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On polynomial approximation for strictly stationary processes.

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§ 1. Introduction and main results.

Consider a Brownian motion $B(t, \omega)$ and form a stochastic process

$$X(t, \omega)=\sum_{p} \sum_{i_1 \cdots i_p} a_{i_1 \cdots i_p} \prod_{\nu=1}^{p}(B(t+s_{i_\nu}, \omega)-B(t+u_{i_\nu}, \omega)). \tag{1.1}$$

Here we shall call the process of the form (1.1) a polynomial process, since this process is a polynomial of the increments of $B(t, \omega)$. We see that a polynomial process is strictly stationary and that it is continuous in probability in the following sense:

$$\lim_{t \to s} P(|X(t, \omega)-X(s, \omega)|>\varepsilon)=0 \tag{1.2}$$

for $\varepsilon>0$ and any $s \in(-\infty, \infty)$.

It is obvious that an arbitrary strictly stationary process continuous in probability is not always a polynomial process, but we can approximate it in a certain sense by polynomial processes. The purpose of this paper is to prove this approximation theorem.

For this purpose, we shall introduce some topologies in the set of stochastic processes. The formal extension of the convergence in law for real random variables is as follows. A sequence of stochastic processes $X_n \equiv\{X_n(t, \omega),-\infty<t<\infty\}$ may be called to converge to the stochastic process $X \equiv\{X(t, \omega),-\infty<t<\infty\}$ in law if any joint distribution of X_n at a finite number of t-values converges to the corresponding one of X in Helly's sense. This definition is inadequate; in fact, even if $X_n \rightarrow X$ and if $X_{n,m} \rightarrow X_n$, we cannot always find a sequence X_{n, m_n} such that $X_{n, m_n} \rightarrow X$. Therefore we shall here introduce a neighborhood system $\{U(X, \varepsilon)\}$ which yields a convergence stronger than the convergence above.

DEFINITION 1. $U(X, \varepsilon)$ is the collection of all stochastic processes $Y \equiv \{Y(t, \omega),-\infty<t<\infty\}$ such that

$$|E e^{i \theta_1 X(t_1, \omega)+\cdots+i \theta_n X(t_n, \omega)}-E e^{i \theta_1 Y(t_1, \omega)+\cdots+i \theta_n Y(t_n, \omega)}|<\varepsilon$$

whenever n, $|\theta_j|$ and $|t_j|$ are all less than ε^{-1}.

Extending the convergence in probability for real random variables we shall say that X_n converges to X in probability if

$$P(|X_n(t, \omega) - X(t, \omega)| > \varepsilon) \to 0$$

for any $\varepsilon > 0$ and $t \in (-\infty, \infty)$. To avoid the same trouble as above we shall further define another neighborhood system $\{V(X, \varepsilon)\}$ which also yields a convergence stronger than the convergence above.

DEFINITION 2. $V(X, \varepsilon)$ is the collection of all stochastic processes Y such that

$$P(|Y(t, \omega) - X(t, \omega)| > \varepsilon) < \varepsilon$$

whenever $|t| < 1/\varepsilon$.

It should be noted that the basic probability space Ω for X_n may or may not vary with n in the Definition 1, while it should be the same for all X_n and X in the Definition 2. It is evident that the V-topology is stronger than the U-topology, because

$$V\left(X, \frac{1}{3n^2}\right) \subset U\left(X, \frac{1}{n}\right).$$

It is also evident that in both topologies, if Y belongs to the ε-neighborhood of X and if Z belongs to the ε'-neighborhood of Y, then Z belongs to the $(\varepsilon+\varepsilon')$-neighborhood of X.

Using these topologies we will state our main results which will be proved in § 3 and § 4.

THEOREM 1. *Let $\{X(t, \omega), -\infty < t < \infty\}$ be strictly stationary, continuous in probability and ergodic in the sense that any measurable functional of the process is constant if it is invariant under the time translation* [**1**][1]. *Then we can form a sequence of polynomial processes which converges to the given process in the U-topology.*

THEOREM 2. *Let $\{X(t, \omega), -\infty < t < \infty\}$ be strictly stationary and continuous in probability. Then we can form a sequence of continuous (in probability), ergodic, strictly stationary processes $\{X_n(t, \omega), -\infty < t < \infty\}, n = 1, 2, \cdots$, such that X_n tends to X in the U-topology as $n \to \infty$. Here the probability space for X_n may vary with n.*

Combining these two theorems we have the following

THEOREM 3. *Let $\{X(t, \omega), -\infty < t < \infty\}$ be strictly stationary and continuous in probability. Then we can form a sequence of polynomial processes which converges to the given process in the U-topology.*

As to Theorem 1, N. Wiener discussed a similar problem on stationary random interval functions in his famous paper "The Homogeneous Chaos"[2] using a very ingenious device which served as a model for our proof.

On the other hand Theorem 2 is closely related to the theorem of J. C.

1) The number in [ ] refers to the references at the end of this paper.
2) See [**2**, Section 12].

Oxtoby and S. M. Ulam [3] that the set of all ergodic transformations is everywhere dense in a certain topology in the set of all invertible measure-preserving transformations.[3] Although we do not here use this theorem itself, we have made use of their idea.

In § 5 we shall discuss Gaussian (strictly) stationary processes continuous in probability. In this case it is enough to consider, as approximating processes, linear processes, namely, processes of the form:

$$X(t, \omega) = \sum_i a_i[B(t+s_i, \omega) - B(t+u_i, \omega)], \tag{1.3}$$

instead of polynomial processes. In fact we can prove the

THEOREM 4. *Given any stationary Gaussian process* $\{X(t, \omega), -\infty < t < \infty\}$ *continuous in probability, we can form a sequence of linear processes which converges to the given process in the U-topology.*

If the covariance function of the given process has an absolutely continuous spectral measure, a much stronger result is known [1, Chap 12] and the above theorem can be readily derived from that results as is shown in § 5.

In § 6, we discuss a similar polynomial approximation for strictly stationary random sequences. We can pass from the continuous time parameter case to the discrete one in the following way. Instead of the differences of Brownian motion $B(t, \omega)$ we use $\{\xi_i(\omega), i = 0, \pm 1 \cdots\}$ which are mutually independent and normally ($N(0, 1)$) distributed. A random sequence $\{X_n(\omega), n = 0, \pm 1 \cdots\}$ of the form,

$$X_n(\omega) = \sum_p \sum_{i_1 \cdots i_p} a_{i_1 \cdots i_p} \prod_{\nu=1}^{p} \xi_{i_\nu + n}(\omega), \tag{1.4}$$

is called here a polynomial sequence.

THEOREM 5. *Let* $\{X_n(\omega); n = 0, \pm 1 \cdots\}$ *be strictly stationary and ergodic. Then we can form a sequence of polynomial sequences* $\{Y_n^{(m)}(\omega), n = 0, \pm 1 \cdots\}$ *which converges to the given random sequence in law, i. e.*

$$\lim_{m \to \infty} | E e^{i\theta_{-n} X_{-n}(\omega) + \cdots + i\theta_n X_n(\omega)} - E e^{i\theta_{-n} Y_{-n}^{(m)}(\omega) + \cdots + i\theta_n Y_n^{(m)}(\omega)} | = 0$$

for any n *and* θ_j.

THEORHM 6. *Let* $\{X_n(\omega), n = 0, \pm 1, \cdots\}$ *be a strictly stationary random sequence. Then we can form a sequence of ergodic (strictly) stationary random sequences* $\{Y_n^{(m)}(\omega), n = 0, \pm 1, \cdots\}$, $m = 0, 1, 2, \cdots$, *which converges to* $\{Y_n(\omega), n = 0, \pm 1, \cdots\}$ *in law, where the probability space for* $Y^{(m)}$ *may vary with* m.

