Journal of the Mathematical Society of Japan Vol. 12, No. 3, July, 1960

An explicit solution of the Schoenflies extension problem.

By William HUEBSCH and Marston MORSE

(Received Feb. 2, 1960)

§ 1. **Introduction.** We shall present a solution of the Schoenflies extension problem with the following properties:

(1) The dependence of the solution on the transformed initial mapping ω will be explicit.

(2) If ω depends continuously (or differentiably) on a parametric point p on an arbitrary paracompact space Σ (or differentiable manifold Σ) the form of the solution will permit us in a later paper to show the continuous (or differentiable) dependence of an extension on the pair (p, x), where $p \in \Sigma$ and (x) is in the euclidean domain of definition of the extension.

(3) The solution will be formally the same in the topological case, $m=0$, as in the differentiable case, $m>0$.

(4) For fixed p the solution is equivalent to that given by (16.20) of [**2**], but will be derived independently without reference to the abstract manifolds which motivated the derivation of an extension both by Mazur in [**1**] and Morse in [**2**]. See also [**5**].

According to an abstract presented in December of 1959 by Dr. Frank Raymond at the Institute for Advanced Study, Morton Brown has prepared a proof *de novo* of the Schoenflies extension theorem in the topological case, based on the "shell" hypothesis of Mazur, omitting the hypothesis of Mazur concerning the semi-linearity of the given mapping near some point of the shell. The subsequent Note of Morse in [**4**], to appear in the Bulletin of the American Mathematical Society, is an extension of a reduction employed in the differentiable case in [**2**] in the Compositio Mathematica and contains, we believe, the first proof that the second Mazur hypothesis can always be satisfied on making a simple reduction of the problem. Thus the prior announcement by Raymond for Brown and the Note by Morse have entirely different ends and methods, and, it is believed, will affect the development of the theory in very different manners.

The methods of the present paper are a refinement of those in [**2**].

§ 2. **A transformation of the problem.** A C^m-diffeomorphism of an open subset of an euclidean n-space or a differentiable n-manifold will be understood in the usual sense when $m>0$.

When $m=0$, a C^m-diffeomorphism shall mean a homeomorphism.

Let E be an euclidean n-space, $n>1$, with rectangular coordinates $(x_1, \cdots, x_n)=(x)$. We represent the point (x) by a vector $\boldsymbol{x}$ whose components are the coordinates of (x) and whose magnitude is denoted by $\|\boldsymbol{x}\|$. Let S_{n-1} be an $(n-1)$-sphere of unit radius in E, with center at a point $\boldsymbol{c}$. Let a be a constant such that $0<a<1$. We introduce the n-shell

$$\delta_a=\{\boldsymbol{x} \mid 1-a<\|\boldsymbol{x}-\boldsymbol{c}\|<1+a\}\,. \tag{2.1}$$

Let φ be a C^m-diffeomorphism, $m \geqq 0$, of δ_a into E. We term m the *index* of φ. If $m=0$, φ is a homeomorphism. Set $\varphi(S_{n-1})=\mathfrak{M}_{n-1}$. We suppose that φ carries points of δ_a interior to S_{n-1} into points of E interior to $\mathfrak{M}_{n-1}$. These conditions on φ are termed the shell hypotheses on φ.

If Σ is a topological $(n-1)$-sphere in E, $\mathring{\mathrm{J}}\Sigma$ shall denote the open interior of Σ and $\mathrm{J}\Sigma$ its closure.

We state the principal theorem.

THEOREM 2.1. *Let φ be a C^m-diffeomorphism of the shell δ_a into E satisfying the above shell hypothesis.*

If κ is a suitably chosen, compact subset of $\mathring{\mathrm{J}}\, S_{n-1}$, which, when $m>0$, includes a suitably chosen point $\boldsymbol{z}$, there exists a homeomorphism Λ_φ of $\delta_a \cup \mathring{\mathrm{J}}\, S_{n-1}$ into E, which is an extension of $\varphi \mid (\delta_a-\kappa)$, and, when $m>0$, is, in addition, a C^m-diffeomorphism into E of

$$(\delta_a \cup \mathrm{J}\, S_{n-1})-\boldsymbol{z}\,.$$

As a consequence,

$$\Lambda_\varphi(\mathrm{J}\, S_{n-1})=\mathrm{J}\, \varphi(S_{n-1})\,. \tag{2.2}$$

When $m>0$ Theorem 2.1 has been established in [**2**], under the weaker hypothesis (cf. [**3**]) that φ is a C^m-diffeomorphism of S_{n-1} onto $\mathfrak{M}_{n-1}$. When $m=0$, the theorem has been reduced by Morse in [**4**] to the theorem of Mazur by showing that no generality is lost if it is assumed that φ reduces to the identity in a neighborhood N_Q of a point Q of S_{n-1}. The present paper will give an independent proof of Theorem 2.1.

The theorem readily implies the following. If the index m of φ is positive then corresponding to an *arbitrary* point $\boldsymbol{z} \in \mathring{\mathrm{J}}\, S_{n-1}$, Λ_φ and κ can be chosen as affirmed. See Theorem 7.1.

As a consequence of the Lemmas of [**4**] we can assume without loss of generality that φ reduces to the identity $\boldsymbol{U}$ on a neighborhood N_Q of the "x_n-pole" Q of S_{n-1}. The x_n-pole Q of S_{n-1}, by definition, is the point on S_{n-1} at which x_n assumes its maximum. We shall further simplify the extension problem by subjecting the space $E-Q$ to a reflection t in an $(n-1)$-sphere S_Q with a center at Q and with diameter $\rho<1$, so small that $\mathrm{J}\, S_Q \subset N_Q$ Under t the image of $S_{n-1}-Q$ is an $(n-1)$-plane Π, through $S_Q \cap S_{n-1}$, on which x_n is constant. This constant will be zero if the origin is chosen properly on the x_n-axis. We suppose the origin so chosen. Setting

$$\psi(t(\boldsymbol{x}))=t(\varphi(\boldsymbol{x})) \qquad (\boldsymbol{x}\in\delta_a-Q) \tag{2.3}$$

we obtain a C^m-diffeomorphism ψ of the subset $t(\delta_a-Q)$ of E into E, such that ψ reduces to the identity $\boldsymbol{U}$ on $E-\mathrm{J}\,S_Q$.

The rectangle K. Let K be the open n-cube of points $\boldsymbol{x}$ of E such that

$$(-1<x_i<1) \qquad (i=1,\cdots,n)\,. \tag{2.4}$$

In [**2**], K was taken as closed. Since $\mathrm{J}\,S_Q\subset K$, ψ reduces to $\boldsymbol{U}$ not only on the complement CK of K, but also on a neighborhood of βK, the boundary of K. The open set on which ψ is defined includes the $(n-1)$-plane Π. The closed set ζ on which ψ is undefined is included in $\mathrm{J}\,S_Q\subset K$. Let ζ' and ζ'' be respectively the closed subsets of ζ on which $x_n<0$ and $x_n>0$. The set ζ' is the image under t of $\mathrm{J}\,S_{n-1}-\delta_a$ and $\zeta''-Q$ the image under t of $C\mathrm{J}\,S_{n-1}-\delta_a$.

The constant d. Let $K_{\nu d}$, $\nu=1,2,3,4$, be the open n-rectangles in K on which

$$(-1+\nu d<x_i<1-\nu d) \qquad (i=1,\cdots,n)\,, \tag{2.5}$$

choosing $d>0$ so small that $\psi=\boldsymbol{U}$ on $K-K_{4d}$. Let the open subrectangles of $K_{\nu d}$ on which $x_n<-\nu d$ and $x_n>\nu d$, respectively, be denoted by H' and H'' for $\nu=1$, and by L' and L'' for $\nu=2$. Let the closures of the subrectangles of $K_{\nu d}$ on which $x_n\leqq-\nu d$ and $x_n\geqq\nu d$, respectively, be denoted by G' and G'' for $\nu=3$, and by Θ' and Θ'' for $\nu=4$. We suppose d so small that

$$\Theta'\supset\zeta',\quad \Theta''\supset\zeta''\,.$$

Note that

$$H'\supset L'\supset G'\supset\Theta',\quad H''\supset L''\supset G''\supset\Theta''\,. \tag{2.6}$$

Set $H=H'\cup H''$, $G=G'\cup G''$, $\Theta=\Theta'\cup\Theta''$.

