z)/H_i$ since the extension H_i is unramified over H_0[19]. Similarly the "radius of convergence" μ can be used for $\bar{H}(z)/\bar{H} = \bar{Z}/\bar{H}$.

Finally, restrict Z/H to be cyclic of prime degree p. Then the completion $\bar{Z}$ is cyclic of degree p over $\bar{H}$. An argument[20], originally due to Kummer, implies that $H^2(\mathcal{Z}, \bar{Z}^*) = 1$, where $\mathcal{Z}$ denotes the common Galois group of $\bar{Z}/\bar{H}$ and Z/H. In order to prove $H^2(\mathcal{Z}, Z^*) = 1$ let $a \in H$. Then $a = N\bar{A}$, $\bar{A} \in \bar{Z}$[21]. The element A is the limit $\lim\limits_{h \to \infty} A_h$ of elements A_h in the approximating field $H(z)$. Hence h may be picked so large that $N(\bar{A}/A_h) \geqq \mu$. Next observe $a = N(A_h)N(\bar{A}/A_h)$ implies that $N(\bar{A}/A_h) = \varepsilon$ lies in H. Since $H = \bigcup H_i$, the unit ε lies in a field $H_i \supseteqq H_0$ of finite degree over F. Therefore, by the above remarks on the unramified prolongations of V_{H_0} to the fields H_i, $\varepsilon = NE$ with $E \in H_j(z)$, according to the preservation of the radius of convergence μ for the Lagrange inversion formula. Thus, $a = N(A_h)N(E)$, and consequently $H^2(\mathcal{Z}, Z^*) = 1$ as asserted.

From now on, unless otherwise stated, F shall denote a complete field whose residue class field possesses for every integer n exactly one cyclic extension of degree n[22].

THEOREM 2. *If K/F is a normal extension of degree n, then its cohomology* group $H^2(\mathcal{K}, K^*)$ *is cycilc and its order divides n.*

Statement (i) asserts that the orders of the cohomology classes are divisors of n. Therefore it suffices to prove that for each prime p dividing n the group $H^2(\mathcal{K}, K^*)$ contains at most one cyclic subgroup of order p. Suppose then that the cohomology group contains a subgroup of type (p, p). Theorem 1 implies that this finite group is split by some unramified finite extension U_m/F. Applying the transfer mapping T_{K,U_m} to the subgroup, there would then exist a non-cyclic subgroup of order p^2 in the group $H^2(\mathcal{U}_m, U_m^*)$. However, the latter group being isomorphic[23] to the norm class groups $F^*/N(U_m^*)$,

18) See [**22**].
19) The fields in question have the same general trace and norm forms.
20) See [**8**].
21) See [**20**, p. 508, Lemma 2].
22) See [**12**], [**13**], [**17**] and [**19**] for the significance of this assumption.
23) For this normalization see [**9**, p. 339], and [**23**, p. 98].

where $N(\cdots)$ denotes the norm taken from U_m to F, is a cyclic group of order m[24]. Consequently $H^2(\mathcal{K}, K^*)$ contains at most one cyclic subgroup of order p.

COROLLARY. *If K/F is a normal extension of degree n, then the order of its norm class group $[F^* : N(K^*)]$ divides n.*