It is not necessary to state Theorem 5 and 6 in terms of the convergence in the U-topology, since this is equivalent to the above convergence in law

3) For a neater proof see P. R. Halmos [4].

for random sequences.

If we combine the above two theorems, we have, corresponding to Theorem 3,

Theorem 7. *For any given strictly stationary random sequence we can form a sequence of polynomial sequences which converges to the given sequence in law.*

We can prove these theorems in the same way as in the case of processes and even more simply.

In conclusion the author wishes to express her sincere thanks to Professor K. Itô for his kind guidance and valuable suggestions.

§ 2. A sequence of random variables $a_n(\omega)$ related to Brownian motion $B(t, \omega)$.

Let $B(t, \omega)$ be Brownian motion. Using the coordinate representation[4] [1] we may assume that ω is the path function of this motion. Let ω_s^+ be the shifted path of ω by s, namely the continuous function whose value at t is equal to the value of ω at $t+s$.

We shall here introduce a sequence of random variables $a_n(\omega)$, $n \geqq 1$, related to $B(t, \omega)$, which will play a fundamental role in the proof of Theorem 1. Let $S(\omega)$ be the set of t-values for which

$$|B(t+1, \omega) - B(t-1, \omega)| > 1 .$$

Because of the continuity of the path functions of $B(t, \omega)$, $S(\omega)$ is open and can therefore be expressed as a denumerable disjoint sum of open intervals $I_i(\omega)$, $i \geqq 1$. Denote with $\mathfrak{J}(\omega)$ the following class of intervals

$$\{I_i(\omega) : |I_i(\omega)| > n, \quad I_i(\omega) \subset (-n, \infty)\} ,^{5)}$$

and put

$$S_n(\omega) = \bigcup_{I_i \in \mathfrak{J}_n} I_i(\omega) .$$

First of all we shall prove that $S_n(\omega)$ is not empty for every n for almost all ω.

Let $f(\omega)$ be a summable function of ω with respect to the Borel field determined by $\{B(t, \omega) - B(s, \omega), -\infty < s < t < +\infty\}$. Then by the ergodicity of $B(t, \omega)$ we have

$$\lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} f(\omega_t^+) \, dt = E\{f(\omega)\}$$

Now put

4) See Chap 1. § 5.

5) $|A|$ = Lebesgue measure of set $A (\subset R^1)$.

$$f(\omega)=\begin{cases}1 & \text{if } |B(t+1,\omega)-B(t-1,\omega)|>1, \quad 0\leqq t<n \\ & \quad\text{and } |B(-n+1,\omega)-B(-n-1,\omega)|\leqq 1 \\ 0 & \text{if otherwise.}\end{cases}$$

Since $f(\omega_t^+)=1$ implies $t\in S_n(\omega)$, we have, for $n\geqq 3$,

$$\text{(2.1)} \qquad \liminf_{T\to\infty}\frac{1}{T}|S_n(\omega)\cap(-n,-n+T)|$$

$$\geqq \lim\frac{1}{T}\int_{-n}^{-n+T}f(\omega_t^+)\,dt=E\{f(\omega)\}$$

$$=P(|B(t+1,\omega)-B(t-1,\omega)|>1,\, 0\leqq t\leqq n\,;\text{and } |B(-n+1,\omega)-B(-n-1,\omega)|\leqq 1)$$

$$=P(|B(t+1,\omega)-B(t-1,\omega)|>1,\, 0\leqq t\leqq n)\; P(|B(-n+1,\omega)-B(-n-1,\omega)|\leqq 1)\,.$$

for almost all ω, so that it is enough to show

$$\text{(2.2)} \qquad P(|B(t+2,\omega)-B(t,\omega)|>1,\; 0\leqq t\leqq n)>0\,.$$

The ω-set in $P(\cdots)$ in (2.2) includes the intersection of the following ω-sets,

$$\{\omega: 2\nu-2<B(t,\omega)-B(0,\omega)<2\nu+1, \quad \nu\leqq t\leqq \nu+1\}$$

$$\nu=0,1,\cdots,n-1\,.$$

By the property of Brownian motion we can easily see that this intersection has positive probability, so that (2.2) is true.

Now we shall define $a_n(\omega)$ by

$$\text{(2.3)} \qquad a_n(\omega)=n+\inf(t:t\in S_n(\omega))\,;$$

we set $a_n(\omega)=\infty$ for convenience if $S_n(\omega)=\phi$. By the above remark $a_n(\omega)$ is finite with probability one.

Next we shall determine the probability law of $a_n(\omega)$. We shall first note that "$a_n(\omega)=t$" is equivalent to the following condition: "$t-n\notin S(\omega)$, $(t-n,t]\subset S(\omega)$ if $0\leqq t\leqq n$, $t-n\notin S(\omega)$, $(t-n,t]\subset S(\omega)$, $(-n,t-n)\cap S_n(\omega)=0$ if $t>n$"; we shall here remark that, if $0\leqq t\leqq n$, then $t-n\notin S(\omega)$ implies $(-n,t-n)\cap S_n(\omega)=\phi$.

We have for $t'+s\leqq n, s>0$ and $t<0$,

$$\text{(2.4)} \qquad P(a_n(\omega)\in(t+s,t'+s))$$

$$=P(\exists h\in(t+s,t'+s)\,;\,h-n\notin S(\omega),\ (h-n,h]\subset S(\omega))$$

$$=P(\exists h\in(t,t')\,;\,h-n\notin S(\omega_s^+),\ (h-n,h]\subset S(\omega_s^+))$$

$$=P(\exists h\in(t,t')\,;\,h-n\notin S(\omega),\ (h-n,h]\subset S(\omega)$$

$$[\text{by the stationarity of } B(t,\omega)]$$

$$=P(a_n(\omega)\in(t,t'))\,,$$

while we get, for $t'+s>n, s>0$ and $t>0$,

$$(2.5)\quad P(a_n(\omega)\in(t+s,t'+s))$$
$$=P(\exists h\in(t+s,t'+s);\ h-n\notin S(\omega),\ (h-n,h]\subset S(\omega),\ (-n,h-n)\cap S_n(\omega)=\phi)$$
$$=P(\exists h\in(t,t');\ h-n\notin S(\omega_s^+),\ (h-n,h]\subset S(\omega_s^+),\ (-n-s,h-n)\cap S_n(\omega_s^+)=\phi)$$
$$\leqq P(\exists h\in(t,t');\ h-n\notin S(\omega_s^+),\ (h-n,h]\subset S(\omega_s^+),\ (-n,h-n)\cap S_n(\omega_s^+)=\phi)$$
$$=P(a_n(\omega)\in(t,t')).$$

(2.4) and (2.5) imply that the probability law of $a_n(\omega)$ is absolutely continuous with respect to Lebesgue measure and that its density function, say f_n, is flat on $[0,n]$ and decreasing on (n,∞).

LEMMA 1. *$f_n(t)$ is expressed as a convex combination of the function $C_y(t)$, $y\geqq n$, with weight $d\sigma_n(y)\equiv y[-df_n(y)]$, where*

$$(2.6)\qquad C_y(t)=\frac{1}{y}\quad \textit{for } 0\leqq t\leqq y,\qquad =0\quad \textit{for } t>y.$$

PROOF.

$$\lim_{y\to\infty} f_n(y)=0 \quad\text{and}\quad \lim_{y\to\infty} yf_n(y)=0,$$

as $f_n(y)$ is non-negative, decreasing and integrable. Therefore

$$f_n(t)=\int_t^\infty -df_n(y)=\int_0^\infty C_y(t)y\,(-df_n(y))$$

and

$$\int_0^\infty y(-df_n(y))=-yf_n(y)\Big|_0^\infty+\int_0^\infty f_n(y)\,dy=1,$$

which implies that $f_n(t)$ is a convex combination of $C_y(t)$ with the weight $y[-df_n(y)]$.