The mapping ω *of* $K-\Theta$. *Let* ω *be the restriction of* ψ *to* $K-\Theta$. Note that $\omega(K-\Theta)\subset K$.

THEOREM 2.2. *If* Ω *is a suitably chosen compact subset of* H' *which includes* G', *and, when* $m>0$, *includes a suitably chosen point* $\boldsymbol{w}$, *there exists a homeomorphism* λ_ω *of* H' *into* E *which is an extension of* $\omega\,|\,(H'-\Omega)$ *and, when* $m>0$, *is in addition a* C^m*-diffeomorphism of* $H'-\boldsymbol{w}$ *into* E. *Moreover*

$$\lambda_\omega(H')=\mathring{\mathrm{J}}\,\omega(\beta H')\,. \tag{2.7}$$

In § 7 we shall show that Theorem 2.2 implies Theorem 2.1. The problem of finding $(\lambda_\omega,\boldsymbol{w},\Omega)$ so as to satisfy Theorem 2.2 will be termed the problem (ω,H',G'). We shall represent λ_ω explicitly.

Theorem 2.2 will be established in § 7.

§ **3. The form of λ_ω.** Let D be an open n-rectangle which is the union of points $\boldsymbol{x} \in E$ such that

$$(-1 < x_i < 9) \qquad (i = 1, \cdots, n). \tag{3.1}$$

Note that D contains the special point P with coordinates $(x) = (8, 0, \cdots, 0)$.

The contraction $\boldsymbol{a}$. Let $\boldsymbol{a}$ be a C^∞-diffeomorphism of D onto H' which leaves L' pointwise invariant. Such a contraction may be shown to exist.

The mapping σ. *In § 7 we shall establish the existence of a homeomorphism* σ *of the open set* CG' *into* E *which extends* $\omega|(H'-G')$. *The mapping* σ *shall be such that* $\sigma(P) = P$ *and* $\sigma = \boldsymbol{U}$ *on some neighborhood of* CD. *In the differentiable case,* $m > 0$, σ *shall be in addition a* C^m-*diffeomorphism of* $CG' - P$ *into* E.

In this section we shall show how to define the mapping λ_ω of Theorem 2.2 in terms of σ and $\boldsymbol{a}$.

The sets $\mathcal{H}' \supset \mathcal{L}' \supset \mathcal{G}'$. These subsets of K are needed throughout the paper and are defined by setting

$$\mathcal{H}' = \mathring{\mathrm{J}}\, \omega(\beta H'), \ \mathcal{L}' = \mathring{\mathrm{J}}\, \omega(\beta L'), \ \mathcal{G}' = \mathrm{J}\, \omega(\beta G'). \tag{3.2}$$

The following partitions will be shown to be a consequence of Lemma 4.1, granting the existence of σ in (3.3) and (3.3)$'$:

$$D = \mathcal{G}' \cup \sigma(D - G'), \tag{3.3}$$

$$D = \mathcal{H}' \cup \sigma(D - H'), \tag{3.3$'$}$$

$$\mathcal{H}' = \mathcal{G}' \cup \omega(H' - G'), \tag{3.4}$$

$$\mathcal{L}' = \mathcal{G}' \cup \omega(L' - G'). \tag{3.5}$$

As a matter of notation we shall write $f_1 \cdot f_2(A)$ in place of $f_1(f_2(A))$.

Since $\boldsymbol{a}$ is a C^∞-diffeomorphism of D onto H', a mapping λ_ω of H' into E can be defined by giving an appropriate definition of $\lambda_\omega \cdot \boldsymbol{a}(\boldsymbol{z})$ for $\boldsymbol{z} \in D$. This is what Theorem 3.1 does.

THEOREM 3.1. (i) *There exists a solution of problem* (ω, H', G') *in which*

$$\lambda_\omega \cdot \boldsymbol{a}(\boldsymbol{z}) = \omega \cdot \boldsymbol{a} \cdot \sigma^{-1}(\boldsymbol{z}) \qquad (\boldsymbol{z} \in \sigma(D - G')), \tag{3.6}$$

$$\lambda_\omega \cdot \boldsymbol{a}(\boldsymbol{z}) = \boldsymbol{z} \qquad (\boldsymbol{z} \in \mathcal{G}'). \tag{3.7}$$

(ii) *The mapping* $\lambda_\omega \boldsymbol{a}$ *is equivalently overdefined by* (3.6) *and the condition*

$$\lambda_\omega \cdot \boldsymbol{a}(\boldsymbol{z}) = \boldsymbol{z} \qquad (\boldsymbol{z} \in \mathcal{L}'). \tag{3.8}$$

That (3.6) and (3.7) define a single-valued $\lambda_\omega \boldsymbol{a}$ over D follows from the fact that D admits the partition (3.3). We prove (α), (β) and (γ).

(α) *The mapping* $\lambda_\omega \boldsymbol{a}$ *is a homeomorphism of* D *onto* $\mathcal{H}'$.

We see that $\lambda_\omega \boldsymbol{a}$ maps $\sigma(D - G')$ homeomorphically onto

$$\omega \cdot \boldsymbol{a}(D - G') = \omega(H' - G')$$

and $\mathcal{G}'$ homeomorphically onto $\mathcal{G}'$. It accordingly maps D biuniquely onto

$$\omega(H'-G')\cup\mathcal{G}'=\mathcal{H}' \qquad \text{[by (3.4)]}.$$

We now show that the mapping (3.8), restricted to $\mathcal{L}'-\mathcal{G}'$, agrees with (3.6). By (3.5) and the definition of σ

$$\mathcal{L}'-\mathcal{G}'=\omega(L'-G')=\sigma(L'-G'). \tag{3.9}$$

Hence $\boldsymbol{z}\in\mathcal{L}'-\mathcal{G}'$ has the form $\boldsymbol{z}=\omega(\boldsymbol{y})=\sigma(\boldsymbol{y})$ with $\boldsymbol{y}\in L'-G'$. Under (3.6), since $\boldsymbol{a}(\boldsymbol{y})=\boldsymbol{y}$ on L',

$$\lambda_\omega\cdot\boldsymbol{a}(\boldsymbol{z})=\omega\cdot\boldsymbol{a}(\boldsymbol{y})=\omega(\boldsymbol{y})=\boldsymbol{z}$$

so that (3.6) and (3.8) agree for $\boldsymbol{z}\in\mathcal{L}'-\mathcal{G}'$. Since $\sigma(D-G')$ and $\mathcal{L}'$ are open subsets of D with D as their union, statement (α) follows.

(β) *A compact subset D_0 of D exists such that $D_0\supset G'\cup P$ and*

$$\lambda_\omega\cdot\boldsymbol{a}(\boldsymbol{z})=\omega\cdot\boldsymbol{a}(\boldsymbol{z}) \qquad (\boldsymbol{z}\in D-D_0). \tag{3.10}$$

Let D_0 be a closed n-rectangle such that $D\supset D_0\supset G'\cup P$, approximating D so closely that $\sigma(\boldsymbol{z})=\boldsymbol{z}$ for $\boldsymbol{z}\in D-D_0$. Statement (β) is then true by virtue of (3.6).

