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On vector differential forms attached to automorphic forms.

Dedicated to Professor Z. Suetuna.

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In recent works [2], [3], it was found that the integral of certain vector differential forms, attached to automorphic forms with respect to a Fuchsian group G, is important in the arithmetic theory of modular correspondences. Those vector differential forms ω are defined on the upper half plane and satisfy the transformation formula

$$\omega\circ\sigma=M(\sigma)\omega \tag{1}$$

for every element σ of the group G, where $M(\sigma)$ is a tensor representation of G. The object of the present paper is to determine all holomorphic forms satisfying this relation (1). M being of degree $2m-1$, we can attach to every cusp form of degree $\leqq 2m$ a holomorphic form ω with the representation M (Theorem 1). Conversely, any holomorphic form satisfying (1) is expressed as a sum of the forms thus obtained from cusp forms of degree $\leqq 2m$; and this expression gives a direct decomposition of the vector space $\mathfrak{F}$ of such holomorphic forms (Theorem 2). Hence the dimension of the vector space $\mathfrak{F}$ is easily obtained if we know the dimension of the linear space of cusp forms for each degree. We note that the integral of the form attached to a cusp form of degree $<2m$ has a period cohomologous to 0, in the sense described in [3]. This fact distinguishes among such forms the forms attached to cusp forms of degree $2m$, which were the object of the investigation in [3].

§ 1. Cusp forms with respect to a Fuchsian group.

Let $\mathcal{H}$ denote the upper half plane, the set of all complex numbers with positive imaginary parts. Every element $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ of $SL(2,\boldsymbol{R})$ operates on $\mathcal{H}$, as usual:

$$\sigma(z)=\frac{az+b}{cz+d};$$

we put

$$J(\sigma,z)=(cz+d)^{-1}.$$

For every differential form ω on $\mathscr{H}$ we shall denote by $\omega\circ\sigma$ the transform of ω by σ; so if ω is expressed in the form $\omega=f(z)dz$ for a function $f(z)$ on $\mathscr{H}$, we have $\omega\circ\sigma=f(\sigma(z))J(\sigma,z)^2dz$.

Let G be a discrete subgroup of $SL(2,\boldsymbol{R})$ such that $SL(2,\boldsymbol{R})/G$ has a finite total volume, measured by an invariant volume element. Then, G, as group of transformations on $\mathscr{H}$, is a Fuchsian group; namely, G operates discontinuously on $\mathscr{H}$ and $\mathscr{H}/G$ has a fundamental domain $\mathscr{D}$ with a finite Poincaré area. If we denote by $\mathscr{H}^*$ the join of $\mathscr{H}$ and the "cusps" of G, the quotient space $\mathscr{H}^*/G$, with a suitable analytic structure, can be regarded as a compact Riemann surface.

A *cusp* of G is the fixed point of a parabolic element of G, which is a real number or the point at infinity ∞. Let s be a cusp of G. Put

$$\rho=\begin{pmatrix}-s & 1\\ -1 & 0\end{pmatrix} \quad\text{or}\quad \rho=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

according as s is a real number or ∞. Then the set of all elements of G having s as fixed point is the free cyclic group generated by an element τ of G, which is of the form

$$\tau=\rho\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\rho^{-1},$$

where h is a positive real number.

Let ν be an integer. We shall understand, by an *automorphic form of degree ν with respect to G*, a function $f(z)$ on $\mathscr{H}$ satisfying the following conditions (A 1-3).

(A 1) *$f(z)$ is meromorphic on $\mathscr{H}$.*

(A 2) *For every $\sigma\in G$, we have $f(\sigma(z))J(\sigma,z)^{\nu}=f(z)$.*

Consider a cusp s of G; the transformation ρ and the positive number h being defined for s as above, we see that, if f satisfies (A 1-2), the function $f(\rho(z))J(\rho,z)^{\nu}$ is invariant under the translation $z\rightarrow z+h$. Hence, if we put

$$q=\exp(2\pi ih^{-1}z),$$

there exists a function $g(q)$ meromorphic in the domain $0<|q|<1$ such that

$$f(\rho(z))J(\rho,z)^{\nu}=g(q).$$

The condition (A 3) is now stated as follows.

(A 3) *For every cusp s of G, the function $g(q)$, defined as above, is meromorphic at $q=0$.*

An automorphic form $f(z)$ with respect to G is called a *cusp form with respect to G*, if the following conditions are satisfied.