Since K/F is solvable there will exist a chain of subfields $F = K_0 \subset K_1 \subset \cdots \subset K_{i-1} \subset K_i \subset \cdots \subset K_r = K$ such that K_i is cyclic of prime degree over K_{i-1}. Denote by $N_{j,i}$ the norm taken from K_j to $K_i > j > i$. The Theorem 1 implies that the corollary is correct for $r=1$, since the cohomology group is isomorphic to the norm class group for a cyclic extension. As induction hypothesis assume that $[K_1^* : N_{i+1,1}(K_{i+1}^*)] \mid [K_{i+1} : K_1]$ for the normal extension K_{i+1}/K_1. Now define A as the subgroup of K_1^* which consists of all elements a for which $N_{1,0}(a) \in N_{i+1,0}(K_{i+1}^*)$. Then $A \supseteq N_{i+1,1}(K_{i+1}^*)$, by the transitivity of the norm, $N_{i+1,0}(\cdots) = N_{1,0}[N_{i+1,1}(\cdots)]$. Consequently $[K_1^* : A]$ divides $[K_1^* : N_{i+1,0}(K_{i+1}^*)]$, and the last index divides $[K_{i+1} : K_1]$ according to the induction hypothesis. Next, considering $N_{1,0}$ as a homomorphism acting on K_1^*, Herbrand's reduction principle for homomorphisms implies $[K_1^* : A] = [N_{1,0}(K_1^*) : N_{i+1,0}(K_{i+1}^*)]$. Consequently $[F^* : N_{i+1,0}(K_{i+1}^*)] = [F^* : N_{1,0}(K_{i+1}^*)][N_{1,0}(K_1^*) : N_{i+1,0}(K_{i+1}^*)]$; therefore $[F^* : N_{i+1,0}(K_{i+1}^*)] = [K_1 : F][N_{1,0}(K_1^*) : N_{i+1,0}(K_{i+1}^*)] = [K_1 : F][K_1^* : A]$ divides $[K_1 : F][K_{i+1} : K_1] = [K_{i+1} : F]$ since K_1 is cyclic over F[25].

THEOREM 3. *If K/F is a normal extension of degree n, then $H^2(\mathcal{K}, K^*)$ is a cyclic group of order n.*

Recalling Theorem 2 it suffices to exhibit a cohomology class of order n for K/F. For this purpose the transfer mapping $T_{U_n,K}$ is used. First, $H^2(\mathcal{U}_n, U_n^*)$ is a cyclic group of order n[26]; second, every cohomology class of U_n/F is split by K[27]. Hence $T_{U_n,K}$ is an isomorphism of $H^2(\mathcal{U}_n, U_n^*)$ into $H^2(\mathcal{K}, K^*)$.

REMARK. For the sake of completeness an outline of the proof for Chevalley's Théorème 0, using cohomology theory, will now be sketched. Suppose that f is a cocycle in a cohomology class of order n in $H^2(\mathcal{U}_n, U_n^*)$. Take $\Lambda_{U_n,U_nK}(f)$. Then this cocycle will be equal to a cocycle $\Lambda_{K,U_nK}(g)$, g a representative of a unique class in $H^2(\mathcal{K}, K^*)$ of $P_{\mathfrak{H}}[\Lambda_{U_n,U_nK}(f)]$ is a coboundary, i.e., if K/F splits f. Next note that U_nK/K is the unramified extension of degree $n/(n,h)$ where h denotes the residue class degree of K/F. Finally note that the cocycle f is equivalent to a normalized cyclic cocycle given as $a=$

24) See [**5**].

25) Note that the proof can be modified so as to yield $[H^2(\mathcal{K}_i, K_i^*) : 1] \mid [K_i : F]$ if the fields K_i/F are assumed to be normal. One uses the connection between the restriction and lift mappings in order to set up the induction. Namely, $[H^2(\mathcal{K}_{i+1}, K_{i+1}^*) : 1] = [P_{\mathfrak{H}}[H^2(\mathcal{K}_{i+1}, K_{i+1}^*) : 1]\,[\Lambda_{K_i,K_{i+1}}[H^2(\mathcal{K}_i, K_i^*)] : 1]$ where $\mathfrak{H}$ denotes the Galois group of K_{i+1}/K_i.

26) See [**5**].

27) See [**3**, p. 142], and [**9**, p. 341].