LEMMA 2. *For any positive l and ε, there exists $n_0(l,\varepsilon)$ such that, if $n\geqq n_0(l,\varepsilon)$,*

$$(2.7)\qquad P(a_n(\omega_s^+)=a_n(\omega)-s,\ -l\leqq s\leqq l)>1-\varepsilon.$$

PROOF. When $s>0$, we shall find some condition equivalent to $a_n(\omega_s^+)=a_n(\omega)-s$. By the definition of $a_n(\omega)$,

$$\text{“}a_n(\omega_s^+)=t\text{”}$$
$$\Leftrightarrow \text{“}t-n\notin S(\omega_s^+),\ (t-n,t]\subset S(\omega_s^+),\ (-n,t-n)\cap S_n(\omega_s^+)=\phi\text{”}$$
$$\Leftrightarrow \text{“}t+s-n\notin S(\omega),\ (t+s-n,t+s]\subset S(\omega),$$
$$(-n+s,t+s-n)\cap S_n(\omega)=\phi\text{”}.^{6)}$$

On the other hand

6) It is evident that $S(\omega_s^+)=S(\omega)-s$ where $S(\omega)-s=\{x;\ s+x\in S(\omega)\}$.

"$a_n(\omega)=t+s$"

$\Leftrightarrow$ "$t+s-n\in S(\omega)$, $(t+s-n, t+s]\subset S(\omega)$, $(-n, t+s-n)\cap S_n(\omega)=\phi$".

Hence $a_n(\omega_s{}^+)=a_n(\omega)-s$ if and only if $(-n, -n+s]\cap S_n(\omega)=\phi$; this condition is also equivalent to $a_n(\omega)\geqq s$. Therefore it is evident that if $s>s'>0$ and if "$a_n(\omega_s{}^+)=a_n(\omega)-s$" then "$a_n(\omega_{s'}^+)=a_n(\omega)-s'$".

When $s<0$, putting $u=-s$, noting $\omega=(\omega^{\pm}_{-u})_u{}^+$ and $\omega_s{}^+=\omega^{\pm}_{-u}$ and applying the previous result to $\omega^{\pm}_{-u}$, we see that "$a_n((\omega^{\pm}_{-u})_u{}^+)=a_n(\omega^{\pm}_{-u})-u$" is equivalent to "$(-n, -n+u)\cap S_n(\omega^{\pm}_{-u})=\phi$", or to "$a_n(\omega^{\pm}_{-u})\geqq u$". And also we shall remark that if $s<s'<0$ and if "$a_n(\omega_s{}^+)=a_n(\omega)-s$" then "$a_n(\omega_{s'}^+)=a_n(\omega)-s'$", because we get $S_n(\omega_s{}^+)\supset S_n(\omega_{s'}^+)$ noting $(S(\omega)-s)\cap(-n, \infty)\supset(S(\omega)-s')\cap(-n, \infty)$. Using the above results we have

$$\begin{aligned}(2.8)\qquad & P(a_n(\omega_s{}^+)=a_n(\omega)-s,\ -l\leqq s\leqq l)\\ &=P(a_n(\omega_l{}^+)=a_n(\omega)-l \text{ and } a_n(\omega^{\pm}_{-l})=a_n(\omega)+l)\\ &\geqq 1-P(a_n(\omega_l{}^+)\neq a_n(\omega)-l)-P(a_n(\omega^{\pm}_{-l})\neq a_n(\omega)+l).\end{aligned}$$

On the other hand we have

$$(2.9)\qquad P(a_n(\omega_l{}^+)\leqq l)\leqq l/n$$

and

$$(2.10)\qquad P(a_n(\omega^{\pm}_{-l})\leqq l)=P(a_n(\omega)\leqq l)\leqq l/n$$

using Lemma 1 and also the stationarity of $B(t,\omega)$ for (2.10). Therefore we have

$$(2.11)\qquad P(a_n(\omega_s{}^+)=a_n(\omega)-s,\ -l\leqq s\leqq l)\geqq 1-\frac{2l}{n},$$

which completes the proof of Lemma 2.

Finally we shall remark that $a_n(\omega_s{}^+)$ is a measurable function of (s,ω) because it is measurable in ω for each s and left continuous in s for each ω by the definition of $a_n(\omega)$.

§ 3. Proof of Theorem 1.

Let $X\equiv\{X(t,\omega^*), -\infty<t<\infty\}$ be strictly stationary, continuous in probability and ergodic. Then there is a process $Y\equiv\{\{Y(t,\omega^*), -\infty<t<\infty\}$ measurable in (t,ω^*) and such that

$$(3.1)\qquad P(X(t,\omega^*)=Y(t,\omega^*))=1,\ -\infty<t<\infty).$$ [7]

We use another symbol ω^* for the probability parameter of X since it may differ from that of the Brownian motion $B(t,\omega)$.

7) See J. L. Doob [**1**, Theorem 2.6 (Chap. 2)].

Define $Y_N=\{Y_N(t,\omega^*),\ -\infty<t<\infty\}$ by

$$Y_N(t,\omega^*)=\begin{cases} Y(t,\omega^*) & \text{for } |Y(t,\omega^*)|\leqq N \\ 0 & \text{for } |Y(t,\omega^*)|>N. \end{cases} \tag{3.2}$$

Then this process is strictly stationary, continuous in the mean and ergodic. Furthermore it converges to Y in the V-topology by the strict stationarity of Y when N tends to infinity.

Next define $Z_{M,N}(t,\omega^*)$ by

$$Z_{M,N}(t,\omega^*)=\frac{M}{2}\int_{-1/M}^{1/M}Y_N(t+s,\omega^*)\,ds\,. \tag{3.3}$$

This process $Z_{M,N}$, which converges to Y_N in V-topology when M tends to infinity, is also strictly stationary, continuous in the mean and ergodic. Furthermore this process $Z_{M,N}$ has the following property of uniform continuity,

$$\begin{aligned} &|Z_{M,N}(t+h,\omega^*)-Z_{M,N}(t,\omega^*)| \\ &\leqq \frac{M}{2}\left|\int_{-1/M+t+h}^{1/M+t+h}Y_N(s)\,ds-\int_{-1/M+t}^{1/M+t}Y_N(s)ds\right|\leqq MNh\,. \end{aligned} \tag{3.4}$$

Therefore we may assume with no loss of generality that the given process is of the type of $Z_{M,N}$. Thus we have reduced our problem to the case that X satisfies the following conditions,

(X. 1) strictly stationary and ergodic,

(X. 2) uniformly bounded, say $|X(t,\omega^*)|\leqq K$,

(X. 3) uniformly continuous, say $|X(t+h,\omega^*)-X(t,\omega^*)|\leqq Ch$,

where K and C are constants independent of t and ω^*.

