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A probabilistic method in Hausdorff moment problem and Laplace-Stieltjes transform.

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§ 0. Introduction.

The purpose of this paper is to show that the proofs of some main representation theorems for moment sequences and Laplace-Stieltjes transforms are obtained probabilistically by making use of the representation theorems in the theory of Martin boundaries induced by Markov processes [**6**].

We prepare, in § 1, the following three topics: (a) the definition of time discrete Markov processes over the denumerable space, (b) the definition and some properties of process harmonic functions and (c) the summary of the theory of Martin boundaries for the above processes. In § 2 we shall construct the Martin boundary for the space-time Markov process attached to the Bernoulli sequence B(1/2). In § 3, using the results of § 2, we shall derive the representation theory for moment sequences which is known as Hausdorff moment problem.

In the last section we shall discuss the representation theory for Laplace-Stieltjes transforms in connection with the space-time Markov process attached to the standard Poisson process. This may be considered as a continuous analogue of § 1 through § 3.

It may be interesting to apply the above method to the representation theory for more general transforms with positive kernels. For example, the convolution transforms [**2**] are expected to have a close relation with the space-time Markov processes attached to additive processes.

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§ 1. The sketch of the theory of Martin boundaries induced by time descrete Markov processes over the denumerable space.

All the results in this section are stated without proof: they shall be discussed systematically in [**6**].

Let E be the denumerable space $\{1, 2, 3, \cdots\}$ with the discrete topology and an extra point ∞ be added to E as an isolated point. We shall denote

$E\cup\infty$ by E^*, the discrete time space $\{0,1,2,3,\cdots,+\infty\}$ by T, the topological Borel field of E^* by $\mathfrak{B}_{E^*}$[1)] and the Borel field of all subsets of T by $\mathfrak{B}_T$. Any mapping from $(T,\mathfrak{B}_T)$ into $(E^*,\mathfrak{B}_{E^*})$ will be denoted by w and the n-th coordinate of w, by w_n, or by $x_n(w)$ or simply by x_n. For each w the *passage time* $\sigma(A;w)$ *for any* $A\in\mathfrak{B}_{E^*}$ is defined by

$$\text{(1.1)}\qquad \sigma(A;w)=\min\{n;x_n(w)\in A\}\quad \text{if } x_n(w)\in A \text{ for some } n\geqq 0,$$
$$=+\infty \qquad \text{otherwise.}$$

Now consider the set W of all the w's which satisfy the two conditions (W. 1) that $x_{+\infty}(w)=\infty$ and (W. 2) that $x_n(w)=\infty$ holds for every $n\geqq\sigma(\{\infty\};w)$.[2)] A general element of W is called a *path function.* Let $\mathfrak{B}$ denote the Borel field generated by the sets $\{w;w_k\in A\}$, where A runs over $\mathfrak{B}_{E^*}$ and k over T. Any mapping $\sigma(w)$ from $(W,\mathfrak{B})$ into $(T,\mathfrak{B}_T)$ is called a *random time.* For a random time σ, we shall now define the *stopped path* w_σ^- and the *shifted path* w_σ^+ as follows:

$$\text{(1.2)}\qquad (w_\sigma^-)_k=w_{\min(\sigma,k)}\quad \text{for except } k=+\infty \text{ and } (w_\sigma^-)_{+\infty}=\infty,$$
$$(w_\sigma^+)_k=w_{\sigma+k}\qquad \text{for every } k\in T.$$

Since $\varphi_\sigma(w)\equiv w_\sigma^-$ is a measurable mapping from $(W,\mathfrak{B})$ into itself, $\mathfrak{B}_\sigma=(\varphi_\sigma)^{-1}\mathfrak{B}$ becomes the Borel subfield of $\mathfrak{B}$. It is clear that the $\mathfrak{B}_n$ for the constant random time n coincides with the Borel field generated by all the sets $\{w;w_k\in A\}$, where k runs over $T_n=\{0,1,2,\cdots,n\}$. Especially, we have $\mathfrak{B}_{+\infty}=\mathfrak{B}$.

A random time σ is called a *Markov time* if $\{w;\sigma(w)\leqq n\}\in\mathfrak{B}_n$ for every $n\in T$. It is easily shown that *every passage time is a Markov time.*

We shall say that the combination $(W,\mathfrak{B},P_x,x\in E^*)$ is a *time discrete Markov process over E* if $(P_x,x\in E^*)$ satisfies the following conditions:

(P. 1) For any fixed x, $P_x(\cdot)$ is a probability measure on $(W,\mathfrak{B})$.

(P. 2) Each x is not fictitious, that is,

$$\text{(1.3)}\qquad P_x\{w;x_0(w)=x\}=1\qquad \text{for every } x\in E^*.$$

(P. 3) (Markov Property) For every $x\in E^*$, every $n\geqq 0$ and every $B\in\mathfrak{B}$,

$$\text{(1.4)}\qquad P_x\{P_x(w;w_n^+\in B\,|\,\mathfrak{B}_n)=P_{x_n}(B)\}=1\,.$$ [3)]

A simple calculation proves that (P. 3) *implies the following property*:

(P. 3)$'$ (Strong Markov Property) For every $x\in E^*$, every Markov time $\sigma\geqq 0$ and every $B\in\mathfrak{B}$,

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1) In our case, $\mathfrak{B}_E{}^*$ coincides with the Borel field of all subsets of E^*.