(A 1′) *$f(z)$ is holomorphic on $\mathscr{H}$.*

(A 3′) *For every cusp s of G, the function $g(q)$, defined as above, is holomorphic and takes the value 0 at $q=0$.*

We denote by $S_\nu(G)$ the set of all cusp forms of degree ν with respect to G. In this paper we shall only deal with the forms of *even* degree.

§ 2. M_n-forms and M_n-vectors.

Let

$$GL(2, \boldsymbol{C}) \ni \sigma \rightarrow M_n(\sigma) \in GL(n+1, \boldsymbol{C})$$

be the representation of $GL(2, \boldsymbol{C})$ by symmetric contravariant tensors of order n, so that the equality

$$\sigma\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix}$$

is led to

$$M_n(\sigma)\begin{pmatrix} u^n \\ u^{n-1}v \\ \vdots \\ uv^{n-1} \\ v^n \end{pmatrix} = \begin{pmatrix} z^n \\ z^{n-1}w \\ \vdots \\ zw^{n-1} \\ w^n \end{pmatrix}.$$

For instance, we have

$$M_n\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & nz & \frac{n(n-1)}{2}z^2 & \cdots & z^n \\ 0 & 1 & (n-1)z & \cdots & z^{n-1} \\ & \cdots & & \cdots & \\ 0 & \cdots & & \cdots & 1 \end{pmatrix}. \tag{2}$$

As this matrix $M_n\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)$ will be often used in our investigation, we denote it briefly by $L_n(z)$:

$$L_n(z) = M_n\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right).$$

We have then, for every $\tau \in SL(2, \boldsymbol{R})$,

$$L_n(\tau(z))^{-1}M_n(\tau)L_n(z) = M_n\left(\begin{pmatrix} J & 0 \\ c & J^{-1} \end{pmatrix}\right), \tag{3}$$

where $J = J(\tau, z) = (cz+d)^{-1}$. In particular, if r is a real number and $\tau = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, we have

$$L_n(\tau(z)) = M_n(\tau)L_n(z). \tag{4}$$

Let f be an automorphic form of degree $n+2$ with respect to G. In [3] we have studied the vector differential form

$$\omega = L_n(z)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix} dz = \begin{pmatrix} fz^n dz \\ fz^{n-1}dz \\ \vdots \\ fdz \end{pmatrix} \tag{5}$$

which satisfies, for every $\sigma \in G$,

$$\omega \circ \sigma = M_n(\sigma)\omega .$$

This is an example of M_n-form, whose definition is given as follows. A column vector of dimension $n+1$

$$\omega = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}$$

is called an *M_n-form with respect to G*, if it satisfies the following conditions (M 1-3).

(M 1) *Each component ω_k is a meromorphic differential form on $\mathcal{H}$.*

(M 2) *For every $\sigma \in G$, we have $\omega \circ \sigma = M_n(\sigma)\omega$.*

Let s be a cusp of G; and let ρ and h be as in § 1. Then, if ω satisfies (M 1-2), we can easily verify, using (4), that the form

$$L_n(z)^{-1}M_n(\rho)^{-1}\omega \circ \rho$$

is invariant under the translation $z \to z+h$. Therefore, if we put $q = \exp(2\pi i h^{-1}z)$, there exist $n+1$ functions $f_0(q), \cdots, f_n(q)$, meromorphic in $0 < |q| < 1$, such that

$$L_n(z)^{-1}M_n(\rho)^{-1}\omega \circ \rho = \begin{pmatrix} f_0(q)dq \\ \vdots \\ f_n(q)dq \end{pmatrix}. \tag{6}$$

Now the condition (M 3) is stated as follows.

(M 3) *For every cusp s of G, the functions $f_k(q)$ defined by* (6) *are meromorphic at $q=0$.*

An M_n-form ω with respect to G is called a *cusp M_n-form with respect to G* if the following conditions (M 1′) and (M 3′) are satisfied.

(M 1′) *Every component of ω is holomorphic on $\mathcal{H}$.*

(M 3′) *For every cusp s of G, the functions $f_k(q)$ defined by* (6) *are holomorphic at $q=0$.*

We can prove that the form ω defined by (5) is an M_n-form with respect to G; it is a cusp M_n-form if and only if $f(z)$ is a cusp form. This fact is a special case of the following Theorem 1. We shall denote by $\mathfrak{F}_n(G)$ the set of all cusp M_n-forms with respect to G.

