J. Math. Soc. Japan
Vol. 13, No. 2, 1961

On almost-analytic tensors of mixed type in a K-space

By Sumio SAWAKI

(Received Nov. 21, 1960)
(Revised Dec. 16, 1960)

§ 0. Introduction.

Let X_n be an n-dimensional differentiable manifold with local coordinates $\{x^i\}$[1]. On this manifold a tensor field $\varphi_j{}^i$ such that

$$\varphi_r{}^i \varphi_j{}^r = -\delta_j^i \tag{0.1}$$

is called an almost-complex structure and a differentiable manifold X_n with such an almost-complex structure is called an almost-complex manifold or an almost-complex space[2].

An almost-complex space X_n with an almost-complex structure satisfying

$$g_{rs}\varphi_j{}^r \varphi_i{}^s = g_{ji} \tag{0.2}$$

where g_{ji} is a positive definite Riemannian metric tensor is called an almost-Hermitian space[3]. In this place, it is easily verified that $\varphi_{ji} = -\varphi_{ij}$ where $\varphi_{ji} = \varphi_j{}^r g_{ri}$.

On the other hand, A. Frölicher[4] proved that there exists an almost-complex structure on the six dimensional sphere S^6, and T. Fukami and S. Ishihara[5] proved that the structure on S^6 is an almost-Hermitian one satisfying

$$\nabla_j \varphi_{ih} + \nabla_i \varphi_{jh} = 0 \tag{0.3}$$

where ∇_j denotes the operator of covariant derivation with respect to the Riemannian connection.

In this paper, by a K-space[6] we shall always mean an n-dimensional almost-Hermitian space satisfying the condition (0.3).

Now, a necessary and sufficient condition that in a compact K-space a vector be almost-analytic (see § 1) has been obtained for a contravariant vector by S. Tachibana in [**10**] and for a covariant vector by the author in [**7**].

---

1) Through this paper the Latin indices run over the values $1, 2, \cdots, n$.
2), 3) K. Yano [**13**, p. 228].
4) A. Frölicher [**3**].
5) T. Fukami and S. Ishihara [**4**].
6) S. Tachibana [**10**].

Recently[7] the author has obtained a necessary and sufficient condition for a contravariant pure tensor or a covariant pure tensor in a compact K-space to be almost-analytic.

The main purpose of this paper is to do exactly the same thing for a pure tensor of mixed type in a compact K-space and to summarise these results. In the last section we shall give a generalization of Bochner's theorem[8] in a compact Kählerian space as an application of these results.

§ 1. Almost-analytic tensors of mixed type.

In an n-dimensional almost-Hermitian space X_n, we consider the operators

$$(1.1)\qquad O_{ih}^{ml}=\frac{1}{2}(\delta_i^m\delta_h^l-\varphi_i^m\varphi_h^l)\,,\qquad {}^*O_{ih}^{ml}=\frac{1}{2}(\delta_i^m\delta_h^l+\varphi_i^m\varphi_h^l)$$

and call a tensor pure (hybrid) in two indices if it is annihilated by transvection of $*O$ (O) on these indices[9]. For instance, if ${}^*O_{i_1 l}^{m j_1}T_{m i_2\cdots i_p}^{l j_2\cdots j_q}=0$, then $T_{i_1\cdots i_p}^{j_1\cdots j_q}$ is called pure in j_1, i_1 and if $O_{i_1 i_2}^{ml}T_{m l i_3\cdots i_p}^{j_1\cdots j_q}=0$, then it is called hybrid in i_1, i_2.

By a pure tensor we mean that it is pure in every pair of indices.

The following propositions which we shall use later on will be easily verified.

PROPOSITION 1. ${}^*O_{ih}^{ml}+O_{ih}^{ml}=A\,,\qquad {}^*O_{tl}^{ms}\,{}^*O_{is}^{th}={}^*O_{il}^{mh}\,,$
$O_{tl}^{ms}O_{is}^{th}=O_{il}^{mh}\,,\qquad {}^*O_{tl}^{ms}O_{is}^{th}=O_{tl}^{ms}\,{}^*O_{is}^{th}=0$
where A is an identity operator.

PROPOSITION 2. ${}^*O_{ih}^{ab}\nabla_j\varphi_{ab}=0\,,\qquad O_{ib}^{ah}\nabla_j\varphi_a{}^b=0\,.$

PROPOSITION 3. *If a tensor is pure (hybrid) in i, j and pure (hybrid) in j, h, then it is pure in i, h, and if it is pure in i, j and hybrid in j, h, then it is hybrid in i, h.*

PROPOSITION 4. *If a tensor is pure and at the same time hybrid in two given indices, then it vanishes.*

PROPOSITION 5. *If a tensor $T_{\cdots h\cdots j\cdots}^{\cdots\cdots\cdots}$ is pure (hybrid) in i, j, then we have*

$$\varphi_h{}^iT_{\cdots i\cdots j\cdots}^{\cdots\cdots\cdots}=\varphi_j{}^iT_{\cdots h\cdots i\cdots}^{\cdots\cdots\cdots}\qquad(-\varphi_j{}^iT_{\cdots h\cdots i\cdots}^{\cdots\cdots\cdots})$$

and if a tensor $T_{\cdots h\cdots}^{\cdots j\cdots}$ is pure (hybrid) in j, i, then

$$\varphi_h{}^iT_{\cdots i\cdots}^{\cdots j\cdots}=\varphi_i{}^jT_{\cdots h\cdots}^{\cdots i\cdots}\qquad(-\varphi_i{}^jT_{\cdots h\cdots}^{\cdots i\cdots})\,.$$

We say that a pure tensor $T_{i_1\cdots i_p}^{j_1\cdots j_q}$ $(p, q\geqq 0)$ is almost-analytic if it satisfies

7) S. Sawaki [8].
8) S. Bochner [2].
9) K. Yano [13, p. 228].

(1.2)
$$\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p} + \varphi_h{}^s\nabla_s(\varphi_t{}^{j_1} T^{t j_2\cdots j_q}_{i_1\cdots i_p}) - \sum_{r=1}^{p} \varphi_h{}^s(\nabla_{i_r}\varphi_s{}^t) T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}$$
$$+ \sum_{r=1}^{q} \varphi_h{}^s(\nabla_t\varphi_s{}^{j_r} - \nabla_s\varphi_t{}^{j_r}) T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p} = 0^{10)} \qquad \text{for} \quad q \geqq 1$$

and

(1.3)
$$\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p} + \varphi_h{}^s\nabla_s(\varphi_{i_1}{}^t T^{j_1\cdots j_q}_{t i_2\cdots i_p})$$
$$- \sum_{r=1}^{p} \varphi_h{}^s(\nabla_{i_r}\varphi_s{}^t) T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p} + \sum_{r=1}^{q} \varphi_h{}^s(\nabla_t\varphi_s{}^{j_r} - \nabla_s\varphi_t{}^{j_r}) T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p} = 0 \qquad \text{for} \quad p \geqq 1$$

where $(\nabla_{i_r}\varphi_s{}^t) T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}$ etc. mean $(\nabla_{i_r}\varphi_s{}^t) T^{j_1\cdots\cdots\cdots\cdots\cdots\cdots j_q}_{i_1\cdots i_{r-1} t i_{r+1}\cdots i_p}$ etc.. These are generalizations of analytic tensors in a Kählerian space[11].

Since $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ is a pure tensor, if $p, q \geqq 1$, then (1.2) and (1.3) are equivalent to each other.

§ 2. Identities in a K-space.