2) $\{\infty\}$ means the set consisting of a single point ∞.

3) $P_x(w;w_n{}^+\in B\,|\,\mathfrak{B}_n)$ is the *conditional probability of the set* $\{w;w_n{}^+\in B\}$ *relative to* $\mathfrak{B}_n$ *under* P_x.

(1.4)′ $$P_x\{P(w\,;w_\sigma^+\in B\,|\,\mathfrak{B}_\sigma)=P_{x_\sigma}(B)\}=1\,.$$

In the sequel any Markov process $(W,\mathfrak{B},P_x,x\in E^*)$ will be denoted simply by x_n.

We shall now introduce several notations and definitions. First we shall define the *transition probabilities of the process* x_n as follows:

(1.5) $$\Pi^n(x,y)=P_x\{w\,;x_n(w)=y\}$$

for every $n\in T$ and every $x,y\in E^*$. According to the Markov property $\Pi^n(x,y)$ is the (x,y) element of the n-th power $(\Pi^*)^n$ of the matrix $\Pi^*=(\Pi^1(x,y)\,;x,y\in E^*)$ for except $n=+\infty$, where $(\Pi^*)^0$ is defined as the identity matrix over E^*.

We shall next denote, by $\mathfrak{R}$, the set of all the finite real valued functions over E^* vanishing at ∞. Further we shall put

$$E_x\{f(w)\}=\int_W f(w)P_x(dw) \quad\text{and}\quad E_x\{f(w)\,;B\}=\int_B f(w)P_x(dw)$$

for any real valued measurable function $f(w)$ on $(W,\mathfrak{B})$ and any $B\in\mathfrak{B}$.

The *Green measure of the process* will be defined by

$$G(x,y)=E_x\{\sum_{n\geq 0}\chi_{\{y\}}(x_n(w))\}\,,$$ [4]

where $\chi_{\{y\}}$ is the indicator function of the one point set $\{y\}$.

DEFINITION 1.1. The Markov process x_n is *conservative* over E if

(1.6) $$P_x\{w\,;\sigma(\{\infty\}\,;w)=+\infty\}=1 \qquad \text{for every } x\in E.$$

DEFINITION 1.2. The point $x\in E^*$ is a *trap* if

(1.7) $$P_x\{w\,;\sigma(E^*-\{x\}\,;w)<+\infty\}=0\,.$$

DEFINITION 1.3. The point $x\in E^*$ is *recurrent* if

(1.8) $$P_x\{w\,;\sigma(\{x\}\,;w_1^+)<+\infty\}=1\,.$$

The matrix Π^* is a *stochastic matrix*[5] over E^* which satisfies $\Pi^1(\infty,\infty)=1$. Conversely, we can see that such any stochastic matrix Π^* over E^* determines uniquely a time discrete Markov process over E which satisfies (1.5), using Kolmogorov extension theorem. On the other hand, Π^* is uniquely determined by the restriction Π to E, which is in general a *substochastic matrix* over E.[6] Thus we may identify a Markov process x_n over E with the above substochastic matrix Π. Further we have the proposition that *Π is stochastic if and only if x_n is conservative.*

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4) $G(x,y)$ may take ∞.

5) $\sum_{y\in E^*}\Pi^1(x,y)=1$ for every $x\in E^*$.

6) $\sum_{y\in E}\Pi^1(x,y)\leqq 1$ for every $x\in E$.

If the point y is not recurrent, we have

(1.9) $$G(x, y)=P_x\{\sigma(\{y\};w)<+\infty\}G(y, y)<\infty$$

for any point x. Conversely, if $G(y, y)$ is finite, y is not recurrent (*see* [5]).

Next we shall give the definition of process harmonic functions and two lemmas useful for later considerations.

DEFINITION 1.4. (a) Let $x\in E$ be not a trap for a given process x_n. Then the function $u\in\mathfrak{N}$ is *harmonic with respect to x_n* (or simply, *x_n-harmonic*) *at* x if

(1.10) $$u(x)=E_x\{u(x_{\sigma(E^*-\{x\};w)}(w))\}.$$

(b) If $x\in E$ is a trap, every $u\in\mathfrak{N}$ is *x_n-harmonic at x.*

(c) If $u\in\mathfrak{N}$ is x_n-harmonic at every x in E, the function is *x_n-harmonic* (*over E*).

Then we have

LEMMA 1.1. *A nonnegative function $u\in\mathfrak{N}$ is x_n-harmonic if and only if*

(1.11) $$u(x)=E_x\{u(x_1(w))\}=\sum_{y\in E}\Pi^1(x, y)u(y)$$

for every x in E.

LEMMA 1.2. (i) *A function $u\in\mathfrak{N}$ can be represented as the difference of two nonnegative x_n-harmonic functions if and only if u is x_n-harmonic and the function defined by*

(1.12) $$u_{+\infty}(x)=\lim_{n\to+\infty}E_x\{|u(x_n)|\}$$ [7]

belongs to $\mathfrak{N}$.

(ii) *If y is accessible from x, that is, $P_x\{\sigma(\{y\};w)<+\infty\}>0$, $u_{+\infty}(x)<\infty$ implies that $u_{+\infty}(y)<\infty$.*

In order to proceed to the theory of Martin boundaries, we shall hereafter assume that the x_n satisfies the following three conditions:

(A. 1) All the points in E are not recurrent.