Considering functions in place of differential forms, we get the following definition. A column vector of dimension $n+1$

$$\mathfrak{g} = \begin{pmatrix} g_0 \\ \vdots \\ g_n \end{pmatrix}$$

is called an *M_n-vector with respect to G*, if it satisfies the following conditions

(V 1-3).

(V 1) *Every component* g_k *is a meromorphic function on* $\mathcal{H}$.

(V 2) *For every* $\sigma \in G$, *we have* $\mathfrak{g}\circ\sigma = M_n(\sigma)\mathfrak{g}$.

The notations s, ρ, h being as above, if $\mathfrak{g}$ satisfies (V 1-2), there exist $n+1$ functions $F_0(q), \cdots, F_n(q)$, meromorphic in $0<|q|<1$, such that

$$L_n(z)^{-1}M_n(\rho)^{-1}\mathfrak{g}\circ\rho = \begin{pmatrix} F_0(q) \\ \vdots \\ F_n(q) \end{pmatrix}. \tag{7}$$

(V 3) *For every cusp* s *of* G, *the functions* $F_k(q)$ *defined by* (7) *are meromorphic at* $q=0$.

An M_n-vector $\mathfrak{g}$ with respect to G is called a *cusp* M_n*-vector with respect to* G if the following conditions (V 1′) and (V 3′) are satisfied.

(V 1′) *Every component of* $\mathfrak{g}$ *is holomorphic on* $\mathcal{H}$.

(V 3′) *For every cusp* s *of* G, *the functions* $F_k(q)$ *defined by* (7) *are holomorphic and take the value* 0 *at* $q=0$.

We shall denote by $\mathfrak{V}_n(G)$ the set of all cusp M_n-vectors with respect to G.

§ 3. Main results.

We shall now state our results in the following theorems, for which the proofs will be given in § 4. We first introduce some notations. For every integer k and a non-negative integer j, we shall write

$$\binom{k}{j} = \begin{cases} 1 & \text{for } j=0, \\ \dfrac{k(k-1)\cdots(k-j+1)}{j!} & \text{for } j>0. \end{cases}$$

Consider a triplet (n, ν, k) of integers such that

i) n is even and non-negative;

ii) ν is even and $-(n-2) \leqq \nu \leqq n+2$;

iii) $0 \leqq k \leqq n - \dfrac{\nu+n-2}{2}$.

For such a triplet (n, ν, k), we put

$$\alpha_{n,\nu,k} = \begin{cases} 0 & \text{for } \nu+k-1<0, \\ \dfrac{\left(k+\dfrac{\nu+n-2}{2}\right)!}{k!\,(\nu+k-1)!} & \text{for } \nu+k-1 \geqq 0 \end{cases}$$

and

$$\gamma_{n,\nu,k} = \begin{cases} 0 & \text{for } \nu+k-1<0, \\ \dfrac{\left(k+\dfrac{\nu+n}{2}\right)!}{k!\,(\nu+k-1)!} & \text{for } \nu+k-1 \geqq 0. \end{cases}$$

For fixed n and ν, we denote $\alpha_{n,\nu,k}$ and $\gamma_{n,\nu,k}$ simply by α_k and γ_k.

LEMMA 1. *Let t be an integer such that $0 \leqq t \leqq n$ and $f_t, f_{t+1}, \cdots, f_n$ be $n-t+1$ meromorphic functions on $\mathfrak{H}$. If*

$$\omega = L_n(z)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_t \\ \vdots \\ f_n \end{pmatrix} dz$$

is an M_n-form with respect to G, then f_t is an automorphic form of degree $2t+2-n$ with respect to G. Moreover, if ω is a cusp M_n-form, f_t is a cusp form.

THEOREM 1. *Let n and ν be two even integers such that $n > 0$ and $-(n-2) \leqq \nu \leqq n+2$; put $\mu = \dfrac{n+2-\nu}{2}$. Then, for every automorphic form f of degree ν with respect to G, the vector differential form*

$$\omega = L_n(z)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_0 f \\ \alpha_1 f' \\ \vdots \\ \alpha_\mu f^{(\mu)} \end{pmatrix} dz \tag{8}$$

is an M_n-form with respect to G, where $\alpha_0 = \alpha_{n,\nu,0}, \cdots, \alpha_k = \alpha_{n,\nu,k}$; $f', \cdots, f^{(\mu)}$ denote the derivatives $df/dz, \cdots, d^\mu f/dz^\mu$; and the number of 0 in the column is $n-\mu$. Moreover, in order that ω is a cusp M_n-form, it is necessary and sufficient that f is a cusp form.