(γ) *The mapping $\lambda_\omega\boldsymbol{a}$ is a C^m-diffeomorphism of $D-P$ into $\mathcal{H}'$.*

Recall that $\sigma(P)=P\in D-G'$. We see from (3.6) that $\lambda_\omega\boldsymbol{a}$ defines a C^m-diffeomorphism of the open set $\sigma(D-G')-P$ into $\mathcal{H}'$. Since $\lambda_\omega\boldsymbol{a}$ defines a C^m-diffeomorphism of the open set $\mathcal{L}'$ onto $\mathcal{L}'$ we conclude that $\lambda_\omega\boldsymbol{a}$ defines a C^m-diffeomorphism of

$$\mathcal{L}'\cup\sigma(D-G')-P=D-P \qquad \text{(cf. (3.3))}$$

into $\mathcal{H}'$. This establishes (γ).

It follows from (α), (β) and (γ) that λ_ω, with $\Omega=\boldsymbol{a}(D_0)$, and with $\boldsymbol{w}=\boldsymbol{a}(P)$ when $m>0$, affords a solution of the problem (ω,H',G'). Under λ_ω, H' is mapped onto $\mathcal{H}'$ in accord with (α).

Method of defining σ. The mapping σ of CG' into E is to extend $\omega\,|\,(H'-G')$ with $\sigma(\boldsymbol{x})=\boldsymbol{x}$ on some neighborhood of CD. It remains to complete the definition of σ over $D-G'$.

To that end use will be made of a radial mapping R of E onto E with fixed point P. Under R the point (x) goes into the point (x') such that

$$x_1'-8=\frac{x_1-8}{2}, \quad x_j'=\frac{x_j}{2} \qquad (j=2,\cdots,n).$$

Let R^r be the r-fold iterate of R with R^0 the identity.

The mapping ω is defined on $K-\Theta$. We regard E as the union of the disjoint sets

$$R^r(\Theta'),\ R^r(\Theta'') \qquad (r=0,1,\cdots) \tag{3.11}$$

and of a complementary set of the form $M \cup P$ where $M \cap P = \phi$. In § 5 we shall give a preliminary extension ω_e of ω over $M \cup P$, leaving ω_e undefined over the sets (3.11), setting $\omega_e(\boldsymbol{x}) = \boldsymbol{x}$ on a neighborhood of CD.

The union of the sets (3.11), omitting Θ', will be written in the form

$$[\Theta'' \cup R^1(\Theta')] \cup [R^1(\Theta'') \cup R^2(\Theta')] \cup \cdots \tag{3.12}$$

where the general term in (3.12) is the union of the pair of sets

$$R^{r-1}(\Theta''),\ R^r(\Theta') \qquad (r = 1, 2, \cdots). \tag{3.13}$$

To define σ we shall include the r-th pair of closed sets (3.13) in an open subset $T_r(K)$ of D so chosen that the sets

$$P,\ G',\ T_1(K),\ T_2(K),\ T_3(K), \cdots \tag{3.14}$$

are disjoint. We first set $\sigma(\boldsymbol{x}) = \omega_e(\boldsymbol{x})$ when $\boldsymbol{x}$ is not in the union of the sets (3.14). We then set $\sigma(P) = P$ and in § 7 face the relatively simple task of defining σ over each set $T_r(K)$ consistently with the definition of ω_e near the boundary of $T_r(K)$.

We shall refer to the sets (3.11) as *lacunary* relative to ω_e and to $T_r(K)$ as the *container* of the lacunary sets (3.13). T_r will be defined in § 6 as a C^∞-diffeomorphism of E onto E. Its definition will not depend on ω.

§4. A partition lemma. In this section we shall establish a lemma which is useful throughout this paper.

Let Σ be a topological $(n-1)$-sphere in E. Recall that E admits a partition $E = G_1 \cup \Sigma \cup G_2$, where G_1 and G_2 are open connected sets, [**6**, p. 450]. Moreover $\beta G_1 = \beta G_2 = \Sigma$, [**6**, p. 456]. One of the sets G_1 and G_2, say G_1, is bounded. Then $\mathring{J}\,\Sigma = G_1$ by definition of $\mathring{J}\,\Sigma$. The set G_1 can be characterized in two ways:

(a) G_1 is the maximal bounded connected set of points which does not meet Σ.

(b) If L is a bounded open subset of E such that $\beta L = \Sigma$ then $L = G_1$. Characterization (b) follows from the preceding laws. For $L \cap G_1$ is closed and open, relative to G_1. It is accordingly G_1 or ϕ. Similarly $L \cap G_2$ is G_2 or ϕ. Since $L \cap G_2 \neq G_2$ it follows that $L = G_1$.

We shall also make use of the easily proved fact that whenever Σ' is a topological $(n-1)$-sphere included in $\mathring{J}\,\Sigma$ then $J\,\Sigma' \subset \mathring{J}\,\Sigma$.

We term $\mathring{J}\,\Sigma$ a *Jordan region.*

Let S be an $(n-1)$-sphere of unit radius. A topological $(n-1)$-sphere Σ will be termed *elementary* if it can be defined by a homeomorphism F of S onto Σ, where F is extensible as a homeomorphism over an open neighborhood N of S. We suppose that F carries interior points of S in N into interior points of Σ. Not every topological $(n-1)$-sphere is elementary.

A border B of $\mathrm{J}\,\Sigma$. Suppose Σ elementary. Let θ be an open spherical shell in N, bounded by $(n-1)$-spheres S' and S'' concentric with S, in N, and with radii $r'<1$ and $r''>1$ respectively. We term $F(\theta)$ a border B of $\mathrm{J}\,\Sigma$. Such a border exists in an arbitrarily small neighborhood of Σ. It is clear that $B\cup\mathrm{J}\,\Sigma$ is a neighborhood of $\mathrm{J}\,\Sigma$ which can be taken within an ε-distance of $\mathrm{J}\,\Sigma$. We note also that

$$B\cup\mathrm{J}\,\Sigma=\mathring{\mathrm{J}}\,F(S'') \tag{4.0}$$

since the left member of (4.0) is open, bounded and has $F(S'')$ as boundary.

Topological invariance of elementary character of Σ. If an elementary $(n-1)$-sphere Σ is subjected to a homeomorphism f defined over an open neighborhood N_1 of Σ, then $f\Sigma$, as we shall see, is again elementary. In fact, if the above neighborhood N of S is sufficiently small, then $F(N)\subset N_1$, so that fF is a homeomorphic mapping of N into E by virtue of which the topological $(n-1)$-sphere $fF(S)$ is elementary.

In proving Lemma 4.1 we shall use the following principle.

(i) *If U and V are non-empty, open subsets of E such that V is connected, $U\subset V$ and $\beta U=\beta V$, then $U=V$.*

Since U is open it is sufficient to show that U is closed relative to V. For a relatively closed-open non-empty subset U of V must equal V. The closure of U relative to V is $\bar{U}\cap V=(\beta U\cup U)\cap V=(\beta V\cap V)\cup(U\cap V)=U\cap V=U$ so that $U=V$.

Lemma 4.1 presupposes Hypotheses (α), (β) and (γ).

Hypothesis (α). *Let r be an integer on a finite range $(r)=(0,1,\cdots,p)$ or on the infinite range $(r)=(0,1,2,\cdots)$. Let Y be a Jordan region in E. For each $r\in(r)$ let X_r be the closure of a Jordan region with $X_r\subset Y$. We suppose βX_r elementary. Set $X=\bigcup_r X_r$. If the range (r) is infinite the sets X_r shall converge to a point $P\in Y-X$.*

As a matter of notational convenience let P be an arbitrary point in CY when (r) is finite. Then $Y\cap P=\phi$ when (r) is finite, while $Y\cap P=P$ when (r) is infinite.

Before making further hypotheses we shall prove (ii).

(ii) *The sets $Y-X-(P\cap Y)$ and $Y-X$ are connected.* Set $Y-X-(P\cap Y)=A$. If A is not connected it admits a partition into two non-empty sets U and V, open relative to A. We shall then show that $Y-P$ is not connected, contrary to fact.