(A. 2) There exists a point c called the *center of the process* such that

(1.13) $$P_c\{\sigma(\{y\};w)<+\infty\}>0$$

holds for any point y in E.

(A. 3) For each $x\in E$, there exists a finite set $F_x\subset E$ such that

(1.14) $$P_x\{w;x_{\sigma(E^*-\{x\};w)}(w)\in F_x\cup\infty\}=1.$$

It is clear from (1.9) that, under the assumption (A. 1), (A. 2) is equivalent to the condition

$$0<G(c, y)<+\infty \qquad \text{for every } y\in E.$$

7) If u is x_n-harmonic, $E_x\{|u(x_n)|\}$ increases with n for every x in E. Thus $u_{+\infty}$ is always well defined if we admit ∞ for its value.

Hence the ratio

$$K(x, y) \equiv \frac{G(x, y)}{G(c, y)} = \frac{P_x\{\sigma(\{y\}; w) < +\infty\}}{P_c\{\sigma(\{y\}; w) < +\infty\}} \tag{1.15}$$

is well defined and is finite for every $x, y \in E$. An infinite sequence $\{y_n; n=1,2,3,\cdots\}$ is called a *fundamental sequence* if $K(x, y_n)$ tends to a nonnegative x_n-harmonic function. Then, using (A. 1) and (A. 3), we can prove that *every fundamental sequence has no limit points in E* and that *any infinite sequence having no limit points in E contains at least one fundamental sequence.*[8] We say that two fundemental sequences y_n, z_n are *equivalent* if $K(x, y_n)$ and $K(x, z_n)$ have the same limit function. Let b denote an equivalence class of sequences and $K(x, b)$[9] denote the common limit function for the fundamental sequences belonging to the class b. We shall also denote the set of all b's by $\mathfrak{M}$, and the set $E \cup \mathfrak{M}$ by $\bar{E}$.

Finally, we shall topologize $\bar{E}$, using the metric

$$\rho(\xi, \eta) = \sum_{x \in E} \frac{|K(x, \xi) - K(x, \eta)|}{1 + |K(x, \xi) - K(x, \eta)|} m(x) \qquad \text{for any } \xi, \eta \in \bar{E}, \tag{1.16}$$

in which $m(x)$ is an arbitrary totally finite measure on E such that $m(x) > 0$ for every x in E. Then we have

THEOREM 1.1. (i) *$\bar{E}$ is a compact metric space.*

(ii) *E is open in $\bar{E}$ and its relative topology is the original discrete one.*

(iii) *$\mathfrak{M}$ is the boundary of E in this ρ-topology.*

(iv) $K(x, \xi)$ is continuous in ξ for each x.

DEFINITION 1.5. The above $\mathfrak{M}$ is the *Martin boundary induced by the Markov process.*

Apparently the above definition of $\mathfrak{M}$ depends on the choice of the center c and the measure m, but we can see that *it does not depend actually.*

DEFINITION 1.6. A nonnegative x_n-harmonic function u is *minimal* if any x_n-harmonic function v such that $0 \leqq v \leqq u$ is a constant multiple of u.

Now denote by $\mathfrak{M}_1$ the set of all points of $\mathfrak{M}$ for which $K(\cdot, b)$ is minimal. In order to state a condition that $K(\cdot, b)$ be minimal, we shall need several preparations.

Consider any nonnegative x_n-harmonic function u over E. Given an arbitrary subset A of E, put

$$u_A^*(x) = E_x\{u(x_{\sigma(A)})\}. \tag{1.17}$$

It will be shown that u_A is well defined for every x and that $u_A^* \leqq u$. Next,

8) In case there exist recurrent points, these statements must be slightly modified.

9) $K(x, b)$ is called the *generalized Poisson kernel induced by the given Markov process.*

let D be an arbitrary closed subset in $\mathfrak{M}$ and, $A_1, A_2, \cdots, a$ decreasing sequence of open subsets in $\bar{E}$ such that $A_n \supset D$ and $\bigcap_{n\geqq 1} \bar{A}_n = D$[10]. Then we shall define u_D as

$$u_D = \lim_{n\to\infty} u^*_{[A_n]}, \tag{1.18}$$

where $[A_n] = \bar{A}_n \cap E$. *u_D is well defined, independently of the choice of A_n, and is a nonnegative x_n-harmonic function which does not exceed u.*

Using the above notations, we have

THEOREM 1.2. (i) *$\mathfrak{M}_1$ is Borel measurable in $\mathfrak{M}$.*

(ii) *A necessary and sufficient condition that a boundary point b belongs to $\mathfrak{M}_1$ is that*

$$K_{\{b\}}(c, b) = 1 \tag{1.19}$$

or equivalently that

$$K_D(c, b) = 1 \tag{1.20}$$

for any closed subset D in $\mathfrak{M}$ containing the point b as an interior point.