Remark that, if $\nu \leqq 0$, we have $\alpha_0 = \alpha_1 = \cdots = \alpha_{-\nu} = 0$. We denote by $\mathfrak{S}^n{}_\nu(G)$ the set of all M_n-forms ω of the form (8), where f is a cusp form of degree ν. If $\nu \leqq 0$, the set $\mathfrak{S}^n{}_\nu(G)$ consists only of the zero element. If $\nu > 0$, we have $\alpha_0 \neq 0$, so that the vector space $\mathfrak{S}^n{}_\nu(G)$ is canonically isomorphic to the vector space $S_\nu(G)$ by the mapping $f \to \omega$.

THEOREM 2. *The vector space $\mathfrak{F}_n(G)$ of all cusp M_n-forms is the direct sum of the vector spaces $\mathfrak{S}^n{}_\nu(G)$ for even ν such that $2 \leqq \nu \leqq n+2$:*

$$\mathfrak{F}_n(G) = \mathfrak{S}^n{}_2(G) + \cdots + \mathfrak{S}^n{}_n(G) + \mathfrak{S}^n{}_{n+2}(G)\,.$$

Hence, if we denote by $d_\nu(G)$ the dimension of the vector space $S_\nu(G)$, the dimension of the vector space $\mathfrak{F}_n(G)$ is equal to

$$d_2(G) + \cdots + d_n(G) + d_{n+2}(G)\,.$$

The number $d_\nu(G)$ is easily obtained by means of Riemann-Roch Theorem.

We note that from Lemma 1 and Theorem 1 follows a result of Bol [1], which asserts the $(n-1)$-th derivative of an automorphic form of degree $-(n-2)$ to be an automorphic form of degree n. In fact, consider the case $\nu=-(n-2)$ in Theorem 1; we have then

$$\alpha_0=\alpha_1=\cdots\alpha_{n-2}=0, \alpha_{n-1}\neq 0;$$

so the vector

$$L_n(z)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_{n-1}f^{(n-1)} \\ \alpha_n f^{(n)} \end{pmatrix} dz$$

is an M_n-form for every automorphic form f of degree $-(n-2)$. Hence, by Lemma 1, $f^{(n-1)}$ is an automorphic form of degree n.

THEOREM 3. *Let the integers n, ν, μ be the same as in Theorem 1. Then, for every automorphic form f of degree ν with respect to G, the vector function*

$$\mathfrak{f}=L_n(z)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma_0 f \\ \gamma_1 f' \\ \vdots \\ \gamma_{\mu-1}f^{(\mu-1)} \end{pmatrix} \tag{9}$$

is an M_n-vector with respect to G, where $\gamma_k=\gamma_{n,\nu,k}$; and the number of 0 *in the column is $n-\mu+1$. Moreover, in order that $\mathfrak{f}$ is a cusp M_n-vector, it is necessary and sufficient that f is a cusp form.*

Denote by $\mathfrak{T}^n{}_\nu(G)$ the set of all M_n-vectors $\mathfrak{f}$ of the form (9), where f is a cusp form of degree ν. We see easily $\mathfrak{T}^n{}_\nu(G)=\{0\}$ for $\nu\leqq 0$ and $\nu=n+2$. If $0<\nu\leqq n$, we have $\gamma_0\neq 0$, so that the vector space $\mathfrak{T}^n{}_\nu(G)$ is canonically isomorphic to the vector space $S_\nu(G)$ by the mapping $f\to\mathfrak{f}$.

THEOREM 4. *The vector space $\mathfrak{B}^n(G)$ of all cusp M_n-vectors is the direct sum of the vector spaces $\mathfrak{T}^n{}_\nu(G)$ for even ν such that $2\leqq\nu\leqq n$:*

$$\mathfrak{B}^n(G)=\mathfrak{T}^n{}_2(G)+\cdots+\mathfrak{T}^n{}_n(G).$$

Now we consider the differential $d\mathfrak{f}$ of an M_n-vector $\mathfrak{f}$. If $\mathfrak{f}$ is an M_n-vector with respect to G, then we can easily prove that $d\mathfrak{f}$ is an M_n-form with respect to G; if $\mathfrak{f}$ is a cusp M_n-vector, then $d\mathfrak{f}$ is a cusp M_n-form. More precisely, we have