Define $\rho_T(\theta_1\cdots\theta_n,t_1\cdots t_n,\omega^*)$ by

$$\rho_T(\theta_1\cdots\theta_n,t_1\cdots t_n,\omega^*)=\frac{1}{T}\int_{-T}^{0}e^{i\sum_{j=1}^{n}\theta_jX(t_j+t,\omega^*)}dt\,. \tag{3.5}$$

For every sample path of X we have

$$\begin{aligned} &|\rho_T(\theta_1\cdots\theta_n,t_1\cdots t_n,\omega^*)-\rho_T(\lambda_1\cdots\lambda_n,r_1\cdots r_n,\omega^*)| \\ &\leqq\sum_{j=1}^{n}\frac{1}{T}\int_{-T}^{0}|e^{i\theta_j[X(t_j+t,\omega^*)-X(r_j+t,\omega^*)]}-1|\,dt+\sum_{j=1}^{n}\frac{1}{T}\int_{-T}^{0}|e^{i(\theta_j-\lambda_j)X(r_j+t,\omega^*)}-1|\,dt\,. \end{aligned} \tag{3.6}$$

But

$$\begin{aligned} &\frac{1}{T}\int_{-T}^{0}|e^{i\theta[X(t+s+h,\omega^*)-X(t+s,\omega^*)]}-1|\,dt \\ &\leqq\frac{1}{T}\int_{-T}^{0}4\sin^2\Big\{\frac{\theta}{2}(X(t+s+h,\omega^*)-X(t+s,\omega^*))\Big\}dt \end{aligned} \tag{3.7}$$

$$\leqq \frac{\theta^2}{T}\int_{-T}^{0}|X(t+s+h,\omega^*)-X(t+s,\omega^*)|^2\,dt \leqq \theta^2C^2h^2\,,$$

$$\frac{1}{T}\int_{-T}^{0}|e^{i(\theta-\lambda)X(t,\omega^*)}-1|\,dt \leqq \frac{(\theta-\lambda)}{T}\int_{-T}^{0}X^2(t,\omega^*)dt \leqq (\theta-\lambda)^2K^2\,. \tag{3.8}$$

Thus we obtain

LEMMA 3. *For any $\varepsilon>0$, there exists $\delta(\varepsilon)$ independent of T such that, if n, $|\theta_i|$, $|\lambda_i|$, $|t_i|$, $|\gamma_i|<1/\varepsilon$ and if $|\theta_i-\lambda_i|$, $|t_i-\gamma_i|<\delta(\varepsilon)$, then*

$$|\rho_T(\theta_1\cdots\theta_n,t_1\cdots t_n,\omega^*)-\rho_T(\lambda_1\cdots\lambda_n,\gamma_1\cdots\gamma_n,\omega^*)|<\varepsilon$$

for all ω^*.

By the ergodicity of X

$$\lim_{T\to\infty}\rho_T(\lambda_1\cdots\lambda_n,\gamma_1\cdots\gamma_n,\omega^*)=Ee^{i\sum_{j=1}^{n}\lambda_jX(\gamma_j,\omega^*)} \tag{ρ.1}$$

for any rational number $\{\lambda_j\}$, $\{\gamma_j\}$ and any n, for almost all ω^*. Fix one of such ω^*, say ω_0^*. Noting that the right side of (ρ.1) is continuous in $(\lambda_1\cdots\lambda_n,\gamma_1\cdots\gamma_n)$ and so uniformly continuous in any bounded region, and using Lemma 3 we get the following

LEMMA 4. *We can determine $T_0(\varepsilon)$ such that, whenever n, $|\theta_i|$, $|t_i|\leqq 1/\varepsilon$, we have*

$$|\rho_T(\theta_1\cdots\theta_n,t_1\cdots t_n,\omega_0^*)-Ee^{i\sum_{j=1}^{n}\theta_jX(t_j,\omega^*)}|<\varepsilon \tag{3.9}$$

for $T>T_0(\varepsilon)$.

Using $a_n(\omega)$ introduced in the previous section we shall define a sequence of stochastic processes $\{F_k(t,\omega),\ -\infty<t<\infty\}$, $k\geqq 1$, on the probability field of Brownian motion $B(t,\omega)$, by

$$F_k(t,\omega)=X(-a_k(\omega_t^+),\omega_0^*)\,. \tag{3.10}$$

This is clearly measurable in (t,ω) and we obtain

LEMMA 5. *$\{F_k(t,\omega),\ -\infty<t<\infty\}$ converges to $\{X(t,\omega^*),\ -\infty<t<\infty\}$ in U-topology when k tends to infinity.*

PROOF. We shall compute the characteristic function of $(F_k(t_1,\omega),F_k(t_2,\omega),\cdots,F_k(t_n,\omega))$:

$$Ee^{i\sum_{j=1}^{n}\theta_jF_k(t_j,\omega)}=Ee^{i\sum_{j=1}^{n}\theta_jX(-a_k(\omega_{t_j}^+),\omega_0^*)}$$

$$=Ee^{i\sum_{j=1}^{n}\theta_jX(-a_k(\omega)+t_j,\omega_0^*)}+\varepsilon(k,\theta_1\cdots\theta_n,t_1\cdots t_n)\,;$$

by Lemma 1 this equals

$$\int_k^\infty d\sigma_k(T)\frac{1}{T}\int_0^T e^{i\sum_{j=1}^{n}\theta_jX(t_j-s,\omega_0^*)}ds+\varepsilon(k,\theta_1\cdots\theta_n,t_1\cdots t_n)$$

$$= \int_k^\infty d\sigma_k(T) \rho_T(\theta_1 \cdots \theta_n, t_1 \cdots t_n, \omega_0^*) + \varepsilon(k, \theta_1 \cdots \theta_n, t_1 \cdots t_n) .$$

Take any positive number ε and fix it. By Lemma 2 we have

$$|\varepsilon(k, \theta_1 \cdots \theta_n, t_1 \cdots t_n)| < \frac{\varepsilon}{2}, \text{ if } |t_i| \leqq \frac{1}{\varepsilon}, \ 1 \leqq i \leqq n,$$

and if $k > n_0\left(\frac{1}{\varepsilon}, \frac{\varepsilon}{4}\right)$.

On the other hand by Lemma 4 we have,

$$\int_k^\infty d\sigma_k(T) \left| \rho_T(\theta_1 \cdots \theta_n, t_1 \cdots t_n, \omega_0^*) - Ee^{i\sum_{j=1}^n \theta_j X(t_j, \omega^*)} \right| < \frac{\varepsilon}{2},$$

if $n, |\theta_j|, |t_j| \leqq \frac{1}{\varepsilon}$ and if $k > T_0\left(\frac{\varepsilon}{2}\right)$. Combining the above results we have

$$\left| Ee^{i\sum_{j=1}^n \theta_j F_k(t_j, \omega)} - Ee^{i\sum_{j=1}^n \theta_j X(t_j, \omega^*)} \right| < \varepsilon$$

if $k > \mathrm{Max}\left(T_0\left(\frac{\varepsilon}{2}\right), n_0\left(\frac{1}{\varepsilon}, \frac{\varepsilon}{4}\right)\right)$ and if $n, |\theta_j|, t \leqq \frac{1}{\varepsilon}$.

This completes the proof of Lemma 5.

Now we shall consider the polynomial expression of $F_k(t, \omega)$. To do this we shall first prove that $F_k(0, \omega)$ can be approximated by polynomials of the differences of $B(t, \omega)$. Although this follows at once from a result by R. H. Cameron and W. T. Martin [**5**] or by K. Itô [**6**, Theorem 4.1][8], we shall show it directly using their idea as much as we need.

Since $F_k(0, \omega)$ is a random variable measurable in $\{B(t+1, \omega) - B(t-1, \omega), -k < t < \infty\}$, it can be expressed as the limit (in the mean square sence) of a sequence of $L_2(\Omega)$-random variables $\{\varphi_j(\omega)\}$, $\varphi_j(\omega)$ being measurable in $\{B(t_i^{(j)}+1, \omega) - B(t_i^{(j)}-1, \omega), i = 1, \cdots, n_j\}$, for some n_j and some $\{t_i^{(j)}\}$. Taking a common subdivision of $(t_i^{(j)}+1, t_i^{(j)}-1)$, say $(s_i^{(j)}, u_i^{(j)})$ $i = 1, \cdots, m$, we can see that $\varphi_j(\omega)$ may be expressed as $\Psi(X_1(\omega), \cdots, X_m(\omega))$ where

$$X_i(\omega) = \frac{B(s_i^{(j)}, \omega) - B(u_i^{(j)}, \omega)}{\sqrt{s_i^{(j)} - u_i^{(j)}}}$$

Ψ being a B-measurable function on R^m. It is clear that

$$E\varphi_j^2(\omega) = \int_{-\infty}^{\infty} \cdots \int \Psi^2(\xi_1, \cdots, \xi_m) \left(\frac{1}{\sqrt{2\pi}}\right)^m e^{-\frac{\xi_1^2}{2} - \cdots - \frac{\xi_m^2}{2}} d\xi_1 \cdots d\xi_m < \infty . \tag{3.11}$$