We shall now give the representation theorems for x_n-harmonic functions. Denoting by $\mathfrak{B}_{\mathfrak{M}_1}$ the Borel field consisting of all ρ-Borel subsets in $\mathfrak{M}_1$, we shall first have

THEOREM 1.3. *If u is nonnegative and x_n-harmonic, there exists uniquely a bounded measure μ[11] on $(\mathfrak{M}_1, \mathfrak{B}_{\mathfrak{M}_1})$ such that*

$$u(x) = \int_{\mathfrak{M}_1} K(x, b)\mu(db) \qquad \textit{for every } x \in E. \tag{1.21}$$

The total mass $\mu(\mathfrak{M}_1)$ is equal to the value $u(c)$. Conversely, given any bounded measure μ on $(\mathfrak{M}_1, \mathfrak{B}_{\mathfrak{M}_1})$, the function defined by the right side of (1.21) *is a nonnegative x_n-harmonic function.*

For the x_n-harmonic functions which are not necessarily nonnegative, combining the above theorem, Lemma 1.2 and Jordan decomposition theorem for bounded signed measures, we have the following

THEOREM 1.4. (i) (Uniqueness of the representation). *A function $u \in \mathfrak{R}$ can have at most one representation* (1.21) *if μ is a bounded signed measure on $(\mathfrak{M}_1, \mathfrak{B}_{\mathfrak{M}_1})$.*

(ii) *In order that a function $u \in \mathfrak{R}$ is expressible in the form of* (1.21) *by means of a bounded signed measure on $(\mathfrak{M}_1, \mathfrak{B}_{\mathfrak{M}_1})$, it is necessary and sufficient that u is x_n-harmonic and that*

$$\lim_{n\to+\infty} E_c\{|u(x_n)|\} < \infty,$$

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10) $\bar{A}_n$ is the closure of A_n in $\bar{E}$.

11) By a *bounded measure*, we shall understand a nonnegative and totally finite regular measure.

§ 2. The construction of the Martin boundary induced by the space-time Markov process attached to the Bernoulli sequence B(1/2).

In this section we shall represent the denumerable space E under consideration by the set of all points (n,i) such that $n \geqq i = 0, 1, 2, \cdots$. Now consider a time discrete Markov process conservative over E which corresponds to the stochastic matrix Π defined by $\Pi^1((n,i), (n+1,i)) = \Pi^1((n,i), (n+1,i+1)) = 1/2$. First we shall remark that this process can be derived from the Bernoulli sequence through space-time consideration.

Let $(\Omega, \mathfrak{F}, P)$ be an abstract probability field. If $\{y_n(\tilde{\omega}); n \geqq 1\}$ is a sequence of random variables on $(\Omega, \mathfrak{F}, P)$ which are mutually independent and each of which satisfies

$$(2.1) \qquad P\{y_n(\tilde{\omega}) = 1\} = p, \qquad P\{y_n(\tilde{\omega}) = 0\} = 1 - p,$$

it is called a *Bernoulli sequence* and denoted by B(p). In the sequel we shall consider B(1/2). For convenience we shall introduce a trivial random variable $y_0(\tilde{\omega}) \equiv 0$. Now put $s_k(\tilde{\omega}) = \sum_{l=1}^{k} y_l(\tilde{\omega})$ and consider a collection of processes defined by

$$(2.2) \qquad \mathfrak{z}_k^{(x)}(\tilde{\omega}) = x + (k, s_k(\tilde{\omega}))$$

for every $x \in E^*$ and $k \in T$; if $x = \infty$ or $k = +\infty$, we shall define $\mathfrak{z}_k^{(x)}(\tilde{\omega}) = \infty$ conventionally. Since the set $\{\tilde{\omega}; \mathfrak{z}^{(x)}(\tilde{\omega}) \in B\}$ belongs to $\mathfrak{F}$ for every x and $B \in \mathfrak{B}$,

$$(2.3) \qquad P_x(B) \equiv P\{\tilde{\omega}; \mathfrak{z}^{(x)}(\tilde{\omega}) \in B\}$$

is well defined. Then it is shown from the definition of B(1/2) that the combination $(W, \mathfrak{B}, P_x, x \in E^*)$ gives the time discrete Markov process over E induced at the beginning of this section. Thus it will be natural to call our process the *space-time Markov process attached to* B(1/2).

In order to get into the concrete construction of the Martin boundary for the above process x_n, we shall first check that our process satisfies the assumptions of § 1. In fact, noting that

$$(2.4) \qquad P_{(n,i)}\{x_k = (m,j)\} = \left(\frac{1}{2}\right)^k \binom{k}{j-i} \quad \text{for } m = n+k,\ j \geqq i, \qquad = 0 \quad \text{otherwise,}$$

where $k \geqq 0$, $(n,i) \in E$ and $(m,j) \in E$, we have

$$(2.5) \qquad P_{(n,i)}\{\sigma(\{n,i\}; w_1^+) < +\infty\} = 0 \qquad \text{for any } (n,i) \in E,$$

$$(2.6) \qquad P_{(n,i)}\{x_{\sigma(E-\{n,i\})} \in F_{(n,i)}\} = 1 \qquad \text{for } F_{(n,i)} = \{(n+1,i); (n+1,i+1)\}$$

and

$$(2.7) \qquad G((n,i),(m,j)) = P_{(n,i)}\{\sigma(\{m,j\}) < +\infty\} = \frac{1}{2^{m-n}}\binom{m-n}{j-i},$$

where the right side is understood to be zero unless $m \geqq n$, $j \geqq i$ and $m-n \geqq j-i$.