Let X_n be an almost-Hermitian space, and let $R_{kji}{}^h$ be the curvature tensor formed by the Riemannian connection. We put

(2.1)
$$R_{ji} = R_{rji}{}^r, \qquad R_{kjih} = R_{kji}{}^r g_{rh}, \qquad R^*{}_{kj} = \frac{1}{2}\varphi^{ab}R_{abrj}\varphi_k{}^r,$$
$$R^{*k}{}_j = R^*{}_{rj}g^{rk}, \qquad R^*{}_k{}^j = R^*{}_{kr}g^{rj}.$$

The identity of Ricci[12] is expressed in the following form for any tensor $T^{j_1\cdots j_q}_{i_1\cdots i_p}$:

(2.2)
$$\nabla_k\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p} - \nabla_h\nabla_k T^{j_1\cdots j_q}_{i_1\cdots i_p}$$
$$= \sum_{r=1}^{q} R_{khs}{}^{j_r} T^{j_1\cdots s\cdots j_q}_{i_1\cdots\cdots j_p} - \sum_{r=1}^{p} R_{khi_r}{}^s T^{j_1\cdots\cdots j_q}_{i_1\cdots s\cdots i_p}.$$

Transvecting (2.2) with φ^{kh}, we have

$$\varphi^{kh}\nabla_k\nabla_h T^{j_1\cdots j_q}_{i_1\cdots p} = \frac{1}{2}\sum_{r=1}^{q} R_{khs}{}^{j_r} T^{j_1\cdots s\cdots j_q}_{i_1\cdots\cdots i_p}\varphi^{kh} - \frac{1}{2}\sum_{r=1}^{p} R_{khi_r}{}^s T^{j_1\cdots\cdots j_q}_{i_1\cdots s\cdots i_p}\varphi^{kh}$$

or denoting i_l for some l $(1 \leqq l \leqq p)$ by t

(2.3)
$$\varphi^{kh}\nabla_k\nabla_h T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}$$
$$= \frac{1}{2}\sum_{r=1}^{q} R_{khs}{}^{j_r} T^{j_1\cdots s\cdots j_q}_{i_1\cdots t\cdots i_p}\varphi^{kh} - \frac{1}{2}\sum_{r=1}^{p} R_{khi_r}{}^s T^{j_1\cdots\cdots\cdots j_q}_{i_1\cdots t\cdots s\cdots i_p}\varphi^{kh}$$

where $\varphi^{kh} = \varphi_r{}^h g_r{}^k$.
If $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ is a pure tensor, transvecting (2.3) with $\varphi_{i_l}{}^t$, we have

10) S. Tachibana [**11**], S. Kotō [**5**] and S. Sawaki [**8**].
11) K. Yano and S. Bochner [**12**].
12) J. A. Schouten [**9**].

$$\varphi_{i_l}{}^t \varphi^{kh} \nabla_k \nabla_h T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p} \tag{2.4}$$

$$= \frac{1}{2} \sum_{r=1}^{q} \varphi^{kh} R_{khs}{}^{j_r} \varphi_t{}^s T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p} - \sum_{r=1}^{p} \varphi^{kh} R_{khi_r}{}^s \varphi_s{}^t T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}$$

$$= \sum_{r=1}^{q} R^*{}_t{}^{j_r} T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p} - \sum_{r=1}^{p} R^{*t}{}_{i_r} T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p} .$$

For the tensor $\varphi_j{}^i$ we have

$$\varphi_h{}^s \nabla_s \nabla^h \varphi_k{}^t = -\frac{1}{2} \varphi^{sh} (\nabla_s \nabla_h \varphi_k{}^t - \nabla_h \nabla_s \varphi_k{}^t) \tag{2.5}$$

$$= -\frac{1}{2} \varphi^{sh} (R_{sha}{}^t \varphi_k{}^a - R_{shk}{}^a \varphi_a{}^t)$$

$$= -R^*{}_k{}^t + R^{*t}{}_k$$

where $\nabla^h = g^{hr}\nabla_r$, and by (0.1) we have easily

$$\varphi_j{}^r \nabla_h \varphi_r{}^i = -\varphi_r{}^i \nabla_h \varphi_j{}^r . \tag{2.6}$$

In the rest of the paper, unless otherwise stated, we shall only consider a K-space. Taking account of (0.3), we get

$$*O^{ab}_{ji} \nabla_a \varphi_{bh} = \nabla_j \varphi_{ih} + \varphi_j{}^a \varphi_i{}^b \nabla_a \varphi_{bh}$$

$$= \nabla_j \varphi_{ih} + \varphi_j{}^a \varphi_i{}^b \nabla_h \varphi_{ab}$$

$$= *O^{ab}_{ji} \nabla_h \varphi_{ab}$$

and hence by virtue of Proposition 2, we find

$$*O^{ab}_{ji} \nabla_a \varphi_{bh} = 0 . \tag{2.7}$$

Moreover from (0.3) we have the following

$$\nabla_r \varphi_j{}^r = 0 . \tag{2.8}$$

Since by (2.7) and Proposition 5 we have $\varphi_i{}^l \nabla_l \varphi_{jh} = \varphi_j{}^l \nabla_i \varphi_{lh}$, the Nijenhuis tensor defined by

$$N_{ji}{}^h = \varphi_j{}^l (\nabla_l \varphi_i{}^h - \nabla_i \varphi_l{}^h) - \varphi_i{}^l (\nabla_l \varphi_j{}^h - \nabla_j \varphi_l{}^h)$$

can be easily written as

$$N_{ji}{}^h = 2\varphi_j{}^l (\nabla_l \varphi_i{}^h - \nabla_i \varphi_l{}^h) . \tag{2.9}$$

By using (0.3) the equation (2.9) turns to

$$N_{ji}{}^h = 4\varphi_j{}^r \nabla_r \varphi_i{}^h \tag{2.10}$$

from which we find

$$N_{j(ih)} = 0 . \tag{2.11}$$

The following properties which are also valid in an almost-complex space can be easily verified.

(2.12) $$*O_{ji}^{ab}N_{ab}{}^{h}=0\,,\qquad O_{ib}^{ah}N_{ja}{}^{b}=0\,.$$ [13]

Furthermore the following relations can be proved:

(2.13) $$R^{*}{}_{ri}=R^{*}{}_{ir}\,,\qquad (\nabla_{i}\varphi_{kj})\nabla_{r}\varphi^{kj}=R_{ir}-R^{*}{}_{ir}\,.$$ [14]

Indeed, since φ^{kj} is hybrid in k,j and $\nabla_{k}\varphi_{ji}$ is pure in k,j because of (2.7), we have by Proposition 4

(2.14) $$(\nabla_{k}\varphi_{ji})\varphi^{kj}=0\,.$$

If we operate ∇_{r} to (2.14), then by making use of the Ricci's identity and anti-symmetry of φ^{kj}, we have

$$\begin{aligned}(\nabla_{i}\varphi_{kj})\nabla_{r}\varphi^{kj}&=-(\nabla_{r}\nabla_{k}\varphi_{ji})\varphi^{kj}\\ &=R^{*}{}_{ri}-2R^{*}{}_{ir}+R_{ri}\,.\end{aligned}$$

As the left hand side is symmetric with respect to i and r, we have

$$R^{*}{}_{ir}=R^{*}{}_{ri}$$

and therefore

$$(\nabla_{i}\varphi_{kj})\nabla_{r}\varphi^{kj}=R_{ir}-R^{*}{}_{ir}\,.$$

Thus by virtue of (2.10), (2.11) and (2.13), we have

(2.15) $$N_{rji}N_{k}{}^{ji}=N_{jir}N^{ji}{}_{k}=16(R_{rk}-R^{*}{}_{rk})$$

where $N_{k}{}^{ji}=N_{kr}{}^{i}g^{rj}$ etc. and from (2.5) we have

(2.16) $$\varphi^{sh}\nabla_{s}\nabla_{h}\varphi_{k}{}^{t}=0\,.$$

§ 3. Lemmas.

In this section we shall give some lemmas which will be used to prove the main theorem of this paper in § 4. Let $T_{i_1\cdots i_p}^{j_1\cdots j_q}$ be a pure tensor in a K-space and we consider the following two cases.

1) The case $p\geqq 0$, $q\neq 1$ or $p\geqq 2$, $q=1$.

If $T_{i_1\cdots i_p}^{j_1\cdots j_q}$ is almost-analytic, then from (1.2) and (1.3) we have respectively

(3.1) $$\begin{aligned}&\nabla_{h}T_{i_1\cdots i_p}^{j_1\cdots j_q}+\varphi_{h}{}^{s}\varphi_{t}{}^{j_1}\nabla_{s}T_{i_1\cdots i_p}^{tj_2\cdots j_q}+\varphi_{h}{}^{s}(\nabla_{t}\varphi_{s}{}^{j_1})T_{i_1\cdots i_p}^{tj_2\cdots j_q}\\ &\quad-\sum_{r=1}^{p}\varphi_{h}{}^{s}(\nabla_{i_r}\varphi_{s}{}^{t})T_{i_1\cdots t\cdots i_p}^{j_1\cdots\cdots j_q}+\sum_{r=2}^{q}\varphi_{h}{}^{s}(\nabla_{t}\varphi_{s}{}^{j_r}-\nabla_{s}\varphi_{t}{}^{j_r})T_{i_1\cdots\cdots i_p}^{j_1\cdots t\cdots j_q}=0\end{aligned}$$

and

(3.2) $$\begin{aligned}&\nabla_{h}T_{i_1\cdots i_p}^{j_1\cdots j_q}+\varphi_{h}{}^{s}\varphi_{i_1}{}^{t}\nabla_{s}T_{ti_2\cdots i_p}^{j_1\cdots j_q}\\ &\quad+\varphi_{h}{}^{s}(\nabla_{s}\varphi_{i_1}{}^{t}-\nabla_{i_1}\varphi_{s}{}^{t})T_{ti_2\cdots i_p}^{j_1\cdots j_q}-\sum_{r=2}^{p}\varphi_{h}{}^{s}(\nabla_{i_r}\varphi_{s}{}^{t})T_{i_1\cdots t\cdots i_p}^{j_1\cdots\cdots j_q}\\ &\quad+\sum_{r=1}^{q}\varphi_{h}{}^{s}(\nabla_{t}\varphi_{s}{}^{j_r}-\nabla_{s}\varphi_{t}{}^{j_r})T_{i_1\cdots\cdots i_p}^{j_1\cdots t\cdots j_q}=0\,.\end{aligned}$$

---

13) K. Yano [**13**].
14) S. Tachibana [**10**].

