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An example on the fundamental conjecture of *GLC*.

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Since 1953, the author has worked on the fundamental conjecture (abbrev. F.C.) of *GLC* [**3**]. Professor Gödel presented to the author a very interesting possible counterexample to F.C. Following the advice of Professor Gödel, the author wishes to prove that F.C. holds on this example. It should be remarked that G. Kreisel [**2**] presented a special case of this example.

1. First, we give some definitions to state the example. Δ_0 is the axiom of natural numbers without the axiom of mathematical induction in *LK*. $e(a)$ is Fregean, which is defined by

$$\forall\varphi(\forall x(x=0 \vdash \varphi[x]) \wedge \forall x(\varphi[x] \vdash \varphi[x']) \vdash \varphi[a]),$$

where a' is the successor of a.

$A_i(a)$ is the arithmetical formula, which means that a is not the Gödel number of a proof-figure in G^iLC (cf. [**5**], appendix) to the contradiction from Δ_0. ($A_i(a)$ can be represented in many ways, e.g. by using recursive functions. Anyway F.C. holds on the example by the following proof.) $B_i(a)$ is the formula of $G^{i+1}LC$, which means, 'if a is the Gödel number of provable formula in $G^{i+1}LC$, then $Tr(a)$,' where $Tr(a)$ is a formula in $G^{i+1}LC$ which means that a is the Gödel number of a true formula. 'Truth' of a formula may be formalized in various ways, subject only to the condition that Δ_0, $e(a) \to B_i(a)$ is provable in $G^{i+1}LC$. C_i is a certain prenex normal form of $\neg e(a) \vee A_i(a)$ or $\neg e(a) \vee B_i(a)$. Then the example is $\Delta_0 \to C_i$, which is provable in $G^{i+1}LC$ (cf. Tarski [**6**]).

To prove F.C. on the example, it is sufficient that we prove the following more general theorems.

THEOREM 1. *Let $\Gamma \to \Delta$ be a sequence in $G^{i+1}LC$. If $\Gamma \to \Delta$ is provable in $G^{i+1}LC$, then $e(a)$, $\Gamma \to \Delta$ is provable without cut in $G^{i+1}LC$.*

THEOREM 2. *Let $\Gamma \to \Delta$ be a sequence in $G^{i+1}LC$, A and B be formulas in $G^{i+1}LC$, one of which is a prenex normal form of another. If $\Gamma \to \Delta$, A is provable without cut, then $\Delta \to \Gamma$, B is provable without cut in $G^{i+1}LC$.*

We prove F.C. on the example from Theorems 1 and 2 as follows: Since $\Delta_0 \to C_i$ is provable in $G^{i+1}LC$ (loc. cit.) $e(a)$, $\Delta_0 \to C_i$ is provable without cut in $G^{i+1}LC$ by Theorem 1. Then $e(a)$, $\Delta_0 \to \neg e(a) \vee A_i(a)$ (or $e(a)$, $\Delta_0 \to \neg e(a) \vee B_i(a)$) is provable without cut by Theorem 2. Hence $\Delta_0 \to \neg e(a) \vee A_i(a)$ (or $\Delta_0 \to$

$\neg e(a) \vee B_i(a))$ is provable without cut. Thus F.C. on the example is proved by Theorem 2.

2. Proof of Theorem 1. We prove the theorem by induction on the number of inferences of the proof-figure $\mathfrak{P}$ to $\Gamma \rightarrow \Delta$. We may and shall assume that every a in $\mathfrak{P}$ is not an eigenvariable. If $\Gamma \rightarrow \Delta$ is of the form $D \rightarrow D$, then clearly $e(a), \Gamma \rightarrow \Delta$ is provable without cut. Let $\Gamma \rightarrow \Delta$ be a lower sequence of an inference $\mathfrak{J}$ and the theorem be proved for the upper sequences of $\mathfrak{J}$. If $\mathfrak{J}$ is not a cut, $e(a), \Gamma \rightarrow \Delta$ is obviously provable without cut. If $\mathfrak{J}$ is a cut of the form

$$\frac{\Gamma_1 \rightarrow \Delta_1, D \qquad D, \Gamma_2 \rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2},$$

we have the proof-figure to $e(a), \Gamma \rightarrow \Delta$ without cut as follows:

