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Perturbation of continuous spectra by unbounded operators, I.

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§ 1. Introduction and theorems.

1. Introduction. Recently Kato proved in [**5**], among others, that the absolutely continuous part of the spectrum of a self-adjoint operator H_0 is stable under the addition of a bounded self-adjoint perturbation V with finite trace norm. So far as we impose the assumption on V irrespective of H_0, this theorem was shown to be the best possible one in the sense that "trace norm" can not be replaced by any other "cross norm" for bounded operators (Kuroda [**9**]). The main purpose of the present paper is to generalize the above mentioned theorem of Kato in another direction so as to include those unbounded perturbations which are *relatively* small with respect to H_0. In this generalized form we can apply it to some problems of differential operators, especially to the Schrödinger operator of quantum mechanics.

On the other hand, the stability of the continuous spectra is closely connected with the asymptotic properties of the family of unitary operators $\{\exp(itH)\exp(-itH_0)\}$, where H is the perturbed operator, in other words, with the existence of the so-called wave and scattering operators in quantum mechanics. The relations between these two seemingly different concepts are given, for example, in the previous paper of the writer (see Kuroda [**10**] and the references given in [**10**]). According to it, the stability of "continuous spectra" is established if we prove the existence of the wave operators, the definition of which will be given in the next paragraph. We shall study the problem from this point of view. The application of our theorem gives an existence proof of these operators in some problems of quantum mechanics.

2. Unitary equivalence and the wave operator. Let $\mathfrak{H}$ be a Hilbert space and H_0 and H self-adjoint operators in $\mathfrak{H}$; let $\mathfrak{M}_0$ and $\mathfrak{M}$ be the absolutely continuous subspaces of $\mathfrak{H}$ with respect to H_0 and H[1]; and let P_0 and P be

1) For the definition of the absolutely continuous subspace, see e.g. Kato [**4**], Kuroda [**10**]. We agree that a "subspace" always means a closed subspace.

(orthogonal) projections on $\mathfrak{M}_0$ and $\mathfrak{M}$, respectively.[2] Put

(1.1) $$U_t = U_t(H, H_0) = \exp(itH)\exp(-itH_0),\ -\infty < t < +\infty\ .$$

The *generalized wave operator* $W_\pm$ is then defined by[3]

(1.2) $$W_\pm = W_\pm(H, H_0) = \underset{t\to\pm\infty}{\text{s-lim}}\ U_t(H, H_0)P_0\ ,$$

whenever the respective limit on the right-hand side exists. When both W_+ and W_- exist the *generalized scattering operator* is defined by[3]

(1.3) $$S = W_+^* W_- = W_+(H, H_0)^* W_-(H, H_0)\ .$$

Some of the fundamental properties of $W_\pm$ and S were investigated in Kuroda [**10**] and the following lemma summarizes those results of [**10**] which will be used frequently in the sequel.

LEMMA 1.1. i) *If $W_+ = W_+(H, H_0)$ exists, W_+ is a partially isometric operator with the initial set $\mathfrak{M}_0$ and the final set contained in $\mathfrak{M}$. Furthermore, W_+ satisfies the relations*

(1.4) $$\exp(itH)W_+ = W_+\exp(itH_0),\ -\infty < t < +\infty,\ HPW_+ = W_+H_0P_0\ .$$

ii) *If $W_+(H, H_0)$ and $W_+(H_0, H)$ exist, then $W_+\mathfrak{H} = \mathfrak{M}$ and the parts of H_0 and H in $\mathfrak{M}_0$ and $\mathfrak{M}$ are unitarily equivalent. Furthermore, we have*

(1.5) $$W_+(H, H_0)^* = W_+(H_0, H)\ .$$

iii) *If $W_\pm(H, H_0)$ and $W_\pm(H_0, H)$ all exist, then the scattering operator S is a partially isometric operator with the initial and the final sets both identical with $\mathfrak{M}_0$; the part of S in $\mathfrak{H}\ominus\mathfrak{M}_0$ is equal to zero and that in $\mathfrak{M}_0$ is unitary.*

iv) *If both $W_+(H_1, H_0)$ and $W_+(H_2, H_1)$ exist, then $W_+(H_2, H_0)$ also exists and we have*

(1.6) $$W_+(H_2, H_0) = W_+(H_2, H_1)\ W_+(H_1, H_0)\ .$$

The same assertions as i), ii) *and* iii) *hold true for W_- in place of W_+.*

By virtue of ii) of this lemma we see that, in order to prove the unitary equivalence of the absolutely continuous parts of H_0 and H, it suffices to prove the existence of $W_\pm(H, H_0)$ and $W_\pm(H_0, H)$. In the following, therefore, we shall be mainly concerned with the problem of finding the sufficient condition for the existence of $W_\pm$ (Theorem 1). We shall also examine the continuity properties of $W_\pm(H, H_0)$ with respect to H and H_0 (Theorem 2). We shall first state those notations and concepts which are needed in the sequel.