In this place, by Propositions 2, 3 and (2.7) $\varphi_h{}^s(\nabla_t\varphi_s{}^{j_1})T^{tj_2\cdots j_q}_{i_1\cdots i_p}$ is hybrid in h, j_1 and hence by Proposition 1 it can be written as

$$\varphi_h{}^s\nabla_t\varphi_s{}^{j_1}T^{tj_2\cdots j_q}_{i_1\cdots i_p}={}^*O^{sj_1}_{ht}\varphi_s{}^a(\nabla_b\varphi_a{}^t)T^{bj_2\cdots j_q}_{i_1\cdots i_p}.$$

Similarly we have

$$\sum_{r=1}^{p}\varphi_h{}^s(\nabla_{i_r}\varphi_s{}^t)T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}={}^*O^{sj_1}_{ht}\sum_{r=1}^{p}\varphi_s{}^a(\nabla_{i_r}\varphi_a{}^b)T^{tj_2\cdots j_q}_{i_1\cdots b\cdots i_p}$$

and

$$\sum_{r=2}^{q}\varphi_h{}^s(\nabla_t\varphi_s{}^{j_r}-\nabla_s\varphi_t{}^{j_r})T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}=-\frac{1}{2}O^{sj_1}_{ht}\sum_{r=2}^{q}N_{sb}{}^{j_r}T^{tj_2\cdots b\cdots j_q}_{i_1\cdots\cdots i_p}$$

because of (2.9).

Thus the equation (3.1) can be written in the following

$$\begin{aligned}(3.3)\qquad &{}^*O^{sj_1}_{ht}[2\nabla_sT^{tj_2\cdots j_q}_{i_1\cdots i_p}+\varphi_s{}^a(\nabla_b\varphi_a{}^t)T^{bj_2\cdots j_q}_{i_1\cdots i_p}\\ &-\sum_{r=1}^{p}\varphi_s{}^a(\nabla_{i_r}\varphi_a{}^b)T^{tj_2\cdots j_q}_{i_1\cdots b\cdots i_p}]-\frac{1}{2}O^{sj_1}_{ht}[\sum_{r=2}^{q}N_{sb}{}^{j_r}T^{tj_2\cdots b\cdots j_q}_{i_1\cdots\cdots i_p}]=0.\end{aligned}$$

If we operate ${}^*O^{hl}_{kj_1}$ and $O^{hl}_{kj_1}$ to (3.3), then we have by Proposition 1 respectively

$$\begin{aligned}(3.4)\qquad &\nabla_hT^{j_1\cdots j_q}_{i_1\cdots i_p}+\varphi_h{}^s\varphi_t{}^{j_1}\nabla_sT^{tj_2\cdots j_q}_{i_1\cdots i_p}\\ &+\varphi_h{}^s(\nabla_t\varphi_s{}^{j_1})T^{tj_2\cdots j_q}_{i_1\cdots i_p}-\sum_{r=1}^{p}\varphi_h{}^s(\nabla_{i_r}\varphi_s{}^t)T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}=0\end{aligned}$$

and

$$(3.5)\qquad \sum_{r=2}^{q}N_{ht}{}^{j_r}T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}=0.$$

Consequently by (3.5), the equation (3.2) turns to

$$\begin{aligned}&\nabla_hT^{j_1\cdots j_q}_{i_1\cdots i_p}+\varphi_h{}^s\varphi_{i_1}{}^t\nabla_sT^{j_1\cdots j_q}_{ti_2\cdots i_p}-\sum_{r=2}^{p}\varphi_h{}^s(\nabla_{i_r}\varphi_s{}^t)T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}\\ &+\varphi_h{}^s(\nabla_s\varphi_{i_1}{}^t-\nabla_{i_1}\varphi_s{}^t)T^{j_1\cdots j_q}_{ti_2\cdots i_p}+\varphi_h{}^s(\nabla_t\varphi_s{}^{j_1}-\nabla_s\varphi_t{}^{j_1})T^{tj_2\cdots j_q}_{i_1\cdots i_p}=0\end{aligned}$$

and then by (2.9) we have

$$\begin{aligned}(3.6)\qquad &\nabla_hT^{j_1\cdots j_q}_{i_1\cdots i_p}+\varphi_h{}^s\varphi_{i_1}{}^t\nabla_sT^{j_1\cdots j_q}_{ti_2\cdots i_p}-\sum_{r=2}^{p}\varphi_h{}^s(\nabla_{i_r}\varphi_s{}^t)T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}\\ &+\frac{1}{2}N_{hi_1}{}^tT^{j_1\cdots j_q}_{ti_2\cdots i_p}-\frac{1}{2}N_{ht}{}^{j_1}T^{tj_2\cdots j_q}_{i_1\cdots i_p}=0.\end{aligned}$$

By the same way as in the preceding arguments we can express (3.6) in the following

$$\begin{aligned}(3.7)\qquad &{}^*O^{st}_{hi_1}[2\nabla_sT^{j_1\cdots j_q}_{ti_2\cdots i_p}-\sum_{r=2}^{p}\varphi_s{}^a(\nabla_{i_r}\varphi_a{}^b)T^{j_1\cdots\cdots j_q}_{ti_2\cdots b\cdots i_p}]\\ &+\frac{1}{2}O^{st}_{hi_1}[N_{st}{}^bT^{j_1\cdots j_q}_{bi_2\cdots i_p}-N_{sb}{}^{j_1}T^{bj_2\cdots j_q}_{ti_2\cdots i_p}]=0.\end{aligned}$$

Operating $*O_{kl}^{hi_1}$ and $O_{kl}^{hi_1}$, we have respectively

$$\nabla_h T_{i_1\cdots i_p}^{j_1\cdots j_q}+\varphi_h{}^s\varphi_{i_1}{}^t\nabla_s T_{ti_2\cdots i_p}^{j_1\cdots j_q}-\sum_{r=2}^{p}\varphi_h{}^s(\nabla_{i_r}\varphi_s{}^t)T_{i_1\cdots t\cdots i_p}^{j_1\cdots\cdots j_q}=0\,, \tag{3.8}$$

and

$$N_{hi_1}{}^t T_{ti_2\cdots i_p}^{j_1\cdots j_q}=N_{ht}{}^{j_1}T_{i_1\cdots i_p}^{tj_2\cdots j_q}\,. \tag{3.9}$$

Next, from (3.5), we have

$$N_{ht}{}^{j_2}T_{i_1\cdots\cdots i_p}^{j_1tj_3\cdots j_q}=-(N_{ht}{}^{j_3}T_{i_1\cdots\cdots\cdots i_p}^{j_1j_2tj_4\cdots j_q}+\cdots+N_{ht}{}^{j_q}T_{i_1\cdots\cdots i_p}^{j_1\cdots j_{q-1}t})\,. \tag{3.10}$$

