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On certain properties of parametric curves.

Dedicated to Professor Z. Suetuna in celebration of his sexagenary birthday.

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Introduction.

We are mainly concerned with continuous parametric curves in the m-dimensional Euclidean space $\boldsymbol{R}^m$, where $m \geqq 2$, and intend to initiate a differential-geometric theory of such curves under general conditions. In principle, differentiability will not be imposed on the curves. We shall accordingly be naturally led to betake ourselves to methods of real function theory. Especially, certain properties of length of parametric curves will be indispensable for our purposes.

The most important quantities in classical differential geometry of curves are obviously curvature and torsion (besides arc length). However, these two are not capable of direct extension to our situation, inasmuch as they are local quantities involving differentiation. We shall therefore take another way and introduce a global quantity, called *bend*, certain of whose properties will constitute the main subject matter of the present paper. In fact, bend is closely related to curvature as we shall presently see in the next paragraph, and its theory is expected to be preparative to our further study. As regards torsion, it may fairly be said that we have obtained virtually no results as yet.

We define a parametric curve in $\boldsymbol{R}^m$ to be a mapping φ of a nonvoid set E of real numbers into $\boldsymbol{R}^m$. We shall regard $\boldsymbol{R}^m$ as a vector space whenever convenient. In the rest of the introduction we shall restrict ourselves for simplicity to curves defined on an interval. Let I be a closed interval. When φ is a regular $\boldsymbol{C}^2$ curve on I, classical theory applies, and we can consider the integral of the curvature of φ with respect to arc length, along the whole curve. We shall call this quantity, *integrated curvature* of φ. As is easily seen, this coincides with the length of the spheric representation of φ.

Returning to the general case we define, for every continuous curve φ on I, two quantities $\Theta(\varphi)$ and $\Omega(\varphi)$ as follows. We denote by $\Theta(\varphi)$ the lower limit of the integrated curvature of regular C^2 curves on I tending uniformly to φ, where existence of such curves will be an easy consequence of the Weierstrass approximation theorem; while $\Omega(\varphi)$ means, roughly speaking, the supremum of the sum of the exterior angles at the vertices, of a variable broken line inscribed in the curve φ. It will then turn out that these definitions are equivalent, so that we have $\Theta(\varphi)=\Omega(\varphi)$ for any continuous curve φ whatsoever defined on a closed interval.

Plainly, the above definition of $\Omega(\varphi)$ will remain meaningful even when φ is any parametric curve on an interval I_0, where φ need not be continuous and I_0 need not be a closed interval. We propose to call $\Omega(\varphi)$, in this wider sense, the *bend* of φ on I_0 (or over I_0).

One will observe at once an analogy between surface area theory and ours, if one compares $\Theta(\varphi)$ with the Lebesgue area, and $\Omega(\varphi)$ with the Geöcze area as given by Cesari in his book *Surface Area*. There are further instances of analogy, but we shall content ourselves with quoting only a few of them, as follows. Let φ be a parametric curve defined on an interval I_0. The quantity $\Omega(\varphi)$ is then invariant under Fréchet equivalence of φ, and lower semicontinuous with respect to φ when the interval I_0 is fixed. Also, if we fix φ and denote by $\Omega(\varphi, J)$ the bend of the restriction of φ to a closed interval J in I_0, it will follow that the interval function $\Omega(\varphi, J)$ is overadditive.

We shall now state the chief result of this paper, restricting ourselves for simplicity to the case of a continuous curve φ defined on an open interval K. As usual, we shall call φ to be *light* iff (= if and only if) it is constant on no subintervals of K. Let then φ be light and consider any fixed point c of K. We can clearly extract from K a sequence of points $\langle t_n; n=1,2,\cdots\rangle$ such that $t_n > t_{n+1} > c$ and $p_n=\varphi(t_n)-\varphi(c)\neq 0$ for every n and that $t_n\to c$ as $n\to\infty$. Now write $q_n=|p_n|^{-1}p_n$, so that $\langle q_n\rangle$ is a sequence of unit vectors. A unit vector will be called *right-hand derived direction* of φ at c iff it is the limit of a converging subsequence of one such sequence $\langle q_n\rangle$. Evidently, φ possesses at least one right-hand derived direction at each point c of K.

This being so, let γ be any parametric curve defined on K and such that $\gamma(t)$ is a right-hand derived direction of φ at every t of K. It should be noted that γ is not necessarily continuous. We define the spheric length of γ as the supremum of the length of a variable spheric broken line, inscribed in the curve γ and consisting of a finite number of minor arcs of great circles on the unit sphere.

We now assert that the bend of φ is then equal to the spheric length of γ.

We shall prove this theorem in § 95 in a slightly more general form, in allowing that the curve γ may fail to be defined at the points of a certain countable subset of K. And we shall conclude with an application of this result to the well-known inequality of Fenchel concerning the magnitude of the integrated curvature of closed space curves.

Chapter I. Properties of $\Theta(\varphi)$.

1. We shall denote by $\boldsymbol{R}$ and $\boldsymbol{N}$, respectively, the set of the real numbers and that of the natural numbers. If not explicitly stated otherwise, all the functions and functionals that we shall consider will be either finite real or else nonnegative (inclusive ∞) and all the intervals considered will be situated in $\boldsymbol{R}$.

The length of a (finite or infinite) interval I will be denoted by $|I|$, and I° will stand for the interior of I. Infinite intervals will not be excluded from our considerations unless the contrary is stated explicitly. Also note that, as usual, we shall only apply the terms *open* and *closed* to finite intervals.

We agree once for all that the letters δ and ε, without or with suffix, should mean positive finite numbers throughout the sequel, even when we do not specify so.

2. An approximation theorem. *Let $f_1, \cdots, f_m$ be m continuous functions on a closed interval I, where $m \geqq 2$. Then, for any given ε, there are m polynomials $P_1, \cdots, P_m$ in t such that*

$$|f_i(t) - P_i(t)| < \varepsilon \qquad (t \in I;\ i = 1, \cdots, m)$$

and that the derivatives $P_1'(t), \cdots, P_m'(t)$ never vanish simultaneously on I.

If the functions f_i are further all $\boldsymbol{C}^2$ on I, we may also require that

$$|f_i'(t) - P_i'(t)| < \varepsilon, \quad |f_i''(t) - P_i''(t)| < \varepsilon \qquad (t \in I;\ i = 1, \cdots, m)\,.$$

Proof. By the Weierstrass approximation theorem, a continuous function on a closed interval can be approximated uniformly by real polynomials, which may clearly be assumed to be nonconstant. Thus we can take m nonconstant polynomials $Q_1, \cdots, Q_m$ such that $|f_i(t) - Q_i(t)| < 2^{-1}\varepsilon$ $(i = 1, \cdots, m)$ on I. Consider now the points τ of I at which $Q_2'(\tau) = \cdots = Q_m'(\tau) = 0$. Plainly they are finite in number. Taking δ so small that $2\delta|I| < \varepsilon$ and defining afresh m polynomials P_i by

$$P_1(t) = Q_1(t) + \delta(t-a), \quad P_i(t) = Q_i(t) \qquad (i = 2, \cdots, m)\,,$$

where a is the left-hand extremity of I, we see that $|f_1(t) - P_1(t)| \leqq |f_1(t) - Q_1(t)| + \delta|I| < \varepsilon$ on I. Moreover, $P_1'(\tau) = Q_1'(\tau) + \delta \neq 0$ for all τ as soon as δ is sufficiently small. This proves the first part of the assertion.

To establish the second part, take δ so small that $2l^2\delta<\varepsilon$ where $l=|I|+1$, find m nonconstant polynomials $A_i(t)$ such that $|f_i''(t)-A_i(t)|<\delta$ on I, and consider the polynomials B_i and Q_i $(i=1,\cdots,m)$ defined by

$$B_i(t)=\int_a^t A_i(u)du+f_i'(a), \qquad Q_i(t)=\int_a^t B_i(u)du+f_i(a).$$

Then $Q_i'(t)=B_i(t)$, $Q_i''(t)=A_i(t)$ for every real t, so that the Q_i are nonconstant polynomials. And we find for $t\in I$ successively that

$$|f_i''(t)-Q_i''(t)|<\delta<2^{-1}\varepsilon,$$

$$|f_i'(t)-Q_i'(t)|=\left|\int_a^t\{f_i''(u)-Q_i''(u)\}du\right|<l\delta<2^{-1}\varepsilon,$$

$$|f_i(t)-Q_i(t)|=\left|\int_a^t\{f_i'(u)-Q_i'(u)\}du\right|<l^2\delta<2^{-1}\varepsilon.$$

From now on we may proceed in the same way as in the first part of the proof. To see that the additional inequalities hold good, we need only notice that, for $t\in I$,

$$|f_1'(t)-P_1'(t)|\leqq|f_1'(t)-Q_1'(t)|+\delta<2^{-1}\varepsilon+\delta<\varepsilon,$$

$$|f_1''(t)-P_1''(t)|=|f_1''(t)-Q_1''(t)|<2^{-1}\varepsilon.$$

3. Parametric curves. We shall be concerned with a fixed Euclidean space $\boldsymbol{R}^m$ of dimension $m\geqq 2$, which we interpret as the set of all m-tuples of real numbers. Whenever convenient, we shall regard $\boldsymbol{R}^m$ as a vector space with the usual definitions for addition, subtraction, and the two kinds of multiplication, scalar and inner. In conformity with this, the elements of $\boldsymbol{R}^m$ will be called points or vectors synonymously, according to circumstances.

We define a *parametric curve*, or simply *curve*, in $\boldsymbol{R}^m$ to be a mapping of a nonvoid subset E of $\boldsymbol{R}$ into $\boldsymbol{R}^m$. We shall usually omit all reference to the containing space $\boldsymbol{R}^m$ in case there is no fear of ambiguity. A parametric curve will be called *continuous* or $\boldsymbol{C}$ iff (= if and only if) the mapping is continuous. The domain of definition E will be an interval in most cases. A curve defined on an interval I will be said to be *differentiable* iff its coordinate functions are differentiable on I, to be $\boldsymbol{C}^n$ for an $n\in\boldsymbol{N}$ iff they are $\boldsymbol{C}^n$ functions on I, and to be a *polynomial curve* or $\boldsymbol{C}^P$ iff they are polynomials in $t\in I$. The symbols $\boldsymbol{C}$, $\boldsymbol{C}^n$, $\boldsymbol{C}^P$ will also be used to denote the respective classes of the relevant curves. When precision is desired, we may write $\boldsymbol{C}(E)$, $\boldsymbol{C}^n(I)$, $\boldsymbol{C}^P(I)$ respectively. A differentiable curve φ on I will be termed *regular* iff $\varphi'(t)$ never vanishes on I, and then the curve $\hat{\varphi}$ on I determined by $\hat{\varphi}(t)=|\varphi'(t)|^{-1}\varphi'(t)$ will be called *spheric representation* of φ. Finally, the restriction of a curve φ defined on a set $E\subset\boldsymbol{R}$ to a nonvoid subset E' of E will be called *subcurve* of φ and denoted by (φ, E').

4. Distance between two curves. We shall understand by the *distance* $\rho(\varphi, \psi)$ between two curves φ and ψ defined on a set $E \subset \boldsymbol{R}$ the supremum of $|\varphi(t)-\psi(t)|$ for $t \in E$. Of course this need not be finite. A sequence $\langle \varphi_n ; n \in \boldsymbol{N} \rangle$ of curves defined on a common set of real numbers will as usual be termed to *converge uniformly* to a curve φ defined on the same set iff $\rho(\varphi, \varphi_n) \to 0$ as $n \to \infty$. We shall express this by writing $\varphi_n \rightrightarrows \varphi$ in contradistinction to $\varphi_n \to \varphi$, the latter meaning pointwise convergence.

5. Suppose that I is a closed interval. The length of a parametric curve φ on I will be denoted by $\mathrm{L}(\varphi)$. Every $\boldsymbol{C}^1$ curve φ on I is rectifiable and we have the length formula $\mathrm{L}(\varphi)=\int_I |\varphi'(t)|\,dt$. If φ is a regular $\boldsymbol{C}^2$ curve on I, then $\hat{\varphi}$ is $\boldsymbol{C}^1$, and direct calculation will show that $|\hat{\varphi}'(t)| = |\varphi'(t) \times \varphi''(t)| \cdot |\varphi'(t)|^{-2}$ and hence that

$$\mathrm{L}(\hat{\varphi}) = \int_I |\varphi'(t) \times \varphi''(t)| \cdot |\varphi'(t)|^{-2} dt ,$$

where we mean, for any pair p, q of vectors in $\boldsymbol{R}^m$, by the symbol $|p \times q|$ the nonnegative quantity $\sqrt{p^2q^2-(pq)^2}$. Needless to say, we may, in case $\boldsymbol{R}^m$ is 3-dimensional, interpret $|p \times q|$ as the actual magnitude of the vector product $p \times q$.

6. Definition of $\Theta(\varphi)$. Let φ be a continuous curve on a closed interval I and let us consider the sequences $\Gamma = \langle \varphi_n ; n \in \boldsymbol{N} \rangle$ of regular $\boldsymbol{C}^2$ curves φ_n defined on I and converging uniformly to φ, where existence of such sequences is a direct consequence of the approximation theorem given in § 2. Let us now define

$$\Theta(\varphi) = \inf_{\Gamma} \varliminf_{n} \mathrm{L}(\hat{\varphi}_n) ,$$

so that we always have $0 \leqq \Theta(\varphi) \leqq \infty$.

For closed intervals J in I we shall denote by $\Theta(\varphi, J)$ the Θ of the subcurve (φ, J). Thus, in particular, $\Theta(\varphi, I) = \Theta(\varphi)$. Similarly, we shall understand by $\mathrm{L}(\varphi, J)$ the length of (φ, J), so that $\mathrm{L}(\varphi, I) = \mathrm{L}(\varphi)$. We shall sometimes use the symbols $\Theta(\varphi, J)$ and $\mathrm{L}(\varphi, J)$ also in the more general case in which the continuous curve φ under consideration is defined on an interval which is not a closed one.

7. *Given a regular $\boldsymbol{C}^2$ curve φ on a closed interval I and given an arbitrary ε, there always exists a regular polynomial curve Π on I such that all of*

$$\rho(\varphi, \Pi), \quad \rho(\varphi', \Pi'), \quad \rho(\varphi'', \Pi''), \quad |\mathrm{L}(\hat{\varphi}) - \mathrm{L}(\hat{\Pi})|$$

are less than ε.

PROOF. In virtue of the approximation theorem of § 2 we can, for any positive number $\eta<\varepsilon$, find a regular polynomial curve Π on I such that all of $\rho(\varphi, \Pi)$, $\rho(\varphi', \Pi')$, $\rho(\varphi'', \Pi'')$ are $<\eta$. Inspection of the formula for $\mathrm{L}(\hat{\varphi})$ given in § 5 then shows that Π will satisfy $|\mathrm{L}(\hat{\varphi})-\mathrm{L}(\hat{\Pi})|<\varepsilon$ as well, so soon as η is sufficiently small.

8. In the definition of $\Theta(\varphi)$ we considered the sequences $\langle\varphi_n\rangle$ of regular $\boldsymbol{C}^2$ curves on I, such that $\varphi_n \rightrightarrows \varphi$. Now let us replace there the class $\boldsymbol{C}^2(I)$ by $\boldsymbol{C}^k(I)$ $(k\in\boldsymbol{N})$ or $\boldsymbol{C}^P(I)$ and let us denote, respectively by $\Theta_k(\varphi)$ or $\Theta_P(\varphi)$, the functionals that will then result in place of $\Theta(\varphi)$. *We shall then have* $\Theta_k(\varphi)=\Theta_P(\varphi)=\Theta(\varphi)$, *provided that* $k\geqq 2$.

REMARK. This will still hold for $k=1$ as we shall see later on (§ 20), but then the proof will not be so simple as here below.

PROOF. From the obvious inclusions $\boldsymbol{C}^2\supset\boldsymbol{C}^k\supset\boldsymbol{C}^P$ we derive at once $\Theta(\varphi)\leqq\Theta_k(\varphi)\leqq\Theta_P(\varphi)$. It therefore suffices to show that $\Theta(\varphi)\geqq\Theta_P(\varphi)$, where we may clearly suppose $\Theta(\varphi)$ finite. Given any real $A>\Theta(\varphi)$ we can, by definition of $\Theta(\varphi)$, take a sequence $\langle\varphi_n; n\in\boldsymbol{N}\rangle$ of regular $\boldsymbol{C}^2$ curves on I such that $\varphi_n\rightrightarrows\varphi$ and that $\underline{\lim}_n \mathrm{L}(\hat{\varphi}_n)<A$. It then follows from the preceding § that there exists a sequence $\langle\Pi_n; n\in\boldsymbol{N}\rangle$ of regular polynomial curves on I which fulfils both $\rho(\varphi_n, \Pi_n)<n^{-1}$ and $|\mathrm{L}(\hat{\varphi}_n)-\mathrm{L}(\hat{\Pi}_n)|<n^{-1}$ for all n. Then $\Pi_n\rightrightarrows\varphi$, and consequently

$$\Theta_P(\varphi)\leqq\underline{\lim}_n \mathrm{L}(\hat{\Pi}_n)=\underline{\lim}_n \mathrm{L}(\hat{\varphi}_n)<A\,.$$

This completes the proof.

9. Characterization of $\Theta(\varphi)$. *Given a continuous curve φ on a closed interval I, write for short $\Theta_0=\Theta(\varphi)$ and let A be an arbitrary real number. Then,*

(i) *If $A>\Theta_0$, there exists for any δ a regular polynomial curve Π on I such that $\rho(\varphi, \Pi)<\delta$ and that $\mathrm{L}(\hat{\Pi})<A$; and further,*

(ii) *If $A<\Theta_0$, then there exists a δ_0 such that we have $\mathrm{L}(\hat{\psi})>A$ for every regular $\boldsymbol{C}^2$ curve ψ on I for which $\rho(\varphi, \psi)<\delta_0$.*

Moreover, Θ_0 is uniquely determined, in the interval $[0, \infty]$, by these two properties.

PROOF. We may begin with (ii), since (i) is an easy consequence of the preceding §. If (ii) were false, there would exist a sequence $\langle\psi_n; n\in\boldsymbol{N}\rangle$ of regular $\boldsymbol{C}^2$ curves on I such that $\psi_n\rightrightarrows\varphi$ and that $\mathrm{L}(\hat{\psi}_n)\leqq A$ for every n. We should then have $\underline{\lim}_n \mathrm{L}(\hat{\psi}_n)\leqq A<\Theta_0$, which is evidently incompatible with the definition of Θ_0. This proves (ii).

Moreover, if each of two distinct values Θ_1 and Θ_2 belonging to $[0, \infty]$ possessed the same two properties as Θ_0 does in (i) and (ii), there would at once arise a contradiction on taking A between Θ_1 and Θ_2. This completes the proof.

10. *If φ is a continuous curve on a closed interval I, there is a sequence $\langle \Pi_n ; n \in \boldsymbol{N} \rangle$ of regular polynomial curves on I such that $\Pi_n \rightrightarrows \varphi$ and $\mathrm{L}(\hat{\Pi}_n) \to \Theta(\varphi)$, as $n \to \infty$.*

Proof. On account of (i) of the preceding §, there exists a sequence $\langle \Pi_n ; n \in \boldsymbol{N} \rangle$ of regular polynomial curves on I such that $\Pi_n \rightrightarrows \varphi$ and that $\mathrm{L}(\hat{\Pi}_n) < \Theta(\varphi) + n^{-1}$ for every n, where possibility of $\Theta(\varphi) = \infty$ is not excluded from consideration. We then see at once, by definition of $\Theta(\varphi)$, that

$$\Theta(\varphi) \leqq \varliminf_n \mathrm{L}(\hat{\Pi}_n) \leqq \varlimsup_n \mathrm{L}(\hat{\Pi}_n) \leqq \Theta(\varphi),$$

and hence that $\Theta(\varphi) = \lim_n \mathrm{L}(\hat{\Pi}_n)$. This completes the proof.