Let B_r be a border of X_r so near to βX_r that the sets $Z_r=\bar{B}_r\cup X_r$ are disjoint and do not meet P or βY. Let U^* be the union of U and the sets Z_r such that βZ_r meets U, and V^* the union of V and the sets Z_r such that

βZ_r meets V. Since βZ_r is connected and included in A, it meets just one of the sets U and V, so that Z_r is added to just one of the sets U and V. We see then that U^* and V^* are disjoint. One sees readily that the sets U^* and V^* are open. They partition $Y-P$ contrary to the connectedness of $Y-P$. Hence A is connected.

The closure of A is given by the formula

$$\mathrm{Cl}(Y-X-(P\cap Y))=\bigcup_r \beta X_r\cup\beta Y\cup(Y-X) \tag{4.1}$$

Thus $A\subset Y-X\subset\bar{A}$ so that $Y-X$ is connected.

Hypothesis (β). *Let f be a homeomorphism into E of a subset Δ of E. We require that*

$$\mathrm{Cl}(Y-X)\subset\Delta \tag{4.2}$$

and that $\Delta=M^*\cup(P\cap Y)$ where M^* is an open subset of E. Thus when (r) is finite Δ is open.

As a consequence of Hypothesis (β) each boundary βX_r is in the open set M^*. Observe further that

$$\beta(Y-X-(P\cap Y))=\bigcup_r \beta X_r\cup\beta Y\cup(Y\cap P)\,. \tag{4.3}$$

The set $Y\cap P$ is included in $Y-X$. Set

$$\mathcal{Y}=\mathring{\mathrm{J}}f(\beta Y)\qquad \mathcal{X}_r=\mathrm{J}f(\beta X_r)\qquad\qquad (r\in(r))\,. \tag{4.4}$$

Hypothesis (γ). *We require that the image $f(\boldsymbol{y})$ of some point $\boldsymbol{y}\in Y-X$ be in $\mathcal{Y}$.*

LEMMA 4.1. *If Hypotheses* (α), (β) *and* (γ) *are satisfied then $\mathcal{Y}$ admits the partition*

$$\mathcal{Y}=\bigcup_r \mathcal{X}_r\cup f(Y-X) \tag{4.5}$$

into disjoint sets $f(Y-X)$ and $\mathcal{X}_r$, $r\in(r)$.

Note that $f(Y-X)$ is connected since $Y-X$ is connected and f is continuous. We continue by proving I to VII.

I. $\beta f(Y-X-(P\cap Y))=\bigcup_r f(\beta X_r)\cup f(\beta Y)\cup f(P\cap Y)$. Set $Y-X-(P\cap Y)=A$. Then A and hence $f(A)$ is open. Since $\mathrm{Cl}\,A$ is a compact subset of E on which the homeomorphism f is defined $f(\mathrm{Cl}\,A)=\mathrm{Cl}\,f(A)$. Now $\beta f(A)=\mathrm{Cl}\,f(A)-f(A)=f(\mathrm{Cl}\,A)-f(A)=f(\mathrm{Cl}\,A-A)=f\beta\,A$. Relation I now follows from (4.3).

II. $f(Y-X)\subset\mathcal{Y}$. At least one point $\boldsymbol{y}\in Y-X$ has an image $f(\boldsymbol{y})\in\mathcal{Y}$ by hypothesis. Moreover $f(Y-X)$ is connected and does not meet $\beta\mathcal{Y}$. Hence II holds.

III. $\mathcal{X}_r\subset\mathcal{Y}$, $r\in(r)$. It follows from II that

$$f(\mathrm{Cl}(Y-X))\subset\mathrm{Cl}\,\mathcal{Y}\,.$$

Hence in particular $f(\beta X_r)\subset \mathrm{Cl}\,\mathcal{Y}$. But βX_r does not meet βY since $X_r\subset Y$ by hypothesis. Thus $f(\beta X_r)\subset \mathcal{Y}$. It follows that $\mathcal{X}_r\subset \mathcal{Y}$.

IV. $\mathcal{X}_r\subset C\mathcal{X}_s$ $r\neq s$. If this inclusion is false then either $\mathcal{X}_r\subset \mathcal{X}_s$ or $\mathcal{X}_s\subset \mathcal{X}_r$ since these sets and their complements are connected and have disjoint boundaries. If $\mathcal{X}_r\subset \mathcal{X}_s$, $\beta\mathcal{X}_r\subset \mathcal{X}_s$. It would follow that a neighborhood of $\beta\mathcal{X}_r$ would be included in $\mathcal{X}_s$. Thus some points, and hence *all* points of $f(Y-X)$ would be in $\mathcal{X}_s$. For $f(Y-X)$ is connected and does not meet $\beta\mathcal{X}_s$. If $f(Y-X)$ were included in $\mathcal{X}_s$, then $f(\beta Y)$ would also be included in $\mathcal{X}_s$ contrary to III. We infer the truth of IV.

V. $f(Y-X)\subset C\mathcal{X}_r$. If this inclusion were false then some points and hence all points of $f(Y-X)$ would be in $\mathcal{X}_r$, implying that $f(\beta Y)\subset \mathcal{X}_r$ contrary to III.

We have thus shown that the sets on the right of (4.5) are disjoint and that

(4.6) $$\bigcup \mathcal{X}_r\cup f(Y-X)\subset \mathcal{Y}.$$

Setting $\bigcup\mathcal{X}_r=\mathcal{X}$ we write (4.6) in the form

(4.7) $$f(Y-X)\subset \mathcal{Y}-\mathcal{X}.$$

VI. *The set* $V=\mathcal{Y}-\mathcal{X}-f(P\cap Y)$ *is connected.* We shall apply (ii) to establish VI, with Y replaced by $\mathcal{Y}$, X_r by $\mathcal{X}_r$ and P by $f(P)$.

Note that each $\beta\mathcal{X}_r$ is an elementary topological $(n-1)$-sphere. For $\beta\mathcal{X}_r=f(\beta X_r)$ by definition, $\beta\mathcal{X}_r$ is elementary by hypothesis, f is a homeomorphism defined over a neighborhood of βX_r, and finally the elementary character of βX_r is preserved under f. Moreover $\mathcal{X}_r\subset\mathcal{Y}$ by III, the $\mathcal{X}_r$ are disjoint by IV, and when (r) is infinite the sets $\mathcal{X}_r$ converge to $f(P)$. Thus (ii) applies and shows that V is connected.

VII. *The inclusion* (4.7) *is an equality.* To prove that (4.7) is an equality is equivalent to proving that the inclusion

(4.8) $$f(Y-X-(Y\cap P))\subset \mathcal{Y}-\mathcal{X}-f(Y\cap P)$$

is an equality.

Let the left and right members of (4.8) be denoted by U and V respectively. U is open since $Y-X-(Y\cap P)$ is open and f a homeomorphism, while V is open by strict analogy with $Y-X-(Y\cap P)$. According to VI, V is connected. Moreover $\beta U=\beta V$, as one sees from I and a formula for βV analogous to (4.3). Thus VII follows from (i).

This establishes the lemma.

COROLLARY 4.1. *Under the conditions of Lemma* 4.1 $\mathcal{Y}$ *admits the partition*

$$\mathcal{Y}=\bigcup_r \mathring{\mathcal{X}}_r\cup f(Y-\mathring{X})$$

Note. Verification of hypotheses of Lemma 4.1.

In each application of Lemma 4.1 the sets βX_r will be boundaries W_r of n-rectangles or images of such boundaries W_r under homeomorphisms τ_r into E of an open subset of E which includes W_r. Each W_r is obviously elementary, and a homeomorphism τ_r defined over a neighborhood of an elementary topological $(n-1)$-sphere yields an elementary image $\tau_r(W_r)$.

The condition (γ) of Lemma 4.1 is the only hypothesis likely to cause difficulty. However each application of Lemma 4.1 will come under one of two cases.