Since $N_{ht}{}^{j_r}$ is pure in h, t because of (2.12) and $T_{i_1\cdots\cdots i_p}^{j_1j_2\cdots t\cdots j_q}$ is pure in j_2, t, by virtue of Proposition 3, the right hand side of (3.10) is pure in h, j_2. On the other hand by (2.12) the left hand side of (3.10) is hybrid in h, j_2. Accordingly, by Proposition 4, we find

$$N_{ht}{}^{j_2}T_{i_1\cdots\cdots i_p}^{j_1tj_3\cdots j_q}=0\,,\qquad N_{ht}{}^{j_3}T_{i_1\cdots\cdots\cdots i_p}^{j_1j_2tj_4\cdots j_q}+\cdots+N_{ht}{}^{j_q}T_{i_1\cdots\cdots i_p}^{j_1\cdots j_{q-1}t}=0$$

and similarly from the last equation, we have

$$N_{ht}{}^{j_3}T_{i_1\cdots\cdots\cdots i_p}^{j_1j_2tj_4\cdots j_q}=0\,,\qquad N_{ht}{}^{j_4}T_{i_1\cdots\cdots\cdots\cdots i_p}^{j_1j_2j_3tj_5\cdots j_q}+\cdots+N_{ht}{}^{j_q}T_{i_1\cdots\cdots i_p}^{j_1\cdots j_{q-1}t}=0\,.$$

Repeating this process, we have

$$N_{ht}{}^{j_r}T_{i_1\cdots\cdots i_p}^{j_1\cdots t\cdots j_q}=0 \qquad \text{for every}\quad r=2,3,\cdots,q\,. \tag{3.11}$$

When $p\geqq 1$, $q\geqq 2$, the left hand side of (3.9) is pure in j_1, j_2 but the right hand side is hydrid in j_1, j_2. Hence by Proposition 4, we have

$$N_{hi_1}{}^t T_{ti_2\cdots i_p}^{j_1\cdots j_q}=N_{ht}{}^{j_1}T_{i_1\cdots i_p}^{tj_2\cdots j_q}=0\,. \tag{3.12}$$

Also for the case $p\geqq 2$, $q=1$, (3.12) holds good. In fact, in this case, from (3.9) we have

$$N_{hi_1}{}^t T_{ti_2\cdots i_p}^{j_1}=N_{ht}{}^{j_1}T_{i_1\cdots i_p}^{t}\,. \tag{3.13}$$

Here the left hand side of (3.13) is hybrid in i_1, i_2 but the right hand side is pure in i_1, i_2. Therefore both members vanish.

Moreover, if we notice that the first definition of the almost-analytic tensor (1.2) or (1.3) is equivalent to respectively

$$\nabla_h T_{i_1\cdots i_p}^{j_1\cdots j_q}+\varphi_h{}^s\nabla_s(\varphi_t{}^{j_m}T_{i_1\cdots\cdots i_p}^{j_1\cdots t\cdots j_q})$$
$$-\sum_{r=1}^{p}\varphi_h{}^s(\nabla_{i_r}\varphi_s{}^t)T_{i_1\cdots t\cdots i_p}^{j_1\cdots\cdots j_q}+\sum_{r=1}^{q}\varphi_h{}^s(\nabla_t\varphi_s{}^{j_r}-\nabla_s\varphi_t{}^{j_r})T_{i_1\cdots\cdots i_p}^{j_1\cdots t\cdots j_q}=0$$

or

$$\nabla_h T_{i_1\cdots i_p}^{j_1\cdots j_q}+\varphi_h{}^s\nabla_s(\varphi_{i_m}{}^t T_{i_1\cdots t\cdots i_p}^{j_1\cdots\cdots j_q})$$
$$-\sum_{r=1}^{p}\varphi_h{}^s(\nabla_{i_r}\varphi_s{}^t)T_{i_1\cdots t\cdots i_p}^{j_1\cdots\cdots j_q}+\sum_{r=1}^{q}\varphi_h{}^s(\nabla_t\varphi_s{}^{j_r}-\nabla_s\varphi_t{}^{j_r})T_{i_1\cdots\cdots i_p}^{j_1\cdots t\cdots j_q}=0\,,$$

then by the same way as in the preceding paragrph we shall have also the following relations:

(3.14) $$N_{hi_m}{}^{t}T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}=0\,,\qquad\text{for every}\quad m=2,3,\cdots,p\,.$$

Now, since our space is a K-space, for $p\geqq 1$ (3.8) turns to

$$\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p}+\varphi_h{}^s\varphi_{i_1}{}^t\nabla_s T^{j_1\cdots j_q}_{ti_2\cdots i_p}+\frac{1}{4}\sum_{r=2}^{p}N_{hi_r}{}^tT^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}=0$$

because of (2.10) and moreover using (3.14) it becomes

(3.15) $$\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p}+\varphi_h{}^s\varphi_{i_1}{}^t\nabla_s T^{j_1\cdots j_q}_{ti_2\cdots i_p}=0\quad\text{or}\quad {}^*O^{st}_{hi_1}\nabla_s T^{j_1\cdots j_q}_{ti_2\cdots i_p}=0\,.$$

For $q\geqq 1$ we get from (3.4)

(3.16) $${}^*O^{sj_1}_{ht}\nabla_s T^{tj_2\cdots j_q}_{i_1\cdots i_p}=0\,.$$

Since $N_{ht}{}^{j_r}T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}=0$ is equivalent to $N_{abt}N^{abj_r}T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}=0$, by (2.15), from (3.11) and (3.12) we have

(3.17) $$(R_t{}^{j_r}-R^*{}_t{}^{j_r})T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}=0\qquad\text{for every}\quad r=1,2,\cdots,q\,.$$

Similarly from (3.12) and (3.14), we have

(3.18) $$(R_{i_r}{}^t-R^*{}_{i_r}{}^t)T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}=0\qquad\text{for every } r=1,2,\cdots,p\,.$$

Thus we have (3.15), (3.16), (3.17) and (3.18) as a necessary condition for a pure tensor in a K-space to be almost-analytic and it is evident that conversely this is also a sufficient condition. Hence we have the following

LEMMA 3.1. *In a* K-*space, a pure tensor* $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ ($p\geqq 0$, $q\neq 1$ *or* $p\geqq 2$, $q=1$) *is almost-analytic if and only if*

(1) $${}^*O^{st}_{hi_1}\nabla_s T^{j_1\cdots j_q}_{ti_2\cdots i_p}=0\ (p\geqq 1)\quad\text{or}\quad {}^*O^{sj_1}_{ht}\nabla_s T^{tj_2\cdots j_q}_{i_1\cdots i_p}=0\ (q\geqq 1)\,,$$

(2) $$(R_t{}^{j_r}-R^*{}_t{}^{j_r})T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}=0\qquad\text{for every}\quad r=1,2,\cdots,q\,,$$

(3) $$(R_{i_r}{}^t-R^*{}_{i_r}{}^t)T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}=0\qquad\text{for every}\quad r=1,2,\cdots,p\,.$$

REMARK 1. In a K-space, if the rank of the matrix $\|R_{ji}-R^*{}_{ji}\|$ is n, then there exists no almost-analytic tensor $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ ($p\geqq 0$, $q\neq 1$ or $p\geqq 2$, $q=1$) other than the zero tensor.

As we remarked in (3.14), the former of the condition (1), for example, can be replaced by

$${}^*O^{st}_{hi_r}\nabla_s T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}=0\qquad\text{for every}\quad r=1,2,\cdots,p$$

which means that $\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p}$ is pure in h, i_r ($r=1,2,\cdots,p$). By the same method as in (3.4), we have for any m ($1\leqq m\leqq q$)

$$\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p}+\varphi_h{}^s\varphi_t{}^{j_m}\nabla_s T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}-\frac{1}{4}N_{ht}{}^{j_m}T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}+\frac{1}{4}\sum_{r=1}N_{hi_r}{}^tT^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}=0$$

from which we get

$${}^*O^{sj_m}_{ht}\nabla_s T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}=0\qquad\text{for every}\quad m=1,2,\cdots,q\,.$$

Thus on taking accont of Proposition 3, we have the following lemma which corresponds to the definition of analytic tensor in a Kählerian space.

LEMMA 3.2. *In a* K-*space, a pure tensor* $T_{i_1\cdots i_p}^{j_1\cdots j_q}$ $(p\geqq 0,\ q\neq 1$ *or* $p\geqq 2,\ q=1)$ *is almost-analytic if and only if*

(1) $\nabla_h T_{i_1\cdots i_p}^{j_1\cdots j_q}$ is a pure tensor,

(2) $(R_t^{\ j_r}-R^*{}_t^{\ j_r})T_{i_1\cdots\cdots i_p}^{j_1\cdots t\cdots j_q}=0$ for every $r=1,2,\cdots,q$,

(3) $(R_{i_r}^{\ \ t}-R^*{}_{i_r}^{\ \ t})T_{i_1\cdots t\cdots i_p}^{j_1\cdots\cdots j_q}=0$ for every $r=1,2,\cdots,p$.