THEOREM 5. *The integers n, ν, μ being as in Theorem* 1, *let f be an auto-*

morphic form of degree ν with respect to G. Define an M_n-form ω and an M_n-vector $\mathfrak{f}$ by (8) *and* (9). *Then we have*

$$d\mathfrak{f} = \mu(n-\mu+1)\omega .$$

Remark that $\mu(n-\mu+1)\neq 0$ if $\mu\geqq 1$. Hence, if $0<\nu\leqq n$, the mapping $\mathfrak{f}\rightarrow d\mathfrak{f}$ gives an isomorphism of $\mathfrak{T}^n{}_\nu(G)$ onto $\mathfrak{S}^n{}_\nu(G)$.

From the last theorem, we can conclude that, if $0<\nu<n+2$ and if $\omega\in\mathfrak{S}^n{}_\nu(G)$, the period of the integral $\int^z\omega$ is cohomologous to 0 in the sense of [**3**]. On the other hand, Theorem 1 of [**3**] claims that the period of $\int^z\omega$ is not cohomologous to 0 for every element $\omega\neq 0$ of $\mathfrak{S}^n{}_{n+2}(G)$. Therefore, we obtain the following result.

THEOREM 6. *Let $\mathfrak{N}_n(G)$ denote the set of all cusp M_n-forms with respect to G, whose integrals have the periods cohomologous to* 0. *Then, the factor space $\mathfrak{F}_n(G)/\mathfrak{N}_n(G)$ is canonically isomorphic to $S_{n+2}(G)$.*

Put, similarly as in [**3**], for $\omega,\eta\in\mathfrak{F}_n(G)$,

$$(\omega,\eta)=i\int_{\mathcal{D}}{}^t\omega P_n\bar{\eta} ,$$

where P_n is the symmetric matrix introduced in § 1 of [**3**] and $\mathcal{D}$ is a fundamental domain of G. Then (ω,η) is a Hermitian form on $\mathfrak{F}_n(G)$. By the above considerations, we see that two subspaces $\mathfrak{S}^n{}_{n+2}(G)$ and $\mathfrak{S}^n{}_2(G)+\cdots+\mathfrak{S}^n{}_n(G)$ of $\mathfrak{F}_n(G)$ are transversal to each other with respect to this form (ω,η), and (ω,η) is a zero form on the latter space, while it is a definite form on the former space (§ 2 of [**3**]).

§ 4. Proofs of Theorems.

LEMMA 2. *Let $f_0,\cdots,f_n$ be $n+1$ meromorphic functions on $\mathcal{H}$; put*

$$\mathfrak{f}=\begin{pmatrix} f_0\\ \vdots\\ f_n\end{pmatrix},\quad \omega=L_n(z)\mathfrak{f}dz .$$

Then, ω satisfies the condition (M 2) *if and only if*

$$(\mathfrak{f}\circ\sigma)J^2=M_n\left(\begin{pmatrix} J & 0\\ c & J^{-1}\end{pmatrix}\right)\mathfrak{f}$$

holds for every $\sigma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in G$, where $J=J(\sigma,z)=(cz+d)^{-1}$.

This follows from the relation (3) of § 2.

Let $\tau=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ be an element of $SL(2,\boldsymbol{R})$ and $J=(cz+d)^{-1}$; we have then

$$M_n\left(\begin{pmatrix} J & 0 \\ c & J^{-1} \end{pmatrix}\right) = J^n \begin{pmatrix} 1 & & & \\ cJ^{-1} & J^{-2} & & \\ \cdots & \cdots & \cdots & \\ c^n J^{-n} & nc^{n-1}J^{-n-1} & \cdots & J^{-2n} \end{pmatrix}. \tag{10}$$

In the matrix (10), the elements above the diagonal are all 0; the $(r+1)$-th diagonal element is J^{n-2r}; and the $(r+1)$-th row is

$$\left(c^r J^{n-r}, \binom{r}{1} c^{r-1} J^{n-r-1}, \binom{r}{2} c^{r-2} J^{n-r-2}, \cdots, J^{n-2r}, 0, \cdots, 0\right). \tag{10'}$$