By the completeness of Hermite polynomials, Ψ can be approximated by a polynomial $\Theta_N(\xi_1 \cdots \xi_m)$ as

8) The number in [ ] refers to the references at the end of this paper and Theorem... is the Theorem to be consulted.

$$\int_{-\infty}^{\infty}\cdots\int |\Psi(\xi_1,\cdots,\xi_m)-\Theta_N(\xi_1,\cdots,\xi_m)|^2\left(\frac{1}{\sqrt{2\pi}}\right)^m e^{-\frac{\xi_1^2}{2}-\cdots-\frac{\xi_m^2}{2}}\,d\xi_1\cdots d\xi_m<\varepsilon$$

so that

$$E|\varphi_j(\omega)-\Theta_N(X_1(\omega),\cdots,X_m(\omega))|^2<\varepsilon\,.$$

By the stationarity of Brownian motion we have

$$E|\varphi_j(\omega_t^+)-\Theta_N(X_1(\omega_t^+),\cdots,X_m(\omega_t^+))|^2<\varepsilon\,.$$

Hence $\{\varphi_j(\omega_t^+),\ -\infty<t<\infty\}$ is approximated by polinomial processes in the V-topology and so in the U-topology.

This completes the proof of Theorem 1.

§ 4. Proof of Theorem 2.

In order to prove our Theorem 2 we may assume that $\Omega=R^R$ and that $X(t,\omega)$ is the value of the path function ω at t, taking the coordinate representation of the given stochastic process.

We shall define the transformation T_t on Ω by

$$T_t\omega=\omega_{-t}^{+},\ \text{i.e. } X(s,T_t\omega)=X(s-t,\omega)\qquad\text{for all } s\,.$$

It is clear that T_t is a 1-1 measure-preserving transformation and $T_{t+s}=T_tT_s$.

Furthermore we may assume that

(4.1) $$\Omega=\{\omega\,;\,\omega\in R^R,\ |X(t,\omega)|\leqq K\quad |X(t+h,\omega)-X(t,\omega)|\leqq Ch\}$$

for suitable positive integers K and C independent of t,h and ω, since we can find a strictly stationary continuous (in probability) process whose sample function belongs to Ω with probability 1 in any V-neighborhood of the given process by forming $Z_{M,N}$ in (3.3). We remark that $T_t\Omega=\Omega$.

Next we will map Ω onto $[0,1]^2$. Put $\Omega_1=\{\omega\,;\,P(\omega)>0\}$ and $\Omega_2=\Omega-\Omega_1$. Ω_1 consists of a countable number of points, say $\{\omega_i\}_{i\leqq 1}$. Let $\boldsymbol{B}$ be the minimum Borel field which makes $\{X(t,\omega),\ -\infty<t<\infty\}$ all measurable. On account of its continuity, E_{ij}, defined by

$$E_{ij}=\{\omega\,;\,X(r_i,\omega)<r_j,\ \omega\in\Omega_2\}$$

and the above $\{\omega_i\}_{i\geqq 1}$ will generate $\boldsymbol{B}$.

We arrange $\{E_{ij}\}$ in linear order to make a simple sequence $\{E_n\}_{n\geqq 1}$. We map Ω onto the Lebesgue measure space $[0,1)$ as follows. $\omega_i(\in\Omega_1)$ is mapped onto $[\sum_{j=1}^{i-1}P_j,\ \sum_{j=1}^{i}P_j)$ where $P_j=P(\omega_j)$. E_1 and $E_1^c\cap\Omega_2$ are mapped onto $[\sum_{j\geqq 1}P_j,\ \sum_{j\geqq 1}P_j+P(E_1))$ and $[\sum_{j\leqq 1}P+P(E_1),1)$ respectively. Next for E_2, $E_1\cap E_2$, $E_1\cap E_2^c$, $E_1^c\cap E_2$ and $E_1^c\cap E_2^c\cap\Omega_2$ are mapped onto $[\sum_{j\leqq 1}P_j,\ \sum_{j\leqq 1}P_j+$

$P(E_1 \cap E_2))$, $[\sum_{j \geqq 1} P_j + P(E_1 \cap E_2)$, $\sum_{j \geqq 1} P_j + P(E_1))$, $[\sum_{j \geqq 1} P_j + P(E_1)$, $\sum_{j \geqq 1} P_j + P(E_1) + P(E_1^c \cap E_2))$ and $[\sum_{j \leqq 1} P + P(E_1) + P(E_1^c \cap E_2), 1))$ respectively and so on.

Denote by Ψ the above set transformation from Ω onto $[0, 1)$. Next we map $[0, 1)$ onto $[0, 1]^2$ by a 1-1 measure-preserving point transformation ν. Then $\nu\Psi$ will be a measure-preserving set transformation from Ω onto $[0, 1]^2$.

Let $\chi_A(\cdot)$ be the indicator function of a set A. Since $X(t, \omega)$ is measurable with respect to $\boldsymbol{B}$, there exists f_t, which is a Borel measurable function on a space of infinite dimension, such that

$$P(X(t, \omega) \neq f_t(\chi_{E_1}(\omega), \cdots, \chi_{\omega_1}(\omega) \cdots)) = 0$$

for any t. Therefore $\{f(\chi_{\nu\Psi E_1}(\tilde{\omega}), \cdots, \chi_{\nu\Psi\omega_1}(\tilde{\omega}) \cdots), -\infty < t < \infty\}$ is a version[9] on $[0, 1]^2$ of the given process and we denote it by $X(t, \tilde{\omega})$, where $\tilde{\omega} \in [0, 1]^2$.

Now we shall modify this transformation for the later use. Firstly we shall remark that if ω_0 has positive probability, then $X(t, \omega_0) = X(0, \omega_0)$ for all t, i.e. ω_0 represents a constant-valued function. Suppose in the contrary that there exists t such that $X(t, \omega_0) \neq X(0, \omega_0)$. Then by the continuity of the path function, we shall have infinitely many different points among $\{T_{-t}\omega_0\}$. Since T_t is a measure-preserving point transformation, this contradicts the boundedness of the probability measure.

If $P(\bigcap_{i=1}^{k} E_i') = 0$, then $\Psi(\bigcap_{i=1}^{k} E_i') = \phi$ where E' means either E or E^c. Denote by A the set of all ω such that $\omega \in \bigcap_{k \geqq 1} E_k'$ where $P(\bigcap_{k=1}^{N} E_k') = 0$ for some N. We shall remark that $P(A) = 0$. Set $\varepsilon_N = \{\bigcap_{k=1}^{N} E_k' ; P(\bigcap_{k=1}^{N} E_k') = 0, P(\bigcap_{k=1}^{N-1} E_k') > 0\}$. Since ε_N is a finite system of sets, we have $P(\bigcup_{\varepsilon_N} (\bigcap_{k=1}^{N} E_k')) = 0$ which implies $P(A) = 0$ as $A = \bigcup_{N} \bigcup_{\varepsilon_N} (\bigcap_{k=1}^{N} E_k')$.

We shall define a 1-1 point transformation φ from $D \equiv \Omega_2 \cap A^c$ onto $[\sum_{j \leqq 1} P_j, 1]$ minus a countable set such that φ induces Ψ on D. Set $F_k' = \Psi E_k'$. φ is defined by

$$\varphi\omega = \lim_{N \to \infty} \overline{\bigcap_{k=1}^{N} F_k'} \qquad \text{if} \qquad \omega \in \bigcap_{k \geqq 1} E_k' \cap D .$$

Since the right side of the above equality is a single point set, φ can be considered as a point transformation on D. $[\sum_{j \leqq 1} P_j, 1] - \varphi(D)$ is a subset of all the endpoints of F_k, $k = 1, 2, \cdots$, and therefore it is at most countable. The

9) $Y(t)$ is called a version of $X(t)$ if any joint distribution of $Y(t)$ at any finite time points is equal to that of $X(t)$.

set $\{\varphi\omega\}$ which has at least two inverse images is a subset of all the endpoints of F_k, $k=1,2,\cdots$, and therefore it is at most countable. Hence denoting the set $[\sum_{j\geq 1} P_j, 1]$ minus some countable set by R, we have a measure-preserving point transformation φ from D to R.