$(\alpha_{\sigma^i}, {}_{\sigma^j}) = \pi^{rs}$, with $s = \left[\frac{i+j}{n}\right] - \left[\frac{i}{n}\right] - \left[\frac{j}{n}\right]$ and r relatively prime to n for a fixed, but arbitrary, choice of the generator σ of the Galois group of U_n/F and a prime element π of F. Then $P_{\mathcal{H}}[\Lambda_{U_nK,K}(a)]$ is equivalent to Π^e, where Π denotes a prime element of K. All along Hasse's result, loc. cit. [8], was used that for unramified cyclic extensions the units of the base field are norms. For this reason, the choice of π and Π is immaterial for the determination of the normalized representatives of the cohomology classes in question. Since e, the ramification degree of K/F, equals $n/(n, h) = [U_nK : K]$, Hasse's Theorem implies that $P_{\mathcal{H}}[\Lambda_{U_nK,K}(a)]$ is a coboundary. Consequently the transfer mapping $T_{U_n,K}$ is an isomorphism of $H^2(\mathcal{U}_n, U_n^*)$ into $H^2(\mathcal{K}, K^*)$. Also note that an element of order $m|n$ in $H^2(\mathcal{U}_n, U_n^*)$ always has the form $\Lambda_{U_m,U_n}(k)$, $k \in H^2(\mathcal{U}_m, U_m^*)$ since the groups in question are cyclic. Then $T_{U_n,K}$ is an isomorphism on $\Lambda_{U_m,U_n}[H^2(\mathcal{U}_m, U_m^*)]$ which coincides with $\Lambda_{U_mK,U_nK}\ T_{U_m,K}$ on $H^2(\mathcal{U}_m, U_m^*)$.

REMARK. Immediate consequences of the above results are

(iv) If $\mathcal{H}$ is a subgroup $\mathcal{K}$, then $H^2(\mathcal{H}, K^*) = P_{\mathcal{H}}[H^2(\mathcal{K}, K^*)]$; and

(v) if $\mathcal{H}$ is a normal subgroup of $\mathcal{K}$ with corresponding field S/F, then $\Lambda_{S,K}[H^2(\mathcal{K}/\mathcal{H}, S^*)] = [H^2(\mathcal{K}, K^*)]^h$, $h = [K:S]$[28].

It is now a relatively simple matter to establish the classical law of reciprocity for abelian extensions K/F and furthermore as a significant corollary the theory of norm residues[29].

For this purpose it is necessary to enlarge upon the previously used results concerning the lift mapping $\Lambda_{S,K}$ of a couple (cohomology class) of a normal subfield S/F belonging to a normal subgroup H of the Galois group of K/F. Suppose that $f(\sigma, \tau)$ is a cocycle of K/F whose cohomology class is $c(\sigma, \tau) = c$. Let $g_{\mathcal{H}}(\sigma) = \prod_{\rho \in \mathcal{H}} f_{\rho,\sigma}$. Then, denoting the order of $\mathcal{H}$ by h and by $N_{K/S}$ the norm from K to S,

$$f(\sigma, \tau)^h = \Lambda_{S,K}\left[N_{K/S}(f(\bar{\sigma}, \bar{\tau}))\frac{g_{\mathcal{H}}(\overline{\sigma\tau})}{g_{\mathcal{H}}(\bar{\sigma}\bar{\tau})}\right] \pmod{B^2(\mathcal{K}, K^*)}$$

where $\bar{\sigma}, \bar{\tau}$ are fixed representatives of σ, τ modulo $\mathcal{H}$, and $\overline{\sigma\tau}$ is the chosen representative of $\bar{\sigma}\bar{\tau}$[30]. This formula implies[31] that

c^h equals the cohomology class of

$d(\sigma, \tau) g_{\mathcal{H}}(\sigma) g_{\mathcal{H}}(\tau)^{\sigma} g_{\mathcal{H}}(\sigma\tau)^{-1}$

28) These relations are important for Nakayama's proof of the limitation theorem of local class field theory. See [**16**, pp. 880-882]. Also [**10**].

29) See [2a], [3], [4], [7], [12] and [13], [19a].