11. Lower semicontinuity of $\Theta(\varphi)$, by which we mean the following. *If φ_n $(n \in \boldsymbol{N})$ and φ are continuous curves on a closed interval I such that $\varphi_n \rightrightarrows \varphi$, then $\Theta(\varphi) \leqq \varliminf_n \Theta(\varphi_n)$.*

Proof. Writing $A = \varliminf_n \Theta(\varphi_n)$ for short, we may clearly suppose A finite. There exists a subsequence $\langle \psi_n ; n \in \boldsymbol{N} \rangle$ of the sequence $\langle \varphi_n \rangle$ such that $\rho(\varphi, \psi_n) < n^{-1}$ and $\Theta(\psi_n) < A + n^{-1}$ for every n. But then, with the help of (i) of § 9, we can make correspond to each $n \in \boldsymbol{N}$ a regular polynomial curve Π_n on I subject to the conditions $\rho(\psi_n, \Pi_n) < n^{-1}$ and $\mathrm{L}(\hat{\Pi}_n) < A + n^{-1}$. So that $\rho(\varphi, \Pi_n) \leqq \rho(\varphi, \psi_n) + \rho(\psi_n, \Pi_n) < 2n^{-1}$ for every n, and hence $\Pi_n \rightrightarrows \varphi$. Therefore $\Theta(\varphi) \leqq \varliminf_n \mathrm{L}(\hat{\Pi}_n) \leqq A$, which clearly completes the proof.

12. Overadditivity of interval functions. Let I be a given interval and let us denote by J a typical closed interval contained in I. Suppose that $F(J)$ is a nonnegative interval function defined on the class of all J. Note that F need not be a finite function, and also that we do not assume I a closed interval.

We shall as usual call F *overadditive* on I iff we have $F(J_1) + \cdots + F(J_n) \leqq F(J)$ for any J whenever $J_1, \cdots, J_n$ are a finite number of non-overlapping closed intervals contained in J. It may be mentioned that if F is overadditive, then F is nondecreasing, i. e. $F(J') \leqq F(J)$ whenever J' and J are closed intervals in I such that $J' \subset J$.

13. Overadditivity of $\Theta(\varphi, J)$. *If φ is a fixed continuous curve on an* *interval I, then $\Theta(\varphi, J)$ is an overadditive interval function for closed intervals J in I.*

Proof. Given a finite number of non-overlapping closed intervals $J_1, \cdots, J_n$ in J, suppose that $\langle \varphi_k ; k \in \boldsymbol{N} \rangle$ is an arbitrary sequence of regular $\boldsymbol{C}^2$ curves defined on J, such that $\varphi_k \rightrightarrows (\varphi, J)$. Then

$$\sum_{i=1}^{n} \Theta(\varphi, J_i) \leqq \sum_{i=1}^{n} \varliminf_{k} \mathrm{L}(\hat{\varphi}_k, J_i) \leqq \varliminf_{k} \sum_{i=1}^{n} \mathrm{L}(\hat{\varphi}_k, J_i) \leqq \varliminf_{k} \mathrm{L}(\hat{\varphi}_k, J),$$

the last step being easily effected by § 5, and hence $\sum_{i=1}^{n} \Theta(\varphi, J_i) \leqq \Theta(\varphi, J)$.

14. Supposing that Δ is a finite set of real numbers, we shall define $\|\Delta\|$ as follows. If Δ is degenerate, i.e. consists of at most one point, we set $\|\Delta\| = 0$. Otherwise we write $\Delta = \{t_0, \cdots, t_n\}$, where $t_0 < \cdots < t_n$, and denote by $\|\Delta\|$ the maximum of the numbers $t_i - t_{i-1}$ for $i = 1, 2, \cdots, n$. Further, in this latter case, the closed intervals $[t_{i-1}, t_i]$ are called to be *pertaining* to Δ.

15. Quasilinear curves. Let I be a closed interval. By a *subdivision* of I we shall mean any finite subset of I containing the endpoints of I. Suppose that Δ is a subdivision of I and write $\Delta = \{t_0, t_1, \cdots, t_n\}$, where $t_0 < t_1 < \cdots < t_n$. A parametric curve φ on I will be termed *quasilinear with reference to* Δ iff it is linear on every pertaining interval $J_i = [t_{i-1}, t_i]$ where $i = 1, 2, \cdots, n$. In view of additivity of length, we then clearly have

$$\mathrm{L}(\varphi) = \sum_{i=1}^{n} \mathrm{L}(\varphi, J_i) = \sum_{i=1}^{n} |\varphi(t_i) - \varphi(t_{i-1})|.$$

Further, φ will simply be called *quasilinear* iff there exists a subdivision Δ of I with respect to which φ is quasilinear. In this latter case we shall call any such Δ *typical subdivision* of I for the quasilinear curve φ, or else *subdivision of quasilinearity* of φ.

16. *Given a continuous curve ψ on a closed interval I, there exists a sequence $\langle \psi_n ; n \in \boldsymbol{N} \rangle$ of $\boldsymbol{C}^1$ curves on I, such that $\psi_n \rightrightarrows \psi$ and that $\mathrm{L}(\psi_n) \to \mathrm{L}(\psi)$.*

Proof. Write $I = [a, b]$ and consider any subdivision $\Delta = \{t_0, \cdots, t_k\}$ of I such that $a = t_0 < t_1 < \cdots < t_k = b$, where $k \geqq 2$. We construct a curve φ on I, which is quasilinear with respect to Δ, by setting $\varphi(t_i) = \psi(t_i)$ for each $i = 0, 1, \cdots, k$. Then $p_i = \varphi'(t)$ is constant on each open interval (t_{i-1}, t_i) for $i = 1, 2, \cdots, k$. Furthermore, it follows easily from continuity of ψ and the equality $\mathrm{L}(\varphi) = \sum_{i=1}^{k} |\varphi(t_i) - \varphi(t_{i-1})|$ of the preceding § that, for any given ε, we can choose the above subdivision Δ in such a manner that $\rho(\psi, \varphi) < \varepsilon$ and that $|\mathrm{L}(\varphi) - \mathrm{L}(\psi)| < \varepsilon$ or $\mathrm{L}(\varphi) > \varepsilon^{-1}$ according as ψ is rectifiable or not.

We consider now the subdivision of I given by

$$\Delta_\delta = \{a, t_1 - \delta, t_1 + \delta, \cdots, t_{k-1} - \delta, t_{k-1} + \delta, b\},$$

where $2\delta < \|\Delta\|$, and construct a curve χ on I, which is quasilinear with respect to Δ_δ, by setting

$$\chi(a) = p_1, \ \chi(b) = p_k, \ \chi(t_i - \delta) = p_i, \ \chi(t_i + \delta) = p_{i+1},$$

where $i = 1, 2, \cdots, k-1$. Putting $\omega(t) = \int_a^t \chi(\tau)d\tau + \varphi(a)$ for $t \in I$, we shall show that we can make $\rho(\varphi, \omega)$ and $|\mathrm{L}(\varphi) - \mathrm{L}(\omega)|$ as small as we please by taking δ small, provided that the subdivision Δ, and hence the curve φ also, is kept fixed. Clearly this will complete the proof.

Write $M = \mathrm{Max}\,|p_i|$ $(i = 1, 2, \cdots, k)$, so that $|\chi(t)| \leqq M$ everywhere on I. Then, denoting by $\varphi'(t)$ the derivative of φ where it exists, and zero where it does not, we see at once that φ' is a curve with measurable coordinate functions, that $|\varphi'(t)| \leqq M$ everywhere on I, and further that $\varphi'(t) = \chi(t)$ on I except at the points of the intervals $[t_i - \delta, t_i + \delta]$, $i = 1, 2, \cdots, k-1$. Hence we find for $t \in I$ that, as required,

$$|\varphi(t) - \omega(t)| = \left| \int_a^t [\varphi'(\tau) - \chi(\tau)] d\tau \right| \leqq 4Mk\delta,$$

$$|\mathrm{L}(\varphi) - \mathrm{L}(\omega)| = \left| \int_I [|\varphi'(\tau)| - |\chi(\tau)|] d\tau \right| \leqq 4Mk\delta.$$

17. *Given a nonvanishing curve ψ on a closed interval I, denote by α the infimum of $|\psi(t)|$ for $t \in I$. Then $\mathrm{L}(\psi) \geqq \alpha \mathrm{L}(\psi_0)$, where ψ_0 is a curve defined on I by $\psi_0(t) = |\psi(t)|^{-1}\psi(t)$.*

Proof. We shall begin by showing that if p, q are any pair of nonvanishing points of $\boldsymbol{R}^m$ such that $|p| \geqq \alpha$ and $|q| \geqq \alpha$, then $|p - q| \geqq \alpha |p_0 - q_0|$, where we write $p_0 = |p|^{-1}p$ and $q_0 = |q|^{-1}q$. In fact,

$$|p - q|^2 = (|p|p_0 - |q|q_0)^2 = |p|^2 + |q|^2 - 2|p| \cdot |q| (p_0 q_0)$$
$$= (|p| - |q|)^2 + |p| \cdot |q| (2 - 2p_0 q_0) \geqq \alpha^2 |p_0 - q_0|^2.$$

This being so, we shall show that $\mathrm{L}(\psi) \geqq \alpha\eta$ whenever η is a real number $< \mathrm{L}(\psi_0)$. For this purpose, we may clearly assume that $\eta > 0$. There then exists a subdivision $\{t_0, t_1, \cdots, t_n\}$ of I such that $t_0 < t_1 < \cdots < t_n$ and that $\sum_{i=1}^n |\psi_0(t_i) - \psi_0(t_{i-1})| > \eta$. But we have, by what has been said above, $|\psi(t_i) - \psi(t_{i-1})| \geqq \alpha |\psi_0(t_i) - \psi_0(t_{i-1})|$ for $i = 1, \cdots, n$. Consequently, as announced above,

$$\mathrm{L}(\psi) \geqq \sum_{i=1}^n |\psi(t_i) - \psi(t_{i-1})| \geqq \alpha\eta.$$

Since $\eta < \mathrm{L}(\psi_0)$ is arbitrary, this leads at once to $\mathrm{L}(\psi) \geqq \alpha \mathrm{L}(\psi_0)$, as required.

REMARK. In case $L(\psi_0)=\infty$ we follow the usual convention concerning product. So that $\alpha\infty$ means ∞ or 0 according as $\alpha>0$ or $\alpha=0$ respectively.

18. Spheric curves. We shall call a curve γ on a set $E\subset \boldsymbol{R}$ to be *spheric* iff $|\gamma(t)|=1$ for every $t\in E$, or what comes to the same thing, iff the locus $\gamma[E]$ of the curve γ is a subset of the unit sphere.

Given a continuous spheric curve γ on a closed interval I, there always exists a sequence $\langle\gamma_n; n\in \boldsymbol{N}\rangle$ of spheric $\boldsymbol{C}^1$ curves on I such that $\gamma_n\rightrightarrows\gamma$ and that $L(\gamma_n)\to L(\gamma)$.

PROOF. In accordance with § 16 we can choose a sequence $\langle\psi_n; n\in \boldsymbol{N}\rangle$ of $\boldsymbol{C}^1$ curves on I such that $\rho(\gamma, \psi_n)<n^{-1}$ for every n and that $L(\psi_n)\to L(\gamma)$. Then every ψ_n is nonvanishing throughout I and so we can associate with each $n\in \boldsymbol{N}$ a spheric curve γ_n defined on I by $\gamma_n(t)=|\psi_n(t)|^{-1}\psi_n(t)$. Thus defined, every γ_n is obviously $\boldsymbol{C}^1$ on I, and it is easy to see that $\gamma_n\rightrightarrows\gamma$. It follows at once from lower semicontinuity of length (cf. III. 3.6 of Radó, *Length and Area*) that $\underline{\lim}_n L(\gamma_n)\geqq L(\gamma)$. But the foregoing section gives $(1-n^{-1})L(\gamma_n)\leqq L(\psi_n)$ for every $n\in \boldsymbol{N}$, whence we derive immediately $\overline{\lim}_n L(\gamma_n)\leqq \overline{\lim}_n L(\psi_n)=L(\gamma)$. We thus obtain $\lim_n L(\gamma_n)=L(\gamma)$, and this completes the proof.

19. *Given a regular $\boldsymbol{C}^1$ curve φ on a closed interval I, there exists a sequence $\langle\varphi_n; n\in \boldsymbol{N}\rangle$ of regular $\boldsymbol{C}^2$ curves on I such that $\varphi_n\rightrightarrows\varphi$, $L(\varphi_n)\to L(\varphi)$, $\hat{\varphi}_n\rightrightarrows\hat{\varphi}$,* $L(\hat{\varphi}_n)\to L(\hat{\varphi})$.

PROOF. Write $\gamma=\hat{\varphi}$ and let $\langle\gamma_n\rangle$ be the sequence attached to the spheric continuous curve γ by the preceding proposition. Since $|\varphi'(t)|$ is positive and continuous on I, there is a sequence $\langle r_n; n\in \boldsymbol{N}\rangle$ of positive $\boldsymbol{C}^1$ functions on I such that $r_n(t)\rightrightarrows|\varphi'(t)|$. We now associate with each $n\in \boldsymbol{N}$ a curve φ_n defined on I by

$$\varphi_n(t)=\int_a^t r_n(\tau)\gamma_n(\tau)d\tau+\varphi(a),$$

where a denotes the left-hand endpoint of I. We then see at once that every φ_n, thus defined, is a regular $\boldsymbol{C}^2$ curve on I fulfilling both $\varphi_n'(t)=r_n(t)\gamma_n(t)$ and $|\varphi_n'(t)|=r_n(t)>0$ everywhere on I. It follows that $\hat{\varphi}_n$ coincides with γ_n for every n, and hence that $\hat{\varphi}_n\rightrightarrows\hat{\varphi}$ and $L(\hat{\varphi}_n)\to L(\hat{\varphi})$. Furthermore,

$$L(\varphi_n)=\int_I r_n(\tau)d\tau\to\int_I|\varphi'(\tau)|d\tau=L(\varphi).$$

Finally, for every n and every $t\in I$,

$$\varphi_n(t)-\varphi(t)=\int_a^t\{r_n(\tau)\gamma_n(\tau)-|\varphi'(\tau)|\gamma(\tau)\}d\tau,$$

where $r_n \rightrightarrows |\varphi'|$ and $\gamma_n \rightrightarrows \gamma$, and hence $r_n \gamma_n \rightrightarrows |\varphi'| \gamma$. Therefore $\varphi_n \rightrightarrows \varphi$, and this completes the proof.

20. We end this chapter with a result which will be of importance later on (§ 50 and § 54).

THEOREM. *Given a continuous curve φ on a closed interval I, let $\Theta_1(\varphi)$ be defined as in* § 8. *Then $\Theta_1(\varphi) = \Theta(\varphi)$.*

PROOF. We need only show that $\Theta(\varphi) \leqq \Theta_1(\varphi)$, the opposite inequality being obvious. We may clearly assume $\Theta_1(\varphi)$ to be finite. By definition of $\Theta_1(\varphi)$ there then exists a sequence $\langle \varphi_n ; n \in \boldsymbol{N} \rangle$ of regular $\boldsymbol{C}^1$ curves on I such that $\varphi_n \rightrightarrows \varphi$ and that $\mathrm{L}(\hat{\varphi}_n) < \Theta_1(\varphi) + n^{-1}$ for every n. But the preceding § gives for each n a regular $\boldsymbol{C}^2$ curve ψ_n on I satisfying both $\rho(\varphi_n, \psi_n) < n^{-1}$ and $|\mathrm{L}(\hat{\varphi}_n) - \mathrm{L}(\hat{\psi}_n)| < n^{-1}$. Thus $\psi_n \rightrightarrows \varphi$, as well as $\mathrm{L}(\hat{\psi}_n) < \Theta_1(\varphi) + 2n^{-1}$ for every n. Hence $\Theta(\varphi) \leqq \underline{\lim}_n \mathrm{L}(\hat{\psi}_n) \leqq \Theta_1(\varphi)$, as required.

Chapter II. The identity $\Theta(\varphi) = \Omega(\varphi)$.

21. Definition of angle. By the *angle* $x \diamond y$ between two nonvanishing vectors x, y of $\boldsymbol{R}^m$ we shall as usual understand $\mathrm{Cos}^{-1}(x_0 y_0)$, where $x_0 = |x|^{-1}x$, $y_0 = |y|^{-1}y$ and where Cos^{-1} means the principal value of the inverse cosine belonging to the interval $[0, \pi]$.

Clearly $x \diamond y = y \diamond x$, and $(\lambda x) \diamond (\mu y) = x \diamond y$ whenever λ, μ are a pair of real numbers with $\lambda\mu > 0$. Again, we have $(Ux) \diamond (Uy) = x \diamond y$ for any orthogonal transformation U of the vector space $\boldsymbol{R}^m$, and further the angle $x \diamond y$ is a continuous function of the combined variable $\langle x, y \rangle$, where $x \neq 0$ and $y \neq 0$.

It should be borne in mind that, as mentioned already in § 3, we always suppose the Euclidean space $\boldsymbol{R}^m$ with which we are concerned to be at least 2-dimensional.

22. Triangular inequality. Although this, as well as the triangular equality of the next §, is well known and in fact rather trivial, we wish to provide them with rigorous proofs, inasmuch as they will be fundamentally important for the sequel.

We have $x \diamond z + y \diamond z \geqq x \diamond y$ for any triple x, y, z of nonvanishing vectors of the space $\boldsymbol{R}^m$.

PROOF. Without loss of generality we may assume that $|x| = |y| = |z| = 1$ and further, by applying a suitable orthogonal transformation if necessary, that $z = \langle 1, 0, \cdots, 0 \rangle$. Writing $x = \langle x_1, x_2, \cdots, x_m \rangle$, $y = \langle y_1, y_2, \cdots, y_m \rangle$, $\alpha = x \diamond z$, and $\beta = y \diamond z$, we find at once that $\cos\alpha = x_1$, $\sin\alpha = \sqrt{x_2^2 + \cdots + x_m^2}$, $\cos\beta = y_1$, $\sin\beta = \sqrt{y_2^2 + \cdots + y_m^2}$, and further, on account of the Schwarz inequality, that

$$\cos(\alpha+\beta)=x_1y_1-\sqrt{x_2^2+\cdots+x_m^2}\ \sqrt{y_2^2+\cdots+y_m^2}$$
$$\leqq x_1y_1+x_2y_2+\cdots+x_my_m=\cos(x\diamond y)\,.$$

Hence $\alpha+\beta\geqq x\diamond y$ provided that $\alpha+\beta\leqq\pi$. But if $\alpha+\beta>\pi$, then trivially $\alpha+\beta>x\diamond y$, and this completes the proof.

23. Triangular equality. *If $x+y=z$ in the above, then $x\diamond z+y\diamond z=x\diamond y$.*

PROOF. We may assume that $x=\langle 1,0,\cdots,0\rangle$, $y=\langle y_1,y_2,0,\cdots,0\rangle$, and $z=\langle z_1,z_2,0,\cdots,0\rangle$, so that $z_1=y_1+1$ and $z_2=y_2$. For if not, we have only to multiply in the first place x,y,z by $|x|^{-1}$ and then to apply a suitable orthogonal transformation of vectors. Writing $\alpha=x\diamond z$, $\beta=y\diamond z$, and $\gamma=x\diamond y$ to shorten our notations, we find readily that $\cos\alpha=|z|^{-1}z_1$, $\sin\alpha=|z|^{-1}|z_2|$, $\cos\gamma=|y|^{-1}y_1$, $\sin\gamma=|y|^{-1}|y_2|$. But we have here $\cos\alpha\geqq\cos\gamma$, or equivalently, $|y|z_1\geqq|z|y_1$. In point of fact, this is evident when $y_1<0<z_1$, and otherwise it follows from the simple fact that

$$z_1^2(y_1^2+y_2^2)-y_1^2(z_1^2+z_2^2)\begin{cases}\geqq 0 & \text{when } y_1\geqq 0\,,\\ \leqq 0 & \text{when } z_1\leqq 0\,.\end{cases}$$

Thus $0\leqq\gamma-\alpha\leqq\pi$, and this, combined with

$$\cos(\gamma-\alpha)=|y|^{-1}|z|^{-1}(y_1z_1+|y_2|\cdot|z_2|)=|y|^{-1}|z|^{-1}(yz)=\cos\beta\,,$$

leads at once to the desired equality.