We shall define F as

$$F(\tilde{\omega})=\begin{cases} f(\varphi^{-1}\nu^{-1}\tilde{\omega}) & \text{for } \tilde{\omega}\in\nu R \\ f(\omega_i) & \text{for } \tilde{\omega}\in\nu[\sum_{j=1}^{i-1}P_j, \sum_{j=1}^{i}P_j)\ i\geq 1\,. \\ 0 & \text{otherwise} \end{cases}$$

where $f(\omega)\equiv X(0,\omega)$. Next define $\tilde{T}_h$ as

$$\tilde{T}_h\tilde{\omega}=\begin{cases} \nu\varphi T_h\varphi^{-1}\nu^{-1}\tilde{\omega} & \text{for } \tilde{\omega}\in\nu\varphi(D\cap T_hD) \\ \tilde{\omega} & \text{otherwise} \end{cases}$$

for any fixed $h>0$. Then the transformation $\tilde{T}_h$ is 1-1 and measure-preserving and $\{F(\tilde{T}_h{}^k\tilde{\omega}),\ k=0,\pm 1,\cdots\}$ is a version of $\{X(kh,\omega),\ k=0,\pm 1,\cdots\}$.

We shall omit $\sim$ in $\tilde{\omega}$ since we shall not refer to the original ω in the sequel.

We shall now introduce some notions on measure-preserving transformation following Halmos. We call a square of the form $\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)\times\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ a diadic square of rank n, $(n=0,1,\cdots,\ i,j=0,1,\cdots,2^n-1)$. We mean by permutation an invertible measure-preserving transformation which maps each diadic square of rank n onto a diadic square of rank n by an ordinary translation. A cyclic permutation of rank n is a permutation that acts as a cyclic permutation on the diadic square of rank n. (This means that there is only one cycle). A subbasic neighborhood of an invertible measure-preserving transformation S is a set of the form:

$$N(s)=N(s:E,\varepsilon)=\{T\colon P(SE\sim TE)<\varepsilon\}$$

where E is a measurable set and $\sim$ means symmetric difference. Then we have

LEMMA 6 (P. R. Halmos)[10]. *Every neighborhood contains cyclic permutations of arbitary high ranks.*

By a diadic step function of rank M we shall understand a function of the form:

$$f(\omega)=\sum_{i,j=0}^{2^M-1}a_{ij}\chi_{A_{ij}}(\omega) \tag{4.2}$$

10) See [4, p. 65].

where $A_{ij}=\left[\frac{i}{2^M}, \frac{i+1}{2^M}\right)\times\left[\frac{j}{2^M}, \frac{j+1}{2^M}\right)$ and $\chi_A(\cdot)$ is the indicator of the set A. For any given measurable function f, there exists a sequence of diadic step functions f_n such that

$$P\left(|f_n(\omega)-f(\omega)|>\frac{1}{n}\right)<\frac{1}{n}.$$

Then the measure-preserving property of $\tilde{T}_h$ implies

$$(4.3)\qquad P\left(|f_n(\tilde{T}_{-h}{}^k\omega)-f(\tilde{T}_{-h}{}^k\omega)|>\frac{1}{n}\right)=P\left(|f_n(\omega)-f(\omega)|>\frac{1}{n}\right)<\frac{1}{n}.$$

Since X satisfies (4.1),

$$(4.4)\qquad P\left(|f_n(\omega)-f_n(\tilde{T}_{-h}\omega)|<\frac{3}{n}\right)<\frac{1}{n}\qquad \text{for}\quad |h|\leqq\frac{1}{2nC}$$

for any n. Fixing n for a while, we shall write $g(\tilde{T}_{-\delta}{}^k\omega)$ and δ for $f_n(\tilde{T}_{-\delta}{}^k\omega)$ and $\frac{1}{2nC}$ respectively. Let M be the rank of g. We use the same notations as in (4.2) for g.

Now we shall discuss some properties of the random sequence $\{g(\tilde{T}_{-\delta}{}^n\omega)$ $n=0,\pm1,\cdots\}$. Write m for $2n^2C$ and define T, B_{ij}^l, $B_{(ij)(i'j')}^l$ and $D_{(ij)(i'j')}^l$ by

$$(4.5)\qquad T\equiv\tilde{T}_{-\delta},\quad B_{ij}^l\equiv T^lA_{ij},\quad B_{(ij)(i'j')}^l\equiv B_{ij}^l\cap A_{i'j'},$$
$$D_{(ij)(i'j')}^l\equiv T^{-l}B_{(ij)(i'j')}^l,\qquad \text{for } |l|\leqq m.$$

Thus we have $A_{ij}=\bigcup_{(i'j')}D_{(ij)(i'j')}^l$ and $D_{(ij)(i'j')}^l\cap D_{(ij)(nm)}^l=\phi$ unless $i'=n$ and $j'=m$. Since the system $\{D_{(ij)(i'j')}^l\}$ is finite, there exists a class of disjoint measurable sets D_i $1\leqq i\leqq k$, such that each $D_{(ij)(i'j')}^l$ can be expressed as the sum of those sets taken from among $\{D_i\}$ and the set T^lD_i is contained in a certain set A_{nm} for any i and any l so that

$$(4.6)\qquad g(T^l\omega)=\text{constant (depending on } l)\qquad \text{for } \omega\in D_i.$$

Using Lemma 6, we can find a cyclic permutation of arbitary high rank $N(>M)$, say Q, such that

$$(4.7)\qquad P(QD_i\sim TD_i)<\varepsilon'\qquad i=1,\cdots,k,$$

where $\varepsilon'=\frac{1}{m^2kn}$. We shall estimate the difference between $g(T^n\omega)$ and $g(Q^n\omega)$. When l is positive, it is clear that

$$(4.8)\qquad P(g(T^l\omega)\neq g(Q^l\omega))\leqq\sum_{i=0}^{l-1}P(g(T^{l-i}Q^i\omega)\neq g(T^{l-i-1}Q^{i+1}\omega))$$

noting that if $\omega\in D_j$ and if $Q\omega\in TD_j$ then $g(T^{l-i-1}Q\omega)=\text{constant}$ by (4.6), we have

$$(4.9)\qquad P(g(T^{l-i}Q^i\omega)\neq g(T^{l-i-1}Q^{i+1}\omega))$$

$$= P(Q^i\omega\,;\, g(T^{l-i-1}T\omega) \neq g(T^{l-i-1}Q\omega))$$
$$= P(g(T^{l-i-1}T\omega) \neq g(T^{l-i-1}Q\omega))$$
$$= \sum_{j=1}^{k} P(g(T^{l-i-1}T\omega) \neq g(T^{l-i-1}Q\omega) : \omega \in D_j)$$
$$\leqq \sum_{j=1}^{k} P(TD_j \sim QD_j) < k\varepsilon' .$$

Therefore the left side of (4.8) is less than $|l|k\varepsilon'$ for $l>0$ and it is also true for $l<0$, so that

$$P(\exists\, |l| \leqq m\,;\, g(Q^l\omega) \neq g(T^l\omega)) \leqq \sum_{l=-m}^{m} |l|k\varepsilon' < \frac{1}{n} . \tag{4.10}$$

Suppose now that $\{F_0 \cdots F_{4^N-1}\}$ is the partition of Ω into diadic squares of rank N such that

$$QF_i = F_{i+1},\ (i \neq 4^N - 1),\quad QF_{4^N-1} = F_0 , \tag{4.11}$$

where $F_0 = \left[0, \frac{1}{2^N}\right) \times \left[0, \frac{1}{2^N}\right)$. Each F_i is contained in some A_{ij} since the rank N of Q is large than M.