30) See the elegant proof of Witt [**21**]; this is formula (18) of [**2**, p. 569]. See also [**9**, pp. 333-334].

31) This is equation (12) of [**2**, p. 568].

where

$$d(\sigma, \tau) = \prod_{\rho \in \mathcal{H}} \frac{f(\tau, \rho\sigma)}{f(\tau, \rho)}.$$

This relation can be written in a simpler form if one observes that a representing cocycle $k(\sigma, \tau)$ in c may be chosen such that $k(\rho, \tau) = 1$ for $\rho \in \mathcal{H}$ and τ in a fixed set of representatives of $\mathcal{K}$ modulo $\mathcal{H}$[32]. Then, using the associativity relations, one finds that the cohomology class of c^h can be represented by the cocycle (vi) $\prod_{\rho \in \mathcal{H}} \frac{k(\tau\rho, \sigma)}{k(\rho, \sigma)^\tau}$. This is possible because the quantities $g_{\mathcal{H}}(\sigma)$ computed for $k(\sigma, \tau)$ turn out to be 1[33].

As a special result a lemma of Chevalley[34] is obtained if the representatives of $\mathcal{K}$ modulo $\mathcal{H}$ can be picked so as to form a subgroup $\mathcal{L}$ of $\mathcal{K}$, in particular, if $\mathcal{K}$ is the direct product of $\mathcal{H}$ and $\mathcal{L}$.

Then

(vii) $c^h = \Lambda_{S,K}(N_{K/S}(f(\sigma^*, \tau^*))$ where $\mathcal{L} = \{\sigma, \tau, \cdots\}$.[35]

THEOREM 4. *Suppose that K/F is an abelian extension of degree n with the Galois group $\mathcal{K} = \{\sigma, \tau, \cdots\}$. Then each cohomology class c of $H^2(\mathcal{K}, K^*)$ determines an isomorphism J_c of K with the norm class group $F^*/N(K^*)$. This isomorphism is given by* $\sigma \to J_c(\sigma) = \prod_{\tau \in \mathcal{K}} f(\tau, \sigma) = g(\sigma) \bmod N(K^*)$[36] *where $f(\tau, \sigma)$ is a cocycle in c.*

For the proof suppose that $\mathcal{K} = \mathcal{K}_i \times \cdots \times \mathcal{K}_s$ is representation of K as a direct product of cyclic groups $\mathcal{K}_i = \{\sigma_i\}$ of respective orders n_i. Corresponding to the subgroups $\hat{\mathcal{K}}_i$, obtained from the direct product by replacing the i-th component by 1, there are s subfields K_i/F which are cyclic of degree n_i and whose Galois groups are isomorphic to $\mathcal{K}_i = \{\sigma_i\}$. These Galois groups are the restrictions of $\mathcal{K}$ to K_i and may be identifield with $\mathcal{K}_i$ in order to avoid complicated notation. Furthermore, using the interpretation of $H^2(\mathcal{K}, K^*)$ by means of group extensions of K^* by $\mathcal{K}$[37], it may be assumed that each cohomology class of $H^2(\mathcal{K}, K^*)$ is represented by a cocycle $f(\sigma, \tau)$ for which $f(\sigma_i^\mu, \sigma_i^\nu) = a_i^{x_i}$, with $x_i = \left[\frac{\mu+\nu}{n_i}\right] - \left[\frac{\mu}{n_i}\right] - \left[\frac{\nu}{n_i}\right]$, and a_i lying in $\check{K}_i = K_1 \cup \cdots \cup K_{i-1} \cup K_{i+1} \cup \cdots \cup K_s$, $1 \leqq i \leqq s$[38]. This normalization implies $g(\sigma_i) =$

32) See [2, p. 262]; and also [23, pp. 94-98].

33) See [2, p. 569].

34) [3, p. 147]; and [21].