24. *Suppose given a pair x,y of vectors of $\boldsymbol{R}^m$.*

(i) *If $|x|=|y|=1$, then*

$$x\diamond y-(x\diamond y)^2\leqq|x-y|\leqq x\diamond y\leqq 2|x-y|\,.$$

PROOF. Writing $x\diamond y=2\theta$ for short, we get $|x-y|^2=2-2\cos 2\theta=4\sin^2\theta$, whence it follows in view of $0\leqq\theta\leqq 2^{-1}\pi$ that $|x-y|=2\sin\theta\leqq 2\theta\leqq 4\sin\theta$, or equivalently, that $|x-y|\leqq x\diamond y\leqq 2|x-y|$. If $\theta<1$ here, we find further that

$$|x-y|=2\sin\theta\geqq 2(\theta-6^{-1}\theta^3)\geqq 2\theta(1-2\theta)\,.$$

But if $\theta\geqq 1$, then trivially $|x-y|>2\theta(1-2\theta)$. We thus have always $|x-y|\geqq x\diamond y-(x\diamond y)^2$.

(ii) *If $|x|>|y|$, then $x\diamond(x+y)<2^{-1}\pi$.* This follows at once from $x(x+y)>|x|\cdot|y|+xy\geqq 0$.

(iii) *If $x,y\neq 0$ and if $x\diamond y\leqq 2^{-1}\pi$, then $x\diamond y=\mathrm{Sin}^{-1}(|x|^{-1}|y|^{-1}|x\times y|)$, where we use the symbol $|x\times y|$ in the sense explained already in* § 5 *and where* Sin^{-1} *represents the principal value of the inverse sine, contained in the interval* $[-2^{-1}\pi,2^{-1}\pi]$. This is evident since, by definition, $|x\times y|=\sqrt{x^2y^2-(xy)^2}$.

25. *Given a sequence* $\langle p_0, p_1, \cdots, p_n\rangle$ $(n \in \boldsymbol{N})$ *of* $n+1$ *nonvanishing vectors of* $\boldsymbol{R}^m$, *let us write* $\omega = \sum_{i=1}^{n} p_{i-1} \diamond p_i$. *Replace now any one of the* p_i, *say* p_k, *by a pair of nonvanishing vectors* p', p'' *such that* $p_k = p' + p''$, *and denote by* $\bar{\omega}$ *the sum constructed from the new sequence* $\langle p_0, \cdots, p_{k-1}, p', p'', p_{k+1}, \cdots, p_n\rangle$ *in precisely the same way as* ω *from the original sequence. Then* $\bar{\omega} \geqq \omega$.

Proof. We may clearly assume that $n \leqq 2$. Considering firstly the case $n=1$, $k=0$, we find on account of §§ 22–23 that

$$\begin{aligned}\bar{\omega} &= p' \diamond p'' + p'' \diamond p_1 = p' \diamond p_0 + p'' \diamond p_0 + p'' \diamond p_1 \\ &\geqq p' \diamond p_0 + p_0 \diamond p_1 \geqq p_0 \diamond p_1 = \omega\,.\end{aligned}$$

Next, the case $n=k=1$ is essentially the same as the first case. It remains to discuss the case $n=2$. Here we suppose that $k=1$, as we plainly may, and obtain, again by §§ 22–23,

$$\begin{aligned}\bar{\omega} &= p_0 \diamond p' + p' \diamond p'' + p'' \diamond p_2 = p_0 \diamond p' + p' \diamond p_1 + p'' \diamond p_1 + p'' \diamond p_2 \\ &\geqq p_0 \diamond p_1 + p_1 \diamond p_2 = \omega\,.\end{aligned}$$

26. Continuation. *Supposing that* $\langle q_0, q_1, \cdots, q_s\rangle$ $(s \in \boldsymbol{N})$ *is a subsequence of the sequence* $\langle p_0, p_1, \cdots, p_n\rangle$ *considered in the above, let us write* $\omega' = \sum_{j=1}^{s} q_{j-1} \diamond q_j$. *Then* $\omega' \leqq \omega$.

Proof. We need only treat the case in which $n=2$, $s=1$, $q_0 = p_0$, $q_1 = p_2$. Then § 22 gives at once

$$\omega = p_0 \diamond p_1 + p_1 \diamond p_2 \geqq p_0 \diamond p_2 = \omega'\,.$$

27. Definition of $\Omega(\varphi, \varDelta)$. Given a parametric curve φ on a set $E \subset \boldsymbol{R}$ and given a finite subset $\varDelta$ of E, we define the quantity $\Omega(\varphi, \varDelta)$ as follows. If $\varDelta$ is degenerate, that is, contains at most one point, we set simply $\Omega(\varphi, \varDelta) = 0$. Otherwise we write $\varDelta = \{t_0, t_1, \cdots, t_n\}$, where $t_0 < t_1 < \cdots < t_n$, and consider the vectors $p_i = \varphi(t_i) - \varphi(t_{i-1})$ for $i = 1, 2, \cdots, n$. If the nonvanishing terms of the sequence $\langle p_i\rangle$ are at most one in number, we put $\Omega(\varphi, \varDelta) = 0$ as before. If they are at least two, we arrange them in a subsequence $\langle q_0, q_1, \cdots, q_s\rangle$ of $\langle p_i\rangle$ and define $\Omega(\varphi, \varDelta) = \sum_{j=1}^{s} q_{j-1} \diamond q_j$.

We remark that, thus defined, $\Omega(\varphi, \varDelta)$ is monotone nondecreasing with respect to the set $\varDelta$, provided the curve φ is kept fixed. Indeed $\varDelta' \subset \varDelta$ implies that $\Omega(\varphi, \varDelta') \leqq \Omega(\varphi, \varDelta)$, as we see easily by §§ 25–26.

28. Definition of bend. Given a parametric curve φ on a set $E \subset \boldsymbol{R}$, let A be any subset of E. We shall call *bend of* φ *on* A and denote by $\Omega(\varphi, A)$, the supremum of $\Omega(\varphi, \varDelta)$ where $\varDelta$ represents the finite subsets of

A. $\Omega(\varphi, E)$ will simply be termed *bend* of φ and we shall usually write for short $\Omega(\varphi)$ for it. Thus $\Omega(\varphi, A)$ is the bend of the subcurve (φ, A), when A is nonvoid. It should be noted that we have neither assumed continuity of φ nor that the set E is an interval, in the foregoing. The curve φ will be called *of bounded bend* iff $\Omega(\varphi) < \infty$.

29. CONTINUATION. A few remarks are relevant here to the preceding definition of bend. Firstly, we clearly have $0 \leqq \Omega(\varphi, A) \leqq \infty$ for every subset A of E. Secondly, the notation $\Omega(\varphi, A)$ is legitimate since, when A is a finite subset of E, the quantity $\Omega(\varphi, A)$ determined according to the preceding definition coincides with $\Omega(\varphi, A)$ as defined in §27, in virtue of the remark given there. Thirdly, $\Omega(\varphi, A)$ is nondecreasing with respect to the set A, provided the curve φ is kept fixed. So that, fourthly and finally, if E is an interval, then $\Omega(\varphi, E)$ equals the supremum of $\Omega(\varphi, J)$ where J represents the closed intervals contained in E.

30. Lower semicontinuity of bend, by which we mean the following. *If $\varphi_n (n \in \boldsymbol{N})$ and φ are parametric curves defined on a set $E \subset \boldsymbol{R}$ and such that $\varphi_n \to \varphi$, i.e. that $\varphi_n(t) \to \varphi(t)$ for each $t \in E$, then $\Omega(\varphi) \leqq \underline{\lim}_u \Omega(\varphi_n)$.*

PROOF. Given an arbitrary real number $\eta < \Omega(\varphi)$, we can find a finite set $\varDelta \subset E$ for which $\Omega(\varphi, \varDelta) > \eta$, and we easily see by continuity of angle (§21) that

$$\underline{\lim}_n \Omega(\varphi_n) \geqq \underline{\lim}_n \Omega(\varphi_n, \varDelta) > \eta ,$$

whence the required result follows at once.

REMARK. It should be noted that the convergence of the sequence $\langle \varphi_n \rangle$ to φ is assumed to be only pointwise, not necessarily uniform, on E.

31. Overadditivity of bend. *Given a curve φ defined on an interval I, suppose that $I_1, \cdots, I_n$ $(n \in \boldsymbol{N})$ are n non-overlapping closed intervals in I. Then $\Omega(\varphi, I_1) + \cdots + \Omega(\varphi, I_n) \leqq \Omega(\varphi)$. This shows in particular that $\Omega(\varphi, J)$ is an overadditive interval function for closed intervals J in I* (cf. §12).

PROOF. We may assume $\Omega(\varphi)$ and consequently each $\Omega(\varphi, I_i)$ also, to be finite. Given an arbitrary ε we can choose a finite set $\varDelta_i \subset I_i$ for each i in such a manner that $\Omega(\varphi, \varDelta_i) > \Omega(\varphi, I_i) - n^{-1}\varepsilon$. It follows that

$$\Omega(\varphi) \geqq \Omega(\varphi, \varDelta_1 \cup \cdots \cup \varDelta_n) \geqq \sum_{i=1}^{n} \Omega(\varphi, \varDelta_i) > \sum_{i=1}^{n} \Omega(\varphi, I_i) - \varepsilon ,$$

which completes the proof.

32. *If I is an interval with a left-hand or right-hand endpoint $c \in I$ and if φ is a parametric curve, defined on I and continuous at the point c, then $\Omega(\varphi, I)$*

$=\Omega(\varphi, I-\{c\})$. *So that, if I is a closed interval and if φ is continuous at the endpoints of I, then $\Omega(\varphi, I)=\Omega(\varphi, I^\circ)$ too, where I° denotes the interior of I as mentioned in* § 1.

Proof. By symmetry we need only deal with the case in which c is the left-hand endpoint of I. Given any real $\eta<\Omega(\varphi)$, we can find a finite nondegenerate set $\varDelta\subset I$ containing the point c and such that $\Omega(\varphi,\varDelta)>\eta$. Write now $\varDelta=\{c, t_1, \cdots, t_n\}$, where $c<t_1<\cdots<t_n$, and take a δ so that $t_0=c+\delta<t_1$. Then $\varDelta'=\{t_0, t_1, \cdots, t_n\}\subset I-\{c\}$, and we find by continuity of the curve φ at c that $\Omega(\varphi,\varDelta')>\eta$ as soon as δ is sufficiently small. We thus get $\Omega(\varphi, I-\{c\})>\eta$ for such δ and this completes the proof.

Remark. It follows at once from the above result that if φ is a curve defined and continuous on an interval I, then $\Omega(\varphi)=\Omega(\varphi, I^\circ)$. Hence we also see, in view of the fourth remark given in § 29, that $\Omega(\varphi)$ equals the supremum of $\Omega(\varphi, J)$ where J represents the closed intervals contained in I°.

33. Inside limit property of interval functions. Let I be an interval and let us write J for a generic closed interval in I. Suppose that $F(J)$ is a nonnegative interval function defined on the class of all J. We shall say that F has *inside limit property* iff for every J and every neighbourhood U of $F(J)$ we can find a closed interval J_0 contained in the interior J° of J, in such a manner that $F(J')\in U$ for any closed interval J' containing J_0 and contained in J. Here we understand by a neighbourhood of ∞ any interval of the form $(c, \infty]$ where c is a positive finite number.

We may observe that if F is nondecreasing, then inside limit property of F is equivalent to asserting that $F(J)$ is for each J the supremum of $F(K)$ where K stands for a variable closed interval contained in J°. When F is a finite function we shall often use the expression *inside continuity* instead of inside limit property.

34. Inside limit property of bend. *Given a curve φ defined and continuous on an interval I, the interval function $\Omega(\varphi, J)$, where J represents the closed intervals contained in I, possesses inside limit property.*

Proof. Let J be an arbitrary closed interval contained in I. In accordance with the equality $\Omega(\varphi, J)=\Omega(\varphi, J^\circ)$ established in § 32 we can choose for any given real number $\eta<\Omega(\varphi, J)$ a finite subset $\varDelta$ of J° such that $\Omega(\varphi,\varDelta)>\eta$. Taking a closed interval J_0 contained in J° and containing $\varDelta$, we find at once, in view of the third remark given in § 29 (monotonity of bend), that

$$\Omega(\varphi, J)\geqq\Omega(\varphi, J')\geqq\Omega(\varphi, J_0)\geqq\Omega(\varphi,\varDelta)>\eta\,,$$

where J' is an arbitrary closed interval containing J_0 and contained in J.

Hence the result.

35. *Suppose that F is a finite nonnegative interval function defined for the closed intervals J contained in a given closed interval $I=[a, b]$. If F is overadditive and inside continuous, then we can find for any given ε a subdivision Δ of I in such a manner that $F(J)<\varepsilon$ for every interval J pertaining to Δ* (cf. § 14).

Proof. We may clearly assume that $F_0=F(I)>0$. Now write $G(x)= F([a, x])$ for each point x of the open interval (a, b). Then $G(x) \leqq F_0- F([x, b])$ by overadditivity of F, and hence $G(x) \rightarrow 0$ as $x \rightarrow a+$ by inside continuity of F. Consequently (a, b) contains points x for which $G(x) \leqq 2^{-1} F_0$. Denoting by c the supremum of such x, we find that $a<c<b$, since $G(x) \rightarrow F_0$ $(x \rightarrow b-)$ on account of inside continuity of F. From the definition of c it follows that $G(x) \leqq 2^{-1} F_0$ whenever $a<x<c$, and hence that $G(c) \leqq 2^{-1} F_0$. Again, if $c<x<b$, we have $F([x, b]) \leqq F_0-G(x)<2^{-1} F_0$, whence $F([c, b]) \leqq 2^{-1} F_0$. To complete the proof, we need now only repeat the above process a certain finite number of times.

36. Corollary. *If φ is a continuous curve of bounded bend on a closed interval I, then for any given ε there is a subdivision Δ of I such that $\Omega(\varphi, J)<\varepsilon$ for every interval J pertaining to Δ.*

Proof. This is an immediate consequence of the preceding section, since $\Omega(\varphi, K)$ is an overadditive inside continuous interval function for closed intervals $K \subset I$ as shown in § 31 and § 34.

Remark. Continuity of φ is essential for the validity of the corollary. To see this, let us define a function $f(t)$ on the interval $[0,1]$ by setting $f(0)=1$ and $f(t)=0$ whenever $0<t \leqq 1$. Consider now the curve φ_0 in the plane $\boldsymbol{R}^2$ defined by $\varphi_0(t)=\langle t, f(t)\rangle$ on $[0,1]$. It is then easy to see that $\Omega(\varphi_0, J)=2^{-1} \pi$ whenever J is a closed interval contained in $[0,1]$ and containing the point $t=0$, and hence that the corollary does not hold for the curve φ_0. The reason for this is of course the discontinuity of φ_0 at $t=0$.

The same curve might have been attached to § 32 as a counter-example showing that continuity of φ at the point c is essential.

37. Definition of length for general curves. We have hitherto considered length only for curves defined on closed intervals. We shall now deal with the case of curves defined on general sets. Suppose that ψ is a parametric curve on a set $E \subset \boldsymbol{R}$ and let Δ be any finite subset of E. Let us introduce the symbol $\mathrm{L}(\psi, \Delta)$ as follows. If Δ is degenerate, we set simply $\mathrm{L}(\psi, \Delta)=0$. Otherwise we write $\Delta=\{t_0, t_1, \cdots, t_n\}$, where $t_0<t_1<\cdots<t_n$, and set $\mathrm{L}(\psi, \Delta) = \sum_{i=1}^n |\psi(t_i)-\psi(t_{i-1})|$. We then define, for the subsets A of E, the *length* of

the curve ψ *on* A by $L(\psi, A) = \sup L(\psi, \Delta)$, where Δ represents the finite subsets of A. Further, $L(\psi, E)$ will be called *length* of the curve ψ and we shall often write simply $L(\psi)$ for it. $L(\psi, A)$ is thus the length of the subcurve (ψ, A), when A is nonvoid.

In case E is a closed interval, the present definition of length is evidently equivalent to the usual one according to which the length of ψ is the supremum of $L(\psi, \Delta)$ where Δ represents generically the subdivisions of E (cf. § 15).

Remarks similar to those given in § 29 in regard to the definition of bend are also relevant here to the present definition of length of general curves.

38. Rectifiability. We shall term a curve ψ in $\boldsymbol{R}^m$ to be *rectifiable* iff $L(\psi) < \infty$. *In order that* ψ *be rectifiable, it is necessary and sufficient that the coordinate functions* $x_i(t)$ $(i = 1, 2, \cdots, m)$ *of* ψ *should be of bounded variation on the set* E *on which* ψ *is defined.*

Proof. This follows directly from the obvious inequalities $V(x_i, \Delta) \leqq L(\psi, \Delta) \leqq \sum_{i=1}^{m} V(x_i, \Delta)$, where V denotes weak variation (see p. 221 of Saks, *Theory of the Integral*) and Δ is any finite subset of E.

39. Spheric length. Suppose that γ is a spheric curve on a set $E \subset \boldsymbol{R}$ (cf. § 18) and replace, in the definition of the symbol $L(\psi, \Delta)$ given in § 37, the letters L and ψ by Λ and γ respectively, and $|\psi(t_i) - \psi(t_{i-1})|$ by $\gamma(t_{i-1}) \diamond \gamma(t_i)$. We then get the definition of the symbol $\Lambda(\gamma, \Delta)$. Further, by replacing L and ψ in the definition of $L(\psi, A)$ by Λ and γ respectively, we obtain the definition of $\Lambda(\gamma, A)$, which will be termed *spheric length* of the curve γ *on* A. $\Lambda(\gamma, E)$ will simply be called *spheric length* of γ and often denoted by $\Lambda(\gamma)$, so that $\Lambda(\gamma, A)$ is the spheric length of the subcurve (γ, A) for nonvoid subsets A of E.

Remarks analogous to those made in § 29 for the definition of bend will also hold good in regard to the present definition of spheric length.

40. Spheric rectifiability. We shall call a spheric curve γ defined on a set $E \subset \boldsymbol{R}$ to be *spherically rectifiable* iff $\Lambda(\gamma) < \infty$. *In order that this be the case, it is necessary and sufficient that* γ *should be rectifiable. In point of fact, we always have* $L(\gamma) \leqq \Lambda(\gamma) \leqq 2L(\gamma)$ *for any spheric curve* γ *whatsoever.*

Proof. On account of (i) of § 24 we have $L(\gamma, \Delta) \leqq \Lambda(\gamma, \Delta) \leqq 2L(\gamma, \Delta)$ for every finite subset Δ of E. Hence the inequalities $L(\gamma) \leqq \Lambda(\gamma) \leqq 2L(\gamma)$.

41. The properties of length and spheric length are to a certain extent similar to those of bend. Indeed, the assertions and proofs of §§ 30-32, 34, 36 will be valid with little alterations on replacing the letters Ω, φ by L, ψ respectively or by Λ, γ respectively.

There is, however, one simple property of ordinary and spheric length, namely *additivity*, which is not shared by bend. This amounts of course to asserting that

$$L(\psi, J_1 \cup J_2) = L(\psi, J_1) + L(\psi, J_2),$$
$$\Lambda(\gamma, J_1 \cup J_2) = \Lambda(\gamma, J_1) + \Lambda(\gamma, J_2),$$

for any pair of abutting closed intervals J_1 and J_2 contained in the interval I on which the curve ψ and the spheric curve γ are defined arbitrarily. The proof is trivial in each case and well known in the case of ordinary length.