Let S be a measure-preserving point transformation which maps $\omega = \left(\frac{n}{2^N} + x, \frac{n'}{2^N} + y\right) \in F_i$ to

$$S\omega = \left(x, \frac{i}{2^N} + y\right) \tag{4.12}$$

under the identification $(x, y+1) = \left(x + \frac{1}{2^N}, y\right)$. Then S carries F_i onto $\left[0, \frac{1}{2^N}\right) \times \left[\frac{i}{2^N}, \frac{i+1}{2^N}\right)$ under the same identification.

Define the transformation Q_t by

$$Q_t(x, y) = \left(x, y + \frac{t}{\delta}\frac{1}{2^N}\right) \tag{4.13}$$

under the identification $(x, y+1) = \left(x + \frac{1}{2^N}, y\right)$ and $(x+1, y) = (x, y)$. Then Q_t is a 1-1 measure-preserving point transformation such that $Q_{t+s} = Q_t Q_s$ and

$$g(S^{-1}Q_\delta S\omega) = g(Q\omega) . \tag{4.14}$$

It is evident that the process $\{g(S^{-1}Q_t S\omega), -\infty < t < \infty\}$ is continuous in probability by definition, but we shall here estimate the fluctuation of $g(S^{-1}Q_t S\omega)$ more precisely. For any t, if $|t-s| \leqq \delta$, then $Q_s(x, y)$ belongs to the same diadic square of rank N as one of $Q_t(x, y)$, $Q_{t+\delta}(x, y)$ and $Q_{t-\delta}(x, y)$ does, so that

$$(4.15)\quad P\left(\exists s: |t-s| \leqq \delta,\ |g(S^{-1}Q_tS\omega)-g(S^{-1}Q_sS\omega)| > \frac{3}{n}\right)$$

$$= P\left(\exists h: |h| \leqq \delta,\ |g(S^{-1}Q_hS\omega)-g(\omega)| > \frac{3}{n}\right)$$

$$\leqq P\left(|g(S^{-1}Q_\delta S\omega)-g(\omega)| > \frac{3}{n}\right) + P\left(|g(S^{-1}Q_{-\delta}S\omega)-g(\omega)| > \frac{3}{n}\right)$$

$$= 2P\left(|g(S^{-1}Q_\delta S\omega)-g(\omega)| > \frac{3}{n}\right)$$

$$\leqq 2\left[P(g(Q\omega) \neq g(T\omega)) + P\left(|g(T\omega)-g(\omega)| > \frac{3}{n}\right)\right] \leqq \frac{6}{n}$$

(by (4.4) and (4.10)).

Next we shall estimate the difference between two processes $X(t, \omega)$ and $g(S^{-1}Q_tS\omega)$. Take any t such that $|t| < m\delta = n$. Then we have

$$(4.16)\quad P\left(|g(S^{-1}Q_tS\omega)-X(t, \omega)| > \frac{6}{n}\right)$$

$$\leqq P\left(|g(S^{-1}Q_tS\omega)-g(Q^l\omega)| > \frac{3}{n}\right) + P(g(Q^l\omega) \neq g(T^l\omega))$$

$$+ P\left(|g(T^l\omega)-X(l\delta, \omega)| > \frac{1}{n}\right) + P\left(|X(l\delta, \omega)-X(t, \omega)| > \frac{1}{n}\right)$$

where $(l-1)\delta \leqq t < l\delta$. Applying (4.3), (4.10) and (4.15) to the right side of this inequality, we have

$$(4.17)\quad P\left(|g(S^{-1}Q_tS\omega)-X(t, \omega)| > \frac{6}{n}\right) < \frac{9}{n},$$

which implies that $\{f_n(S^{-1}Q_tS\omega), -\infty < t < \infty\}$ belongs to $V\left(X, \frac{9}{n}\right)$.

Next we shall show that there is a sequence of ergodic processes which converges to $g(S^{-1}Q_tS\omega)$ in the V-topology.

Define the transformation R_t^α by modifying Q_t slightly as follows

$$(4.18)\quad R_t^\alpha(x, y) = \left(x, y + \frac{t}{\delta}\frac{1}{2^N}\right)$$

under the identification $(x, y+1) = (x+\alpha, y)$ and $(x+1, y) = (x, y)$. $\{R_t^\alpha\}$ is ergodic for any irrational α [7].

We define $\{G_i^\alpha, i = 0, \pm 1, \pm 2, \cdots\}$ and $\{H_i, i = 0, \pm 1, \pm 2, \cdots\}$ by

$$(4.19)\quad \begin{aligned} &R_\delta^\alpha G_i^\alpha = G_{i+1}^\alpha, \qquad &&G_0 = \left[0, \frac{1}{2^N}\right) \times \left[0, \frac{1}{2^N}\right), \\ &Q_\delta H_i = H_{i+1}, \qquad &&H_0 = \left[0, \frac{1}{2^N}\right) \times \left[0, \frac{1}{2^N}\right). \end{aligned}$$

Then we have $SF_i = H_i$ mod (4^N) and

$$P(H_i \sim G_i^{\alpha}) = 2\left|\frac{1}{2^N} - \alpha\right| \left|\left[\frac{i}{2^N}\right]\right| \frac{1}{2^N}. \tag{4.20}$$

Hence

$$\begin{aligned} &P(g|S^{-1}Q_tS\omega) \neq g(S^{-1}R_t^{\alpha}S\omega)) \\ &= \sum_{i=0}^{4^N-1} P(g(S^{-1}Q_tS\omega) \neq g(S^{-1}R_t^{\alpha}S\omega)\,;\, \omega \in F_i) \\ &\leqq P(H_i \sim G_i^{\alpha})4^N \leqq 2\left|\frac{1}{2^N} - \alpha\right| \left(\frac{|t|}{2^N\delta} + 1\right)2^N \end{aligned} \tag{4.21}$$

where $(i-1)\delta \leqq t < i\delta$. For irrational α such that

$$\left|\frac{1}{2^N} - \alpha\right| < \frac{1}{2\left(\frac{n}{2^N\delta} + 1\right)2^N} \frac{1}{n},$$

the right side of (4.21) is less than $\frac{1}{n}$ if $|t| < n$. This implies that $\{g(S^{-1}R_t^{\alpha}S\omega), -\infty < t < \infty\}$ belongs to $V\left(X, \frac{10}{n}\right)$.

Finally we can easily show that this ergodic process is continuous in probability, because for any positive ε,

$$\begin{aligned} &P(\exists h\,;\, |h| \leqq \delta\varepsilon,\ g(S^{-1}R_{t+h}^{\alpha}S\omega) \neq g(S^{-1}R_t^{\alpha}S\omega)) \\ &\leqq P(g(S^{-1}R_{t+\delta\varepsilon}^{\alpha}S\omega) \neq g(S^{-1}R_t^{\alpha}S\omega) + P(g(S^{-1}R_{t-\delta\varepsilon}^{\alpha}S\omega) \neq g(S^{-1}R_t^{\alpha}S\omega)) \\ &= 2P(g(S^{-1}R_{\delta\varepsilon}^{\alpha}S\omega) \neq g(\omega)) \leqq 2.4^N \cdot \frac{1}{2^N} \cdot \frac{1}{2^N} \frac{\delta\varepsilon}{\delta} = 2\varepsilon. \end{aligned} \tag{4.22}$$

This completes the proof of Theorem 2.

§ 5. On the case of Gaussian processes.