REMARK 2. By an $^*O$-space[15] we mean an n-dimensional almost-Hermitian space satisfying $^*O_{ji}^{ab}\nabla_a\varphi_{bh}=0$. An $^*O$-space is a more general space than a K-space, because by (2.7) a K-space is an $^*O$-space. As we can see the preceding paragraph, in an $^*O$-space a pure tensor $T_{i_1\cdots i_p}^{j_1\cdots j_q}$ $(p\geqq 0,\ q\neq 1$ or $p\geqq 2,\ q=1)$ is almost-analytic if and only if

(1) $^*O_{hi_1}^{st}\nabla_s T_{ti_2\cdots i_p}^{j_1\cdots j_q}-\sum_{r=2}^{p}\varphi_h^{\ s}(\nabla_{i_r}\varphi_s^{\ t})T_{i_1\cdots t\cdots i_p}^{j_1\cdots\cdots j_q}=0$ $(p\geqq 1)$ or (3.4) $(q\geqq 1)$,

(2) $N_{ht}^{\ \ j_r}T_{i_1\cdots\cdots i_p}^{j_1\cdots t\cdots j_q}=0$ for every $r=1,2,\cdots,q$,

(3) $N_{hi_r}^{\ \ \ t}T_{i_1\cdots t\cdots i_p}^{j_1\cdots\cdots j_q}=0$ for every $r=1,2,\cdots,p$.

If the rank of the matrix $\|N^{ab}{}_jN_{abi}\|$ is n, then there exists no almost-analytic tensor $T_{i_1\cdots i_p}^{j_1\cdots j_q}$ $(p\geqq 0,\ q\neq 1$ or $p\geqq 2,\ q=1)$ other than the zero tensor.

2) The case $p=q=1$.

Let $T_i^{\ j}$ be an almost-analytic tensor. In this case we can not make use of the relations (3.17) and (3.18). But since (3.8) and (3.9) hold good, we have

$$\nabla_h T_i^{\ j}+\varphi_h^{\ s}\varphi_i^{\ t}\nabla_s T_t^{\ j}=0\,, \tag{3.19}$$

$$N_{ht}^{\ \ j}T_i^{\ t}-N_{hi}^{\ \ t}T_t^{\ j}=0\,. \tag{3.20}$$

On the other hand, we have from (3.1)

$$\nabla_h T_i^{\ j}+\varphi_h^{\ s}\varphi_t^{\ j}\nabla_s T_i^{\ t}+\varphi_h^{\ s}(\nabla_t\varphi_s^{\ j})T_i^{\ t}-\varphi_h^{\ s}(\nabla_i\varphi_s^{\ t})T_t^{\ j}=0$$

or using (2.10)

$$\nabla_h T_i^{\ j}+\varphi_h^{\ s}\varphi_t^{\ j}\nabla_s T_i^{\ t}-\frac{1}{4}N_{ht}^{\ \ j}T_i^{\ t}+\frac{1}{4}N_{hi}^{\ \ t}T_t^{\ j}=0\,,$$

from which we have by (3.20)

$$\nabla_h T_i^{\ j}+\varphi_h^{\ s}\varphi_t^{\ j}\nabla_s T_i^{\ t}=0\,. \tag{3.21}$$

Consequently we see that (3.19) and (3.20) are equivalent to (3.21) and (3.20). Thus we have the following

LEMMA 3.3. *In a* K-*space, a pure tensor* $T_i^{\ j}$ *is almost-analytic if and only if*

(1) $^*O_{ht}^{sj}\nabla_s T_i^{\ t}=0\,,$

(2) $N_{ht}^{\ \ j}T_i^{\ t}-N_{hi}^{\ \ t}T_t^{\ j}=0$

where (1) *may be replaced by* $^*O_{hi}^{st}\nabla_s T_t^{\ j}=0$.

15) S. Kotō [5].

LEMMA 3.4. *In a* K-*space, a pure tensor* $T_i^{\ j}$ *is almost-analytic if and only if*

(1) $\nabla_h T_i^{\ j}$ *is pure tensor,*

(2) $N_{ht}^{\ \ j} T_i^{\ t} - N_{hi}^{\ \ t} T_t^{\ j} = 0$.

REMARK 3. In an *O-space, a pure tensor is almost-analytic if and only if

(1) ${}^*O_{hi}^{st} \nabla_s T_t^{\ j} = 0$,

(2) $N_{ht}^{\ \ j} T_i^{\ t} - N_{hi}^{\ \ t} T_t^{\ j} = 0$.

§ 4. Main theorem.

First by using Lemma 3.1 and Lemma 3.3 we shall prove the following two theorems.

THEOREM 4.1. *In a compact* K-*space, a necessary and sufficient condition that a pure tensor* $T_{i_1 \cdots i_p}^{j_1 \cdots j_q}$ ($p \geqq 0$, $q \neq 1$ *or* $p \geqq 2$, $q = 1$) *be almost-analytic is that it satisfies*

(1) $$\nabla^h \nabla_h T_{i_1 \cdots i_p}^{j_1 \cdots j_q} + \sum_{r=1}^{q} R_t^{\ j_r} T_{i_1 \cdots\cdots i_p}^{j_1 \cdots t \cdots j_q} - \sum_{r=1}^{p} R_{i_r}^{\ \ t} T_{i_1 \cdots t \cdots i_p}^{j_1 \cdots\cdots j_q} = 0,$$

(2) $$(R_t^{\ j_r} - R^*{}_t^{\ j_r}) T_{i_1 \cdots\cdots i_p}^{j_1 \cdots t \cdots j_q} = 0 \qquad \text{for every } r = 1, 2, \cdots, q,$$

(3) $$(R_{i_r}^{\ \ t} - R^*{}_{i_r}^{\ \ t}) T_{i_1 \cdots t \cdots i_p}^{j_1 \cdots\cdots j_q} = 0 \qquad \text{for every } r = 1, 2, \cdots, p.$$

PROOF. If $T_{i_1 \cdots i_p}^{j_1 \cdots j_q}$ is almost-analytic, then from Lemma 3.1 we have (2), (3) and

(4.1) $$-P_{h i_1 \cdots i_p}^{j_1 \cdots j_q} \overset{\text{def}}{=} \nabla_h T_{i_1 \cdots i_p}^{j_1 \cdots j_q} + \varphi_h^{\ s} \varphi_{i_1}^{\ \ t} \nabla_s T_{t i_2 \cdots i_p}^{j_1 \cdots j_q} = 0 \qquad \text{for } p \geqq 1.$$

Operating ∇_h to (4.1) and using (2.8) we have

(4.2) $$\nabla^h \nabla_h T_{i_1 \cdots i_p}^{j_1 \cdots j_q} + \varphi_h^{\ s} (\nabla^h \varphi_{i_1}^{\ \ t}) \nabla_s T_{t i_2 \cdots i_p}^{j_1 \cdots j_q} + \varphi_h^{\ s} \varphi_{i_1}^{\ \ t} \nabla^h \nabla_s T_{t i_2 \cdots i_p}^{j_1 \cdots j_q} = 0.$$

By virtue of (2.4) and (2.13), (4.2) can be written as

(4.3) $$\nabla^h \nabla_h T_{i_1 \cdots i_p}^{j_1 \cdots j_q} + \varphi_h^{\ s} (\nabla^h \varphi_{i_1}^{\ \ t}) \nabla_s T_{t i_2 \cdots i_p}^{j_1 \cdots j_q} + \sum_{r=1}^{q} R^*{}_t^{\ j_r} T_{i_1 \cdots\cdots i_p}^{j_1 \cdots t \cdots j_q} - \sum_{r=1}^{p} R^*{}_{i_r}^{\ \ t} T_{i_1 \cdots t \cdots i_p}^{j_1 \cdots\cdots j_q} = 0.$$

On the other hand, operating ∇_s to

$$(\nabla^h \varphi_{i_1}^{\ \ t}) T_{t i_2 \cdots i_p}^{j_1 \cdots j_q} = 0$$

which is equivalent to

$$(R_{i_1}^{\ \ t} - R^*{}_{i_1}^{\ \ t}) T_{t i_2 \cdots i_p}^{j_1 \cdots j_q} = 0$$

and transvecting with $\varphi_h^{\ s}$, we have

(4.4) $$\varphi_h^{\ s} (\nabla_s \nabla^h \varphi_{i_1}^{\ \ t}) T_{t i_2 \cdots i_p}^{j_1 \cdots j_q} + \varphi_h^{\ s} (\nabla^h \varphi_{i_1}^{\ \ t}) \nabla_s T_{t i_2 \cdots i_p}^{j_1 \cdots j_q} = 0.$$