Furthermore, if the same curves ψ and γ are continuous at some point of the interval I and if ψ is rectifiable and γ spherically rectifiable, then the interval functions $L(\psi, J)$ and $\Lambda(\gamma, J)$, defined and additive for the closed intervals J in I, are both continuous at that particular point. This will follow at once from the analogues of § 32 for ordinary and spheric length, in view of additivity and monotonity of both $L(\psi, J)$ and $\Lambda(\gamma, J)$.

Certain of the simpler properties of ordinary length have already been made use of without proofs in the preceding chapter (§§ 5, 15, 16, 18).

42. Tangent directions and derived directions. Suppose that φ is a parametric curve in $\boldsymbol{R}^m$ defined on an interval I and that a point c of I is fixed. We shall call a unit vector p of $\boldsymbol{R}^m$ *tangent direction* of the curve φ at the point c iff for any given ε we can find a δ such that we have both $\varphi(a) \neq \varphi(b)$ and $\{\varphi(b) - \varphi(a)\} \diamond p < \varepsilon$ whenever $c \in [a, b] \subset I$ and $b - a < \delta$. Here p is plainly uniquely determined.

Again, a unit vector $q \in \boldsymbol{R}^m$ will be termed a *derived direction* of the curve φ at c iff for any given ε we can find a closed interval $[a, b] \subset I$ such that $c \in [a, b]$, $b - a < \varepsilon$, $\varphi(a) \neq \varphi(b)$, and that $\{\varphi(b) - \varphi(a)\} \diamond q < \varepsilon$. Clearly q need not be uniquely determined in this case, so that the curve φ may have more than one derived directions at c.

43. Spherically representable curves and C^* curves. Given a curve φ and a spheric curve γ, both of them defined on an interval I and situated in the space $\boldsymbol{R}^m$, we shall call γ *spheric representation* of φ iff $\gamma(t)$ is the tangent direction of φ at every $t \in I$. Here γ is plainly uniquely determined by φ, and we write $\gamma = \hat{\varphi}$. By a *spherically representable curve* we shall understand one possessing spheric representation. When φ is a regular differentiable curve, the present definition of $\hat{\varphi}$ evidently reduces to that given in § 3.

A parametric curve φ on an interval I will be termed to be $\boldsymbol{C}^*$ on I iff it is continuous and spherically representable. The symbol $\boldsymbol{C}^*$ will also be used to denote the class of the relevant curves. We may write $\boldsymbol{C}^*(I)$ when precision is desired.

44. Direction curves. Let φ be a curve defined on an interval I. A spheric curve γ defined on a set $E \subset I$ will be termed *direction curve* of φ on E iff $\gamma(t)$ is a derived direction of φ at every point t of E. Needless to say, a curve may possess more than one direction curves on a given set. In case φ is spherically representable on I, the spheric representation $\hat{\varphi}$ is evidently the unique direction curve of φ on I.

45. *Suppose that φ is a curve defined on an interval I and further that γ, a spheric curve on a set $E \subset I$, is a direction curve of φ. Then $\Omega(\varphi) \geqq \Lambda(\gamma) \geqq \mathrm{L}(\gamma)$.*

Proof. In view of § 40 it is enough to show that $\Omega(\varphi) > \eta$ for every real number η less than $\Lambda(\gamma)$. We may plainly assume that $\eta \geqq 0$. According to the definition of $\Lambda(\gamma)$ there then exists a finite nondegenerate set Δ in E subject to the condition $\Lambda(\gamma, \Delta) > \eta$. Let us write $\Delta = \{t_0, t_1, \cdots, t_n\}$, where $t_0 < t_1 < \cdots < t_n$. Given an arbitrary δ, we can associate with each t_i a closed interval $J_i = [a_i, b_i] \subset I$ in such a manner that $t_i \in J_i$, $|J_i| = b_i - a_i < \delta$, $\varphi(a_i) \neq \varphi(b_i)$, and that $\{\varphi(b_i) - \varphi(a_i)\} \diamond \gamma(t_i) < \delta$. When δ is small, the intervals $J_0, J_1, \cdots, J_n$ are evidently disjoint. And it is easy to see that, taking δ still smaller if necessary, we can ensure that $\Omega(\varphi, \Delta') > \eta$ where $\Delta' = \{a_0, b_0, \cdots, a_n, b_n\}$. We thus get $\Omega(\varphi) > \eta$, which completes the proof.

46. *If φ is a continuous curve defined on a closed interval I and if we denote, for each $n \in \boldsymbol{N}$, by Δ_n the subdivision of I consisting of $n+1$ equidistant points of I, then, for every finite subset Δ of I,*

$$\Omega(\varphi, \Delta) \leqq \varliminf_n \Omega(\varphi, \Delta_n).$$

Proof. We replace each point p of Δ by that point q of Δ_n which is nearest p and denote the resulting set by Δ_n' (if there are two nearest points for a p, then q may be either one). It follows at once by continuity of φ that $\Omega(\varphi, \Delta) \leqq \varliminf_n \Omega(\varphi, \Delta_n')$. But since $\Delta_n' \subset \Delta_n$ we have $\Omega(\varphi, \Delta_n') \leqq \Omega(\varphi, \Delta_n)$ for every n, which in conjunction with the foregoing inequality gives the desired result.

47. *If φ is a regular $\boldsymbol{C}^2$ curve defined on a closed interval I and if we denote for every $n \in \boldsymbol{N}$ by Δ_n the same subdivision of I as in the preceding section, then $\Omega(\varphi, \Delta_n) \to \mathrm{L}(\hat{\varphi})$ as $n \to \infty$.*

PROOF. Suppose n large and write $\Delta = \{t_0, t_1, \cdots, t_n\}$, where $t_0 < t_1 < \cdots < t_n$. We shall understand by the letters i, j, k variable integers subject to the conditions $0 \leqq i \leqq n$, $1 \leqq j \leqq n$, $1 \leqq k \leqq n-1$ respectively, and by θ and ϑ indiscriminately certain vectors and numbers, respectively, that tend uniformly to 0 as $n \to \infty$. Writing for short $\varphi_i = \varphi(t_i)$, $\varphi_i' = \varphi'(t_i)$, $\varphi_i'' = \varphi''(t_i)$, $h = n^{-1}|I|$, $\delta_j = \varphi_j - \varphi_{j-1}$, $\sigma_j = h^{-1}\delta_j$, we easily find successively that

$$\delta_j = h\varphi_j' - 2^{-1}h^2\varphi_j'' + h^2\theta \neq 0, \quad \sigma_j = \varphi_j' - 2^{-1}h\varphi_j'' + h\theta \neq 0\,,$$

$$|\sigma_j| = |\varphi_j'| + \vartheta, \ |\sigma_j|^{-1} = |\varphi_j'|^{-1} + \vartheta, \ |\sigma_{k+1}| = |\varphi_k'| + \vartheta, \ |\sigma_{k+1}|^{-1} = |\varphi_k'|^{-1} + \vartheta\,,$$

$$\sigma_{k+1} - \sigma_k = (\varphi_{k+1}' - \varphi_k') - 2^{-1}h(\varphi_{k+1}'' - \varphi_k'') + h\theta = h\varphi_k'' + h\theta\,,$$

$$|\sigma_{k+1} - \sigma_k| = \vartheta < |\varphi_k'| + \vartheta = |\sigma_k|\,.$$

On account of (ii) and (iii) of § 24 it follows that

$$\delta_k \diamond \delta_{k+1} = \sigma_k \diamond \sigma_{k+1} = \mathrm{Sin}^{-1}(|\sigma_k|^{-1} \cdot |\sigma_{k+1}|^{-1} \cdot |\sigma_k \times \sigma_{k+1}|)\,.$$

In order to transform the last expression, we utilize the identity $|p \times q| = |p \times (q-p)|$, which holds for any pair of vectors p, q of $\boldsymbol{R}^m$. This may be verified by direct computation as follows:

$$|p \times (q-p)|^2 = p^2(q-p)^2 - \{p(q-p)\}^2 = p^2q^2 - (pq)^2 = |p \times q|^2\,.$$

We thus find that

$$|\sigma_k \times \sigma_{k+1}| = |\sigma_k \times (\sigma_{k+1} - \sigma_k)| = |(\varphi_k' + \theta) \times (h\varphi_k'' + h\theta)|$$
$$= h|(\varphi_k' + \theta) \times (\varphi_k'' + \theta)| = h|\varphi_k' \times \varphi_k''| + h\vartheta\,,$$

where the last step is effected with the help of the simple inequality $|\sqrt{u} - \sqrt{v}| \leqq \sqrt{|u-v|}$ which holds for any pair u, v of nonnegative real numbers. In view of

$$|\sigma_k|^{-1} \cdot |\sigma_{k+1}|^{-1} = (|\varphi_k'|^{-1} + \vartheta)(|\varphi_k'|^{-1} + \vartheta) = |\varphi_k'|^{-2} + \vartheta\,,$$

it follows now successively that

$$\delta_k \diamond \delta_{k+1} = \mathrm{Sin}^{-1}(h|\varphi_k'|^{-2} \cdot |\varphi_k' \times \varphi_k''| + h\vartheta)$$
$$= h|\varphi_k'|^{-2} \cdot |\varphi_k' \times \varphi_k''| + h\vartheta\,,$$
$$\Omega(\varphi, \Delta_n) = \sum_k \delta_k \diamond \delta_{k+1} = \sum_k h|\varphi_k'|^{-2} \cdot |\varphi_k' \times \varphi_k''| + hn\vartheta$$
$$= \int_I |\varphi'(t)|^{-2} \cdot |\varphi'(t) \times \varphi''(t)|\,dt + \vartheta = \mathrm{L}(\hat{\varphi}) + \vartheta\,.$$

This completes the proof.

REMARK. The above equality $|p \times q| = |p \times (q-p)|$ may be extended to the form $|p \times q| = |p \times (\lambda p + q)|$, which is valid for any p, q of $\boldsymbol{R}^m$ and any real λ. This will be used later on (§ 53).

48. COROLLARY. *If φ is a regular $\boldsymbol{C}^2$ curve on a closed interval I, then*

$\Omega(\varphi) \leqq L(\hat{\varphi})$. This follows directly from the preceding two sections.

49. *If φ is a continuous curve on a closed interval I, then* $\Omega(\varphi) \leqq \Theta(\varphi)$.

PROOF. Consider the sequences $\langle \varphi_n ; n \in \boldsymbol{N} \rangle$ of regular $\boldsymbol{C}^2$ curves on I such that $\varphi_n \rightrightarrows \varphi$. Then, by § 48 and § 30 (lower semicontinuity of Ω),

$$\Theta(\varphi) = \inf \varliminf_n L(\hat{\varphi}_n) \geqq \inf \varliminf_n \Omega(\varphi_n) \geqq \Omega(\varphi)\,,$$

which completes the proof.

50. *If φ is a regular $\boldsymbol{C}^1$ curve on a closed interval I, then* $\Theta(\varphi) = \Omega(\varphi) = L(\hat{\varphi}) = \Lambda(\hat{\varphi})$.

PROOF. On writing $\varphi_n = \varphi$ $(n \in \boldsymbol{N})$ it follows at once from § 20 and § 45 that

$$\Theta(\varphi) = \Theta_1(\varphi) \leqq \varliminf_n L(\hat{\varphi}_n) = L(\hat{\varphi}) \leqq \Lambda(\hat{\varphi}) \leqq \Omega(\varphi)\,,$$

which conjointly with the foregoing section gives the asserted result.

51. Light curves. We shall term a curve φ on an interval I to be *light on* a subinterval J of I iff it is constant on no subintervals of J. When this is the case for $J = I$, we shall simply say that φ is *light*. By a *light subinterval* of I we shall understand one on which φ is light. If $I_1, \cdots, I_n$ are a finite number of intervals in I that together cover I, then clearly a curve φ on I is light iff it is light on every I_i. We may observe that we have considered general, not necessarily closed, intervals throughout this §.

52. *Let φ be a quasilinear curve* (§ 15) *on a closed interval I. If $\varDelta$ is any typical subdivision of I for φ, then $\Omega(\varphi, \varDelta) = \Omega(\varphi)$. Moreover we can uniformly approximate over I the curve φ to any degree of precision by a light quasilinear curve ψ on I such that $\Omega(\psi) = \Omega(\varphi)$.*

PROOF. The first part of the assertion is obvious since we have $\Omega(\varphi, \varDelta) = \Omega(\varphi, \varDelta \cup \varDelta') = \Omega(\varphi, \varDelta')$ for any pair $\varDelta, \varDelta'$ of subdivisions of quasilinearity of φ. To prove the second part, let us fix a typical subdivision $\varDelta$. We may clearly assume that φ is nonconstant on I and that there are no two abutting intervals pertaining to $\varDelta$ (see §14) on both of which φ is constant simultaneously.

Taking a δ such that $2\delta < \| \varDelta \|$, we shall define a curve ψ on $I = [a, b]$ as follows. Suppose that $J = [\alpha, \beta]$ is an interval pertaining to $\varDelta$ such that φ is constant on J. If $\beta < b$, then the interval, pertaining to $\varDelta$ and abutting J on the right of J, must be light. We then determine the restriction of ψ to $J' = [\alpha, \beta + \delta]$ so as to be linear on J' with $\psi(\alpha) = \varphi(\alpha)$ and $\psi(\beta + \delta) = \varphi(\beta + \delta)$. If $\beta = b$, then there exists on the left of J a light interval pertaining to $\varDelta$ and abutting J, and we determine the restriction of ψ to

$J' = [\alpha - \delta, \beta]$ in a similar way. Finally, we put simply $\psi(t) = \varphi(t)$ for each point t of I that belongs to no J' of the aforesaid two types.

Thus constructed the curve ψ is evidently quasilinear and light, and it is easy to see that $\Omega(\psi) = \Omega(\varphi)$. Moreover, this curve will clearly approximate over I the given curve φ to any extent of precision as soon as δ is sufficiently small.

53. *If φ is a quasilinear curve defined on a closed interval I, then there exists for any given ε a regular C^1 curve χ on I of bounded bend such that $\rho(\varphi, \chi) < \varepsilon$ and $|\Omega(\varphi) - \mathrm{L}(\hat{\chi})| < \varepsilon$.*

PROOF. In view of the preceding section we may assume that φ is light. Let $\Delta = \{t_0, t_1, \cdots, t_n\}$ be a subdivision of $I = [a, b]$ of quasilinearity for φ, where $n \geqq 2$ and $a = t_0 < t_1 < \cdots < t_n = b$. Then $\varphi'(t)$ is constant on each open interval (t_{i-1}, t_i) $(i = 1, 2, \cdots, n)$, and we denote by p_i this constant value. We may assume here that p_i and p_{i+1} are linearly independent for each $i = 1, 2, \cdots, n-1$. For otherwise we need only take suitably, instead of φ, another curve sufficiently near φ and quasilinear with respect to the subdivision Δ.

Consider now the subdivision Δ_δ of I given by

$$\Delta_\delta = \{a, t_1 - \delta, t_1 + \delta, \cdots, t_{n-1} - \delta, t_{n-1} + \delta, b\},$$

where $2\delta < \|\Delta\|$. We define a nonvanishing curve ψ on I, which is quasilinear with reference to Δ_δ, by putting

$$\psi(a) = p_1, \ \psi(b) = p_n, \ \psi(t_i - \delta) = p_i, \ \psi(t_i + \delta) = p_{i+1},$$

where $i = 1, 2, \cdots, n-1$. Then, writing

$$\chi(t) = \int_a^t \psi(\tau) d\tau + \varphi(a) \qquad \text{for} \quad t \in I,$$

we shall show that this curve χ, which is clearly regular C^1 on I, satisfies the required conditions as soon as δ is sufficiently small.

Write $M = \mathrm{Max}\,|p_i|$ $(1 \leqq i \leqq n)$, so that $|\psi(t)| \leqq M$ on I. Let us denote by $\varphi'(t)$ the derivative of φ where it exists, and zero where it does not. Then $\varphi'(t)$ is evidently a measurable function on I with $|\varphi'(t)| \leqq M$, and the points $t \in I$ at which $\psi(t) \neq \varphi'(t)$ constitute a set of measure $< 2n\delta$. Given any ε, we therefore have on I, for small δ,

$$|\chi(t) - \varphi(t)| = \left| \int_a^t \{\psi(\tau) - \varphi'(\tau)\} d\tau \right| \leqq 4Mn\delta < \varepsilon,$$

and the condition $\rho(\chi, \varphi) < \varepsilon$ is satisfied.

Furthermore, writing $t = (t_i - \delta) + 2\delta\lambda$ on each $J_i = [t_i - \delta, t_i + \delta]$ for $i = 1, 2, \cdots, n-1$, so that

$$\psi(t) = p_i + \lambda q_i \qquad (q_i = p_{i+1} - p_i),$$

we easily find, noting the remark of § 47, that

$$\mathrm{L}(\hat{\chi}, J_i) = \int_{J_i} \frac{|\chi'(t)\times\chi''(t)|}{|\chi'(t)|^2}\,dt = \int_0^1 \frac{|\psi(t)\times\psi'(t)|}{|\psi(t)|^2}\frac{dt}{d\lambda}\,d\lambda$$

$$= \int_0^1 \frac{|p_i\times p_{i+1}|}{p_i^2+2\lambda(p_iq_i)+\lambda^2q_i^2}\,d\lambda = \mathrm{Cos}^{-1}\frac{p_ip_{i+1}}{|p_i|\cdot|p_{i+1}|} = p_i\diamond p_{i+1},$$

and therefore, by the first part of § 52, that

$$\mathrm{L}(\hat{\chi}) = \sum_{i=1}^{n-1}\mathrm{L}(\hat{\chi}, J_i) = \Omega(\varphi, \Delta) = \Omega(\varphi).$$

54. *We have* $\Theta(\varphi)=\Omega(\varphi)$ *for any quasilinear curve* φ *on a closed interval* I.

PROOF. Since $\Theta(\varphi)\geqq\Omega(\varphi)$ by § 49, it is enough to show that $\Theta(\varphi)\leqq\Omega(\varphi)$. The preceding section enables us to choose a sequence $\langle\varphi_n;\, n\in \boldsymbol{N}\rangle$ of regular $\boldsymbol{C}^1$ curves on I of bounded bend, such that $\varphi_n\rightrightarrows\varphi$ and $\mathrm{L}(\hat{\varphi}_n)\to\Omega(\varphi)$. Then § 20 gives $\Theta(\varphi)\leqq\underline{\lim}_n\, \mathrm{L}(\hat{\varphi}_n)=\Omega(\varphi)$.

55. We shall now establish the following theorem, which is the aim of the present chapter.

THEOREM. *We have* $\Theta(\varphi)=\Omega(\varphi)$ *for any continuous curve* φ *on a closed interval* I.

PROOF. In view of § 49 it suffices to show that $\Theta(\varphi)\leqq\Omega(\varphi)$. We take a sequence $\langle\Delta_n;\, n\in\boldsymbol{N}\rangle$ of subdivisions of I such that $\|\Delta_n\|\to 0$ and $\Omega(\varphi,\Delta_n)\to\Omega(\varphi)$. Let us denote for each n by φ_n the curve which is quasilinear with respect to Δ_n and coincides with φ at all points of Δ_n. Then clearly $\varphi_n\rightrightarrows\varphi$ by continuity of φ, and moreover, it follows from the preceding § and the first part of § 52 that $\Theta(\varphi_n)=\Omega(\varphi_n)=\Omega(\varphi_n,\Delta_n)=\Omega(\varphi,\Delta_n)$ for every n. Hence, by lower semicontinuity of Θ, $\Theta(\varphi)\leqq\underline{\lim}_n\,\Theta(\varphi_n)=\Omega(\varphi)$.

Chapter III. Further properties of bend.

56. Suppose that A is a nonvoid set of real numbers. By a *δ-net* in A, where δ is a given positive number, we mean as usual a nonvoid finite subset M of A such that the distance of every point of A from M is less than δ.