$$\begin{array}{c}
\dfrac{\dfrac{\dfrac{\Gamma_1, e(a) \rightarrow \Delta_1, D}{\Gamma_1, e(a) \rightarrow \Delta_1, b=0 \vdash D}}{\Gamma_1, e(a) \rightarrow \Delta_1, \forall x(x=0 \vdash D)} \qquad \dfrac{\dfrac{D \rightarrow D}{\rightarrow D \vdash D}}{\rightarrow \forall x(D \vdash D)}}{\dfrac{\Gamma_1, e(a) \rightarrow \Delta_1, \forall x(x=0 \vdash D) \wedge \forall x(D \vdash D) \qquad D, \Gamma_2, e(a) \rightarrow \Delta_2}{\dfrac{\forall x(x=0 \vdash D) \wedge \forall x(D \vdash D) \vdash D, \Gamma_1, e(a), \Gamma_2, e(a) \rightarrow \Delta_1, \Delta_2}{\dfrac{\forall \varphi(\forall x(x=0 \vdash \varphi[x]) \wedge \forall x(\varphi[x] \vdash \varphi[x']) \vdash \varphi[a]), \Gamma_1, e(a), \Gamma_2, e(a) \rightarrow \Delta_1, \Delta_2}{\dfrac{\text{Some exchanges and contractions}}{e(a), \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2}}}}}
\end{array}$$

where b is assumed not to be contained in $\mathfrak{P}$. The proof is thus concluded.

3. The rest of this paper is devoted to a proof of Theorem 2. We give first some definitions.

3.1. Definition 1. A proof-figure $\mathfrak{P}$ is called '*split*', if and only if all the formulas in $\mathfrak{P}$ are divided into two classes, which are called the first and second classes, and the following conditions are fulfilled:

3.1.1. The two formulas in a beginning sequence belong to the different classes.

3.1.2. All formulas belonging to a fibre simultaneously belong to the same class.

3.1.3. The two cut-formulas of a cut belong to the different classes.

A sequence $\mathfrak{S}$ is called 'provable splitly', if there exists a split proof-figure to $\mathfrak{S}$. 'A sequence $\mathfrak{S}$ is reducible splitly to sequences $\mathfrak{S}_1, \cdots, \mathfrak{S}_m$,' will mean 'if $\mathfrak{S}_1, \cdots, \mathfrak{S}_m$ are provable splitly without cut, then $\mathfrak{S}$ is also provable splitly without cut.' 'A proof-figure $\mathfrak{P}$ is reduced splitly to proof-figures $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$' will mean '$\mathfrak{P}$ is reduced to $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ and if $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ are split, then $\mathfrak{P}$ is also split.' (Cf. [3, § 4].)

3.2. DEFINITION 2. A proof-figure $\mathfrak{P}$ is called '*naive*', if and only if the following conditions are satisfied:
3.2.1. Every inference $\forall$ left in $\mathfrak{P}$ is of the form

$$\frac{F(\alpha), \Gamma \to \Delta}{\forall \varphi F(\varphi), \Gamma \to \Delta} \quad \text{or} \quad \frac{F(a), \Gamma \to \Delta}{\forall x F(x), \Gamma \to \Delta}.$$

3.2.2. Every inference $\exists$ right in $\mathfrak{P}$ is of the form

$$\frac{\Gamma \to \Delta, F(\alpha)}{\Gamma \to \Delta, \exists \varphi F(\varphi)} \quad \text{or} \quad \frac{\Gamma \to \Delta, F(a)}{\Gamma \to \Delta, \exists x F(x)}.$$

3.3. DEFINITION 3. A cut $\mathfrak{J}$ in a proof-figure is called '*semi-explicit*', if and only if there exists a cut-formula A of $\mathfrak{J}$ satisfying the following conditions:
3.3.1. Let $\mathfrak{S}$ be any beginning sequence. If one formula of $\mathfrak{S}$ is an ancestor (cf. [**3**]) of A or A itself, then another formula of $\mathfrak{S}$ is explicit (cf. [**3**]).
3.3.2. Let B be an ancestor of A or A itself. If B is related to a beginning formula (cf. [**3**, 2.5] for related formulas), then every leading formula (cf. [**4**, 1.2]) of B is a beginning formula or a weakening formula.

A proof-figure $\mathfrak{P}$ is called '*semi-explicit*', if and only if every cut of $\mathfrak{P}$ is semi-explicit.
3.4. LEMMA. If $\mathfrak{S}$ is an end-sequence of a semi-explicit proof-figure $\mathfrak{P}$, then there is a proof-figure $\mathfrak{Q}$ such that $\mathfrak{Q}$ contains no cut and its end-sequence is $\mathfrak{S}$. Moreover, if $\mathfrak{P}$ is split (or naive), then so is $\mathfrak{Q}$.

PROOF. The former part of this lemma is proved by [**4**, § 3 and § 4], in which 'separative' is replaced by 'semi-explicit'. The other part can be easily seen in tracing the reduction.

In the following we say that a sequence $A \to B$ is elementary, if there is a split and naive proof-figure which contains no cut to $A \to B$.

We have clearly the following two propositions.