Case I. In this case the maximum value of the coordinate x_n on $\overline{Y}$ is assumed exclusively on an $(n-1)$-rectangle k, while the maximum value of x_n on $\overline{q}$ is assumed exclusively on $f(k)$, and $f(k)$ is an $(n-1)$-rectangle. Moreover, as the reader will verify, the mapping f is linear on a neighborhood of k relative to $\overline{Y}$ and maps this neighborhood onto a neighborhood of $f(k)$ relative to $\overline{Y}$.

Case II. This case is like Case I, with minimum replacing maximum.

The partitions (3.4) and (3.5) follow from Lemma 4.1 on setting $f=\omega$, $\Delta=K-\Theta$, $X_0=G'$, and $Y=H'$ and L' respectively. The partition (3.3)$'$ follows from Corollary 4.1 on setting $f=\sigma$, $\Delta=\mathrm{C}G'$, $X_0=\overline{H'}$ and $Y=D$. The boundaries of the rectangles involved are obviously elementary topological $(n-1)$-spheres. Hypothesis (γ) is verified as indicated in the Note. The partition (3.3) follows from (3.3)$'$ and (3.4).

§ 5. **A preliminary extension ω_e of ω.** To follow the program outlined at the end of § 3 we shall first make an easy extension of ω over a subset M of E now to be defined.

The set M. The space E admits a trivial partition,

$$E=\bigcup_{r=0}^{\infty} R^r(K)\cup A\cup P, \qquad [P=(8,0,\cdots,0)] \tag{5.1}$$

provided A is suitably chosen. The mapping ω is defined on $K-\Theta$. An open subset M of E is now defined by the disjoint union,

$$M=\bigcup_{r=0}^{\infty} R^r(K-\Theta)\cup A\,. \tag{5.2}$$

Definition of ω_e on M. Let ω_e be defined on $R^r(K-\Theta)$, $r=0,1,\cdots$, as the "*transform*" of $\omega|(K-\Theta)$ under R_r. That is, set

$$\omega_e(\boldsymbol{x})=R^r\cdot\omega\cdot R^{-r}(\boldsymbol{x}) \qquad [\boldsymbol{x}\in R^r(K-\Theta)]\,. \tag{5.3}$$

The definition of ω_e over M is completed by setting $\omega_e(\boldsymbol{x})=\boldsymbol{x}$ for $\boldsymbol{x}\in A$.

To describe ω_e we refer to the subrectangle K_d of K defined in § 2.

LEMMA 5.1. *The mapping ω_e of M is a C^m-diffeomorphism of M into E which reduces to the identity on A and on the "border" $R^r(K-K_d)$ of $R^r(K)$*

and maps $R^r(K-\Theta)$ into $R^r(K)$.

Since ω reduces to the identity over $K-K_d$, ω_e reduces to the identity over $R^r(K-K_d)$. The image of $R^r(K-\Theta)$ under ω_e is

$$\omega_e \cdot R^r(K-\Theta) = R^r \cdot \omega(K-\Theta) \subset R^r(K) .$$

Since ω_e reduces to the identity on A and on the border $R^r(K-K_d)$ of $R^r(K-\Theta)$ we infer that ω_e is locally a C^m-diffeomorphism on M. It is biunique. For the image of A is A under the identity. Moreover $R^r(K-\Theta)$ is mapped homeomorphically into $R^r(K)$, while the sets $R^r(K)$ and A are disjoint in accord with (5.1).

The extension of ω_e over $M \cup P$. The mapping ω_e of M into E can be extended as a homeomorphism over $M \cup P$ by setting $\omega_e(P) = P$.

To verify this, note that this extension is biunique since $\omega_e(M)$ does not meet P in accord with Lemma 5.1. We now show that ω_e is continuous on $M \cup P$ at P.

Let ρ_r be the maximum distance of a point of $R^r(\bar{K})$ from P. Then $\rho_r \to 0$ as $r \uparrow \infty$. Let $\boldsymbol{c}$ be the vector representing P. If $\boldsymbol{x}$ is in $M \cup P$ but not in one of the sets $R^r(K)$ let $\mu(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{c}\|$. If $\boldsymbol{x}$ is in $R^r(K-\Theta)$ then $\omega_e(\boldsymbol{x})$ is in $R^r(K)$ in accord with Lemma 5.1, and we set $\mu(\boldsymbol{x}) = \rho_r$. In any case

$$\| \omega_e(\boldsymbol{x}) - \omega_e(\boldsymbol{c}) \| \leqq \mu(\boldsymbol{x}) .$$

As $\boldsymbol{x}$ tends to $\boldsymbol{c}$, $\mu(\boldsymbol{x})$ tends to zero so that ω_e is continuous on $M \cup P$ at P.

The proof that ω_e^{-1} is continuous at P is similar.

Program for defining $\sigma \mid T_r(K)$. In § 6 we shall define a C^∞-diffeomorphism T_r of E onto E such that $T_r(K)$ is a *container* of the lacunary sets (3.13). We shall start with the restriction of ω_e to a border $T_r(K-K_d)$ of $T_r(K)$ and extend this restriction of ω_e over all of $T_r(K)$ as our definition of σ over $T_r(K)$.

Partitions of K. Sets $\mathcal{H}''$ and $\mathcal{G}''$ analogous to the sets $\mathcal{H}'$ and $\mathcal{G}'$ already defined are introduced by setting

(5.4) $$\mathcal{H}'' = \mathring{J}\, \omega(\beta H'') \qquad \mathcal{G}'' = J\, \omega(\beta G'') .$$

Set

(5.5) $$\mathcal{H} = \mathcal{H}' \cup \mathcal{H}'' \qquad \mathcal{G} = \mathcal{G}' \cup \mathcal{G}'' .$$

Our partitions of K are

(5.6) $$K = \mathcal{H}' \cup \mathcal{H}'' \cup \omega(K-H) ,$$

(5.7) $$K = \mathcal{G}' \cup \mathcal{G}'' \cup \omega(K-G) ,$$

consequences of Corollary 4.1 and Lemma 4.1, respectively, setting $f = \omega_e$ and $\Delta = M$. (Cf. 5.2.) Relation (3.4) is analogous to a similar relation in which H' is replaced by H''. This new relation and (3.4) give the equality

$$\mathcal{H} = \mathcal{G}' \cup \mathcal{G}'' \cup \omega(H-G)\,. \tag{5.8}$$

This is a partition of $\mathcal{H}$. For the sets on the right of (5.7) are disjoint and hence those on the right of (5.8) are disjoint.

§ 6. **The mapping T.** In § 3 we have referred to a set $T_r(K)$ as the *container* of the pair of lacunary sets in (3.13). In this section we shall define T_r and certain associated sets and mappings, starting with a definition of $T = T_1$.

The mapping $T = T_1$. If B is a bounded subset of E let Int B denote the smallest n-rectangle Π with faces parallel to the coordinate $(n-1)$-planes and such that $\Pi \supset B$. Referring to the constant d introduced in § 2, let Z' and Z'' be respectively the open subsets of K on which $x_n < -d$ and $x_n > d$. In § 9 of [**2**] we have denoted the sets Z' and Z'' by H' and H'' respectively. In the notation of the present paper the results of [**2**, § 9], imply the existence of a C$^\infty$-diffeomorphism T of E onto E such that

$$T(\boldsymbol{x}) = \boldsymbol{x} \qquad (\boldsymbol{x} \in Z''), \tag{6.1}$$

$$T(\boldsymbol{x}) = R(\boldsymbol{x}) \qquad (\boldsymbol{x} \in Z'), \tag{6.2}$$

$$RT(\bar{K}) \cap T(\bar{K}) = \phi\,, \tag{6.3}$$

$$T(\bar{K}) \subset \mathrm{Int}[\bar{K} \cup R\bar{K}]\,, \tag{6.4}$$

while the sign of the x_n-coordinate of $T(\boldsymbol{x})$ is that of the x_n-coordinate of $\boldsymbol{x}$.