35) The asterisk in σ^* indicates the restriction of σ to the subfield S. Furthermore, also note that the cohomology class is to be taken on the right side of the equation.

36) See [15] for a model in which algebras are used.

37) See [23, pp. 100-103].

38) See [23, pp. 102-3].

$\prod_{\tau \in \mathcal{K}_i} f(\sigma_i, \tau) = a_i$. But then $g(\sigma_i) = N_{\check{K}_i/F}(a_i)$[39]. Next apply relation (vii) to the cohomology class c, letting $\hat{\mathcal{K}}_i = \mathcal{H}$, $\mathcal{K}_i = \mathcal{L}$, then

$$\begin{aligned} c^{m_i} &= \Lambda_{K_i,K}(N_{K/K_i}(f(\sigma_i^\mu, \sigma_i^\nu)) \\ &= \Lambda_{K_i,K}(N_{\check{K}_i/F}(f(\sigma_i^\mu, \sigma_i^\nu)) , \end{aligned}$$

where $0 \leqq \mu, \nu \leqq n_i, 1 \leqq i \leqq s$[40]. Finally, the assumption that c has order n is used. Noting Remark (v) it follows that c^{m_i} has order n_i if c has order n. This implies that $N_{\check{K}_i/F}(a_i) = g(\sigma_i)$, computed for the normalized representative $f(\sigma, \tau)$, has the precise order n_i modulo $N(K^*)$. The coset $g(\sigma_i)$ modulo $N(K^*)$ does not depend on the normalized representative $f(\sigma, \tau)$ of the cohomology class c[41]. Consequently $g(\sigma_i)^t \in N_{K_i/F}(K_i^*)$ if and only if $t \equiv 0 \pmod{n_i}$. Moreover, note that the elements $N_{\check{K}_i/F}(a_i)N(K_i^*)$ lie in $N_{\check{K}_i/F}(\check{K}_i^*)$ by the transitivity property of the norm. Assume now that $g(\prod_{i=1}^{s} \sigma_i^{y_i})$, computed for a class c of order n, lies in $N(K^*)$. Then $g(\sigma_i)^{y_i}$ lies in $N_{K_i/F}(K_i^*)$, $1 \leqq i \leqq s$, because $g(\sigma_j) \in N_{\check{K}_j/F}(K_j^*) \subseteqq N_{K_i/F}(K_i)$, for all j distinct from a given fixed i. This implies $y_i \equiv 0 \pmod{n_i}$, thus $\prod_{i=1}^{s} \sigma_i^{y_i} = 1$. Hence $g(\sigma)$ establishes an isomorphism of $\mathcal{K}$ into the norm class group $F^*/N(K^*)$ whose order is at most equal to $[K:F]$ according to the corollary of Theorem 2. This means that $g(\sigma)$ determines an isomorphism J_c as asserted.

THEOREM 5. *If K/F is a normal extension with finite Galois group $\mathcal{K} = \{\sigma, \cdots\}$ whose commutator group is $\mathcal{K}'$, K'/F denoting the corresponding subfield, then each cohomology class c of order $[K:F]$ in $H^2(\mathcal{K}, K^*)$ determines by $\sigma \to J_c(\sigma)$ an isomorphism of $\mathcal{K}/\mathcal{K}'$ with the norm class group $F^*/N(K^*)$*[42].

As in the preceding proof details of argument are simplified if cohomology classes are represented by suitably normalized cocycles. In this case the normalization (see page 240) shall be used. Suppose therefore that the cohomology class c of order n is represented by the cocycle $k(\sigma, \tau)$ loc. cit. Next, letting $u = [\mathcal{K}' : 1]$, $c^u = \Lambda_{K',K}(d'(\sigma', \tau'))(\text{mod } B^2(\mathcal{K}, K^*))$ where $d'(\sigma, \tau')$ denotes the cocycle of $H^2(\mathcal{K}', K^*)$ which is determined by (vi) since the latter depends on the cosets of $\mathcal{K}$ modulo $\mathcal{K}'$. Consider now for the cocycle $d'(\sigma', \tau')$ in $Z^2(\mathcal{K}/\mathcal{K}', K'^*)$ the corresponding function $g(\cdots)$, that is $g'(\sigma') = \prod_{\tau'} d'(\tau', \sigma')$. Then the mapping $\sigma' \to g'(\sigma')N(K'^*)$[43] is by the preceding theorem an isomorphiscm of

39) See footnote 11).