(2.7) implies that the point (0,0) is the unique center of our process. Hence we have

$$K((n,i),(m,j)) = \frac{P_{(n,i)}\{\sigma(\{m,j\}) < +\infty\}}{P_{(0,0)}\{\sigma(\{m,j\}) < +\infty\}} = 2^n \frac{(m-n)!\,j!\,(m-j)!}{m!(m-n-j+i)!\,(j-i)!}. \tag{2.8}$$

Now consider an infinite sequence (m_k, j_k) having no limit points in E such that

$$\lim_{k\to\infty} \frac{j_k}{m_k} = 1-b \tag{2.9}$$

for a suitable $0 \leqq b \leqq 1$. Then using Stirling formula, it is shown that such a sequence is a fundamental sequence with the limit function $2^n b^{n-i}(1-b)^i$, and conversely that every fundamental sequence has the property (2.9) for some $0 \leqq b \leqq 1$. Moreover it is also clear that, given any fixed $0 \leqq b \leqq 1$, there exists a fundamental sequence (m_k, j_k) which satisfies (2.9). Thus we may consider that the Martin boundary $\mathfrak{M}$ induced by the space-time process x_n coincides with the interval $[0,1]$ as a set. Hence we shall denote a Martin boundary point by $b \in [0,1]$ and the generalized Poisson kernel $2^n b^{n-i}(1-b)^i$ by $K((n,i),b)$. We shall now show that the relative ρ-topology in $\mathfrak{M}$ defined by (1.16) coincides with the ordinary one in $[0,1]$. In fact this is easily proved, noting that $K((n,i),b)$ is continuous as a function of b with respect to the ordinary topology in $[0,1]$ for every (n,i) and that

$$\rho(b,b') > \frac{2|b-b'|}{1+2|b-b'|} m(\{1,0\}) > \frac{2}{3} m(\{1,0\})|b-b'|. \tag{2.10}$$

Next we shall prove that $\mathfrak{M} = \mathfrak{M}_1$, namely, that $K((n,i),b)$ is minimal x_n-harmonic for every $b \in [0,1]$. For this purpose, according to Theorem 1.2, it is enough to show that

$$K_D((0,0),b) = 1 \tag{2.11}$$

holds for any closed set $D = [b-\varepsilon, b+\varepsilon] \cap [0,1]$, where ε is an arbitrary positive number.

Now we shall denote by A_n the set of all the boundary points b' in the interval $\left(b-\varepsilon-\frac{1}{n},\ b+\varepsilon+\frac{1}{n}\right)$ and all the points (m,j) in E with $m \geqq n$ and

$$1-b-\varepsilon-\frac{1}{n} < \frac{j}{m} < 1-b+\varepsilon+\frac{1}{n}.$$

Then it is evident that $A_n \supset D$ and $\bigcap_{n\geqq 1} \bar{A}_n = D$. Therefore using the results in § 1, we have

$$1 = K((0,0),b) \geqq K_D((0,0),b) = \lim_{n\to\infty} K^*_{[A_n]}((0,0),b), \tag{2.12}$$

and

$$K^*_{[A_n]}((0,0),b) \geqq E_{(0,0)}\{K(x_{\sigma(A_n)},b)\,;\,\sigma(A_n)=n\} \tag{2.13}$$
$$\geqq \sum_{1-b-\varepsilon \leqq i/n \leqq 1-b+\varepsilon} b^{n-i}(1-b)^i \binom{n}{i}.$$

Now applying the law of large numbers, we can see that the last term of (2.13) tends to 1 as n increases to infinity. This proves (2.11).

Finally, noting that for a function u over E

$$E_{(0,0)}(|u(x_n)|) = 2^{-n}\sum_{i=0}^{n}|u(n,i)|\binom{n}{i}, \tag{2.14}$$

and applying the theorems of §1 to our case, we can summarize our results in the following

THEOREM 2.1. *Let x_n be the space-time Markov process attached to the Bernoulli sequence* B(1/2).

(i) *The Martin boundary induced by x_n is equivalent to the interval* $[0,1]$ *with the ordinary topology.*

(ii) *The generalized Poisson kernel* $K((n,i),b)$ *is the function* $2^n b^{n-i}(1-b)^i$ *and is minimal x_n-harmonic for any fixed* $b\in[0,1]$.[12]

(iii)[13] *Every nonnegative x_n-harmonic function u over E can be written in the form*

$$u(n,i) = 2^n\int_0^1 b^{n-i}(1-b)^i d\mu(b) \qquad \textit{for every } (n,i)\in E, \tag{2.15}$$

where μ is a bounded measure on $([0,1],\mathfrak{B}_{[0,1]})$ *which is uniquely determined by u and whose total mass* $\mu([0,1])$ *is equal to* $u(0,0)$. *Conversely, given any bounded measure μ on* $([0,1],\mathfrak{B}_{[0,1]})$, *the function defined by the right side of* (2.15) *is nonnegative and x_n-harmonic.*

(iv) *A function $u\in\mathfrak{R}$ can have at most one representation* (2.15) *if μ is a bounded signed measure on* $([0,1],\mathfrak{B}_{[0,1]})$.

(v) *A function $u\in\mathfrak{R}$ can be written in the form of* (2.15) *by means of a bounded signed measure on* $([0,1],\mathfrak{B}_{[0,1]})$, *if and only if u is x_n-harmonic and* (2.14) *is bounded in n.*

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12) The assertions (i), (ii) implies that $\mathfrak{B}_{\mathfrak{M}_1}$ defined in §1 coincides with the Borel field $\mathfrak{B}_{[0,1]}$ consisting of all the ordinary Borel subsets in $[0,1]$.