The mapping T_{r+1}. We define a C$^\infty$-diffeomorphism T_r of E onto E by setting

$$T_{r+1} = R^r T \qquad (r = 0, 1, \cdots) \tag{6.5}$$

and note the basic relations,

$$T_r(\bar{K}) \cap T_p(\bar{K}) = \phi, \qquad Z' \cap T_p(K) = \phi \qquad (r > p > 0) \tag{6.6}$$

of which the first relations follow from (6.3) and (6.4). The relation in (6.6) involving Z' is the relation (9.25)$''$ of [**2**]. It follows from (6.4) that $T(K)$ does not meet P. Hence $T_r(K)$ does not meet P.

Partitions used in defining σ. Recalling that Z' is open, G' closed, and $Z' \supset G'$, the relations (6.6) and the above remarks concerning P imply a partition,

$$E = \bigcup_{r=1}^{\infty} T_r(K) \cup G' \cup P \cup L\,, \tag{6.7}$$

for suitable choice of L. It is by virtue of (6.7) that our definition of σ over CG' in § 7 is single-valued. Cf. (3.14). In defining σ over $T_r(K)$ we shall make use of the partition, cf. (5.7),

$$T_r(K) = T_r(K - \mathcal{G}) \cup T_r(\mathcal{G}') \cup T_r(\mathcal{G}'')\,. \tag{6.8}$$

We note the trivial partition,

$$T_r(K)=T_r(K-\Theta)\cup T_r(\Theta')\cup T_r(\Theta'') \tag{6.9}$$

$$=T_r(K-\Theta)\cup R^r(\Theta')\cup R^{r-1}(\Theta''). \tag{6.10}$$

According to (6.10), $T_r(K)$ is a container of the lacunary sets (3.13).

The partition (6.8) of $T_r(K)$ leads naturally to a covering of $T_r(K)$ by open sets,

$$T_r(K)=T_r(K-\mathcal{G})\cup T_r(\mathcal{H}')\cup T_r(\mathcal{H}''), \tag{6.11}$$

noting that $\mathcal{H}'\cup\mathcal{H}''\supset\mathcal{G}'\cup\mathcal{G}''$ in accord with (5.8). It is on the basis of the covering (6.11) that we shall define σ on $T_r(K)$.

The mapping α_r. We set

$$\alpha_r(x)=T_r\cdot\omega^{-1}\cdot T_r^{-1}(x) \qquad [x\in T_r(K-\mathcal{G})]. \tag{6.12}$$

Under α_r, $T_r(K-\mathcal{G})$ is mapped onto $T_r(K-G)$ as follows from the relation

$$K-G=\omega^{-1}(K-\mathcal{G}) \qquad (\text{cf. } (5.7)). \tag{6.13}$$

To further orient our procedures it will be useful to term the subsets of $T_r(K-\mathcal{G})$,

$$T_r(\mathcal{H}-\mathcal{G}),\quad T_r(K-K_d),$$

the *inner* and *outer borders* of $T_r(K-\mathcal{G})$, noting that

$$K-K_d\subset\omega(K-G)=K-\mathcal{G}$$

by virtue of (5.7). We prove the following lemma.

LEMMA 6.1. *The mapping α_r reduces to the identity on the outer border of $T_r(K-\mathcal{G})$.*

This is an immediate consequence of the fact that $\omega(x)=x$ for $x\in K-K_d$, and the definition of α_r in (6.12).

The inner border of $T_r(K-\mathcal{G})$ is the union of two disjoint sets (cf. (5.6)),

$$T_r(\mathcal{H}'-\mathcal{G}'),\quad T_r(\mathcal{H}''-\mathcal{G}''), \tag{6.14}$$

in terms of which we state the following lemma.

LEMMA 6.2. *On the inner border of $T_r(K-\mathcal{G})$ the mapping $\omega_e\alpha_r$ is independent of ω, reducing to the mappings,*

$$R^rT_r^{-1},\qquad R^{r-1}T_r^{-1} \tag{6.15}$$

on the respective sets (6.14).

By definition of α_r

$$\omega_e\cdot\alpha_r(x)=\omega_e\cdot T_r\cdot\omega^{-1}\cdot T_r^{-1}(x) \qquad [x\in T_r(K-\mathcal{G})]. \tag{6.16}$$

In evaluating this mapping on the first set in (6.14) one is led to evaluate $\omega_e\cdot T_r\omega^{-1}$ on $\mathcal{H}'-\mathcal{G}'$. The definition of ω_e on $R^r(K-\Theta)$, namely $\omega_e=R^r\omega R^{-r}$, and relations,

$$T_r = R^{r-1}T, \qquad T|H' = R|H',$$

imply that on $\mathcal{H}' - \mathcal{G}' = \omega(H' - G')$,

$$\omega_e T_r \omega^{-1} = \omega_e R^{r-1} T \omega^{-1} = \omega_e R^r \omega^{-1} = R^r . \tag{6.17}$$

Thus

$$\omega_e \cdot \alpha_r(\boldsymbol{x}) = R^r T_r^{-1}(\boldsymbol{x}) \qquad [\boldsymbol{x} \in T_r(\mathcal{H}' - \mathcal{G}')]$$

as stated in the Lemma. The evaluation of $\omega_e \alpha_r$ on the second set in (6.14) is similar, using the definition $T|H'' = \boldsymbol{U}|H''$.

§ 7. **The definition of** σ. Lemmas 6.1 and 6.2 suggest the following.

LEMMA 7.1. *Consistent with the definition of ω_e on the outer border of $T_r(K-\mathcal{G})$ we can define σ on $T_r(K)$ as a mapping of $T_r(K)$ into E such that*

$$\sigma(\boldsymbol{x}) = \omega_e \cdot \alpha_r(\boldsymbol{x}) \qquad [\boldsymbol{x} \in T_r(K - \mathcal{G})], \tag{7.0}$$

$$\sigma(\boldsymbol{x}) = R^r \cdot T_r^{-1}(\boldsymbol{x}) \qquad [\boldsymbol{x} \in T_r(\mathcal{H}')], \tag{7.1}$$

$$\sigma(\boldsymbol{x}) = R^{r-1} T_r^{-1}(\boldsymbol{x}) \qquad [\boldsymbol{x} \in T_r(\mathcal{H}'')]. \tag{7.2}$$

Then σ defines a C^m-diffeomorphism of $T_r(K)$ into E such that

$$\sigma T_r(K) = \mathring{\mathrm{J}}\, \omega_e(\beta T_r(K)). \tag{7.3}$$

To show that a mapping of an open subset of E into E is a C^m-diffeomorphism it is sufficient to show that it is locally a C^m-diffeomorphism and is biunique. It follows from Lemmas 6.1 and 6.2 that σ is single-valued on $T_r(K)$, agrees with ω_e on the outer border of $T_r(K-\mathcal{G})$, and is locally a C^m-diffeomorphism. Applying σ to the sets in the partition (6.8) of $T_r(K)$ one sees that

$$\sigma \cdot T_r(K) = \omega_e \cdot T_r(K - G) \cup R^r(\mathcal{G}') \cup R^{r-1}(\mathcal{G}'') \tag{7.4}$$

where the sets on the right of (7.4) are the (1-1)-images under σ of the respective sets on the right of (6.8). Hence the mapping σ of $T_r(K)$ is a C^m-diffeomorphism if (7.4) is a partition.

Proof that (7.4) *is a partition.* We shall apply Lemma 4.1 setting $(r) = (0, 1)$ and

$$Y = K, \quad X_0 = G', \quad X_1 = G'', \quad f = \omega_e T_r, \quad \Delta = T_r^{-1}(M). \tag{7.5}$$

The condition (4.2) on Δ takes the form

$$\mathrm{Cl}\, T_r(K - G) \subset M. \tag{7.6}$$

To verify (7.6) recall that CM, as defined in (5.2), is the union of P, Θ' and the lacunary sets (3.13). The set $\mathrm{Cl}\, T_r(K-G)$ does not meet P, nor Θ' in accord with (6.6), nor the lacunary sets $R^r(\Theta')$, $R^{r-1}(\Theta'')$ by virtue of (6.10), nor the remaining lacunary sets by (6.6). Thus (7.6) holds. The condition (γ) for the applicability of Lemma 4.1 is verified as in the Note at the end of § 4.