We shall now prove Lemma 1. Suppose that $f_0 = \cdots = f_{t-1} = 0$ in Lemma 2 and $\omega = L_n(z)\mathfrak{f}dz$ is an M_n-form with respect to G. Then, by Lemma 2 and by (10), we have, for every $\sigma \in G$,

$$(f_t \circ \sigma) J(\sigma, z)^{2t+2-n} = f_t;$$

so f_t satisfies the condition (A 2) for $\nu = 2t+2-n$. Let s be a cusp of G; ρ, h and q being defined for s as in § 2, there exist $n+1$ meromorphic functions $g_0(q), \cdots, g_n(q)$ in $|q| < 1$, such that

$$L_n(z)^{-1} M_n(\rho)^{-1} L_n(\rho(z))(\mathfrak{f} \circ \rho) J(\rho, z)^2 dz = \begin{pmatrix} g_0(q)dq \\ \vdots \\ g_n(q)dq \end{pmatrix}.$$

By the relation (3), putting $J = J(\rho, z) = (cz+d)^{-1}$, we have

$$M_n\left(\begin{pmatrix} J & 0 \\ c & J^{-1} \end{pmatrix}\right)^{-1} (\mathfrak{f} \circ \rho) J^2 = 2\pi i h^{-1} q \begin{pmatrix} g_0(q) \\ \vdots \\ g_n(q) \end{pmatrix},$$

so that by (10),

$$(f_t \circ \rho) J^{2t+2-n} = 2\pi i h^{-1} q g_t(q). \tag{11}$$

This shows that f_t satisfies (A 3). Hence f_t is an automorphic form of degree $2t+2-n$ with respect to G. Furthermore, if ω is a cusp M_n-form, f_t must be holomorphic on $\mathcal{H}$, since f_t is the $(t+1)$-th component of $L_n(-z)\omega/dz$; and as $g_t(q)$ is holomorphic at $q=0$ by virtue of (M 3′), the relation (11) shows that f_t satisfies (A 3′). This completes the proof of Lemma 1.

LEMMA 3. *If f is an automorphic form of degree ν with respect to G, we have, for every $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$,*

$$(f^{(k)} \circ \sigma) J^2 = \sum_{j=0}^{k} \binom{k}{j} \binom{\nu+k-1}{j} j!\, c^j J^{j+2-2k-\nu} f^{(k-j)},$$

where $J = J(\sigma, z) = (cz+d)^{-1}$.

This is easily obtained by the induction on k.

Now we shall prove Theorem 1. Notations being as in that theorem, by

Lemma 2, ω satisfies the condition (M 2) if we have, for every $\sigma\in G$,

$$J^2\begin{pmatrix}0\\ \vdots\\ 0\\ \alpha_0 f\circ\sigma\\ \alpha_1 f'\circ\sigma\\ \vdots\\ \alpha_\mu f^{(\mu)}\circ\sigma\end{pmatrix}=M_n\left(\begin{pmatrix}J & 0\\ c & J^{-1}\end{pmatrix}\right)\begin{pmatrix}0\\ \vdots\\ 0\\ \alpha_0 f\\ \alpha_1 f'\\ \vdots\\ \alpha_\mu f^{(\mu)}\end{pmatrix}, \tag{12}$$

where $J=J(\sigma,z)=(cz+d)^{-1}$. Put $t=n-\mu$. By (10), we see that the first t components of the vectors in both sides are equal to 0; and by (10′), if $r\geqq t$, the $(r+1)$-th component of the vector on the right hand side of (12) is equal to

$$\sum_{u=t}^{r}\binom{r}{u}c^{r-u}J^{n-r-u}\alpha_{u-t}f^{(u-t)}; \tag{13}$$

hence the equality (12) is proved if we show that (13) is equal to $J^2\alpha_{r-t}f^{(r-t)}\circ\sigma$. By Lemma 3, we have

$$\begin{aligned}J^2\alpha_{r-t}f^{(r-t)}\circ\sigma&=\alpha_{r-t}\sum_{j=0}^{r-t}\binom{r-t}{j}\binom{\nu+r-t-1}{j}j!\,c^jJ^{j+2-2(r-t)-\nu}f^{(r-t-j)}\\&=\alpha_{r-t}\sum_{u=t}^{r}\binom{r-t}{r-u}\binom{\nu+r-t-1}{r-u}(r-u)!\,c^{r-u}J^{e(u)}f^{(u-t)},\end{aligned}$$

where $e(u)=r-u+2-2(r-t)-\nu$. Since $\nu=2t-(n-2)$, we have $e(u)=n-r-u$. On the other hand, we can easily verify

$$\alpha_{r-t}\binom{r-t}{r-u}\binom{\nu+r-t-1}{r-u}(r-u)!=\alpha_{u-t}\binom{r}{u}.$$