40) Note that $N_{\check{K}_i/F}(f(\sigma_i^u, \sigma_i^\nu)) = h(\mu, \nu)$ is a normalized cocycle in $H^2(\mathcal{K}_i, K_i^*)$, one has $h(\mu, \nu) = N_{\check{K}_i/F}(h(\mu, \nu)) = [N_{\check{K}_i/F}(a_i)]^{x_i}$. Above passage to cohomology classes is tacitly assumed.

41) See foot note 11).

42) See [**15**].

43) $N(K'^*)$ denotes the group of norms of elements ot K'^* taken down to F.

the factor commutator group $\mathcal{K}/\mathcal{K}'$ with $F^*/N(K'^*)$, since $d'(\sigma', \tau')$ has order $n/u = [\mathcal{K}' : 1]$ according to (v). Using (vi) and the associativity relations one finds $g'(\sigma') = \prod_{\tau^*} \prod_{\rho \in \mathcal{K}'} k(\rho\tau^*, \sigma)[N_{K'/F}(\prod_{\rho \in \mathcal{K}'} k(\sigma, \rho))]^{-1} = g(\sigma)[N_{K'/F}(\prod_{\rho \in \mathcal{K}'} k(\sigma, \rho))]^{-1}$, where τ^* varies over a set of representatives of $\mathcal{K}$ modulo $\mathcal{K}'$ and where σ' is the coset of σ modulo $\mathcal{K}'$. Consequently $g(\sigma) \not\equiv g(\tau) \pmod{N(K'^*)}$ if σ and τ do not belong to the same coset modulo $\mathcal{K}'$. But this implies a fortiori $g(\sigma) \not\equiv g(\tau)$ (mod $N(K^*)$), for $N(K^*) \subseteqq N(K'^*)$. Since passage to another representative of a cohomology class c implies a change modulo norms, it follows that the above congruence statements are valid for arbitrary representatives of the class c. Finally (iv) and (v) imply by a strictly cohomological argument[44] that $N(K^*) = N(K'^*)$. Hence $\sigma \to g(\sigma)$ determines indeed an isomorphism from $\mathcal{K}/\mathcal{K}'$ to $F^*/N(K^*)$.

It is now easy to develop the classical theory of the norm residue symbol. For this purpose suppose that Z/F is a cyclic extension of degree n with the Galois group $\mathcal{Z}$. According to Theorem 1 the elements of the cohomology group $H^2(\mathcal{Z}, Z^*)$ are split by the unramified extension U_n/F of degree n. Moreover, $T_{U_n,Z}[H^2(\mathcal{U}_n, U_n^*)] = H^2(\mathcal{Z}, Z^*)$[45]. Next suppose that c_n is the (canonical) cohomology class in $H^2(\mathcal{U}_n, U_n^+)$ which is determined by the cocycle $p(\zeta^a, \zeta^b) = \pi^x$, $x = \left[\frac{a+b}{n}\right] - \left[\frac{a}{n}\right] - \left[\frac{b}{n}\right]$ where π denotes a prime element of F and ζ is the Frobenius automorphism of U_n/F for which $V(a^\zeta - a) > 0$ for all a in the valuation ring of U_n[46]. The basic theorem of Hasse[47] implies that the cohomology class c_n does not depend on the choice of the prime element π[48]. Furthermore denote by $f\langle a\rangle$, $a \in F^*$, the normalized cocycle $f(\sigma^a, \sigma^b) = a^x$, x defined as before, σ a generator of the Galois group $\mathcal{Z}$.