In accord with the definitions of $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}$ and $\mathcal{Y}$ in § 4 and of T_r, σ_r and f,

$$\mathcal{X}_0 = \mathrm{J} f(\beta X_0) = \mathrm{J}\, \omega_e \cdot R^r(\beta G') = \mathrm{J}\, R^r \cdot \omega(\beta G') = R^r(\mathcal{G}') \, , \tag{7.7}$$

$$\mathcal{X}_1 = \mathrm{J} f(\beta X_1) = \mathrm{J}\, \omega_e \cdot R^{r-1}(\beta G'') = \mathrm{J}\, R^{r-1} \cdot \omega(\beta G'') = R^{r-1}(\mathcal{G}'') \, , \tag{7.8}$$

$$\mathcal{Y} = \mathring{\mathrm{J}} f(\beta Y) = \mathring{\mathrm{J}}\, \omega_e T_r(\beta K) = \mathring{\mathrm{J}}\, \omega_e(\beta T_r(K)) \, , \tag{7.9}$$

$$f(Y-X) = f(K-G) = \omega_e T_r(K-G) \, . \tag{7.10}$$

Lemma 4.1 gives the partition

$$\mathcal{Y} = f(Y-X) \cup \mathcal{X}_0 \cup \mathcal{X}_1 \, . \tag{7.11}$$

The sets on the right of (7.11) are the respective sets on the right of (7.4). Hence (7.4) is a partition.

By virtue of (7.9) and the equality of the right members of (7.4) and (7.11), (7.3) holds.

Completion of definition of σ. The mapping σ is to be defined over CG'. It has been defined on the disjoint sets $T_r(K)$, $r = 1, 2, \cdots$. The mapping ω_e is defined over E except on the sets

$$R^r(\Theta'), \quad R^r(\Theta'') \qquad (r = 0, 1, 2, \cdots) \tag{7.12}$$

each of which, except for Θ', is included in the domain on which σ has been defined. Since $G' \supset \Theta'$ one can accordingly complete the definition of σ over CG' by setting $\sigma(\boldsymbol{x}) = \omega_e(\boldsymbol{x})$ at all points of CG' where σ is not yet defined. This leads to Lemma 7.2.

LEMMA 7.2. *If one defines σ on the sets $T_r(K)$ as in Lemma 7.1 and at each other point $\boldsymbol{x}$ of CG' sets $\sigma(\boldsymbol{x}) = \omega_e(\boldsymbol{x})$, then σ is a homeomorphism of CG' into E and defines a C^m-diffeomorphism of $CG' - P$ into E.*

We commence with a proof of I.

I. *The mapping σ is continuous at P.*

We have already seen in § 5 that ω_e is continuous on $M \cup P$ at P. It is accordingly sufficient to prove that the restriction of σ to the union of P and the sets $T_r(K)$ is continuous at P. But this follows at once from the fact that the sets $T_r(K)$ converge to P as $r \uparrow \infty$, and hence the sets $\sigma \cdot T_r(K)$, as can be seen from (7.3), recalling the continuity of ω_e. Thus the sets $\sigma \cdot T_r(K)$ converge to P with $T_r(K)$ as $r \uparrow \infty$.

Since the definition of σ on the outer border of $T_r(K - \mathcal{G})$ is consistent with the definition of ω_e on this border it follows that σ defines a C^m-diffeomorphism into E of some neighborhood of each point of CG', excepting at most P. It remains to prove that σ is a homeomorphism. Since $\sigma(\boldsymbol{x}) = \boldsymbol{x}$ for $\boldsymbol{x}$ in a neighborhood of CD it is sufficient to prove II.

II. *The mapping σ defines a homeomorphism of $D - G'$ into D.*

We shall apply Lemma 4.1 with $Y=D$, $f=\omega_e$, $\Delta=M\cup P$ and (r) infinite. Set

$$X_0=G', \quad X_r=T_r(\bar{K}), \quad \mathcal{X}_r=\mathrm{J}\,\omega_e(\beta T_r(K)) \qquad (r>0). \tag{7.13}$$

In accord with the definition of σ

$$\begin{aligned}\sigma(D-G') &= \bigcup_{r=1}^{\infty}\sigma\cdot T_r(\bar{K})\cup\omega_e[D-G'-\bigcup_{r=1}^{\infty}T_r(\bar{K})]\\ &= \bigcup_{r=1}^{\infty}\mathcal{X}_r\cup\omega_e[D-\bigcup_{r=0}^{\infty}X_r],\end{aligned} \tag{7.14}$$

making use of (7.3). To show that $\sigma|(D-G')$ is biunique it is sufficient to show that (7.14) is a partition of $\sigma(D-G')$.

Proof that (7.14) *is a partition.* Consistent with the notation of § 4 set

$$\mathcal{X}_0=\mathcal{G}', \quad X=\bigcup_{r=0}^{\infty}X_r. \tag{7.15}$$

We shall use Lemma 4.1 to show that D admits the partition

$$D=\bigcup_{r=0}^{\infty}\mathcal{X}_r\cup\omega_e(D-X). \tag{7.16}$$

The right member of (7.16) is identical with the right member of (7.14) except for the addition of the set $\mathcal{X}_0$. If (7.16) is a partition, (7.14) is then necessarily a partition.

In the application of Lemma 4.1 one verifies (4.2) in the form $\mathrm{Cl}(Y-X)\subset M\cup P$. This follows at once from the relation $\mathring{X}=\bigcup_{r=0}^{\infty}\mathring{X}_r\supset \mathrm{C}(M\cup P)$. Cf. proof of (7.6). The condition (γ) for the applicability of Lemma 4.1 is satisfied since P is in $Y-X$ and $\omega_e(P)=P\in D=Y=\mathcal{Y}$. We infer that (7.16), and hence (7.14), is a partition.

The continuity of the inverse of σ at P is readily established taking account of the fact that ω_e is a homeomorphism, mapping P into P, that each set $\sigma\cdot T_r(\bar{K})$ is bounded from P and the sets $T_r(\bar{K})$ converge to P as $r\uparrow\infty$. We conclude that σ is bicontinuous and biunique and hence a homeomorphism.

This completes the proof of Lemma 7.2.

Now that the existence of σ is established, Theorem 3.1 implies the existence of a solution $(\lambda_\omega, w, \Omega)$ of problem (ω, H', G') in Theorem 2.2. The domain of definition of λ_ω is H'. The image of H' under λ_ω is $\mathcal{H}'$. The mapping λ_ω is a homeomorphism into E. It is a C^m-diffeomorphism on $H'-\boldsymbol{w}$ and is an extension of $\omega|(H'-\Omega)$. We complete the necessary information concerning λ_ω by proving the following.

(π) *The set* $\mathcal{H}'$ *does not meet the point* $Q\in S_{n-1}$.

Under the reflection t in § 2 the subspace of E on which $x_n < 0$, is mapped onto $\mathring{\mathrm{J}}\, S_{n-1}$. Since $x_n < 0$ on the closure of H', the set $t(H')$ and its boundary (which we denote by Σ) are included in $\mathring{\mathrm{J}}\, S_{n-1}$. Moreover Σ, S_{n-1} and the set $\mathring{\mathrm{J}}(S_{n-1}) - \mathrm{J}(\Sigma)$ are included in the shell δ_a. Recall that

$$\varphi(\Sigma) \subset \mathring{\mathrm{J}}\, \varphi(S_{n-1}) \tag{7.17}$$

since φ carries points of δ_a interior to S_{n-1} into points interior to $\mathring{\mathrm{J}}\, \varphi(S_{n-1})$. Since $\varphi(Q) = Q \in S_{n-1}$, Q is in $\varphi(S_{n-1})$. Hence $\mathring{\mathrm{J}}\, \varphi(\Sigma)$ does not meet Q. It follows that $\mathring{\mathrm{J}}\, t \cdot \varphi(\Sigma)$ does not meet Q. But

$$t\varphi(\Sigma) = t \cdot \varphi \cdot t(\beta H') = \omega(\beta H') = \beta \mathcal{H}' \tag{7.18}$$

in accord with the definition of ω and the definition of $\mathcal{H}'$ in (3.2). Since $\mathring{\mathrm{J}}\, \beta \mathcal{H}' = \mathcal{H}'$ we conclude that $\mathcal{H}'$ does not meet Q.