13) Let $\mathfrak{F}_n$ be the Borel field generated by $s_1(\tilde{\omega}), s_2(\tilde{\omega}),\cdots\cdots, s_n(\tilde{\omega})$ and u, a function belonging to $\mathfrak{R}$. Then according to (2.4) and the Markov property of the sequence $\{s_n(\tilde{\omega})\,;\,n\geqq 0\}$, the condition that $u(n,i)$ is a nonnegative x_n-harmonic function is equivalent to the condition that $\{u(n,s_n),\mathfrak{F}_n, n\geqq 0\}$ is a nonnegative martingale. Hence the assertion (iii) answers to the following problem: *Under what condition is* $\{u(n,s_n),\mathfrak{F}_n, n\geqq 0\}$ *a nonnegative martingale?*

§ 3. The solution for Hausdorff moment problem.

Hausdorff moment problem is the representation theory for moment sequences. In this section we shall show that the solution for the problem is easily derived from Theorem 2.1.

First we shall give the statement of the problem following D. V. Widder [7] except some change of notations.

DEFINITION 3.1. A sequence of (real) numbers, $\{f(n); n \geqq 0\}$, is a *moment sequence* if there exists a bounded signed measure μ on $([0,1], \mathfrak{B}_{[0,1]})$ such that for every $n \geqq 0$

$$f(n) = \int_0^1 b^n d\mu(b). \tag{3.1}$$

We shall now denote by Δ the difference operator:

$$\Delta f(n) = f(n+1) - f(n). \tag{3.2}$$

Then we have

$$\Delta^k f(n) = \overbrace{\Delta \cdot \Delta \cdots \Delta}^{k} f(n) = \sum_{m=0}^{k} (-1)^m \binom{k}{m} f(n+k-m) \quad \text{for every } k \geqq 0, \tag{3.3}$$

where Δ^0 is the identity operator by definition.

DEFINITION 3.2. A sequence of numbers, $\{f(n); n \geqq 0\}$ is *completely monotonic* if for every $n \geqq 0$ and every $k \geqq 0$

$$(-1)^k \Delta^k f(n) \geqq 0. \tag{3.4}$$

The main part of the representation theory for moment sequences consists of the following

THEOREM 3.1.[14] *A sequence of numbers can have at most one representation* (3.1) *if μ is a bounded signed measure on* $([0,1], \mathfrak{B}_{[0,1]})$.

THEOREM 3.2.[15] *A sequence of numbers is a moment sequence with a bounded measure on* $([0,1], \mathfrak{B}_{[0,1]})$ *if and only if it is completely monotonic.*

THEOREM 3.3.[16] *A sequence of numbers is a moment sequence if and only if there exists a constant L such that*

$$\sum_{i=0}^{n} |\Delta^i f(n-i)| \binom{n}{i} < L \tag{3.5}$$

for every $n \geqq 0$.

We shall now show that Theorems 3.1, 3.2 and 3.3 are, respectively, reduced to the statements (iv), (iii) and (v) of Theorem 2.1.

PROOF OF THEOREMS. Let x_n be the Markov process of § 2 and $u(n, i)$ an

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14) D. V. Widder [7], p. 60, Theorem 6.1.
15) Ibid. p. 108, Theorem 4a.
16) Ibid. p. 103, Theorem 2b.

x_n-harmonic function over $E=\{(n,i)\,;\,n\geqq i=0,1,2,\cdots\}$. Then, for the sequence of numbers defined by

$$f(n)=2^{-n}u(n,0) \qquad \text{for } n\geqq 0, \tag{3.6}$$

we have

$$(-1)^i\Delta^i f(n-i)=2^{-n}u(n,i) \qquad \text{for every } (n,i)\in E. \tag{3.7}$$

This shows that any x_n-harmonic function is uniquely determined by those values on the n-axis. Conversely, given any sequence of numbers $\{f(n)\,;\,n\geqq 0\}$, the function $u(n,i)$ over E defined by the right side of (3.7) is x_n-harmonic.

Now consider a moment sequence

$$f(n)=\int_0^1 b^n d\mu(b) \qquad \text{for } n\geqq 0. \tag{3.8}$$

Then the x_n-harmonic function u obtained by (3.7) is written in the form (2.15) for the above μ. Therefore u determines μ uniquely as a bounded signed measure (Theorem 2.1. (iv)). This proves Theorem 3.1.

Moreover, for the above function u, (2.14) is bounded in n (Theorem 2.1. (v)). Hence the moment sequence satisfies (3.5). Next, assume that a sequence $f(n)$ satisfies (3.5). Then, for the x_n-harmonic function u defined by (3.7), (2.14) does not exceed L for every $n\geqq 0$. Therefore u has the representation (2.15) (use Theorem 2.1. (v) again). This shows that $f(n)$ is a moment sequence. Thus Theorem 3.3 was proved.

Finally we shall prove Theorem 3.2. Let $f(n)$ be a moment sequence with a bounded measure μ. Then the function u given by (2.15) for the same μ is a nonnegative x_n-harmonic function and satisfies (3.7). This proves that $f(n)$ is completely monotonic. Conversely, assume that $f(n)$ is completely monotonic. Then the function u obtained by (3.7) is nonnegative and x_n-harmonic. Hence u can be written in the form (2.15) by means of a bounded measure (Theorem 2.1. (iii)). Consequently the completely monotonic sequence is a moment sequence with a bounded measure. This completes the proof of Theorem 3.2.

§ 4. A probabilistic approach to the representation theory for Laplace-Stieltjes transforms.