The norm residue symbol $(Z/F, a)$ is defined as the automorphism σ^α where $\bar{f}\langle a\rangle = f\langle a\rangle B^2(\mathcal{Z}, Z^*) = T_{U_n,K}(c_n)^\alpha$[49]. This definition implies immediately[50]:

44) See [**16**].

45) See for example [**9**, Theorem 5.2, p. 341, and Theorem 2].

46) The automorphism ζ need not be the classical Frobenius automorphism, any generator of the Galois group of the algebraic completion of the residue class field F may be used for its definition. See [**18**, p. 163].

47) See [8].

48) Note that for a divisor m of n, $c_n^m = \Lambda_{U_k,U_n}(c_k)$, $k = n/m$, c_k the canonical class for U_k/F. Furthermore, $P_{\mathcal{H}}[p(\zeta^a, \zeta^b)]$, $\mathcal{H}$ the subgroup of order m in U_n, determines the canonical class of $H^2(\mathcal{H}, U_n^*)$, U_n/U_k.

49) The automorphism σ^α is uniquely defined by a as an element of $\mathcal{Z}$. Note that the independence of the particular generating element σ may be proved as follows: Using a and σ, set $f\langle a\rangle = f\langle a\rangle\sigma$ and denote by $c(a, \sigma)$ the corresponding cohomology class. Next denote by $\tau = \sigma^y$ another generator of $\mathcal{Z}$. Define similarly $c(a, \tau)$. Then, by a simple computation, $c(a, \sigma) = c(a, \tau)^y$. Using the transfer mapping, suppose that $T_{Z,U_n}[c(a, \tau)] = c_n^\beta$, c_n the canonical class for U_n/F, therefore $c_n^\beta = c_n^{\alpha y}$. Con-

(viii) $(Z/F, ab)=(Z/F, a)(Z/F, b)$, for $T_{U_n,K}$ is an isomorphism, and all elements of $H^2(\mathcal{Z}, Z^*)$ are representable isomorphically by cosets of F^* modulo $N_{Z/F}(Z^*)$;

(ix) $(Z/F, a)=1$ if and only if $a \in N_{Z/F}(Z^*)$.

This follows from Theorem 3 and the connection between the non-zero elements of F and the normalized representatives of the cohomology classes relative to some generator of the Galois group; and

(x) the restriction of $(Z/F, a)$ to a subfield Z'/F is equal to $(Z'/F, a)$ if the restriction of σ to Z'/F is used to define $f\langle a\rangle B^2(\mathcal{Z}/\mathcal{Z}', Z'^*)$, $\mathcal{Z}'$ the Galois group of Z/Z'.

The last statement holds because $\bar{f}\langle a\rangle^m = \Lambda_{Z',Z}[\bar{f}\langle a\rangle]$,[51] where $f'\langle a\rangle$ is the normalized cocycle for Z/Z' which is determined by $a \in F^*$. Furthermore, $\bar{f}\langle a\rangle^m = [T_{U_n,Z}(c_n)]^{\alpha m}$, but then $\bar{f}\langle a\rangle^m = [T_{U_n,Z}(\Lambda_{U_k,U_m}(c_k))]^{\alpha}$[52].

Now suppose that K/F is an abelian extension which is the union of s cyclic subfields K_i/F with respective Galois groups $\mathcal{K}_i=\{\sigma_i\}$[53].

Then $(K/F, a)$ is defined as $\prod_{i=1}^{s}(K_i/F, a)$.