On account of (2.16), from (4.4) it follows that

(4.5) $$\varphi_h^{\ s} (\nabla^h \varphi_{i_1}^{\ \ t}) \nabla_s T_{t i_2 \cdots i_p}^{j_1 \cdots j_q} = 0.$$

Consequently, (4.3) becomes

$$\nabla^h\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p}+\sum_{r=1}^{q} R^*{}_t{}^{j_r}T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}-\sum_{r=1}^{p} R^*{}_{i_r}{}^t T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}=0\,,$$

or using (2) and (3)

$$\nabla^h\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p}+\sum_{r=1}^{q} R_t{}^{j_r}T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}-\sum_{r=1}^{p} R_{i_r}{}^t T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}=0\,. \tag{4.6}$$

In order to prove the converse, we consider a pure tensor $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ satisfying (1), (2) and (3), and writing out the square of $P^{j_1\cdots j_q}_{hi_1\cdots i_p}$ we have

$$\frac{1}{2}P^{j_1\cdots j_q}_{hi_1\cdots i_p}P^{hi_1\cdots i_p}_{j_1\cdots j_q}=(\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p})\nabla^h T^{i_1\cdots i_p}_{j_1\cdots j_q}+\varphi_h{}^s\varphi_{i_1}{}^t(\nabla_s T^{j_1\cdots j_q}_{ti_2\cdots i_p})\nabla^h T^{i_1\cdots i_p}_{j_1\cdots j_q}$$

where $P^{hi_1\cdots i_p}_{j_1\cdots j_q}=P^{b_1\cdots b_q}_{sa_1\cdots a_p}g^{sh}g^{a_1i_1}\cdots g^{a_pi_p}g_{b_1j_1}\cdots g_{b_qj_q}$ etc..
Thus we have

$$\begin{aligned}\frac{1}{2}P^{j_1\cdots j_q}_{hi_1\cdots i_p}P^{hi_1\cdots i_p}_{j_1\cdots j_q}&+\nabla^h(T^{i_1\cdots i_p}_{j_1\cdots j_q}P^{j_1\cdots j_q}_{hi_1\cdots i_p})\\ &=\frac{1}{2}P^{j_1\cdots j_q}_{hi_1\cdots i_p}P^{hi_1\cdots i_p}_{j_1\cdots j_q}+(\nabla^h T^{i_1\cdots i_p}_{j_1\cdots j_q})P^{j_1\cdots j_q}_{hi_1\cdots i_p}+T^{i_1\cdots i_p}_{j_1\cdots j_q}\nabla^h P^{j_1\cdots j_q}_{hi_1\cdots i_p}\\ &=T^{i_1\cdots i_p}_{j_1\cdots j_q}\nabla^h P^{j_1\cdots j_q}_{hi_1\cdots i_p}\,.\end{aligned} \tag{4.7}$$

By Green's theorem, from (4.7), we have

$$\int_{X_n}\Big[T^{i_1\cdots i_p}_{j_1\cdots j_q}\nabla^h P^{j_1\cdots j_q}_{hi_1\cdots i_p}-\frac{1}{2}P^{j_1\cdots j_q}_{hi_1\cdots i_p}P^{hi_1\cdots i_p}_{j_1\cdots j_q}\Big]d\sigma=0 \tag{4.8}$$

where $d\sigma$ means the volume element of the K-space X_n.
(4.8) shows that if $\nabla^h P^{j_1\cdots j_q}_{hi_1\cdots i_p}=0$ i. e. if (4.2) holds good, then we have

$$P^{j_1\cdots j_q}_{hi_1\cdots i_p}=0\,.$$

But from (1), (2) and (3), we have (4.2). Thus from Lemma 3.1 it follows that $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ is almost-analytic. Also for $q\geqq 1$ exactly the same method can be applied. q. e. d.

THEOREM 4.2. *In a compact* K-*space, a necessary and sufficient condition that a pure tensor* T_i^j *be almost-analytic is that it satisfies*

(1) $\nabla^h\nabla_h T_i^j+R_t^j T_i^t-R_i^t T_t^j=0\,,$

(2) $N_{st}{}^j\nabla^s T_i^t=0\,,$

(3) $N_{ht}{}^j T_i^t-N_{hi}{}^t T_t^j=0$

where the condition (2) *may be replaced by* $N^s{}_i{}^t\nabla_s T_t^j=0$.

PROOF. Let T_i^j be an almost-analytic tensor. From Lemma 3.3, we have

$$-P_{hi}{}^j\overset{\text{def}}{=}\nabla_h T_i^j+\varphi_h{}^s\varphi_t{}^j\nabla_s T_i^t=0 \tag{4.9}$$

and (3).
Operating ∇^h to (4.9) and making use of (2.4) and (2.8), we have

(4.10) $$-\nabla^h P_{hi}{}^j = \nabla^h\nabla_h T_i{}^j + \varphi_h{}^s(\nabla^h\varphi_t{}^j)\nabla_s T_i{}^t + \varphi_h{}^s\varphi_t{}^j\nabla^h\nabla_s T^t$$
$$= \nabla^h\nabla_h T_i{}^j - \frac{1}{4} N^s{}_t{}^j\nabla_s T_i{}^t + R^*{}_t{}^j T_i{}^t - R^*{}_i{}^t T_t{}^j = 0 .$$

$\nabla_s T_i{}^t$ is pure in s, t because of $P_{hi}{}^j = 0$ but $N^s{}_t{}^j$ is hybrid in s, t because of (2.12) and therefore by Proposition 4 we have $N^s{}_t{}^j\nabla_s T_i{}^t = 0$.

Consequently, from (4.10) we have

(4.11) $$\nabla^h\nabla_h T_i{}^j + R^*{}_t{}^j T_i{}^t - R^*{}_i{}^t T_t{}^j = 0$$

and

(4.12) $$N_{st}{}^j\nabla^s T_i{}^t = 0 .$$

Moreover, on multiplying (3) by $N^{hi}{}_k$ and using (2.15) we get

$$16(R_k{}^t - R^*{}_k{}^t)T_t{}^j - N^{hi}{}_k N_{ht}{}^j T_i{}^t = 0$$

or

(4.13) $$16(R_i{}^t - R^*{}_i{}^t)T_t{}^j - N^{hr}{}_i N_{ht}{}^j T_r{}^t = 0 .$$

Similarly multiplying (3) by $N^h{}_{kj}$, we get

(4.14) $$N_{hi}{}^r N^{hj}{}_s T_r{}^s - 16(R_t{}^j - R^*{}_t{}^j)T_i{}^t = 0 .$$

From (4.13) and (4.14), we have

(4.15) $$R^*{}_t{}^j T_i{}^t - R^*{}_i{}^t T_t{}^j = R_t{}^j T_i{}^t - R_i{}^t T_t{}^j$$

and therefore (4.11) turns to

(4.16) $$\nabla^h\nabla_h T_i{}^j + R_t{}^j T_i{}^t - R_i{}^t T_t{}^j = 0 .$$

To prove the converse, let $T_i{}^j$ be a pure tensor and suppose that it satisfies (1), (2) and (3). Calculating the square of $P_{hi}{}^j$, we have easily the following

(4.17) $$\frac{1}{2} P_{hi}{}^j P^{hi}{}_j + \nabla^h(T^i{}_j P_{hi}{}^j) = T^i{}_j\nabla^h P_{hi}{}^j .$$

Hence by virtue of Green's theorem, we have

(4.18) $$\int_{X_n}\Big[T^i{}_j\nabla^h P_{hi}{}^j - \frac{1}{2} P_{hi}{}^j P^{hi}{}_j\Big]d\sigma = 0$$

which shows that in a compact K-space $\nabla^h P_{hi}{}^j = 0$ is equivalent to $P_{hi}{}^j = 0$.

On the other hand, from (1), (2) and (3), we have (4.10). Thus by virtue of Lemma 3.3, $T_i{}^j$ becomes almost-analytic.