Now let φ *be a continuous curve on a finite interval* I *and* η *any real number* $<\Omega(\varphi)$. *There then exists a* δ *such that* $\Omega(\varphi,\Delta)>\eta$ *for any δ-net* Δ *in* I.

REMARK. Continuity of φ is essential for the validity of the result as is shown by the example given in the remark of § 36.

PROOF. If this were false, we could find a real number $\eta_0<\Omega(\varphi)$ and a sequence $\langle\Delta_n;\, n\in\boldsymbol{N}\rangle$ of finite subsets of I such that, for each n, the set Δ_n is an n^{-1}-net in I satisfying $\Omega(\varphi,\Delta_n)\leqq\eta_0$. We shall show that $\Omega(\varphi,J)\leqq\eta_0$

for any given closed interval J in I°. For this purpose, we may clearly assume that every Δ_n is nondegenerate and that J is contained for every n in the sum S_n of the intervals pertaining to Δ_n. Now let us construct, for each n, a curve φ_n on S_n which is quasilinear with respect to Δ_n and which coincides with the given curve φ on Δ_n. It would then follow at once from continuity of φ that $(\varphi_n, J) \rightrightarrows (\varphi, J)$. We should therefore obtain $\Omega(\varphi, J) \leqq \underline{\lim}_n \Omega(\varphi_n, J)$. On the other hand we have, for every n,

$$\Omega(\varphi_n, J) \leqq \Omega(\varphi_n) = \Omega(\varphi_n, \Delta_n) = \Omega(\varphi, \Delta_n) \leqq \eta_0 ,$$

which conjointly with the foregoing inequality would yield $\Omega(\varphi, J) \leqq \eta_0$ for every J, as announced. Hence we should get $\Omega(\varphi) = \sup \Omega(\varphi, J) \leqq \eta_0$ by virtue of the remark of § 32. But this would contradict $\eta_0 < \Omega(\varphi)$, and the proof is complete.

57. Lebesgue and Fréchet equivalences. Let φ and ψ be two curves on the intervals I and J respectively. We shall term φ and ψ as usual to be *Lebesgue equivalent* (or *L-equivalent*) iff there exists a homeomorphic mapping h of I onto J such that $\psi(h(t)) = \varphi(t)$ for every point $t \in I$. Again, the curves will be said as usual *Fréchet equivalent* (or *F-equivalent*) iff, given any ε, we can find a homeomorphic mapping h_ε of I onto J in such a manner that $|\psi(h_\varepsilon(t)) - \varphi(t)| < \varepsilon$ for every $t \in I$.

Lebesgue and Fréchet equivalences are clearly reflexive, symmetric, and transitive, so that they are equivalence relations in the usual sense. Also, Lebesgue equivalence implies Fréchet equivalence. It is further easily seen that if φ and ψ are F-equivalent as above, we can find a sequence $\langle \varphi_n; n \in N \rangle$ of curves on I that are L-equivalent to ψ, such that $\varphi_n \rightrightarrows \varphi$. If φ and ψ are both of them continuous or spheric in addition, then the curves φ_n may evidently be supposed all of them respectively continuous or spheric, too.

58. *If φ and ψ are F-equivalent as in the above, then* $\Omega(\varphi) = \Omega(\psi)$ *and* $\mathrm{L}(\varphi) = \mathrm{L}(\psi)$. *If moreover the two curves are spheric, then* $\Lambda(\varphi) = \Lambda(\psi)$.

PROOF. We shall only deal with the first inequality, the others admitting similar treatments. We note first that if φ and ψ are L-equivalent, then $\Omega(\varphi) = \Omega(\psi)$ is an immediate consequence of the definition of bend, since every homeomorphic mapping of an interval onto another is necessarily a strictly monotone function.

This being so, suppose φ and ψ Fréchet equivalent and take the sequence $\langle \varphi_n \rangle$ of the preceding section. Then $\Omega(\varphi_n) = \Omega(\psi)$ by what has just been said, and we find, by lower semicontinuity of bend, that $\Omega(\varphi) \leqq \underline{\lim}_n \Omega(\varphi_n) = \Omega(\psi)$. By symmetry we get $\Omega(\psi) \leqq \Omega(\varphi)$, which completes the proof.

59. Fréchet distance. Given two curves φ and ψ defined on homeomorphic intervals I and J respectively, we shall call *Fréchet distance* (or *F-distance*) between φ, ψ and denote by $d_F(\varphi, \psi)$, the infimum of the distance $\rho(\varphi, \chi)$ where the curve χ is defined on I by $\chi(t) = \psi(h(t))$, h representing the homeomorphic mappings of I onto J. Clearly φ is Fréchet equivalent to ψ iff $d_F(\varphi, \psi) = 0$.

If $\varphi, \varphi_1, \varphi_2, \cdots$ are parametric curves defined respectively on the homeomorphic intervals $I, I_1, I_2, \cdots$ and such that $d_F(\varphi, \varphi_n) \to 0$ as $n \to \infty$, then $\Omega(\varphi) \leqq \underline{\lim}_n \Omega(\varphi_n)$ and $\mathrm{L}(\varphi) \leqq \underline{\lim}_n \mathrm{L}(\varphi_n)$. If moreover all the curves are spheric, then $\Lambda(\varphi) \leqq \underline{\lim}_n \Lambda(\varphi_n)$.

Proof. We may restrict ourselves to the case of bend. There are homeomorphic mappings h_n of I onto I_n such that $\rho(\varphi, \chi_n) \to 0$, or in other words, that $\chi_n \rightrightarrows \varphi$, where the curves χ_n are defined on I by $\chi_n(t) = \varphi_n(h_n(t))$. Then χ_n and φ_n are L-equivalent for every n, so that $\Omega(\chi_n) = \Omega(\varphi_n)$. Hence

$$\Omega(\varphi) \leqq \underline{\lim}_n \Omega(\chi_n) = \underline{\lim}_n \Omega(\varphi_n) .$$

60. *If φ is a nonconstant curve on a closed interval $[a, b]$ and if $\varphi(a) = \varphi(b)$, then $\Omega(\varphi) \geqq \pi$.*

Proof. Take a point c such that $a < c < b$ and that $\varphi(c) \neq \varphi(a)$. Writing $\varDelta = \{a, c, b\}$ and $p = \varphi(c) - \varphi(a)$, we see at once that $\Omega(\varphi) \geqq \Omega(\varphi, \varDelta) = p \diamond (-p) = \pi$.

61. *If φ is a light curve defined and periodic on the straight line $\boldsymbol{R}$ with a period $\omega > 0$, then $\Omega(\varphi, J) \geqq 2\pi$ for every interval J such that $|J| > \omega$.*

Proof. We may plainly assume that J is a closed interval $[a, b]$. Take a δ such that $\delta < \omega < b - a - \delta$ and that $\varphi(a + \delta) \neq \varphi(a)$. This is possible, since φ is light and since $|J| = b - a > \omega$. Then, writing $\varDelta = \{a, a + \delta, a + \omega, a + \omega + \delta\}$ and $p = \varphi(a + \delta) - \varphi(a)$, we immediately obtain

$$\Omega(\varphi, J) \geqq \Omega(\varphi, \varDelta) = p \diamond (-p) + (-p) \diamond p = 2\pi .$$

62. *If φ is a spherically representable curve* (see § 43) *defined on a closed interval $[a, b]$ and such that $\varphi(a) = \varphi(b)$, then $\Omega(\varphi) \geqq 2\pi - \hat{\varphi}(a) \diamond \hat{\varphi}(b)$.*

Proof. Given an arbitrary ε, we can find a δ in such a manner that $a' = a + \delta < b - \delta = b'$, $p = \varphi(a') - \varphi(a) \neq 0$, $q = \varphi(b) - \varphi(b') \neq 0$, and that $p \diamond q < \hat{\varphi}(a) \diamond \hat{\varphi}(b) + \varepsilon$. Writing $\varDelta = \{a, a', b', b\}$ and $r = \varphi(b') - \varphi(a')$, so that $p + q + r = 0$, we distinguish two cases: If $r = 0$, then $q = -p$, and hence $\Omega(\varphi, \varDelta) = p \diamond q = 2\pi - p \diamond q$. But if $r \neq 0$, then it follows from § 25 that

$$\begin{aligned} \Omega(\varphi, \varDelta) + p \diamond q &= p \diamond r + r \diamond q + q \diamond p \\ &\geqq p \diamond (r + q) + (r + q) \diamond p = 2\pi . \end{aligned}$$

We thus get, in both cases, $\Omega(\varphi, \Delta) \geqq 2\pi - p \diamond q$, whence we find $\Omega(\varphi) > 2\pi - \hat{\varphi}(a) \diamond \hat{\varphi}(b) - \varepsilon$. This completes the proof since ε is arbitrary.

63. *Given a parametric curve φ on a closed interval $I = [a, b]$, suppose that $p = \varphi(b) - \varphi(a) \neq 0$ and that $\Omega(\varphi) < \varepsilon < 2^{-1}\pi$. Then $\mathrm{L}(\varphi) \cos \varepsilon < |p|$. Further, if $a \leqq \alpha < \beta \leqq b$ and $q = \varphi(\beta) - \varphi(\alpha) \neq 0$, then $p \diamond q < \varepsilon$.*

PROOF. Let $\Delta = \{t_0, t_1, \cdots, t_n\}$ be any subdivision of I where $a = t_0 < t_1 < \cdots < t_n = b$, and write $\varphi_i = \varphi(t_i) - \varphi(t_{i-1})$ for $i = 1, 2, \cdots, n$. Since $p = \varphi_1 + \cdots + \varphi_n \neq 0$, the sequence $\langle \varphi_1, \cdots, \varphi_n \rangle$ contains nonvanishing terms. Let us arrange all of them in a subsequence $\langle p_1, \cdots, p_k \rangle$ of $\langle \varphi_i \rangle$. We then find by § 26, for every pair of indices i, j not exceeding k, that $p_i \diamond p_j \leqq \Omega(\varphi) < \varepsilon$, and hence that $p_i p_j > |p_i| \cdot |p_j| \cos \varepsilon$. By summing the last inequality over i and j, we get $p^2 > (|p_1| + \cdots + |p_k|)^2 \cos \varepsilon$, and this implies that

$$|p| > (|\varphi_1| + \cdots + |\varphi_n|) \sqrt{\cos \varepsilon} = \mathrm{L}(\varphi, \Delta) \sqrt{\cos \varepsilon}.$$

The first inequality of the assertion now follows immediately.

To derive the second inequality, take the above Δ so as to contain α and β. Then q is the sum of a certain subsequence $\langle q_1, \cdots, q_l \rangle$ of $\langle p_i \rangle$. By summing the inequality $p_i q_j > |p_i| \cdot |q_j| \cos \varepsilon$ over $1 \leqq i \leqq k$ and $1 \leqq j \leqq l$, we obtain

$$pq > (\sum_i |p_i|)(\sum_j |q_j|) \cos \varepsilon \geqq |p| \cdot |q| \cos \varepsilon.$$

This implies that $p \diamond q < \varepsilon$ and so completes the proof.

64. *Every continuous curve φ on a closed interval I is rectifiable whenever it is of bounded bend.*

PROOF. By § 36 there exists a subdivision Δ of I such that $\Omega(\varphi, J) < 3^{-1}\pi$ for every interval $J = [a, b]$ pertaining to Δ. It follows from § 60 for each J that either $\varphi(a) \neq \varphi(b)$, or else that φ is constant throughout J. In the former case we find by the preceding section that $\mathrm{L}(\varphi, J) < 2|\varphi(b) - \varphi(a)|$, while in the latter case we have $\mathrm{L}(\varphi, J) = 0$. So that $\mathrm{L}(\varphi) = \sum \mathrm{L}(\varphi, J) < \infty$ (see § 41), and this completes the proof.

REMARK. We shall show by an example that the assertion will cease to hold if we drop continuity of φ. We define a finite function f on $[0, 1]$ by setting $f(0) = 0$ and $f(t) = t^{-1}$ for $0 < t \leqq 1$. It is then easy to see that the curve φ_0 in $\boldsymbol{R}^2$, defined on $[0, 1]$ by $\varphi_0(t) = \langle t, f(t) \rangle$, is nonrectifiable, though $\Omega(\varphi_0)$ is finite.

65. Given a curve φ on an interval I, we shall frequently use the notation $\varphi(J)$ to mean $\varphi(\beta) - \varphi(\alpha)$, where $J = [\alpha, \beta]$ is any closed interval in I. Thus defined, $\varphi(J)$ is an *additive vector-valued function* of the interval J, that is, we have $\varphi(J_1 \cup J_2) = \varphi(J_1) + \varphi(J_2)$ whenever J_1 and J_2 are a pair of abutting closed intervals contained in I. We shall say that the point function $\varphi(t)$ and the interval function $\varphi(J)$ *correspond* to each other. The function $\varphi(J)$ is not to be confused with the locus of the curve φ, i.e. the set of the points $\varphi(t)$. After Saks, *Theory of the Integral*, we shall write $\varphi[J]$ for the latter. We shall adopt similar notations and terminology for real functions defined on an interval.

66. *Suppose that* φ *is a spherically representable curve in* $\boldsymbol{R}^m$ (§ 43) *on an interval* I, *and let there be given a unit vector* p *of* $\boldsymbol{R}^m$ *and an* $\varepsilon < 2^{-1}\pi$ *in such a manner that* $\hat{\varphi}(t) \diamond p < \varepsilon$ *at every point* t *of* I. *Then the curve* φ *is biunique on* I, *and we have* $\varphi(J) \diamond p < \varepsilon$ *and* $\mathrm{L}(\varphi, J) \cos \varepsilon \leqq \varphi(J)p$, *where* J *stands for a generic closed interval in* I.

Remark. It is not clear whether or not the sharper inequality $\mathrm{L}(\varphi, J) \times \cos \varepsilon < \varphi(J)p$ holds for every J.

Proof. Take any point c of I and consider the intervals J that contain c and are so short as to ensure $\varphi(J) \neq 0$ as well as $\varphi(J) \diamond \hat{\varphi}(c) + \hat{\varphi}(c) \diamond p < \varepsilon$. The last inequality implies, by triangular inequality, that $\varphi(J) \diamond p < \varepsilon$, and hence it follows that

$$\varphi(J)p = |\varphi(J)| \cos(\varphi(J) \diamond p) > |\varphi(J)| \cos \varepsilon . \tag{1}$$

We shall now show that $\varphi(J)p > |\varphi(J)| \cos \varepsilon$ holds for every closed interval J in I. Suppose, if possible, that this were not the case for some J, say J'. We could then determine, by the method of successive bisection, a descending sequence $\langle J_n ; n \in \boldsymbol{N} \rangle$ of closed intervals in J', such that $|J_n| = 2^{-n}|J'|$ and $\varphi(J_n)p \leqq |\varphi(J_n)| \cos \varepsilon$ for every n. There would then exist a point common to all J_n. But this would clearly contradict (1), since $|J_n| \to 0$.

We thus have $\varphi(J)p > |\varphi(J)| \cos \varepsilon$ for every J, and this shows in particular that $\varphi(J)$ never vanishes. The curve φ is therefore biunique over I, and thus (1) is meaningful and valid for every J, so that we always have $\cos(\varphi(J) \diamond p) > \cos \varepsilon$, or what amounts to the same, $\varphi(J) \diamond p < \varepsilon$.

Moreover, if $\bar{J}$ is a fixed closed interval in I and if $\varDelta$ is any subdivision of $\bar{J}$, then, summing $|\varphi(J)| \cos \varepsilon < \varphi(J)p$ over the intervals J pertaining to $\varDelta$, we get $\mathrm{L}(\varphi, \varDelta) \cos \varepsilon < \varphi(\bar{J})p$. It follows at once that $\mathrm{L}(\varphi, \bar{J}) \cos \varepsilon \leqq \varphi(\bar{J})p$, and this completes the proof.

67. *Given a* $\boldsymbol{C}^*$ *curve* φ *on an interval* I (§ 43), *suppose that* $\hat{\varphi}$ *is rectifiable. Then* $\hat{\varphi}$ *is necessarily continuous on* I.

Proof. Take a fixed closed interval $J_0 = [\alpha, \beta]$ in I. Since $\hat{\varphi}$ is rectifiable, there exists a right-hand limit p of $\hat{\varphi}(t)$ at α (cf. § 38), and p is clearly a unit vector. We then can find, for any given $\varepsilon < 2^{-1}\pi$, a closed interval J_ε containing α and contained in J_0, and such that $\hat{\varphi}(t) \diamond p < \varepsilon$ as well as $\varphi(t) \neq$

$\varphi(\alpha)$, for every $t \in J_\varepsilon - \{\alpha\}$. In virtue of continuity of φ it follows at once from the preceding section that φ is biunique on J_ε and that $\varphi(J) \diamond p \leqq \varepsilon$ for every closed interval J in J_ε. But if such a J contains α and is sufficiently short, then $\hat{\varphi}(\alpha) \diamond \varphi(J) < \varepsilon$, and hence $\hat{\varphi}(\alpha) \diamond p \leqq \hat{\varphi}(\alpha) \diamond \varphi(J) + \varphi(J) \diamond p < 2\varepsilon$. This implies that $\hat{\varphi}(\alpha) \diamond p = 0$, or in other words, that $p = \hat{\varphi}(\alpha)$. Recalling how we have taken J_ε in the above, we find that the curve $\hat{\varphi}$ is right-hand continuous at α. Similarly we show that it is left-hand continuous at β, and this completes the proof since $J_0 = [\alpha, \beta]$ is arbitrary.

Remark. The $\boldsymbol{C}^*$ condition cannot be replaced by spheric representability, as we see at once from the example given in the remark of § 64.

68. Theorem. *We have $\Omega(\varphi) = \Lambda(\hat{\varphi}) = \mathrm{L}(\hat{\varphi})$ for every $\boldsymbol{C}^*$ curve φ on an interval I.*

Remark. This is an essential generalization of part of § 50.

Proof. In view of § 45 it is enough to show that $\Omega(\varphi) \leqq \mathrm{L}(\hat{\varphi})$. Further, the final remark of § 29 and its analogue for length enable us to restrict ourselves to the case in which I is a closed interval.

We may plainly suppose $\hat{\varphi}$ rectifiable, so that $\hat{\varphi}$ is a continuous curve by the foregoing section. Given an $\varepsilon < 2^{-1}\pi$ we can therefore find a δ such that $\hat{\varphi}(t_1) \diamond \hat{\varphi}(t_2) < \varepsilon$ for any two points t_1 and t_2 of I satisfying $|t_1 - t_2| < \delta$. Now let J denote a generic closed interval in I, in the sequel. If $|J| < \delta$, it follows from § 66 that the subcurve (φ, J) is biunique and has nonvanishing finite length $\mathrm{L}(\varphi, J)$ satisfying $\mathrm{L}(\varphi, J) \cos \varepsilon \leqq |\varphi(J)| \leqq \mathrm{L}(\varphi, J)$, and further that $\varphi(J) \diamond \hat{\varphi}(t) < \varepsilon$ for every $t \in J$. Hence we also see that $\mathrm{L}(\varphi) < \infty$.

Let us now take a point function $S(t)$ defind on I and corresponding to the additive interval function $\mathrm{L}(\varphi, J)$ (see § 65). Then S is clearly continuous, and we see by the above that it is strictly increasing on I. Hence $s = S(t)$ is a homeomorphic mapping of the interval I onto the closed interval $I_0 = S[I]$.

This being so, we define a curve ψ on I_0 by $\psi(s) = \varphi(t)$ where $s = S(t)$, $t \in I$. From what has been said above concerning (φ, J), we then infer easily that ψ is a regular $\boldsymbol{C}^1$ curve on I_0 with $\psi'(s) = \hat{\varphi}(t)$. But this implies that $\hat{\psi}(s) = \hat{\varphi}(t)$, whence we see that $\hat{\varphi}$ and $\hat{\psi}$ are Lebesgue equivalent. On the other hand φ and ψ are evidently so, too. We thus have, by § 58, $\Omega(\varphi) = \Omega(\psi)$, $\mathrm{L}(\hat{\varphi}) = \mathrm{L}(\hat{\psi})$, and $\Lambda(\hat{\varphi}) = \Lambda(\hat{\psi})$. This completes the proof on account of § 50.