THEOREM 6. *Suppose that K/F is an abelian extension with the Galois group $\mathcal{K}=\{\sigma, \cdots\}$. Then the mapping $a \rightarrow (K/F, a)=\sigma$ is an isomorphism of the norm class group $F^*/N(K^*)$ on $\mathcal{K}$. Furthermore, $(K/F, g(\sigma))=\sigma$ if $g(\sigma)$ is computed for a cocycle in the canonical cohomology class $T_{U_n,K}(c_n)=c_K$ of $H^2(\mathcal{K}, K^*)$.*

The proof can be achieved by a slight extension of the arguments leading to Theorem 4. Now normalize $c=c_K \in H^2(\mathcal{K}, K^*)$ to be the principal cohomology class $T_{U_n,K}(c_n)$ which is to be represented by the normalized couple $f=f(\sigma, \tau)$[55]. Then $f^{m_i}=\Lambda_{\check{K}_i,K}(N_{\check{K}_i/F}(f(\sigma_i^{\mu}, \sigma_i^{\nu}))$ (mod $B^2(\mathcal{K}, K^*)$).

Furthermore, by definition of the norm residue symbol and the fact that the cohomology class of f^{m_i} is the lift $\Lambda_{K_i,K}(c_{K_i})$ of the principal class c_{K_i} of K_i/F, it follows that $(K_i/F, g(\sigma_i))=\sigma_i$. Furthermore, according to the normalization of the cocycle f, the norm $N_{\check{K}_i/F}(a_i)=g(\sigma_i)$ lies in the norm group $N_{\check{K}_i/F}(\check{K}_i^*)$. But this implies that $(\check{K}_i/F, g(\sigma_i))=1$ since $N_{\check{K}_i/F}(\check{K}_i^*) \subseteq N_{K_1/F}(K_1^*/ \cap \cdots \cap N_{K_{i-1}}(K_{i-1}^*) \cap N_{K_{i+1}/F}(K_{i+1}^*) \cap \cdots \cap N_{K_s/F}(K_s^*) = H_1 \cap \cdots \cap H_{i-1} \cap H_{i+1} \cap \cdots \cap H_s$ accord-

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sequently $\alpha \equiv \beta y$ (mod n), i.e., $\sigma^{\alpha}=\tau\beta$. This is a cohomological version of [**3**, p. 145].

50) This rule simply expresses the fact that cohomology classes form a group and that the transfer mapping is an isomorphism.

51) See for example [**9**, p. 334]. The bar denotes passage to the cohomology class.

52) See footnote 48).

53) See the proof of Theorem 4 for notation and identification of the groups K_i.

54) This is the classical law of reprocity of local class field theory. See, for example, [**5**] and [**7**].

55) See the notation in the proof of Theorem .4 The coccle $f(\sigma_i^{\mu}, \sigma_i^{\nu})$ is determined by the element $g(\sigma_i)=N_{\check{K}_i/F}(a_i)$ as before.

ing to the special result (ix) for cyclic extensions. Therefore, according to the definition of the norm refidue symbol for abelian extensions $(K/F, g(\sigma_i)) = \sigma_i$. Consequently (iii) implies $(K/F, a_\sigma) = \sigma$ for every a_σ lying in the coset of $g(\sigma)$ modulo $N(K^*)$. This means that the kernel H of the norm residue mapping of K/F equals $\bigcap_{i=1}^{s} H_i$, for each element of the Galois group $\mathcal{K}$ is the image of an element of F^*. Therefore $[F^* : H] = [\mathcal{K} : 1] = [K : F]$. Finally, $N(K^*) \subseteqq \bigcap_{i=1}^{s} N_{K_i/F}(K_i^*) = \bigcap_{i=1}^{s} H_i = H$ and hence $N(K^*) = H$ since $[F^* : N(K^*)] = [K : F]$ according to Theorem 4 and the initial choice of the principal cocycle[56].

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56) The uniqueness proof of Chevalley [3, p. 150], for $(K/F, a)$, i.e., independence of the representation of the field K as the union of cyclic subfields K_i carries over without change because of the existence of the element $g(\sigma)$ for given σ. See the above proof and that of Theorem 5, p. 31.

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