Finally we must show that (2) may be replaced by $N^s{}_i{}^t\nabla_s T_t{}^j = 0$. In fact, if $T_i{}^j$ is almost-analytic, from Lemma 3.3 we have also

(4.19) $$\nabla_h T_i{}^j + \varphi_h{}^s\varphi_i{}^t\nabla_s T_t{}^j = 0 .$$

Operating ∇^h to (4.19) and using (2.4) and (2.8), we have

$$\nabla^h\nabla_h T_i{}^j + \varphi_h{}^s(\nabla^h\varphi_i{}^t)\nabla_s T_t{}^j + R^*{}_t{}^j T_i{}^t - R^*{}_i{}^t T_t{}^j = 0$$

or by (2.10) and (4.15)

(4.20) $$\nabla^h\nabla_h T_i^{\ j} - \frac{1}{4} N^{s\ t}_{\ i}\nabla_s T_t^{\ j} + R_t^{\ j} T_i^{\ t} - R_i^{\ t} T_t^{\ j} = 0\,.$$

Hence by (4.16), we find

(4.21) $$N^{s\ t}_{\ i}\nabla_s T_t^{\ j} = 0\,.$$

For the converse, the same method as that used in the preceding paragraph can be applied.

Summarising these results and the result obtained by S. Tachibana for a contravariant vector, we have the following main theorem of this paper.

THEOREM 4.3. *In a compact* K-*space, a necessary and sufficient condition that a pure tensor* $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ $(p, q \geqq 0)$ *be almost-analytic is that it satisfies*

I. *In case* $p=0$, $q=1$

(1) $$\nabla^h\nabla_h v^j + R_t^{\ j} v^t = 0\,,$$

(2) $$(R_t^{\ j} - R^{*}{}_t^{\ j}) v^t + \frac{1}{2} N_{ab}^{\ \ j}\nabla^a v^b = 0\,.$$

II. *In case* $p=q=1$

(1) $$\nabla^h\nabla_h T_i^{\ j} + R_t^{\ j} T_i^{\ t} - R_i^{\ t} T_t^{\ j} = 0\,,$$

(2) $$N_{st}^{\ \ j}\nabla^s T_i^{\ t} = 0\,,$$

(3) $$N_{ht}^{\ \ j} T_i^{\ t} - N_{hi}^{\ \ t} T_t^{\ j} = 0\,,$$

where (2) *may be replaced by* $N^{s\ t}_{\ i}\nabla_s T_t^{\ j} = 0$.

III. *In other cases*

(1) $$\nabla^h\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p} + \sum_{r=1}^{q} R_t^{\ j_r} T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p} - \sum_{r=1}^{p} R_{i_r}^{\ \ t} T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p} = 0\,,$$

(2) $$(R_t^{\ j_r} - R^{*}{}_t^{\ j_r}) T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p} = 0 \quad \text{for every} \quad r = 1, 2, \cdots, q\,,$$

(3) $$(R_{i_r}^{\ \ t} - R^{*}{}_{i_r}^{\ \ t}) T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p} = 0 \quad \text{for every} \quad r = 1, 2, \cdots, p\,.$$

As a corollary to this theorem, we have

THEOREM 4.5.[16] *In a compact* K-*space, a necessary and sufficient condition that a contravariant pure tensor* $T^{j_1\cdots j_q}$ $(q \geqq 2)$ *be almost-analytic is that it satisfies*

(1) $$\nabla^h\nabla_h T^{j_1\cdots j_q} + \sum_{r=1}^{q} R_t^{\ j_r} T^{j_1\cdots t\cdots j_q} = 0\,,$$

(2) $$(R_t^{\ j_r} - R^{*}{}_t^{\ j_r}) T^{j_1\cdots t\cdots j_q} = 0 \quad \text{for every} \quad r = 1, 2, \cdots, q\,.$$

THEOREM 4.6.[17] *In a compact* K-*space, a necessary and sufficient condition that a covariant pure tensor* $T_{i_1\cdots i_p}$ $(p \geqq 1)$ *be almost-analytic is that it satisfies*

(1) $$\nabla^h\nabla_h T_{i_1\cdots i_p} - \sum_{r=1}^{p} R_{i_r}^{\ \ t} T_{i_1\cdots t\cdots i_p} = 0\,,$$

(2) $$(R_{i_r}^{\ \ t} - R^{*}{}_{i_r}^{\ \ t}) T_{i_1\cdots t\cdots i_p} = 0 \quad \text{for every} \quad r = 1, 2, \cdots, p\,.$$

Since in a Kählerian space $R_{ji} = R^{*}{}_{ji}$ and $N_{ji}^{\ \ h} = 0$[18], we have

THEOREM 4.7.[19] *In a compact Kählerian space, a necessary and sufficient*

---

16), 17) S. Sawaki [8].

18) K. Yano [**13**].

19) K. Yano [**13**], S. Tachibana [**11**] and S. Sawaki and S. Kotō [**6**].

condition that a pure tensor $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ $(p, q \geqq 0)$ *be almost-analytic is that it satisfies*

$$\nabla^h\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p} + \sum_{r=1}^{q} R_t{}^{j_r} T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p} - \sum_{r=1}^{p} R_{i_r}{}^t T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p} = 0 .$$

§ 5. Applications.

Some applications of Theorem 4.3 in which tensors are covariant or contravariant have been given in a previous paper [8]. Especially, we have given applications to a harmonic tensor and a Killing tensor. In this place, we shall state a generalization of Bochner's theorem as another application of Theorem 4.3.

Let $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ be an almost-analytic tensor in a K-space. If we put $\Phi = T^{j_1\cdots j_q}_{i_1\cdots i_p} T^{i_1\cdots i_p}_{j_1\cdots j_q}$, then the Laplacian of Φ can be written as

$$\Delta\Phi = 2[(\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p})\nabla^h T^{i_1\cdots i_p}_{j_1\cdots j_q} + (\nabla^h\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p}) T^{i_1\cdots i_p}_{j_1\cdots j_q}$$

and substituting

$$\nabla^h\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p} = \sum_{r=1}^{p} R_{i_r}{}^t T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p} - \sum_{r=1}^{q} R_t{}^{j_r} T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}$$

into (5.1), we have

$$\Delta\Phi = 2[(\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p})\nabla^h T^{i_1\cdots i_p}_{j_1\cdots j_q} + G\{T\}] \tag{5.2}$$

where

$$G\{T\} = \Big(\sum_{r=1}^{p} R_{i_r}{}^t T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p} - \sum_{r=1}^{q} R_t{}^{j_r} T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}\Big) T^{i_1\cdots i_p}_{j_1\cdots j_q} .$$

Thus, by Bochner's lemma[20], we have the following

THEOREM 5.1.[21] *In a compact* K-*space, if an almost-analytic tensor* $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ $(p, q \geqq 0)$ *satisfies the inequality:*

$$G\{T\} \geqq 0 ,$$

then we must have $G\{T\} = 0$ *and* $\nabla^h T^{j_1\cdots j_q}_{i_1\cdots i_p} = 0$.

Furthermore, if, at every point of the space, we denote by M and m the algebraically largest and the smallest eigenvalues of the matrix $\| R_{ji} \|$ respectively, then we have

$$G\{T\} \geqq (pm - qM) T^{j_1\cdots j_q}_{i_1\cdots i_p} T^{i_1\cdots i_p}_{j_1\cdots j_q}$$

and hence we have

THEOREM 5.2.[22] *In a compact* K-*space, if* M *and* m *have the meaning just stated and if*

$$pm - qM \geqq 0 ,$$

then every almost-analytic tensor $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ $(p, q \geqq 0)$ *must satisfy* $\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p} = 0$.

20) S. Bochner [**1**] or K. Yano and S. Bochner [**12**, p. 30].

21), 22), For Kählerian case, see S. Bochner [**2**].

If $pm-qM\geqq 0$ everywhere and $pm-qM>0$ somewhere, then there exists no almost-analytic tensor other than the zero tensor.

As a corollary to this theorem, we can state

THEOREM 5.3.[23] *In a compact Einstein* K*-space, there exists no almost-analytic tensor $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ other than the zero tensor, if either R is positive and $p>q$ or R is negative and $p<q$, where $R=g^{ji}R_{ji}$.*

Also, for $R=0$ or $p=q$, every almost-analytic tensor must have vanishing covariant derivative.