69. Fenchel [2] proved in 1929 the following inequality on the integrated curvature of closed space curves. *Suppose that $\varphi(s)$ is a $\boldsymbol{C}^2$ curve in $\boldsymbol{R}^3$ defined and periodic on $\boldsymbol{R}$ with a period $\omega > 0$, and further that s is the length parameter for φ, so that we have $|\varphi'(s)| = 1$ and $\hat{\varphi}(s) = \varphi'(s)$ at every point $s \in \boldsymbol{R}$.*

Then the integrated curvature of φ over J, given by $\int_J |\varphi''(s)|\,ds = \mathrm{L}(\hat{\varphi}, J)$, is $\geqq 2\pi$ for any closed interval J of length ω.

He also observed marginally that the $\boldsymbol{C}^2$ condition may be replaced by a weaker one that *the curve φ be twice differentiable, with the second derivative φ'' summable over any finite interval.* Here again $\int_J |\varphi''(s)|\,ds = \mathrm{L}(\hat{\varphi}, J)$, by the Lebesgue theory of curve length, as Fenchel remarked further.

We shall now give a far-reaching extension of the Fenchel inequality.

Theorem. *If φ is a $\boldsymbol{C}^*$ curve in $\boldsymbol{R}^m$ defined on a closed interval $[a, b]$ and such that $\varphi(a) = \varphi(b)$, then*

$$\mathrm{L}(\hat{\varphi}) \geqq 2\pi - \hat{\varphi}(a) \diamond \hat{\varphi}(b) .$$

Proof. This is an immediate consequence of § 62 and § 68.

Remark. We do not know if the $\boldsymbol{C}^*$ condition can be replaced by spheric representability. A similar remark applies also to § 68.

70. *Given a real continuous function $f(t)$ on an interval I, suppose that M is a nonvoid subset of I such that $f[I-M]$ is a countable set and that if $[a, b] \subset I$ and $a \in M$, then there exists a point $c \in \boldsymbol{R}$ satisfying both $a < c < b$ and $f(a) < f(c)$. Then the function f is nondecreasing on I.*

Moreover, f increases strictly on I if $I - M$ is countable.

Proof. We shall begin by showing that $f(\alpha) \leqq f(\beta)$ for every $[\alpha, \beta]$ in I. Suppose if possible that we had $f(\alpha) > f(\beta)$ for some $[\alpha, \beta]$, and take an $x_0 \in \boldsymbol{R}$ not belonging to $f[I-M]$ and such that $f(\alpha) > x_0 > f(\beta)$. Denoting by t_0 the greatest value of t of $[\alpha, \beta]$ for which $f(t) = x_0$, we see at once that $t_0 \in M$ and that $t_0 < \beta$. By hypothesis there then exists a $t_1 \in \boldsymbol{R}$ satisfying both $t_0 < t_1 < \beta$ and $f(t_1) > f(t_0)$. But $f(t_0) = x_0 > f(\beta)$, and hence we can find a $t_2 \in \boldsymbol{R}$ such that $t_1 < t_2 < \beta$ and that $f(t_2) = x_0$. This contradicts the definition of t_0. The function f is thus nondecreasing on I.

We shall now show that it increases strictly on I, when $I - M$ is countable. Suppose, on the contrary, that f were constant on some $[\alpha, \beta]$ in I, and take a point t_3 of M satisfying $\alpha < t_3 < \beta$. By hypothesis we could then choose a $t_4 \in \boldsymbol{R}$ such that $t_3 < t_4 < \beta$ and that $f(t_3) < f(t_4)$. This contradicts constancy of f on $[\alpha, \beta]$ and thus completes the proof.

Remark. Continuity of φ cannot be removed without destroying the validity of the assertion, even in the case in which $M = I$. This is shown by considering the function f_0 defined on the open interval $(0, 1)$ by setting $f_0(t) = 0$ for rational t and $f_0(t) = t$ for irrational t.

71. *Given a continuous curve φ on an interval I and a positive finite constant A, suppose that M is a subset of I such that $I - M$ is countable and such* *that if* $[a,b]\subset I$ *and* $a\in M$, *then there exists a point* $c\in \boldsymbol{R}$ *satisfying both* $a<c<b$ *and* $|\varphi(c)-\varphi(a)|<A(c-a)$. *Then* $|\varphi(\beta)-\varphi(\alpha)|<A(\beta-\alpha)$ *for every interval* $[\alpha,\beta]$ *in* I.

PROOF. This proof is very similar to that given in the preceding section. We shall begin by showing the weaker inequality $|\varphi(\beta)-\varphi(\alpha)|\leqq A(\beta-\alpha)$. Suppose, if possible, that we had the contrary for some $[\alpha,\beta]$. Writing $F(t)=|\varphi(t)-\varphi(\alpha)|-A(t-\alpha)$ for $t\in I$, so that $F(\alpha)=0<F(\beta)$, we then can find a positive number ε smaller than $F(\beta)$ and not belonging to the countable set $F[I-M]$. On account of continuity of the function F, there exist points t of $[\alpha,\beta]$ at which $F(t)=\varepsilon$. Denoting by t_0 the largest of such t, we see at once that $t_0<\beta$ and that $t_0\in M$. So that, by hypothesis, we can take a $t_1\in \boldsymbol{R}$ satisfying both $t_0<t_1<\beta$ and $|\varphi(t_1)-\varphi(t_0)|<A(t_1-t_0)$, and it follows that

$$F(t_1)\leqq F(t_0)+|\varphi(t_1)-\varphi(t_0)|-A(t_1-t_0)<\varepsilon\,.$$

But there is then a $t_2\in \boldsymbol{R}$ such that $t_1<t_2<\beta$ and that $F(t_2)=\varepsilon$. This contradicts the definition of t_0, and the weaker inequality is thus established.

To obtain $|\varphi(\beta)-\varphi(\alpha)|<A(\beta-\alpha)$ for every $[\alpha,\beta]$ in I, let us suppose on the contrary that we had $|\varphi(\beta)-\varphi(\alpha)|=A(\beta-\alpha)$ for some $[\alpha,\beta]$. It then follows that $|\varphi(\beta')-\varphi(\alpha')|=A(\beta'-\alpha')$ for any $[\alpha',\beta']$ in $[\alpha,\beta]$, for indeed

$$\begin{aligned}|\varphi(\beta)-\varphi(\alpha)|&\leqq|\varphi(\beta)-\varphi(\beta')|+|\varphi(\beta')-\varphi(\alpha')|+|\varphi(\alpha')-\varphi(\alpha)|\\&\leqq A(\beta-\beta')+A(\beta'-\alpha')+A(\alpha'-\alpha)=A(\beta-\alpha)\,.\end{aligned}$$

Take now a point t_3 of M for which $\alpha<t_3<\beta$. By hypothesis we then can choose a $t_4\in \boldsymbol{R}$ such that $t_3<t_4<\beta$ and that $|\varphi(t_4)-\varphi(t_3)|<A(t_4-t_3)$. This is clearly incompatible with what has just been said above, and thus the proof is complete.

REMARK. Continuity of φ is essential for the validity of the assertion even when $M=I$. This is seen at once by considering the curve φ_0 in $\boldsymbol{R}^2$ defined on $\boldsymbol{R}$ by $\varphi_0(t)=\langle t,f_0(t)\rangle$, where $f_0(t)=0$ for rational t and $f_0(t)=1$ for irrational t. In fact, there exists for any $[a,b]$ a point c such that $a<c<b$ and that $|\varphi_0(c)-\varphi_0(a)|<2(c-a)$, since we need only take c rational or irrational according as a is respectively rational or irrational. Nevertheless, the conclusion of the assertion is clearly false.

72. Intervals endless on the right or left. An interval without a largest element [smallest element] will be termed *right-hand endless* or *endless on the right* [*left-hand endless* or *endless on the left*]. By an *endless interval* we shall understand one which is endless on both sides, or what amounts to the same, one which is an open set in $\boldsymbol{R}$. Open intervals are evidently endless but not *vice versa*, for we do not count infinite intervals as open ones even when they are open sets in $\boldsymbol{R}$ (see § 1).

73. Right-hand [or left-hand] derived directions and direction curves. Given a curve φ in $\boldsymbol{R}^m$ on an interval I endless on the right, we shall say that a unit vector p in $\boldsymbol{R}^m$ is a *right-hand derived direction* of the curve φ at a point c of I, iff we can find, for any δ, a closed interval $J \subset I$ with the left-hand extremity c and such that $|J| < \delta$, $\varphi(J) \neq 0$, and that $\varphi(J) \diamond p < \delta$. Furthermore, a spheric curve γ in $\boldsymbol{R}^m$ defined on a set $E \subset I$ will be called a *right-hand direction curve* on E of the curve φ iff the vector $\gamma(t)$ is a right-hand derived direction of φ at every $t \in E$.

It goes without saying that γ need not be uniquely determined when φ and E are given. So that the curve φ might have more than one right-hand direction curves on the set E. Evidently a similar remark applies to right-hand derived directions at a given point c.

We define the concepts of *left-hand derived direction* and *left-hand direction curve* in an analogous way. Indeed we have only to replace the words right and left respectively by left and right in the above.

74. We shall now prove a proposition which has a strong resemblance to that of § 66 and will be needed later on (§ 83). We may, however, observe that neither of the two propositions includes the other. In point of fact, we assumed the curve under consideration in § 66 to be spherically representable, while we are concerned with a continuous curve in the following result.

Given a continuous curve φ in $\boldsymbol{R}^m$ on an interval I endless on the right and a right-hand direction curve γ of φ on a set $M \subset I$, suppose that $I-M$ is coutable and that there exist an $\varepsilon < 2^{-1}\pi$ and a unit vector p of $\boldsymbol{R}^m$ in such a manner that $\gamma(t) \diamond p < \varepsilon$ for every $t \in M$. Then φ is biunique on I, and we have $\varphi(J) \diamond p < \varepsilon$ and $\mathrm{L}(\varphi, J) \cos \varepsilon \leqq \varphi(J)p$, where J denotes a generic closed interval in I.

PROOF. Let us consider the continuous function $f(t) = \varphi(t)p$ on I. For any $[a, b] \subset I$ with $a \in M$, there exists, by hypothesis, an interval $K = [a, c]$ such that $a < c < b$, $\varphi(K) \neq 0$ and that

$$\varphi(K) \diamond p \leqq \varphi(K) \diamond \gamma(c) + \gamma(c) \diamond p < \varepsilon \, .$$

Consequently we find that

$$f(K) = \varphi(K)\, p = |\varphi(K)| \cos(\varphi(K) \diamond p) > |\varphi(K)| \cos \varepsilon > 0 \, . \tag{1}$$

Therefore, by § 70, the function f increases strictly on I, and so its range $I_0 = f[I]$ must be an interval. It follows also that φ is biunique over I.

We no wdefine a curve ψ on I_0 by setting $\psi(\tau) = \varphi(t)$ for $\tau = f(t)$ $(t \in I)$.

For any $[a_0, b_0] \subset I_0$ with $a_0 \in f[M]$ there then exists, by (1), an interval $K_0 = [a_0, c_0]$ such that $a_0 < c_0 < b_0$ and that $|\psi(K_0)| \cos \varepsilon < |K_0|$. Also, the set $I_0 - f[M]$ is countable since it coincides with the countable set $f[I-M]$. It follows immediately from § 71 that $|\psi(J_0)| \cos \varepsilon < |J_0|$ for every closed interval J_0 in I_0. In other words, we have $|\varphi(J)| \cos \varepsilon < f(J) = \varphi(J)p$ for every closed interval J in I. We thus have $\varphi(J) \diamond p < \varepsilon$ for every J.

From now on we may proceed precisely as in the final paragraph of § 66, and the proof is thus complete.

Remark. We cannot remove continuity of φ even in the special case $M = I$. This is shown by considering the same curve φ_0 as given in the remark of § 71. In fact, defining the spheric curve γ_0 on $\boldsymbol{R}$ by setting $\gamma_0(t) = p_0 = \langle 1, 0 \rangle$ for every t, we see at once that γ_0 is a right-hand direction curve of φ_0 and that $\gamma_0(t) \diamond p_0 = 0$ for every t. However, the asserted inequalities are clearly false.

75. We shall conclude this chapter with a result which completes the preceding proposition in some minor point, though we shall have no occasion to use it in the sequel. Our interest lies chiefly in the method of proof, as we shall make essential use of the Lebesgue theory.

We have, in the preceding proposition, the sharper inequality $\mathrm{L}(\varphi, J) \cos \varepsilon < \varphi(J)p$ *for every* J.

Proof. In virtue of the change of parameter $\tau = f(t)$ utilized above, we may suppose from the first that $|\varphi(J)| \cos \varepsilon < \varphi(J)p = |J|$ for every J. Thus φ is absolutely continuous on every J and so is derivable almost everywhere on I. Note that, if c is an endpoint of the interval I, then we understand by derivability of φ at c onesided derivability. We shall denote by $\varphi'(t)$, the derivative of φ where it is derivable, and zero where it is not. In view of the hypothesis $\varphi(J)p = |J|$ we then find successively that, almost everywhere on I,

$$\varphi'(t)\,p = 1, \quad \varphi'(t) = |\varphi'(t)|\,\gamma(t), \quad |\varphi'(t)| \cos \varepsilon < |\varphi'(t)|\,\gamma(t)\,p = 1\,.$$

Now, for every J, $\mathrm{L}(\varphi, J) = \int_J |\varphi'(t)|\,dt$ by a theorem due to Tonelli (see p. 123 of Saks, *Theory of the Integral*). Therefore

$$\mathrm{L}(\varphi, J) \cos \varepsilon = \int_I |\varphi'(t)| \cos \varepsilon \, dt < \int_J dt = |J| = \varphi(J)\,p\,,$$

which completes the proof.

Chapter IV. Representation of bend as spheric length.

76. *We have* $\mathrm{L}(\gamma) = \Lambda(\gamma)$ *for every continuous spheric curve* γ *defined on an interval* I.

First Proof. In view of § 40 we need only show that $\Lambda(\gamma) \leqq L(\gamma)$. The analogues of the final remark of § 29 for ordinary and spheric length allows us to assume that I is a closed interval. So that γ is uniformly continuous over I. Given any $\varepsilon < 1$ we can therefore find a δ such that if t_1 and t_2 are any points of I satisfying $|t_1 - t_2| < \delta$ and if we write for short $\gamma_1 = \gamma(t_1)$ and $\gamma_2 = \gamma(t_2)$, then $|\gamma_1 - \gamma_2| < 2^{-1}\varepsilon$. On account of (i) of § 24 we then get successively $\gamma_1 \diamond \gamma_2 < \varepsilon$ and $|\gamma_1 - \gamma_2| \geqq (1-\varepsilon)(\gamma_1 \diamond \gamma_2)$. It follows that $L(\gamma, \varDelta) \geqq (1-\varepsilon)\Lambda(\gamma, \varDelta)$ for any finite subset $\varDelta$ of I such that $\|\varDelta\| < \delta$. We find at once that $L(\gamma) \geqq (1-\varepsilon)\Lambda(\gamma)$, whence $L(\gamma) \geqq \Lambda(\gamma)$, as required.

Second Proof. Fixing a point c of I and writing $\varphi(t) = \int_c^t \gamma(\tau) d\tau$ for $t \in I$, we have $\varphi'(t) = \gamma(t)$ everywhere on I. Thus φ is regular and $\boldsymbol{C}^1$ on I and $\hat{\varphi} = \gamma$. Hence the result by § 50, since we may assume I a closed interval.

77. Right-hand [or left-hand] tangent directions and spheric representations. Given a parametric curve φ in $\boldsymbol{R}^m$ on an interval I endless on the right, we shall say that a unit vector p of $\boldsymbol{R}^m$ is the *right-hand tangent direction* of the curve φ at a point c of I, iff for any given ε there exists a δ such that we have both $\varphi(J) \neq 0$ and $\varphi(J) \diamond p < \varepsilon$ whenever J is a closed interval in I whose left-hand endpoint is c and whose length is less than δ. When this is the case, obviously p is uniquely determined and is at the same time the unique right-hand derived direction of φ at c (see § 73).

Again, a spheric curve γ in $\boldsymbol{R}^m$ defined on I will be termed *right-hand spheric representation* of φ and denoted by the symbol φ^R, iff the unit vector $\gamma(t)$ is the right-hand tangent direction of the curve φ at every point t of I. Further, by a *right-hand spherically representable curve* we shall understand one possessing right-hand spheric representation.

By interchanging the words right and left with left and right respectively in the above, we obtain the corresponding definitions for the left-hand concepts and for the notation φ^L.

It is worth while to observe that, even when a curve φ possesses a uniquely determined right-hand derived direction q at a point c, we cannot on that account infer that q is the right-hand tangent direction of φ at c. This may be seen by simple examples.

78. *If a curve φ, defined on an endless interval I, is spherically representable on both sides and if $\varphi^R = \varphi^L$, then φ is spherically representable* (see § 43) *and we have $\hat{\varphi} = \varphi^R = \varphi^L$.*

Proof. Let us write $\psi = \varphi^R = \varphi^L$ for short. Given a point c of I and an arbitrary $\varepsilon < 2^{-1}\pi$ we can find a δ for which we have both $\varphi(J) \neq 0$ and $\varphi(J) \diamond \psi(c) < \varepsilon$ whenever J is a closed interval in I with $|J| < \delta$ and such that c is one of the extremities of J.

Now let $K=[a,b]$ be any closed interval with length $<\delta$, contained in I and containing the point c. We shall show that $\varphi(K)\neq 0$ and that $\varphi(K)\diamond\psi(c)<3\varepsilon$. For this purpose we may plainly assume that $a<c<b$. Writing $J_1=[a,c]$ and $J_2=[c,b]$, we see that $\varphi(J_1)$ and $\varphi(J_2)$ are nonvanishing vectors such that $\varphi(J_i)\diamond\psi(c)<\varepsilon$ $(i=1,2)$. So that $\varphi(J_1)\diamond\varphi(J_2)\leqq\varphi(J_1)\diamond\psi(c)+\varphi(J_2)\diamond\psi(c)<2\varepsilon<\pi$ and consequently $\varphi(K)=\varphi(J_1)+\varphi(J_2)\neq 0$. It therefore follows in virtue of § 23 that

$$\varphi(K)\diamond\varphi(J_1)=\varphi(J_1)\diamond\varphi(J_2)-\varphi(K)\diamond\varphi(J_2)\leqq\varphi(J_1)\diamond\varphi(J_2)<2\varepsilon,$$

whence we get $\varphi(K)\diamond\psi(c)\leqq\varphi(K)\diamond\varphi(J_1)+\varphi(J_1)\diamond\psi(c)<3\varepsilon$. Thus ψ is the spheric representation of φ.

79. $\boldsymbol{C^R}$, $\boldsymbol{C^L}$ and $\boldsymbol{C^{RL}}$ curves. A parametric curve φ defined on an interval I endless on the right [or on the left] will be called to be $\boldsymbol{C^R}$ [or $\boldsymbol{C^L}$] on I iff it is both continuous and right-hand [or left-hand] spherically representable on I. We shall further term a curve defined on an endless interval I to be $\boldsymbol{C^{RL}}$ on I iff it is both $\boldsymbol{C^R}$ and $\boldsymbol{C^L}$ on I.

The symbols $\boldsymbol{C^R}$, $\boldsymbol{C^L}$ and $\boldsymbol{C^{RL}}$ will also be used to denote the respective classes of the relevant curves. Thus $\boldsymbol{C^{RL}}(I)$ is the intersection of $\boldsymbol{C^R}(I)$ and $\boldsymbol{C^L}(I)$ for every endless interval I.