The representation theorems for Laplace-Stieltjes transforms which are the continuous analogues of the theorems in § 3 are the following

THEOREM 4.1.[17] *A real valued function $f(t)$ over $[0,+\infty)$ can have at most one Laplace-Stieltjes transform representation*

17) D. V. Widder [7], p. 63, Theorem 6.3.

(4.1) $$f(t)=\int_0^{+\infty} e^{-bt}d\mu(b)$$

with a bounded signed measure μ *on* $([0,+\infty),\mathfrak{B}_{[0,+\infty)})$, *where* $\mathfrak{B}_{[0,+\infty)}$ *is the topological Borel field of* $[0,+\infty)$.

THEOREM 4.2.[18] *A real valued function* $f(t)$ *over* $[0,+\infty)$ *is a Laplace-Stieltjes transform of a bounded measure on* $([0,+\infty),\mathfrak{B}_{[0,+\infty)})$, *if and only if it is completely monotonic in* $[0,+\infty)$, *that is, it is continuous in* $[0,+\infty)$ *and satisfies*

(4.2) $$(-1)^i\left(\frac{d}{dt}\right)^i f(t)\geqq 0$$

for every $t\in(0,+\infty)$ *and every nonnegative integer* i.

THEOREM 4.3.[19] *A real valued function* $f(t)$ *over* $[0,+\infty)$ *is a Laplace-Stieltjes transform of a bounded signed measure on* $([0,+\infty),\mathfrak{B}_{[0,+\infty)})$ *if and only if it is continuous in* $[0,+\infty)$ *and there exists a constant* L *such that*

(4.3) $$\sum_{i=0}^{\infty}\left|\left(\frac{d}{dt}\right)^i f(t)\right|\frac{t^i}{i!}<L$$

for every $t\in(0,+\infty)$.

We shall here prove these theorems by the same probabilistic idea as we did for the moment problem. For this purpose we shall first give a simple remark concerning the theory of Martin boundaries for time continuous Markov processes and second we shall prove a theorem (Theorem 4.4) which is analogous to Theorem 2.1, constructing the Martin boundary for the space-time Markov process attached to the standard Poisson process. Finally we shall show that Theorem 4.4. implies Theorems 4.1, 4.2 and 4.3.

THE THEORY OF MARTIN BOUNDARIES FOR TIME CONTINUOUS MARKOV PROCESSES. The results in § 1 are true with some modification for a time continuous Markov process x_t over a separable locally compact space E[20] which satisfies some restricted conditions. For simplicity we shall only state the definition of process harmonic functions.

DEFINITION 4.1. Let $\mathfrak{R}$ be the set of all the finite real valued continuous functions vanishing at ∞.

(a) In case $x\in E$ is not a trap, the function $u\in\mathfrak{R}$ is *x_t-harmonic at x* if there exists an open set U containing x such that

(4.4) $$u(x)=E_x\{u(x_{\sigma(V^c;\,w)}(w))\}$$

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18) Ibid. p. 160, Theorem 12a.
19) Ibid. p. 308, Theorem 13.
20) See [**5**] for the definition of such process.

for every open set V ($\bar{V}\subset U$).

(b) If $x\in E$ is a trap, every $u\in\mathfrak{R}$ is *x_t-harmonic at x.*

(c) If $u\in\mathfrak{R}$ is x_t-harmonic at every x in E, the function is *x_t-harmonic (over E).*

THE CONSTRUCTION OF THE MARTIN BOUNDARY FOR THE SPACE-TIME PROCESS ATTACHED TO THE STANDARD POISSON PROCESS. Now consider as E the set of all the points (t, i), where t runs over $[0, +\infty)$ for $i=0$ and over $(0, +\infty)$ for $i=1, 2, 3, \cdots$. We shall determine the topology in E by the neighborhoods $U_{(t,i)}(\varepsilon)=\{(t', i)\,;\,|t-t'|<\varepsilon\}$. Further consider a time continuous Markov process conservative over E whose transition probabilities are given by

$$P\{r, (t, i), (s, j)\}=e^{-r}\frac{r^{j-i}}{(j-i)!} \qquad \text{if } s=t+r,\ j\geqq i, \tag{4.5}$$
$$=0 \qquad \text{otherwise.}$$

The existence of such process is proved by the same arguments as in Hunt's paper [**3**, II, pp. 354–355]. This process will be also constructed from the standard Poisson process[21] by the same space-time consideration as in § 2. Thus we shall name our process the *space-time Morkov process attached to the standard Poisson process.*

We shall now apply the theory of Martin boundaries to the above process. Since a simple computation shows

$$P_{(t,i)}\{\sigma(\{s, j\}\,;\,w)<+\infty\}=e^{-(s-t)}\frac{(s-t)^{j-i}}{(j-i)!} \qquad \text{for } s\geqq t,\ j\geqq i, \tag{4.6}$$

the point $(0, 0)$ is the unique center of our process and we have

$$K((t, i), (s, j))=\frac{P_{(t,i)}\{\sigma(\{s, j\})<+\infty\}}{P_{(0,0)}\{\sigma(\{s, j\})<+\infty\}}=e^t\,\frac{(s-t)^{j-i}j\,!}{s^j(j-i)\,!}. \tag{4.7}$$