Moreover, if a K-space ($n>4$) is conformally flat, then the curvature tensor has the following form[24]

$$R_{kjih}=\frac{1}{n-2}(g_{kh}R_{ji}-g_{jh}R_{ki}+R_{kh}g_{ji}-R_{jh}g_{ki})$$
$$-\frac{R}{(n-1)(n-2)}(g_{kh}g_{ji}-g_{jh}g_{ki}).$$

Hence we have

$$R^*{}_{ji}=\frac{1}{n-2}\left(2R_{ji}-\frac{R}{n-1}g_{ji}\right),$$

from which it follows that

$$R_{ji}-R^*{}_{ji}=\frac{1}{n-2}\left\{(n-4)R_{ji}+\frac{R}{n-1}g_{ji}\right\}.^{25)} \tag{5.3}$$

Consequently, if $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ ($p\geqq 0$, $q\neq 1$ or $p\geqq 2$, $q=1$) is an almost-analytic tensor, then by III of Theorem 4.3 we find

$$(n-4)R_t{}^{j_r}T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}+\frac{R}{n-1}T^{j_1\cdots j_q}_{i_1\cdots i_p}=0, \tag{5.4}$$

$$(n-4)R_{i_r}{}^{t}T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}+\frac{R}{n-1}T^{j_1\cdots j_q}_{i_1\cdots i_p}=0 \tag{5.5}$$

and therefore we have

$$G\{T\}=(\sum_{r=1}^{p}R_{i_r}{}^{t}T^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}-\sum_{r=1}^{q}R_t{}^{j_r}T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p})T^{i_1\cdots i_p}_{j_1\cdots j_q}$$
$$=\frac{q-p}{(n-1)(n-4)}RT^{j_1\cdots j_q}_{i_1\cdots i_p}T^{i_1\cdots i_p}_{j_1\cdots j_q}.$$

On the other hand it is known that in a conformally flat K-space the scalar curvature R is non-negative.[26] Accordingly, if $q\geqq p\geqq 0$ and $q\geqq 2$, then we have $G\{T\}\geqq 0$.

Thus, from Theorem 5.1, we have

23) For Kählerian case, see S. Bochner [**2**].
24) K. Yano and S. Bochner [**12**, p. 78].
25), 26) S. Tachibana [**10**].

THEOREM 5.4.[27] *When a compact* K-*space* ($n>4$) *is conformally flat, then for an almost-analytic tensor* $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ ($q\geqq p\geqq 0,\ q\geqq 2$) *we have*

$$\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p}=0\,.$$

Next we shall consider the case where the space in consideration is not necessarily compact.
If $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ ($p\geqq 0,\ q\neq 1$ or $p\geqq 2,\ q=1$) is an almost-analytic tensor, then by Lemma 3.1, (5.4) and (5.5) hold good. Multiplying (5.4) and (5.5) by $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ we have respectively

$$(n-4)R_t{}^{j_r}T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}T^{i_1\cdots i_p}_{j_1\cdots j_q}+\frac{R}{n-1}T^{j_1\cdots j_q}_{i_1\cdots i_p}T^{i_1\cdots i_p}_{j_1\cdots j_q}=0 \tag{5.6}$$

and

$$(n-4)R_{i_r}{}^tT^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}T^{i_1\cdots i_p}_{j_1\cdots j_q}+\frac{R}{n-1}T^{j_1\cdots j_q}_{i_1\cdots i_p}T^{i_1\cdots i_p}_{j_1\cdots j_q}=0\,. \tag{5.7}$$

Thus we have the following

THEOREM 5.5.[28] *Let a* K-*space* ($n\geqq 4$) *is conformally flat. If the Ricci's form is positive definite, then there exists no almost-analytic tensor* $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ ($p\geqq 0$, $q\neq 1$ *or* $p\geqq 2$ $p=1$) *other than the zero tensor.*

We remark here that in a conformally flat K-space ($n\geqq 4$) the Ricci's form can not be negative definite.[29]

Moreover, from III of Theorem 4.3 and Theorem 5.4, we have

THEOREM 5.6. *When a compact* K-*space* ($n>4$) *is conformally flat, a necessary and sufficient condition that a pure tensor* $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ ($p\geqq 0,\ q\neq 1$ *or* $p\geqq 2,\ q=1$) *be almost-analytic is that it satisfies*

$$\nabla^h\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p}-\frac{q-p}{(n-1)(n-4)}RT^{j_1\cdots j_q}_{i_1\cdots i_p}=0\,, \tag{1}$$

$$R_t{}^{j_r}T^{j_1\cdots t\cdots j_q}_{i_1\cdots\cdots i_p}+\frac{R}{(n-1)(n-4)}T^{j_1\cdots j_q}_{i_1\cdots i_p}=0 \qquad \text{for every}\quad r=1,2,\cdots,q, \tag{2}$$

$$R_{i_r}{}^tT^{j_1\cdots\cdots j_q}_{i_1\cdots t\cdots i_p}+\frac{R}{(n-1)(n-4)}T^{j_1\cdots j_q}_{i_1\cdots i_p}=0 \qquad \text{for every}\quad r=1,2,\cdots,p. \tag{3}$$

If $p=q\geqq 2$, *then the condition* (1) *can be replaced by*

$$\nabla_h T^{j_1\cdots j_q}_{i_1\cdots i_p}=0\,.$$

Finally we shall consider a K-space of constant curvature. Then the curvature has the following form

$$R_{kjih}=\frac{R}{n(n-1)}(g_{ji}g_{kh}-g_{jh}g_{ki})$$

from which we have

27), 28) For the case $q\geqq 2,\ p=0$, these two theorems hold good in an $^*O$-space which is conformally flat because of Theorem 5. 4 in [8].

29) S. Tachibana [**10**].

$$R_{ki}=\frac{R}{n}g_{ki}{}^{30)} \quad \text{and} \quad R^*{}_{ki}=\frac{R}{n(n-1)}g_{ki} \tag{5.8}$$

where R is an absolute constant. Hence if $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ $(p\geqq 0,\ q\neq 1$ or $p\geqq 2,\ q=1)$ is an almost-analytic tensor, then we have from Lemma 3.1

$$\frac{(n-2)R}{n(n-1)}T^{j_1\cdots j_q}_{i_1\cdots i_p}=0\,. \tag{5.9}$$

Thus we have the following

THEOREM 5.7.[31] *Let a* K-*space* $(n\geqq 4)$ *is of constant curvature. If* $R\neq 0$ *then there exists no almost-analytic tensor* $T^{j_1\cdots j_q}_{i_1\cdots i_p}$ $(p\geqq 0,\ q\neq 1$ *or* $p\geqq 2,\ q=1)$ *other than the zero tensor.*

In this place we shall remark the following fact. If $R=0$, then from (5.8) we have $R_{ki}=R^*{}_{ki}=0$, so our K-space becomes a Kählerian space.[32] On the other hand it is known that there does not exist a K-space $(n\geqq 4)$ of constant curvature with $R<0$.[33] Hence when a K-space $(n\geqq 4)$ of constant curvature has non-vanishing scalar curvature, it is non-Kählerian and has a positive curvature.

Niigata University

References

[1] S. Bochner, Vector fields and Ricci curvature, Bull. Amer. Math. Soc., **52** (1946), 776–797.

[2] S. Bochner, Tensor fields and Ricci curvature in Hermitian metric, Proc. Nat. Acad. Sci. U. S. A., **37** (1951), 704–706.

[3] A. Frölicher, Zur Differentialgeometrie der Komplexen Structuren, Math. Ann., **129** (1955), 50–95.

[4] T. Fukami and S. Ishihara, Almost-Hermitian structure on S^6, Tôhoku Math. J., **7** (1955), 151–156.

[5] S. Kotō, Some theorems on almost Kählerian spaces, J. Math. Soc. Japan, **12** (1960), 422–433.

[6] S. Sawaki and S. Kotō, On the analytic tensor in a compact Kaehler space, J. Fac. Niigata Univ. Ser. I, **1** (1958), 77–84.

[7] S. Sawaki, On almost-analytic vectors in almost-Hermitian spaces, J. Fac. Niigata Univ. Ser. I, **3** (1960), 23–32.

[8] S. Sawaki, On almost-analytic tensors in *O-spaces, Tôhoku Math. J., **13** (1961), 154–178.

[9] J. A. Schouten, Ricci calculus, Springer, 1954.

---

30) K. Yano and S. Bochner [**12**, p. 21].

31) For the case $q\geqq 2,\ p=0$, this theorem is valid in an *O-space which is of constant curvature because of Theorem 5. 4 in [8].

32) S. Kotō [**5**].

33) S. Tachibana [**10**].

[10] S. Tachibana, On almost-analytic vectors in certain almost-Hermitian manifolds, Tôhoku Math. J., **11** (1959), 351-363.
[11] S. Tachibana, Analytic tensor and its generalization, Tôhoku Math. J., **12** (1960), 208-221.
[12] K. Yano and S. Bochner, Curvature and Betti numbers, Ann. of Math. Studies **32**, 1953.
[13] K. Yano, The theory of Lie derivatives and its applications, Amsterdam, 1955.