80. *Every light continuous curve φ defined and of bounded bend on an interval I endless on the right is $\boldsymbol{C^R}$ on I.*

PROOF. Take a fixed point c of I. Given an arbitrary $\varepsilon<2^{-1}\pi$ we can, in virtue of § 36, find a closed interval $J=[c,c+\delta]$ in I for which $\Omega(\varphi,J)<\varepsilon$. Since φ is light, it follows from § 60 that φ is biunique on J. Consequently we infer from § 63 that $\varphi(K)\diamond\varphi(J)<\varepsilon$ for every closed interval K in J, and the result follows easily.

REMARK. It is probable that continuity of φ can be dropped simultaneously from hypothesis and conclusion of the above proposition, so that φ is now only asserted to be right-hand spherically representable. But then the proof will become considerably longer.

81. *Every finite real function f defined and of bounded variation on a set $E\subset\boldsymbol{R}$ has a finite right-hand [left-hand] limit at every right-hand [left-hand] accumulation point of E.*

Again, every curve φ defined and rectifiable on a set $E\subset\boldsymbol{R}$ has a finite right-hand [left-hand] limit at every right-hand [left-had] accumulation point of E.

PROOF. By the lemma on p. 221 of Saks, *Theory of the Integral*, the function f coincides on E with a function which is of bounded variation on the whole $\boldsymbol{R}$. Hence the first result. The second result is an immediate consequence of the first and § 38.

82. *If ψ is a parametric curve on a set $E \subset \boldsymbol{R}$ and if we write $F(I) = \mathrm{L}(\psi, I^\circ E)$ for each closed interval $I \subset \boldsymbol{R}$, then F is a nonnegative overadditive interval function on $\boldsymbol{R}$ possessing inside limit property.*

The same result holds for $G(I) = \Lambda(\gamma, I^\circ E)$ in case γ is a spheric curve on E.

So that it follows at once from § 35 *that if ψ is rectifiable, then given any closed interval I and any ε we can find a subdivision Δ of I in such a manner that $F(J) < \varepsilon$ for every interval J pertaining to Δ. Similarly for the function G, if γ is spherically rectifiable.*

Remark. Note that, by § 40, spheric rectifiability of a spheric curve is equivalent to its rectifiability.

Proof. Let $I_1, \cdots, I_n$ $(n \in \boldsymbol{N})$ be n non-overlapping closed intervals in I and take a finite set Δ_i in each $I_i{}^\circ E$. Then $F(I) \geqq \mathrm{L}(\psi, \Delta_1 \cup \cdots \cup \Delta_n) \geqq \sum_{i=1}^{n} \mathrm{L}(\psi, \Delta_i)$, and hence, taking the supremum of the sum, we get $F(I) \geqq F(I_1) + \cdots + F(I_n)$. Thus F is overadditive.

Next let η be any real number $< F(I)$. We shall show that there is a closed interval J in I° for which $F(J) > \eta$. We may clearly assume that $\eta \geqq 0$. There then exists a nondegenerate finite set $\Delta = \{t_0, \cdots, t_k\}$ in $I^\circ E$ such that $t_0 < \cdots < t_k$ and that $\mathrm{L}(\psi, \Delta) > \eta$. Taking δ so small that $J = [t_0 - \delta, t_k + \delta] \subset I^\circ$ we find that $\Delta \subset J^\circ E$ and hence that $F(J) \geqq \mathrm{L}(\psi, \Delta) > \eta$, as announced. Thus $F(I)$ is the supremum of $F(K)$ where K is a generic closed interval in I°, i. e. F has inside limit property.

The assertion for G is proved in an entirely similar way.

83. *Given a continuous curve φ on an endless interval I and a subset M of I such that $I - M$ is countable, suppose that there is a rectifiable curve γ defined on M and which is a right-hand direction curve* (§ 73) *on M of φ. Then φ is $\boldsymbol{C}^{RL}$ on I.*

If, further, ϑ is any direction curve (§ 44) *on I of φ, then $\vartheta(t+) = \varphi^R(t)$ and $\vartheta(t-) = \varphi^L(t)$ at each $t \in I$. So that, in particular, the curves φ^R and φ^L are continuous, respectively on the right and on the left, on I.*

Proof. Noting that every point of I is a left-hand accumulation point of M, we see that the curve γ has by § 81 a finite left-hand limit $\gamma(t-)$ at every $t \in I$. Let now c be a fixed point of I. Given an arbitrary $\varepsilon < 2^{-1}\pi$, we can, by the preceding section, find a δ such that $J = (c - \delta, c) \subset I$ and that $\Lambda(\gamma, JM) < 2^{-1}\varepsilon$. Thus $\gamma(t') \diamond \gamma(t'') < 2^{-1}\varepsilon$ for any pair t', t'' of points of JM, and so $\gamma(t) \diamond \gamma(c-) \leqq 2^{-1}\varepsilon$ for every $t \in JM$. Further, the set $J - JM$ is countable and the subcurve (γ, JM) of γ is a right-hand direction curve on JM of (φ, J). Hence, by § 74, the curve φ is biunique on J and we have $\varphi(K)\diamond\varphi(c-)<\varepsilon$ for every closed interval $K\subset J$.

We shall now show that $\varphi(t)\neq\varphi(c)$ for every $t\in J$. Suppose on the contrary that we had $\varphi(t_1)=\varphi(c)$ for some $t_1\in J$. Fixing a point t_2 of (t_1, c) and writing $J_1=[t_1, t_2]$, $J_2=[t_2, c]$, and $J'=[t_2, t]$ where $t_2<t<c$, we get

$$\varphi(J_1)\diamond\varphi(J')\leqq\varphi(J_1)\diamond\gamma(c-)+\varphi(J')\diamond\gamma(c-)<2\varepsilon\,.$$

By making $t\to c$ here, we find successively that $\varphi(J')\to\varphi(J_2)=-\varphi(J_1)$, $\varphi(J_1)\diamond\varphi(J')\to\varphi(J_1)\diamond\varphi(J_2)=\pi$, and finally that $\pi\leqq 2\varepsilon$, which contradicts the assumption $\varepsilon<2^{-1}\pi$.

The curve φ is thus biunique on $(c-\delta, c]$ and we have $\varphi(K)\diamond\gamma(c-)\leqq\varepsilon$ for any closed interval K in $(c-\delta, c]$. This shows firstly that $\gamma(c-)$ is the left-hand tangent direction of the curve φ at c, so that φ is $\boldsymbol{C}^L$ on I since the point c was taken arbitrarily. Secondly, if ϑ is any direction curve on I of φ, we deduce at once that $\vartheta(t)\diamond\gamma(c-)\leqq\varepsilon$ for every $t\in J$ and hence that $\vartheta(c-)=\gamma(c-)=\varphi^L(c)$. We thus have $\vartheta(t-)=\varphi^L(t)$ everywhere on I.

It can be shown in a similar way that φ is a $\boldsymbol{C}^R$ curve on I, that $\varphi^R(t)=\gamma(t+)$ for every $t\in I$, and that $\vartheta(t+)=\varphi^R(t)$ $(t\in I)$ whenever ϑ is a direction curve on I of φ. But the proof will be shorter since we can dispense with the verification of $\varphi(t)\neq\varphi(c)$ for every $t\in J$. Indeed § 74 will be directly applicable to an interval of the form $[c, c+\delta)$.

REMARK. One might suspect that dropping continuity hypothesis of φ would only result in corresponding deletion of continuity from the conclusion, so that we could at least assert that φ is spherically representable on both sides on I. However, the example given in the remark of § 71 shows that this is not true.

84. *Suppose that ψ is a right-hand continuous curve on a finite interval I which is endless on the right. Given any real number $\eta<\mathrm{L}(\psi)$, there then exists a δ such that $\mathrm{L}(\psi, \Delta)>\eta$ for every δ-net Δ in I (cf. § 56).*

Moreover, if ψ is spheric in addition, then a similar result holds for spheric length, too.

PROOF. Take an ε such that $\eta+\varepsilon<\mathrm{L}(\psi)$. We can find a nondegenerate finite set $\Delta_0\subset I$ such that $\mathrm{L}(\psi, \Delta_0)>\eta+\varepsilon$. Here we may clearly suppose that $\Delta_0\subset I^\circ$, since ψ is right-hand continuous. For the same reason there is a δ such that if Δ is any δ-net in I, then to each point c of Δ_0 there corresponds biuniquely an interval $J=[a, b]$ pertaining to Δ such that $a\leqq c<b$ and that

$$|\psi(a)-\psi(c)|+|\psi(c)-\psi(b)|<|\psi(J)|+n^{-1}\varepsilon\,,$$

where n denotes the number of the points of the set Δ_0. It follows that

$$\eta+\varepsilon<\mathrm{L}(\psi,\Delta_0)\leqq\mathrm{L}(\psi,\Delta\cup\Delta_0)<\sum_K|\psi(K)|+\varepsilon=\mathrm{L}(\psi,\Delta)+\varepsilon,$$

where K represents the intervals pertaining to Δ. Hence $\mathrm{L}(\psi,\Delta)>\eta$, as required.

We can establish similarly the result for spheric length.

85. *If ψ is a right-hand continuous curve on an interval I endless on the right and if M is a dense subset of I, then* $\mathrm{L}(\psi, M)=\mathrm{L}(\psi)$. *Moreover, if ψ is spheric in addition, then* $\Lambda(\psi, M)=\Lambda(\psi)$.

Proof. In virtue of a change of variable $\tau=f(t)$, where f is a real function which is continuous, bounded, and strictly increasing on I, we may suppose from the first that I is a finite interval endless on the right. Then the results follow immediately from the preceding section, since we can, for any δ, take δ-nets Δ in I such that $\Delta\subset M$.

86. Vector-valued interval functions. Let J denote a generic closed interval contained in a given interval I. By a *vector-valued interval function* on I we shall understand a mapping of the class $\{J\}$ into $\boldsymbol{R}^m$. Such a function μ will be termed *additive* iff we have $\mu(J_1\cup J_2)=\mu(J_1)+\mu(J_2)$ whenever J_1 and J_2 are two abutting closed intervals in I. Again, we shall call μ *continuous on J* iff given any ε there is a $\delta=\delta(\varepsilon)$ such that $|\mu(J')|<\varepsilon$ for any closed interval J' in J with length $|J'|<\delta$. We shall further say that μ is *continuous on I* iff it is continuous on every J in I.

If φ is a curve on the interval I, then the interval function $\varphi(J)$ as defined in § 65 is clearly a vector-valued additive interval function in the above sense, and the curve $\varphi(t)$ is continuous on I iff $\varphi(J)$ is continuous on I in the above sense.

On the other hand, given any vector-valued additive interval function μ on I, there exist infinitely many curves φ on I that correspond to μ in the sense of § 65, i. e. $\mu(J)=\varphi(J)$ for every J. Any two of such curves φ obviously differ only by a constant, and hence $\Omega(\varphi)$ and $\mathrm{L}(\varphi)$ are uniquely determined by μ.

87. Given a curve φ on an interval I we shall agree, whenever convenient, to understand the zero vector by $\varphi(E)$ for every degenerate set E of I. In case $E=\{c\}$ is one-pointic, we have accordingly to distinguish strictly between $\varphi(\{c\})$ and $\varphi(c)$.

Given two intervals I and I_0, suppose that φ is a continuous curve defined on I_0, and that f is a continuous nondecreasing mapping of I into I_0. Then the vector-valued interval function ξ defined on I by $\xi(J)=\varphi(f[J])$, where J is a *generic closed interval in I, is continuous and additive on I.*

PROOF. This is evident since ξ corresponds to the point function $\varphi(f(t))$.

88. *Suppose that $\mathfrak{M}$ is a nonvoid countable class of closed intervals contained in a given interval I, and further that to each interval K of $\mathfrak{M}$ there is attached a continuous curve ν_K on K in such a manner that $A=\sum_K \mathrm{O}(\nu_K)<\infty$, where summation is extended over $\mathfrak{M}$ and where $\mathrm{O}(\nu_K)$ denotes the oscillation of ν_K over K, i.e. the diameter of the set $\nu_K[K]$. We define a vector-valued interval function ν on I by $\nu(J)=\sum_K \nu_K(JK)$, where J is a generic closed interval in I. Then ν is additive and continuous on I.*

PROOF. If J_1 and J_2 are a pair of adjacent closed intervals in I, then J_1K and J_2K are so too for each K provided neither of them is degenerate. Hence

$$\nu(J_1\cup J_2)=\sum_K \nu_K(J_1K\cup J_2K)=\sum_K \nu_K(J_1K)+\sum_K \nu_K(J_2K)=\nu(J_1)+\nu(J_2),$$

and so ν is additive.

To show the continuity of ν, let us take, for any given ε, a nonvoid finite subclass $\mathfrak{N}$ of $\mathfrak{M}$ such that $\sum_K' \mathrm{O}(\nu_K)>A-\varepsilon$, where the dash means that the summation is extended over $\mathfrak{N}$. Denoting by n the power of $\mathfrak{N}$, we see at once that there exists a δ such that $|\nu_K(L)|<n^{-1}\varepsilon$ whenever L is a closed interval contained in some $K\in\mathfrak{N}$ and having length $|K|<\delta$. It follows that, for every closed interval $J\subset I$ with length $|J|<\delta$,

$$|\nu(J)|\leqq \sum_K' |\nu_K(JK)|+\varepsilon<\sum_K' n^{-1}\varepsilon+\varepsilon=2\varepsilon.$$

This completes the proof.

89. *Given two endless intervals I and I_0, suppose that φ is a $\boldsymbol{C}^R$ curve defined on I_0, that φ^R is rectifiable (so that φ is also $\boldsymbol{C}^L$ on I_0 by § 83), and further that f is a continuous nondecreasing nonbiunique mapping of I onto I_0. Assume furthermore that $\varphi^R(u)=\varphi^L(u)$ for $u\in I_0$ whenever $f^{-1}(u)$, the inverse image of the point u under the mapping f, is one-pointic. By hypothesis there are points u of I_0 whose inverse images $K=f^{-1}(u)$ are not one-pointic and are therefore closed intervals in I. Denote by $\mathfrak{M}$ the class of all such K and suppose that there is given for each $K=[\alpha,\beta]$ a curve ν_K belonging to $\boldsymbol{C}^*(K)$ and such that we have, writing $f[K]=\{u\}$,*

$$\hat{\nu}_K(\alpha)=\varphi^L(u),\ \hat{\nu}_K(\beta)=\varphi^R(u),\ \Lambda(\hat{\nu}_K)=\hat{\nu}_K(\alpha)\diamond\hat{\nu}_K(\beta).$$

Moreover, let $\sum_K \mathrm{O}(\nu_K)$ be convergent, where (and subsequently) summation is extended over $\mathfrak{M}$, which is clearly a countable class.

Under these conditions, let us define, for each closed interval J in I,

$$\nu(J)=\sum_K \nu_K(JK), \quad \mu(J)=\varphi(f[J])+\nu(J),$$

so that μ *is, by* §§ 87-88, *a vector-valued additive continuous interval function on* I. *Now, if we denote by* ψ *any curve defined on* I *and corresponding to* μ (cf. § 86), *then* ψ *is a* $\boldsymbol{C}^*(I)$ *curve and so we get, by* § 68,

$$\Omega(\psi)=\mathrm{L}(\hat{\psi})=\Lambda(\hat{\psi}).$$

Proof. We remark first that $\hat{\nu}_K$ is, for each K of $\mathfrak{M}$, a continuous curve on K by § 67. This being so, we shall begin by constructing a spheric curve $\tilde{\mu}$ on I as follows. Let t be any point of I and write $\Phi(t)=f^{-1}(u)$ where $u=f(t)$. If $\Phi(t)=\{t\}$, we define $\tilde{\mu}(t)$ to be the common value $\varphi^R(u)=\varphi^L(u)$. Otherwise $\Phi(t)\in\mathfrak{M}$, and we put $\tilde{\mu}(t)=\hat{\nu}_K(t)$ where $K=\Phi(t)$.

We shall show in the sequel that $\tilde{\mu}(c)$ is the right-hand tangent direction of the curve ψ at each $c\in I$. This being evident if c differs from the largest element of $\Phi(c)$, we may suppose the contrary to be the case. Then the function f is not constant on any interval $[c, c+\delta]$ contained in I. Consequently, denoting by t_1 and t_2 respectively the smallest and the largest element of $\Phi(t)$ where $c<t\in I$, we see at once that $c<t_1\leqq t$. It follows that for any $[c, c_1]\subset I$ there exists a subinterval $[c, c_2]\subset[c, c_1]$ such that $c<t_1\leqq t_2<c_1$ whenever $c<t<c_2$.

Now, by § 83 and by rectifiability of φ^R, we can associate with any given $\varepsilon<2^{-1}\pi$ an interval $[c, c_1]\subset I$ such that every point t of the open interval (c, c_1) satisfies both $\varphi^R(u)\diamond\varphi^R(d)<4^{-1}\varepsilon$ and $\varphi^L(u)\diamond\varphi^R(d)<4^{-1}\varepsilon$, where (and subsequently) we write $u=f(t)$ and $d=f(c)$. Let us take, for this point c_1, the point c_2 whose existence we showed just now. Suppose now that $K=\Phi(t)=[t_1, t_2]\in\mathfrak{M}$ for a point t such that $c<t<c_2$. Noting that $\tilde{\mu}(c)=\varphi^R(d)$, $\hat{\nu}_K(t_1)=\varphi^L(u)$, and that $\hat{\nu}_K(t_2)=\varphi^R(u)$, we then find successively, for every $\tau\in K$, that

$$\hat{\nu}_K(\tau)\diamond\hat{\nu}_K(t_1)\leqq\Lambda(\hat{\nu}_K)=\hat{\nu}_K(t_1)\diamond\hat{\nu}_K(t_2)<2^{-1}\varepsilon,$$
$$\hat{\nu}_K(\tau)\diamond\tilde{\mu}(c)\leqq\hat{\nu}_K(\tau)\diamond\hat{\nu}_K(t_1)+\hat{\nu}_K(t_1)\diamond\tilde{\mu}(c)<\varepsilon. \tag{1}$$

We may suppose that the point c_2, considered above and corresponding to the given ε, is so near the point c that, if $J=[c, c_3]$ is any closed interval such that $c<c_3<c_2$ and if we write $Q=f[J]$ for brevity, then $\varphi(Q)\neq 0$ and $\varphi(Q)\diamond\varphi^R(d)<\varepsilon$. Since $\varphi^R(d)=\tilde{\mu}(c)$, the last inequality implies that

$$\varphi(Q)\tilde{\mu}(c)>|\varphi(Q)|\cos\varepsilon. \tag{2}$$

Now let us fix such an interval J. The definition of μ gives $\mu(J)=\varphi(Q)+\nu(J)$, where $\nu(J)=\sum_K \nu_K(JK)$. For each $K\in\mathfrak{M}$ we have two cases to distinguish: If JK is nondegenerate, then JK is clearly a closed interval in (c, c_2), and $K=\Phi(t)$ for each $t\in JK$. Hence we deduce at once from (1) and § 66 that $\nu_K(JK) \neq 0$ and that $\nu_K(JK) \diamond \tilde{\mu}(c) < \varepsilon$, whence

$$\nu_K(JK)\tilde{\mu}(c) \geqq |\nu_K(JK)| \cos \varepsilon . \tag{3}$$

On the other hand, if JK is degenerate, then $\nu_K(JK)=0$ and hence (3) still holds good.

On summing (3) over all $K \in \mathfrak{M}$ it follows that

$$\nu(J)\tilde{\mu}(c) \geqq (\cos \varepsilon) \sum_K |\nu_K(JK)| \geqq (\cos \varepsilon) \,|\sum_K \nu_K(JK)| = |\nu(J)| \cos \varepsilon ,$$

and this in combination with (2) gives

$$\mu(J)\tilde{\mu}(c) = \varphi(Q)\tilde{\mu}(c) + \nu(J)\tilde{\mu}(c) > (|\varphi(Q)| + |\nu(J)|) \cos \varepsilon \geqq |\mu(J)| \cos \varepsilon .$$

We therefore find in the first place that $\mu(J) \neq 0$ and then that $\mu(J) \diamond \tilde{\mu}(c) < \varepsilon$. Thus $\tilde{\mu}(c)$ is the right-hand tangent direction of the curve ψ at the point $c \in I$.