Proof of Theorem 2.1. To satisfy Theorem 2.1 it is necessary to define the elements $(\Lambda_\varphi, \boldsymbol{z}, \kappa)$ appearing in Theorem 2.1.

Since $\Omega \subset H'$

$$t(\Omega) \subset \mathring{\mathrm{J}}\, \Sigma \subset \mathring{\mathrm{J}}\, S_{n-1}\,. \tag{7.19}$$

Set $t(\Omega) = \kappa$ and $t(\boldsymbol{w}) = \boldsymbol{z}$. Since $\boldsymbol{w} \in \Omega$ we have $\boldsymbol{z} \in \kappa$.

According to the definition of ω and λ_ω, for $\boldsymbol{y} \in t(H' - \Omega) = \mathring{\mathrm{J}}\, \Sigma - \kappa$,

$$\varphi(\boldsymbol{y}) = t \cdot \omega \cdot t(\boldsymbol{y}) = t \cdot \lambda_\omega \cdot t(\boldsymbol{y})\,. \tag{7.20}$$

One can define Λ_φ over the set,

$$\mathring{\mathrm{J}}\, S_{n-1} \cup \delta_a = \mathring{\mathrm{J}}\, \Sigma \cup \delta_a = \kappa \cup \delta_a\,, \tag{7.21}$$

by setting

$$\Lambda_\varphi(\boldsymbol{y}) = \varphi(\boldsymbol{y}) \qquad (\boldsymbol{y} \in \delta_a - \mathring{\mathrm{J}}\, \Sigma)\,, \tag{7.22)$'$$

$$\Lambda_\varphi(\boldsymbol{y}) = t \cdot \lambda_\omega \cdot t(\boldsymbol{y}) \qquad (\boldsymbol{y} \in \mathring{\mathrm{J}}\, \Sigma)\,. \tag{7.22)$''$$

We must verify that $\Lambda_\varphi(\boldsymbol{y})$ is well defined in (7.22)$''$, taking account of the domains of definition of t and λ_ω. Now $t\, \mathring{\mathrm{J}}(\Sigma) = H'$. By virtue of (2.7) and the above statement (π),

$$\lambda_\omega(H') = \mathcal{H}', \qquad \lambda_\omega(H') \cap Q = \phi\,. \tag{7.23}$$

By virtue of (7.21) and (7.20) Λ_φ is equivalently defined by setting

$$\Lambda_\varphi(\boldsymbol{y}) = \varphi(\boldsymbol{y}) \qquad (\boldsymbol{y} \in \delta_a - \kappa)\,, \tag{7.24)$'$$

$$\Lambda_\varphi(\boldsymbol{y}) = t \cdot \lambda_\omega \cdot t(\boldsymbol{y}) \qquad (\boldsymbol{y} \in \kappa)\,. \tag{7.24)$''$$

The definitions of $\Lambda_\varphi(\boldsymbol{y})$ in (7.22) and in (7.24) are obviously the same, except at most when $\boldsymbol{y} \in \mathring{\mathrm{J}}\, \Sigma - \kappa$. For such points $(\boldsymbol{y})$, $\Lambda_\varphi(\boldsymbol{y})$ is defined both in (7.22)$''$ and in (7.24)$''$, and these definitions are the same by virtue of (7.20).

Each point $(\boldsymbol{y})$ at which Λ_φ is defined is in $\mathring{\mathrm{J}}\,\Sigma$, or in the interior of $\delta_a - \mathring{\mathrm{J}}\,\Sigma$ in (7.22), except points $(\boldsymbol{y}) \in \Sigma$. The latter points are interior to $\delta_a - \kappa$ in (7.24)$'$. Hence Λ_φ is locally a homeomorphism. It is locally a C^m-diffeomorphism except at most at $\boldsymbol{z}$.

The mapping Λ_φ is an extension of $\varphi \mid (\delta_a - \kappa)$ by virtue of (7.24)$'$. That Λ_φ is biunique may be seen as follows. The restricted mappings defined by (7.22)$'$ and (7.22)$''$ are separately biunique. One verifies readily that the sets

$$\Lambda_\varphi(\mathring{\mathrm{J}}\,\Sigma), \qquad \Lambda_\varphi(\delta_a - \mathring{\mathrm{J}}\,\Sigma)$$

are included in the interior of $\varphi(\Sigma)$ and its complement respectively, and are accordingly disjoint. The biuniqueness of Λ_φ follows. The relation (2.2) is a consequence of the fact that $\Lambda_\varphi(\mathring{\mathrm{J}}\,S_{n-1})$ is a maximal connected bounded subset of E which does not meet $\varphi(S_{n-1})$. Cf. proof of (4) in [**4**].

This establishes Theorem 2.1.

The case $m > 0$. It follows from the results of [**3**] and [**2**, § 5], that when $m > 0$ the existence of a C^m-diffeomorphism Φ of an $(n-1)$-sphere S_{n-1} onto an $(n-1)$-manifold $\mathcal{M}_{n-1}$ in E implies the existence of a C^m-diffeomorphism φ of a suitably chosen shell δ_a into E, where S_{n-1} is included in δ_a and φ extends Φ over δ_a. Hence one obtains Theorem 0.1 of [**2**] as a corollary of Theorem 2.1 of this paper.

We here state another and equivalent formulation of Theorem 2.1. This new formulation meets the needs of our forthcoming paper on "The dependence of the Schoenflies extension on an accessory parameter," Journal d' Analyse Mathématique.

We suppose here that W is an open neighborhood of an $(n-1)$-sphere S in E, and let φ be a C^m-diffeomorphism, $m \geqq 0$, of W into E which maps points of W exterior to S (interior to S) into points exterior to $\varphi(S)$ (interior to $\varphi(S)$). Let Z denote the center of S.

THEOREM 7.1. *If N is a sufficiently small open neighborhood of S there exists an extension Λ_φ of the restriction*

$$\varphi \mid (N \cup (W - \mathrm{J}\,S))$$

of φ which is a homeomorphism of $W \cup J\,S$ into E and which, when $m > 0$, is, in addition, a C^m-diffeomorphism into E of

$$(W \cup \mathrm{J}\,S) - Z.$$

If the theorem is satisfied by N and Λ_φ it is also satisfied by N_1 and Λ_φ, where N_1 is any open neighborhood of S such that $N_1 \subset N$. For a proof of Theorem 7.1 see a forthcoming Note: Proc. Nat. Acad. Sci. U. S. A.

Institute for Advanced Study.

References

[1] Mazur, B., On embeddings of spheres, Bull. Amer. Math. Soc., **65** (1959), 59-65.

[2] Morse, M., Differentiable mappings in the Schoenflies theorem, Compositio Mathematica, **14** (1959), 83-151.

[3] Morse, M., On elevating manifold differentiability, Jubilee of the Indian Math. Soc., (1960), to be published.

[4] Morse, M., A reduction of the Schoenflies extension problem, Bull. Amer. Math. Soc., **66** (1960), 113-115.

[5] Morse, M., Differentiable mappings in the Schoenflies theorem, Proc. Nat. Acad. Sci. U. S. A., **44** (1958), 1068-1072.

[6] Alexandroff-Hopf, Topologie, Julius Springer, Berlin, 1935.