PROPOSITION 1. *If* $A \to B$ *and* $C \to D$ *are elementary, then* $\neg B \to \neg A$, $A \wedge C \to B \wedge D$ *and* $A \vee C \to B \vee D$ *are elementary. If* $F(\alpha) \to G(\alpha)$ *or* $F(a) \to G(a)$ *is elementary, then* $\forall \varphi F(\varphi) \to \forall \varphi G(\varphi)$ *and* $\exists \varphi F(\varphi) \to \exists \varphi G(\varphi)$, *or* $\forall x F(x) \to \forall x G(x)$ *and* $\exists x F(x) \to \exists x G(x)$ *are elementary.*

PROPOSITION 2. *Let* A *be a formula and* B *be a formula obtained from* A *by moving a quantifier outside the scope of a logical symbol* $\wedge, \vee$ *or* $\neg$. *Then* $A \to B$ *and* $B \to A$ *are elementary. Especially the following sequences are elementary.*

$$\forall \varphi (F(\varphi) \wedge G(\varphi)) \to \forall \varphi F(\varphi) \wedge \forall \varphi G(\varphi) \qquad \forall \varphi F(\varphi) \wedge \forall \varphi G(\varphi) \to \forall \varphi (F(\varphi) \wedge G(\varphi))$$

$$\forall \varphi (F(\varphi) \vee A) \to \forall \varphi F(\varphi) \vee A \qquad \forall \varphi F(\varphi) \vee A \to \forall \varphi (F(\varphi) \vee A)$$

$$\forall \varphi \neg F(\varphi) \to \neg \exists \varphi F(\varphi) \qquad \neg \exists \varphi F(\varphi) \to \forall \varphi \neg F(\varphi) \qquad \text{etc.}$$

To prove Theorem 2, we have only to prove the following propositions.

PROPOSITION 3. *Let $\Gamma \to \Delta$, A be a sequence which is provable without cut and $A \to B$ be an elementary sequence. Then $\Gamma \to \Delta$, B is provable without cut.*

PROOF. Since $\Gamma \to \Delta$, A is provable without cut, there exists a proof-figure $\mathfrak{P}_1$ satisfying the following conditions:

1. $\mathfrak{P}_1$ has no cut.

2. Every beginning sequence of $\mathfrak{P}_1$ has no proper logical symbol. There exists also a split and naive proof-figure $\mathfrak{P}_2$ to $A \to B$, satisfying the same conditions as $\mathfrak{P}_1$.

It is easily seen that the following proof-figure is semi-explicit, whence the proposition follows.

$$\dfrac{\begin{matrix}\mathfrak{P}_1\\ \Downarrow\\ \Gamma \to \Delta, A\end{matrix} \qquad \begin{matrix}\mathfrak{P}_2\\ \Downarrow\\ A \to B\end{matrix}}{\Gamma \to \Delta, B}$$

PROPOSITION 4. *If $A \to B$ and $B \to C$ are elementary sequences, then $A \to C$ is elementary.*

PROOF. We shall apply the reduction in the proof of Proposition 3 to the proof-figure

$$\dfrac{\begin{matrix}\mathfrak{P}_1\\ \Downarrow\\ A \to B\end{matrix} \qquad \begin{matrix}\mathfrak{P}_2\\ \Downarrow\\ B \to C\end{matrix}}{A \to C}$$

where each of $\mathfrak{P}_1$, $\mathfrak{P}_2$ is split, naive and has no cut, and has no beginning sequence with proper logical symbols. Then the proposition follows from Lemma.

PROPOSITION 5. *Let A be a formula and B be a prenex normal form of A. Then $A \to B$ and $B \to A$ are elementary.*

PROOF. This follows from Propositions 1, 2 and 4 by induction on the number n of proper logical symbols in A. If $n = 0$, the proposition is trivial. If $n = m+1$, and the proposition is proved in case $n \leqq m$, we prove it by dividing cases according to the outermost logical symbol of A. Since every case is easily treated, we show here only the following case as an example: A is of the form $\neg A_1$, and $\forall \varphi B_1(\varphi)$ and $B_2(\alpha)$ are prenex normal forms of A_1 and $\neg B_1(\alpha)$ respectively. Then $A \to B$ is elementary; By the hypothesis of induction $\neg B_1(\alpha) \to B_2(\alpha)$ is elementary. Then $\exists \varphi \neg B_1(\varphi) \to \exists \varphi B_2(\varphi)$ and $\neg \forall \varphi B_1(\varphi) \to \exists \varphi \neg B_1(\varphi)$ are elementary (by Prop. 1 and 2), whence $\neg \forall \varphi B_1(\varphi) \to \exists \varphi B_2(\varphi)$ is elementary (by Prop. 4). Since $\forall \varphi B_1(\varphi) \to A_1$ is elementary by the hypothesis of induction, $\neg A_1 \to \neg \forall \varphi B_1(\varphi)$ is elementary (by Prop. 1). Then $\neg A_1 \to \exists \varphi B_2(\varphi)$ is elementary.

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