This proves the equality (10). Hence ω satisfies (M 2). The condition (M 1) is of course satisfied. Now consider a cusp s of G. Since ω satisfies (M 1-2), ρ and q being as in § 1, there exist $n+1$ meromorphic functions $f_0(q),\cdots,f_n(q)$ in $0<|q|<1$ such that

$$L_n(z)^{-1}M_n(\rho)^{-1}\omega\circ\rho=\begin{pmatrix}f_0(q)dq\\ \vdots\\ f_n(q)dq\end{pmatrix}.$$

By (A 3), there exists a meromorphic function $g(q)$ in $|q|<1$ such that

$$f(\rho(z))=g(q)J(\rho,z)^{-\nu}. \tag{14}$$

Differentiating this successively, we get, for every k,

$$f^{(k)}(\rho(z))=J^{a(k)}\sum_u F_{ku}(q)z^u, \tag{15}$$

where $a(k)$ is an integer and the $F_{ku}(q)$ are meromorphic functions in $|q|<1$. Comparing both sides of the equality

$$(16)\qquad \begin{pmatrix} f_0(q)dq \\ \vdots \\ f_n(q)dq \end{pmatrix} = L_n(z)^{-1}M_n(\rho)^{-1}L_n(\rho(z))\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_0 f\circ\rho \\ \vdots \\ \alpha_\mu f^{(\mu)}\circ\rho \end{pmatrix} J^2\frac{h}{2\pi i}\frac{dq}{q},$$

we observe that $f_k(q)$ is written in the form

$$(17)\qquad f_k(q)=J^{b(k)}\sum_u H_{ku}(q)z^u .$$

where $b(k)$ is an integer and the $H_{ku}(q)$ are meromorphic functions in $|q|<1$. Hence there exists an integer m such that

$$\lim_{q\to 0} q^m f_k(q)=0$$

for every k. This shows that the $f_k(q)$ are meromorphic at $q=0$. Thus we have proved that ω is an M_n-form. Furthermore, suppose that f is a cusp form. Then the function $g(q)$ of (14) takes the value 0 at $q=0$; so, in the expression (15), we may assume that the $F_{ku}(q)$ take the value 0 at $q=0$. Comparing again both sides of (16), we see that the functions $H_{ku}(q)$ in the expression (17) are holomorphic at $q=0$, so that we have

$$\lim_{q\to 0} q f_k(q)=0$$

for every k. This shows that the $f_k(q)$ are holomorphic at $q=0$. Hence ω is a cusp M_n-form. We can similarly show that if ω is a cusp M_n-form, f satisfies (A 3′). Theorem 1 is then completely proved.

We can prove Theorem 3 in a quite similar way. We shall now prove Theorem 5. Differentiating both sides of

$$L_n(z+w)=L_n(z)L_n(w)$$

with respect to w, and then putting $w=0$, we obtain

$$(18)\qquad L_n'(z)=L_n(z)L_n'(0)\,.$$

From (2) we see that

$$(19)\qquad L_n'(0)=\begin{pmatrix} 0 & n & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-1 & \cdots & 0 & 0 \\ & & \cdots & & \cdots & \cdots \\ 0 & \cdots & & \cdots & 2 & 0 \\ 0 & \cdots & & \cdots & 0 & 1 \\ 0 & \cdots & & \cdots & 0 & 0 \end{pmatrix}.$$

Notations being as in Theorem 5, we have, using (18) and (19),

$$d\mathfrak{f}=d\left[L_n(z)\begin{pmatrix}0\\ \vdots\\ 0\\ \gamma_0 f\\ \vdots\\ \gamma_{\mu-1}f^{(\mu-1)}\end{pmatrix}\right]=\left[L_n'(z)\begin{pmatrix}0\\ \vdots\\ 0\\ \gamma_0 f\\ \vdots\\ \gamma_{\mu-1}f^{(\mu-1)}\end{pmatrix}+L_n(z)\begin{pmatrix}0\\ \vdots\\ 0\\ \gamma_0 f'\\ \vdots\\ \gamma_{\mu-1}f^{(\mu)}\end{pmatrix}\right]dz$$

$$=L_n(z)\begin{pmatrix}0\\ \vdots\\ 0\\ \alpha_0'f\\ \alpha_1'f'\\ \vdots\\ \alpha_\mu'f^{(\mu)}\end{pmatrix}dz\,,$$

where $\alpha_0'=\mu\gamma_0$, $\alpha_1'=(\mu-1)\gamma_1+\gamma_0, \cdots, \alpha_{\mu-1}'=\gamma_{\mu-1}+\gamma_{\mu-2}$, $\alpha_\mu'=\gamma_{\mu-1}$. We can easily verify $\alpha_k'=\mu(n-\mu+1)\alpha_k$ for $0\leqq k\leqq\mu$. This proves Theorem 5.