For the given process $\{X(t, \omega), -\infty < t < \infty\}$ we may assume $EX(t, \omega) = 0$ without loss of generality.

Let $\rho(t)$ be the covariance function of this process and consider the spectral decomposition

$$\rho(t) = \int_{-\infty}^{\infty} e^{2\pi it\lambda} dF(\lambda). \tag{5.1}$$

Firstly we discuss the case in which the spectral measure F is absolutely continuous with respect to Lebesgue measure, say $F' = f$.

Set

$$g(s) = \int_{-\infty}^{\infty} e^{2\pi i\lambda s} f^{1/2}(\lambda)\, d\lambda \tag{5.2}$$

then

$$\rho(t)=\int_{-\infty}^{\infty}g(s)g(s+t)\,ds\,. \tag{5.3}$$

This implies that the process $\{\tilde{X}(t,\omega), -\infty<t<\infty\}$ defined by

$$\tilde{X}(t,\omega)=\int_{-\infty}^{\infty}g(s)\,d_sB(s+t,\omega)\,, \tag{5.4}$$

(this integral is the Wiener integral) is Gaussian with the same probability law as the given process.

We remark that if $g_1, g_2, \cdots$ is a sequence of $L_2(R^1)$-function which converges to g in the $L_2(R^1)$-sense, then the sequence of $\{X^{(n)}(t,\omega), -\infty<t<\infty\}$, defined by

$$X^{(n)}(t,\omega)=\int_{-\infty}^{\infty}g_n(s)\,d_sB(s+t,\omega)\,, \tag{5.5}$$

converges to $(\tilde{X}(t,\omega)-\infty<t<\infty\}$ in the mean because

$$E(X^{(n)}(t,\omega)-X(t,\omega))^2=\int_{-\infty}^{\infty}(g_n(s)-g(s))^2ds \underset{n\to\infty}{\longrightarrow} 0\,. \tag{5.6}$$

Hence the covariance function $\rho_n(t)$ of $\{X^{(n)}(t,\omega). -\infty<t<\infty\}$, converges to $\rho(t)$ uniformly in t. Therefore the characteristic functional, determined by the covariance function

$$Ee^{i\sum_{j=1}^{k}\theta_jX^{(n)}(t_j,\omega)}=e^{-\frac{1}{2}\sum_{i,j=1}^{k}\rho_n(t_i-t_j)\theta_i\theta_j} \tag{5.7}$$

tends to that of the process $\{X(t,\omega), -\infty<t<\infty\}$ uniformly in $(t_1,\cdots,t_k)$.

If we take a sequence of step functions for g_n, $n\geqq 1$, then we can see that Theorem 4 is true for our special case.

For general spectral measure we approximate F by a sequence of absolutely continuous spectral measures $\{F_n\}$ so that $\lim_{n\to\infty}F_n(\lambda)=F(\lambda)$ at all continuity points of F. Then the sequence of covariance functions of the $\{F_n\}$ converges to $\rho(t)$ uniformly in every finite interval. Therefore the sequence of stochastic processes of $\{F_n\}$ converges to $\{X(t,\omega), -\infty<t<\infty\}$ in the U-topology. This implies that Theorem 4 is true for general Gaussian processes.

§ 6. Proof of Theorems 5 and 6.

Since we can carry out the proof of Theorems 5 and 6 as we did for processes in the previous sections, we shall here give only a brief sketch of the proof.

Corresponding to $a_n(\omega)$ defined for Brownian motion $B(t,\omega)$ in § 2, we shall define $a_n(\omega)$ for the random sequence $\xi_n(\omega)$, in the following way:

(6.1) $\quad a_n(\omega)=n+\min\{i;\, i\geqq -n,\, |\xi_i(\omega)|\leqq 1,\, |\xi_{i+1}(\omega)|>1\,,\cdots,\, |\xi_{i+n}(\omega)|>1\}\,.$

It is evident that $P(a_n(\omega)<\infty)=1$, because

$$(6.2)\qquad P(a_n(\omega)=\infty)$$

$$\leqq P(\exists i\geqq -n,\, |\xi_i(\omega)|\leqq 1)+\prod_{k=-n}^{\infty} P(\exists k_n : |\xi_{k_n}(\omega)|\leqq 1,\ kn\leqq k_n<(k+1)n)$$

$$=0\,.$$

We can easily verify the following properties of the probability law of $a_n(\omega)$ which we shall here denote by $P_n(i)=P(a_n(\omega)=i)$;

(a. 1) $\quad P_n(i)$ is constant for $0\leqq i\leqq n$ and decreasing for $i\geqq n+1$.

(a. 2) $\quad P_n(i)=\sum_{k=n+1}^{\infty} C_k(i)k(P_n(k-1)-P_n(k))$

where $\qquad C_k(i)=\frac{1}{k}\,,\ 0\leqq i\leqq k-1, \qquad =0,\quad i\geqq k\,.$

(a. 3) For any positive m and ε, there exists $n_0(m,\varepsilon)$ such that, if $k\geqq n_0(m,\varepsilon)$

$$P(a_k(\omega_i^+)=a_k(\omega)-i,\quad |i|\leqq m)>1-\varepsilon\,.$$

First we shall restrict ourselves to the case that $|X_n(\omega^*)|\leqq N$ for some N independent of n and ω^*. Take a proper sample path, say $\{X_n(\omega_0^*),\, n=0,\pm 1\cdots\}$, as follows

$$(6.3)\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{l=-n}^{0} e^{i\sum_{j=-k}^{k} r_jX_j(\omega_0^*)}=Ee^{i\sum_{j=-k}^{k} r_jX_j(\omega^*)}$$

for any rational r_j and k.

By the boundness of the random sequence it is evident for above $\{X_n(\omega_0^*),\, n=0,\pm 1\cdots\}$

$$(6.4)\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{l=-n}^{0} e^{i\sum_{j=-k}^{k}\theta_jX_j(\omega_0^*)}=Ee^{i\sum_{j=-k}^{k}\theta_jX_j(\omega^*)}$$

for any θ_j and k.

Define $F_k(n,\omega)$ by

$$(6.5)\qquad F_k(n,\omega)=X_{-a_k(\omega_n^+)}(\omega_0^*)\,.$$

Corresponding to Lemma 4 we have

LEMMA 7. *The sequence* $\{F_k(n,\omega),\, n=0,\pm 1\cdots\}$ *converges to* $\{X_n(\omega^*),\, n=0,\pm 1\cdots\}$ *in law.*

For the general strictly stationary and ergodic sequence $\{X_n(\omega^*),\, n=0,\pm 1\cdots\}$ we define $\{Y_n^{(N)}(\omega^*),\, n=0,\pm 1,\cdots\}$ by

$$(6.6)\qquad Y_n^{(N)}(\omega^*)=\begin{cases} X_n(\omega^*) & \text{for } |X_n(\omega^*)|\leqq N \\ 0 & \text{otherwise.}\end{cases}$$

Then $\{Y_n^{(N)}(\omega^*), n=0, \pm 1 \cdots\}$ is strictly stationary, bounded and ergodic for each N and converges to the given random sequence in probability as $N\rightarrow\infty$. Since Lemma 7 is true for $\{Y_n^{(N)}(\omega^*), n=0, \pm 1 \cdots\}$, it is also true for $\{X_n(\omega^*), n=0, \pm 1 \cdots\}$.

On the approximation of $\{F_k(n, \omega), n=0, \pm 1 \cdots\}$ by the polynomial sequence we can make similar discussion as in § 3.

For the proof of Theorem 6 we may assume that $X_n(\omega^*)$ is written as $X_n(\omega^*)=f(T^{-n}\omega^*)$ by a measure preserving transformation T on the unit interval $[0, 1)$ associated with Lebesque measure. It is enough to apply the theorem of J. C. Oxtoby and S. M. Ulam to this transformation T in order to prove Theorem 6.

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