By symmetry we see that $\tilde{\mu}(c)$ is the left-hand tangent direction of ψ at c. Moreover, μ is continuous on I. Hence ψ is $\boldsymbol{C}^*$ on I by § 78, and we have $\hat{\psi} = \tilde{\mu}$. This completes the proof.

90. Continuation. *Under the same hypotheses, we have further* $\Lambda(\hat{\psi}) = \Lambda(\varphi^R) = \Lambda(\varphi^L)$.

Proof. Let U and V be any given neighbourhoods of $\Lambda(\hat{\psi})$ and $\Lambda(\varphi^R)$ respectively, where we understand by a neighbourhood of ∞ any interval of the form $(c, \infty]$, c being a finite positive number. There then exists finite sets $\Delta \subset I$ and $\Delta_0 \subset I_0$ such that $\Lambda(\hat{\psi}, \Delta) \in U$ and that $\Lambda(\varphi^R, \Delta_0) \in V$. We may suppose here that $\Delta_0 \subset f[\Delta]$ and hence that $\Lambda(\varphi^R, f[\Delta]) \in V$, for otherwise we have only to enlarge Δ in a suitable manner. For the same reason we may suppose further that if K is an interval of the class $\mathfrak{M}$ and if Δ contains one at least of the extremities of K, then Δ does both of them. Moreover, we may delete from Δ all the points that lie in the interior of some such $K=[a, b]$. In point of fact, this causes no influence on $f[\Delta]$ and we have, on the other hand, for any subdivision Δ' of K,

$$\Lambda(\hat{\psi}, \Delta') \leqq \Lambda(\hat{\psi}, K) = \Lambda(\hat{\nu}_K) = \hat{\nu}_K(a) \diamond \hat{\nu}_K(b) = \hat{\psi}(a) \diamond \hat{\psi}(b) \leqq \Lambda(\hat{\psi}, \Delta') ,$$

so that $\Lambda(\hat{\psi}, \Delta') = \Lambda(\hat{\psi}, \{a, b\})$. Finally, it is also evident that we may assume $f[\Delta]$ nondegenerate.

This being so, let u be a generic point of $f[\Delta]$. If $f^{-1}(u) = \{t\}$ is one-pointic, then $\hat{\psi}(t) = \varphi^R(u) = \varphi^L(u)$ by hypothesis. While if $f^{-1}(u) = [t_1, t_2]$ is an interval of the class $\mathfrak{M}$, then $\hat{\psi}(t_1) = \varphi^L(u)$ and $\hat{\psi}(t_2) = \varphi^R(u)$. Denoting by $J = [\alpha, \beta]$ a typical interval pertaining to $f[\Delta]$, we consequently obtain

$$\Lambda(\hat{\psi}, \Delta) = \sum_J \varphi^R(\alpha) \diamond \varphi^L(\beta) + \sum_u \varphi^L(u) \diamond \varphi^R(u) .$$

Now we know by § 83 that $\varphi^L(u)=\varphi^R(u-)$ for every $u\in f[\Delta]$. Hence, by making $\delta\rightarrow 0$ in the obvious inequalities

$$\Lambda(\varphi^R, f[\Delta])\leqq\sum_J\varphi^R(\alpha)\diamond\varphi^R(\beta-\delta)+\sum_u\varphi^R(u-\delta)\diamond\varphi^R(u)\leqq\Lambda(\varphi^R),$$

where $\delta<\|f[\Delta]\|$ is so small as to ensure $u-\delta\in I_0$ for every $u\in f[\Delta]$, we derive at once $\Lambda(\varphi^R, f[\Delta])\leqq\Lambda(\hat{\psi},\Delta)\leqq\Lambda(\varphi^R)$. It follows that $\Lambda(\hat{\psi},\Delta)\in V$ and hence that the neighbourhoods U and V intersect. Since these intervals have been chosen arbitrarily at the beginning, we must find that $\Lambda(\hat{\psi})=\Lambda(\varphi^R)$, and this, in conjunction with $\Lambda(\hat{\psi})=\Lambda(\varphi^R)$, which follows by symmetry, completes the proof.

91. *Every finite nondecreasing function f, defined on a set $E\subset\boldsymbol{R}$ and such that $f[E]$ is an interval, is continuous on E.*

Proof. Fixing a point c of E, we show that, given any ε, there exists a δ such that $f(t)<f(c)+\varepsilon$ whenever t is a point of E fulfilling $t<c+\delta$. Indeed, this is evident when $f(c)$ is the largest element of $f[E]$. When this is not the case, then we can, by hypothesis, find a δ fulfilling both $c+\delta\in E$ and $f(c+\delta)<f(c)+\varepsilon$, and this δ clearly has the required propertly. The function f is thus right-hand continuous on E. In the same way we see that it is left-hand continuous on E, and this completes the proof.

92. *Suppose that to each point u of a given interval I_0 there is attached a linear continuum $\Psi(u)\subset\boldsymbol{R}$ in such a manner that $\Psi(u_1)<\Psi(u_2)$ whenever u_1 and u_2 are a pair of points of I_0 such that $u_1<u_2$. Here we understand by $\Psi(u_1)<\Psi(u_2)$ that $t_1<t_2$ for any $t_1\in\Psi(u_1)$ and any $t_2\in\Psi(u_2)$. Let us denote by P the sum of all the sets $\Psi(u)$. For each point t of P there plainly exists one and only one point $u\in I_0$ for which $t\in\Psi(u)$, and we write f for the function that associates with t this uniquely determined point u. Then f is a continuous nondecreasing function on P.*

Proof. Write $u_1=f(t_1)$ and $u_2=f(t_2)$ for any pair t_1, t_2 of points of P such that $t_1<t_2$. Then $t_1\in\Psi(u_1)$, $t_2\in\Psi(u_2)$, and hence $u_1\leqq u_2$, since otherwise we should have $\Psi(u_2)<\Psi(u_1)$ and in particular $t_2<t_1$. Thus f is nondecreasing on P. Since $f[P]=I_0$ is an interval, continuity of f is an immediate consequence of the preceding §.

93. *Given a finite strictly increasing function g on an endless interval I_0, let us attach to each point u of I_0 a linear continuum $\Psi(u)$ by setting $\Psi(u)=\{g(u)\}$ or $\Psi(u)=[g(u-), g(u+)]$ according as g is respectively continuous or discontinuous at the point u. Then $\Psi(u)$ increases together with u, and the sum P of all the sets $\Psi(u)$ is an endless interval.*

Proof. We first remark that $\Psi(u)$ is, by definition, the set of the points

t satisfying $g(u-) \leqq t \leqq g(u+)$ and that, in particular, $g(u) \in \Psi(u)$. Now, given any pair of points u_1, u_2 of I_0 fulfilling $u_1 < u_2$, take a u_0 such that $u_1 < u_0 < u_2$. Then $g(u_1+) < g(u_0) < g(u_2-)$, and hence $\Psi(u_1) < \Psi(u_2)$. This proves the first half of the assertion.

The second half will be established if we show that the set P is convex, for then P, being a nondegenerate set, must be an interval and clearly an endless one. For this purpose suppose that $t_1 \in \Psi(u_1)$, $t_2 \in \Psi(u_2)$, and that $t_1 < t_2$. We have to show that $J = [t_1, t_2] \subset P$. But this is evident when $u_1 = u_2$, since then $J \subset \Psi(u_1) \subset P$. Hence we assume that $u_1 \neq u_2$, so that we must have $u_1 < u_2$ and $\Psi(u_1) < \Psi(u_2)$.

Suppose now, if possible, that there were a point $t_0 \in J$ that did not belong to P. Then $\Psi(u_1)$ and $\Psi(u_2)$ will lie respectively on the left and on the right of t_0, and so in particular $g(u_1) < t_0 < g(u_2)$. We define two disjoint subsets G_1 and G_2 of I_0 as follows. G_1 is the set of the points $u \in I_0$ for which $g(u) < t_0$. G_2 is defined similarly, only that we require $g(u) > t_0$ instead. These sets are then nonvoid, since $u_1 \in G_1$ and $u_2 \in G_2$. Further, they cover together the interval I_0, and moreover, if u is any point of G_1, then all the elements of I_0 that are smaller than u plainly belong to G_1. G_1 and G_2 thus constitute a Dedekind cut for I_0, and so we have the alternatives: either G_1 possesses a largest element u_0 or else G_2 possesses a smallest element u_0. We may plainly restrict ourselves to the former case and then find that $g(u_0-) \leqq g(u_0) < t_0 \leqq g(u_0+)$, since all points of I_0 that are situated on the right of u_0 belong to G_2. We thus arrive at the contradiction $t_0 \in \Psi(u_0) \subset P$. This completes the proof.

94. *If E is a countable set contained in a given endless interval I_0, then there exists a continuous nondecreasing function f, defined on an endless interval I and mapping I onto I_0, and such that the inverse image $f^{-1}(u)$ of a point u of I_0 is nondegenerate, and hence a closed interval, when and only when $u \in E$.*

PROOF. We may clearly suppose E nonvoid. It is enough to construct a finite strictly increasing function g defined on I_0 and such that E coincides with the set of the points of discontinuity of g. In fact, applying the preceding two sections to g, we get immediately the required function f.

For this purpose we choose in the first place a finite positive function h defined on E and with the property that $\sum_v h(v)$ is convergent, where (and subsequently) v stands for a generic point of E. Let us now define $g(u) = u + \sum_{v<u} h(v)$ for each $u \in I_0$, on the usual understanding that a void sum $\sum$ means zero. It follows at once that, if c is any fixed point of I_0 and if δ is so small that $c \pm \delta$ both belong to I_0, then

$$g(c+\delta) - g(c-\delta) = 2\delta + \sum_{c-\delta \leqq v < c+\delta} h(v) > 0\,.$$

Hence g is strictly increasing on I, and further, by making $\delta \to 0$, we find that $g(c+)-g(c-)=h(c)>0$ when $c \in E$ and that $g(c+)-g(c-)=0$ when $c \in I_0 - E$. This completes the proof.

95. We are now in a position to prove the principal result of the present paper, which has already been announced, though in a somewhat simplified form, in the introduction.

THEOREM. *Given an interval I_0 endless on the right and a subset M of I_0 such that $I_0 - M$ is countable, suppose that φ is a continuous curve on I_0 and that γ, a spheric curve defined on M, is a right-hand direction curve of φ. Then $\Omega(\varphi)=\Lambda(\gamma)$, or in other words, the bend of φ is equal to the spheric length of γ.*

PROOF. Since $\Omega(\varphi) \geqq \Lambda(\gamma)$ by § 45, we need only show that $\Omega(\varphi) \leqq \Lambda(\gamma)$, where we may clearly suppose the curve γ rectifiable. We may assume further that I_0 is an endless interval. For if not, we should merely have to replace I_0 by its interior, in view of the remark given in § 32 and also of monotonity of spheric length. It follows from § 83 that φ is $\boldsymbol{C}^{RL}$ on I_0, and from § 85 that $\Lambda(\gamma)=\Lambda(\varphi^R)$, since we must have $\gamma(u)=\varphi^R(u)$ for every $u \in M$ and since φ^R is continuous on the right by § 83.

Thus our task comes to proving $\Omega(\varphi) \leqq \Lambda(\varphi^R)$ under the hypotheses that φ is a $\boldsymbol{C}^{RL}$ curve on an endless interval I_0 and that φ^R is rectifiable. But if φ^R is in addition continuous on I_0, then φ is $\boldsymbol{C}^*$ on I_0 by § 78 and § 83, and hence $\Omega(\varphi)=\Lambda(\hat{\varphi})=\Lambda(\varphi^R)$ by § 68. We may therefore suppose in the sequel that the set E of the points of discontinuity of φ^R is nonvoid. On the other hand E is plainly countable. Moreover, it follows from § 83 that $\varphi^R(u)=\varphi^L(u)$ whenever $u \in I_0 - E$.

This being so, let us construct, in conformity with the preceding section, a continuous nondecreasing mapping f of an endless interval I onto I_0 such that $f^{-1}(u)$ $(u \in I_0)$ is a closed interval iff $u \in E$, and let us denote such a closed interval by K generically. Now, given any real number $\eta < \Omega(\varphi)$, we can choose a finite subset $\varDelta_0$ of I_0 in such a manner that $\Omega(\varphi, \varDelta_0) > \eta$. Taking a finite subset $\varDelta$ of I such that $f[\varDelta] \supset \varDelta_0$ and defining a curve ξ on I by $\xi(t)=\varphi(f(t))$ for $t \in I$, we see at once that $\Omega(\xi, \varDelta) \geqq \Omega(\varphi, \varDelta_0) > \eta$.

We now attach to each interval K a curve ν_K of the class $\boldsymbol{C}^*(K)$ in such a manner that

$$\hat{\nu}_K(\alpha)=\varphi^L(d),\ \hat{\nu}_K(\beta)=\varphi^R(d),\ \Lambda(\hat{\nu}_K)=\hat{\nu}_K(\alpha) \diamond \hat{\nu}_K(\beta),$$

where we write $K=[\alpha, \beta]$ and $f[K]=\{d\}$. For this purpose we have two cases to distinguish, according as $\varphi^L(d)+\varphi^R(d)$ vanishes or not. When it vanishes, we choose a unit vector p which is perpendicular to $\varphi^L(d)$ and we define, for $t \in K$,

$$\nu_K(t) = \rho_K \int_c^t \Big[\varphi^L(d) \cos \frac{\pi(\tau-\alpha)}{\beta-\alpha} + p \sin \frac{\pi(\tau-\alpha)}{\beta-\alpha} \Big] d\tau ,$$

where ρ_K is a positive constant and c is a fixed point of K. Then ν_K is clearly $\boldsymbol{C}^*$ on K and we find, by § 76 and § 5, that

$$\Lambda(\hat{\nu}_K) = \mathrm{L}(\hat{\nu}_K) = \int_K |\nu_K'(\tau)|^{-2} \cdot |\nu_K'(\tau) \times \nu_K''(\tau)| \, d\tau = \pi = \varphi^L(d) \diamond \varphi^R(d) .$$

Hence ν_K fulfils the required conditions. On the other hand, when $\varphi^L(d) + \varphi^R(d) \neq 0$, we define, for $t \in K$,

$$\nu_K(t) = \rho_K \int_c^t \Big[\varphi^L(d) \frac{\beta-\tau}{\beta-\alpha} + \varphi^R(d) \frac{\tau-\alpha}{\beta-\alpha} \Big] d\tau ,$$

with the same meanings of ρ_K and c as above. Here again, ν_K is $\boldsymbol{C}^*$ on K, and $\Lambda(\hat{\nu}_K) = \mathrm{L}(\hat{\nu}_K)$ by § 76. But direct computation gives $\mathrm{L}(\hat{\nu}_K) = \varphi^L(d) \diamond \varphi^R(d)$, the details being exactly the same as at the end of § 53. Thus ν_K has the required properties in both cases.

Adjusting the positive constants ρ_K, involved in the definition of ν_K, so as to make $\sum_K \mathrm{O}(\nu_K) < \delta$ where δ is any given positive number, we now apply the results of §§ 89-90 to our present situation. We thus obtain a $\boldsymbol{C}^*(I)$ curve ψ such that $\psi(J) = \xi(J) + \nu(J)$, where J is any closed interval in I, while the curve ξ has been defined above and ν is a vector-valued interval function on I given by $\nu(J) = \sum_K \nu_K(JK)$, so that $|\nu(J)| \leqq \sum_K \mathrm{O}(\nu_K) < \delta$. Recalling how we have fixed the finite set $\Delta \subset I$ in the above, we easily see that, if δ is sufficiently small, then we must have $\Omega(\psi, \Delta) > \eta$. But $\Lambda(\hat{\psi}) = \Omega(\psi) \geqq \Omega(\psi, \Delta)$ by § 89, while § 90 gives $\Lambda(\hat{\psi}) = \Lambda(\varphi^R)$. We have thus shown that $\Lambda(\varphi^R) > \eta$ for any real number $\eta < \Omega(\varphi)$, and hence that $\Omega(\varphi) \leqq \Lambda(\varphi^R)$. This completes the proof.

Remark. Continuity of φ is essential for the validity of the theorem, as is shown by the example given in the remark of § 71. We may also observe that, under the hypotheses of the theorem, the curve φ must be light on I_0. For if φ were constant on an open subinterval I' of I_0, then φ could possess a right-hand derived direction at no points of the interval I', which is clearly a contradiction.

96. Theorem. *Let I_0 be an interval which is endless on the right. In order that a light continuous curve φ defined on I_0* (cf. § 51) *be of bounded bend, it is necessary and sufficient that φ should be a $\boldsymbol{C}^R(I_0)$ curve for which φ^R is rectifiable. If this condition is satisfied, then $\Omega(\varphi) = \Lambda(\varphi^R)$.*

Proof. This follows directly from § 80 and the preceding section.

Remark. If we only assume that φ is a light curve on I_0 possessing rectifiable right-hand spheric representation, then we cannot infer that φ is of bounded bend. To see this, we have merely to consider the curve φ_0 in the plane $\boldsymbol{R}^2$ defined on the real line $\boldsymbol{R}$ by $\varphi_0(t)=\langle t,[t]\rangle$, where $[t]$ means as usual the notation of Gauss.

97. We shall conclude the present research with a theorem which, together with the one proved at the end of § 69, constitutes an essential extension of the Fenchel inequality (cf. [**2**]).

THEOREM. *Let φ be a light continuous curve defined on $\boldsymbol{R}$, periodic with a period $\omega>0$. Then clearly φ possesses at least one right-hand direction curve γ on $\boldsymbol{R}$ which is periodic with period ω. For any such γ, we assert that $\Lambda(\gamma,I)\geqq 2\pi$, where I is an arbitrary closed interval of length ω.*

PROOF. We shall begin by showing that $\Lambda(\gamma,I)$ is independent of the choice of I. For this purpose, let $I=[a,b]$ and $I'=[a',b']$, where $b-a=b'-a'=\omega$. In view of periodicity of γ, we may suppose without loss of generality that $a<a'<b$. Then $\Lambda(\gamma,[a,a'])=\Lambda(\gamma,[b,b'])$, and so it follows from additivity of spheric length that

$$\begin{aligned}\Lambda(\gamma,I')&=\Lambda(\gamma,[a',b])+\Lambda(\gamma,[b,b'])\\&=\Lambda(\gamma,[a',b])+\Lambda(\gamma,[a,a'])=\Lambda(\gamma,I)\,.\end{aligned}$$

To establish $\Lambda(\gamma,I)\geqq 2\pi$, we may plainly suppose that $\Lambda(\gamma,I)<\infty$. By additivity of Λ, the curve γ is then rectifiable on every finite interval. Hence γ is continuous except at most at the points of a countable set. We may thus assume that the extremities a,b of I are points of continuity of γ. Now write, for the real numbers $t>a$, $F(t)=\Lambda(\gamma,J)$, where $J=[a,t]$. We then see from the last paragraph of § 41 that F is continuous at b and hence that $\Lambda(\gamma,I)=F(b)=F(b+)$. But it follows from § 95 and § 61 that $F(t)\geqq\Lambda(\gamma,J^\circ)=\Omega(\varphi,J^\circ)\geqq 2\pi$, whenever $t>b$. Consequently $F(b+)\geqq 2\pi$, and this completes the proof.

REMARK. Continuity of φ cannot be removed as is seen at once by considering the example given in the remark of § 71.

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References

[1] Lamberto Cesari, Surface Area (No. 35 of Ann. of Math. Studies), Princeton Univ. Press, 1956.

[2] Werner Fenchel, Über Krümmung und Winding geschlossener Raumkurven, Math. Ann., 101 (1929), 238-252.

[3] Tibor Radó, Length and Area (Vol. 30 of Amer. Math. Soc. Colloquium Publications), Amer. Math. Soc., 1948.

[4] Stanisław Saks, Theory of the Integral, Second Revised Edition Hafner Publishing Company, 1937.