It remains to prove Theorem 2 and Theorem 4. We need for that purpose

LEMMA 4. *Suppose that the Fuchsian group G has no cusp. Let n be a positive even integer and $r=\frac{n}{2}$. Then there is no cusp M_n-form ω with respect to G of the type*

$$\omega=L_n(z)\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ f_r\\ \vdots\\ f_n\end{pmatrix}dz\,,$$

where $f_r, \cdots, f_n$ are meromorphic functions on $\mathfrak{H}$.

PROOF. First we remark that f_r must be everywhere holomorphic on $\mathfrak{H}$. By lemma 2 and by (10′), we have, for every $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in G$,

$$f_r(\sigma(z))J(\sigma,z)^2=f_r(z)+rcJ(\sigma,z)\,.$$

Put $\eta=f_r(z)dz$. Then η is a holomorphic differential form on $\mathfrak{H}$ satisfying

$$\eta\circ\sigma=\eta-rd(\log J(\sigma,z)) \tag{20}$$

for every $\sigma\in G$. Consider the integral of η along the boundary $\mathfrak{B}$ of a fundamental domain of G; then we find, taking account of the relation (20),

$$\frac{2}{r}\int_{\mathcal{B}}\eta = 2g-2+\sum_{\lambda}\left(1-\frac{1}{m_\lambda}\right), \tag{21}$$

where g is the genus of the Riemann surface $\mathcal{H}/G$ and the m_λ denote the orders of ramification at the elliptic points of G. It is well known that the number on the right hand side of (21) is positive. On the other hand, as η is holomorphic, we must have $\int_{\mathcal{B}}\eta=0$; thus we are led to contradiction if we assume the existence of a cusp M_n-form of the type described in our lemma.

Now we are ready to prove Theorem 2. First we remark that $S_\nu(G)=\{0\}$ for $\nu<0$ and $S_0(G)=\{0\}$ or $=\boldsymbol{C}$ according as G has a cusp or not. Let ω be a cusp M_n-form with respect to G; put

$$L_n(z)^{-1}\omega=\begin{pmatrix} f_0(z) \\ \vdots \\ f_n(z)\end{pmatrix}dz\,.$$

Let t be the first integer such that $f_t\neq 0$. Then, by Lemma 1, f_t is a cusp form with respect to G of degree $2t-n+2$. By the above remark, we must have $t\geqq\frac{n-2}{2}$. If $t=\frac{n-2}{2}$, f_t is a cusp form of degree 0; then, G has no cusp and f_t is a constant. This is impossible, however, in view of Lemma 4. Hence we have $2t-n+2>0$. Put $\nu=2t-n+2$. Then we have $\alpha_{n,\nu,0}\neq 0$; put $f=\alpha_{n,\nu,0}{}^{-1}f_t$. Let η_ν be the cusp M_n-form defined for the cusp form f by (8). Then we see that the first $t+1$ components of $L_n(z)^{-1}(\omega-\eta_\nu)$ are all 0. Applying the same argument to the form $\omega-\eta_\nu$, we can find an element $\eta_{\nu+2}$ of $\mathfrak{S}^n{}_{\nu+2}(G)$ such that the first $t+2$ components of $L_n(z)^{-1}(\omega-\eta_\nu-\eta_{\nu+2})$ are all 0. Repeating this procedure, we get the expression

$$\omega=\sum_{\lambda=\nu}^{n+2}\eta_\lambda$$

where η_λ is an element of $\mathfrak{S}^n{}_\lambda(G)$ for every λ. It is easy to see that this expression gives a decomposition of the vector space $\mathfrak{F}_n(G)$ as the direct sum of the vector spaces $\mathfrak{S}^n{}_\lambda(G)$ for $2\leqq\lambda\leqq n+2$. Thus we have proved Theorem 2. Theorem 4 can be $\mathfrak{S}^n{}_\lambda$ proved in a quite similar way.

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