Then using the same consideration as in § 2, it is shown that the Martin boundary for our process is equivalent to the half line $[0, +\infty]$ with the ordinary topology and that the generalized Poisson kernel $K((t, i), b)$ is the function $e^te^{-bt}b^i$ which is understood to be identically zero except at $(0, 0)$ for $b=+\infty$. Here we shall remark that the boundary point $b\in[0, +\infty]$ corresponds to a class of all the fundamental sequences (s_k, j_k) in E with

$$\lim_{k\to\infty}\frac{j_k}{s_k}=b. \tag{4.8}$$

Moreover $K((t, i), b)$ proves to be minimal x_t-harmonic as a function of (t, i) except for $b=+\infty$. Since $K((0, 0), +\infty)=1$ by definition, $K((t, i), +\infty)$ is not

21) By the *standard Poisson process* we shall understand a Poisson process whose average holding time is equal to 1 and whose sample paths are (almost all) increasing in isolated jumps of unit magnitude (see Doob [**1**]).

x_t-harmonic at $(0, 0)$.[22] Hence we have $\mathfrak{M}_1 = [0, +\infty)$.

Now let u denote any x_t-harmonic function and $\sigma(w)$ the passage time for the complement of the ε-neighborhood of the point $(t, i) \in E$. From the definition of x_t-harmonic functions we have

$$(4.9) \qquad u(t, i) = E_{(t,i)}\{u(x_{\sigma(w)}(w))\} = \int_t^{t+\varepsilon} u(s, i+1)e^{-(s-t)}ds + u(t+\varepsilon, i)e^{-\varepsilon}$$

for every $(t, i) \in E$ and any $\varepsilon > 0$. Hence it is clear that u satisfies

$$(4.10) \qquad \mathfrak{G}u(t, i) \equiv \frac{\partial}{\partial t} u(t, i) + u(t, i+1) - u(t, i) = 0^{23)}$$

for every $(t, i) \in E$ except for $(0, 0)$. Conversely, if u is continuous in E and satisfies (4.10) in $E - \{0, 0\}$, a simple calculation shows that u is x_t-harmonic. In particular, (4.10) implies that any x_t-harmonic function is infinitely differentiable as a function of t in $(0, +\infty)$.

We shall now sum up our results in

THEOREM 4.4. *Let x_t be the space-time Markov process attached to the standard Poisson process.*

(i) *The Martin boundary induced by x_t is equivalent to the half line $[0, +\infty]$ with the ordinary topology.*

(ii) *The generalized Poisson Kernel $K((t, i), b)$ is the function $e^t e^{-bt} b^i$ and is minimal x_t-harmonic for any fixed $b \in [0, +\infty)$.*

(iii) *Every nonnegative x_t-harmonic function u can be represented in the form*

$$(4.11) \qquad u(t, i) = e^t \int_0^{+\infty} e^{-bt} b^i d\mu(u) \qquad \textit{for every } (t, i) \in E,$$

where μ is a bounded measure on $([0, +\infty), \mathfrak{B}_{[0,+\infty)})$ whose total mass $\mu([0, +\infty))$ is equal to $u(0, 0)$. Such μ is determined by u uniquely. Conversely, given any bounded measure μ on $([0, +\infty), \mathfrak{B}_{[0,+\infty)})$, the function given by the right side of (4.11) *is a nonnegative x_t-harmonic function.*

(iv) *A function $u \in \mathfrak{R}$ can have at most one representation* (4.11) *if μ is a bounded signed measure on $([0, +\infty), \mathfrak{B}_{[0,+\infty)})$.*

(v) *A function $u \in \mathfrak{R}$ is expressible in the form of* (4.11) *by means of a bounded signed measure on $([0, +\infty), \mathfrak{B}_{[0,+\infty)})$, if and only if u is x_t-harmonic and*

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22) In general some boundary points may correspond to limit functions which are not x_t-harmonic, as $+\infty$ in this process, while this does not occur in the special cases as in § 2. See [**6**] as to details.

23) Roughly speaking, this operator $\mathfrak{G}$ is the generator of our process.

(4.12) $$E_{(0,0)}\{|u(x_t)|\} = e^{-t}\sum_{i=0}^{\infty}|u(t,i)|\frac{t^i}{i!}$$

is bounded in t.

(vi) *A function u is x_t-harmonic if and only if it is continuous in E and satisfies* (4.10) *for every point in E except for* (0, 0).

PROOF OF THEOREMS 4.1, 4.2 AND 4.3. Let x_t be the above Markov process and $u(t, i)$ an x_t-harmonic function. Then if we define the function $f(t)$ by

(4.13) $$f(t) = e^{-t}u(t, 0) \qquad \text{for } t \in [0, +\infty)\,,$$

we can see from Theorem 4.4, (vi) that

(4.14) $$(-1)^i\left(\frac{d}{dt}\right)^i f(t) = e^{-t}u(t, i)$$

holds for every $t \in (0, +\infty)$ and $i = 1, 2, \cdots$. Hence the x_t-harmonic function u is uniquely determined by the values of $u(t, 0)$. Conversely, if $f(t)$ is continuous in $[0, +\infty)$ and has derivatives of all orders in $(0, +\infty)$, the function $u(t, i)$ over E defined by (4.13) and (4.14) is x_t-harmonic (use Theorem 4.4, (vi) again).

From this remark it is easily shown that Theorems 4.1, 4.2 and 4.3 are, respectively, reduced to the assertions (iv), (iii) and (v) of Theorem 4.4. Since our arguments are quite similar to those in § 3, the details will be omitted.

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