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2) We agree in the following that $\mathfrak{M}_0$, $\mathfrak{M}$, P_0 and P have the same meaning as defined above and we use these notations without any comment.

3) Kuroda [**10**, § 3.1].

3. Schmidt and trace class. Let $\boldsymbol{B}$ be the set of all bounded linear operators on $\mathfrak{H}$ to $\mathfrak{H}$. We denote by $\|A\|$, $\|A\|_2$ and $\|A\|_1$ the ordinary norm, the Schmidt norm and the trace norm of an operator $A\in\boldsymbol{B}$, respectively.[4] More precisely, $\|A\|$, $\|A\|_2$ and $\|A\|_1$ are given respectively by $\|A\|=\sup\|A\varphi\|/\|\varphi\|$,

$$\|A\|_2=(\sum_{\nu}\|A\varphi_\nu\|^2)^{1/2}, \tag{1.7}$$

where $\{\varphi_\nu\}$ is an arbitrary complete orthonormal set of $\mathfrak{H}$, and

$$\|A\|_1=\||A|^{1/2}\|_2^2, \quad \text{where} \quad |A|=(A^*A)^{1/2}. \tag{1.8}$$

The fundamental relations between these norms are as follows:

$$\begin{cases} \|A\|\leqq\|A\|_2\leqq\|A\|_1, \\ \|AB\|_i\leqq\|A\|\,\|B\|_i, \quad \|AB\|_i\leqq\|A\|_i\|B\|, \quad i=1,2, \\ \|AB\|_1\leqq\|A\|_2\|B\|_2, \\ \|A^*\|=\|A\|, \quad \|A^*\|_i=\|A\|_i, \quad i=1,2. \end{cases} \tag{1.9}$$

The sets of all $A\in\boldsymbol{B}$ with finite $\|A\|_2$ and with finite $\|A\|_1$ are called the *Schmidt class* and the *trace class* and denoted by $\boldsymbol{S}$ and $\boldsymbol{T}$, respectively. Clearly $\boldsymbol{T}\subset\boldsymbol{S}\subset\boldsymbol{B}$ and $\boldsymbol{S}$, $\boldsymbol{T}$ form two sided ideals of $\boldsymbol{B}$. $\boldsymbol{S}$ consists solely of completely continuous operators. By (1.8) $A\in\boldsymbol{T}$ if and only if $|A|^{1/2}\in\boldsymbol{S}$. We denote by $\boldsymbol{T}_s$ the set of all self-adjoint elements of $\boldsymbol{T}$. Let $A\in\boldsymbol{T}_s$ and let $\lambda_1, \lambda_2,\cdots$ be the sequence of all non-zero eigenvalues of A (degenerate eigenvalues being repeated). Then $\|A\|_1$ is given by

$$\|A\|_1=\sum_{k=1}^{\infty}|\lambda_k|. \tag{1.10}$$

4. Theorems.

THEOREM 1. *Let H_0 be a self-adjoint operator in $\mathfrak{H}$ and let V be a symmetric operator in $\mathfrak{H}$ such that*

$$\begin{cases} \mathfrak{D}\equiv\mathfrak{D}(H_0)\subset\mathfrak{D}(V)^{5)} \quad \textit{and} \\ \|Vu\|\leqq a\|H_0u\|+b\|u\| \quad \textit{for any } u\in\mathfrak{D}, \end{cases} \tag{1.11}$$

where a and b are constants such that $0\leqq a<1$ and $0\leqq b$. Then $H=H_0+V$ is self-adjoint. If in addition

$$|V|^{1/2}(H_0-\zeta_0)^{-1}\in\boldsymbol{S} \tag{1.12}$$

for some number $\zeta_0\in\Delta(H_0)$[5]*, then $W_\pm(H,H_0)$ and $W_\pm(H_0,H)$ exist.*

THEOREM 2. *Let H_0 be a fixed self-adjoint operator and let V and V_n, $n=1,2,\cdots$, be symmetric operators satisfying the conditions* (1.11) *and* (1.12) *with*

4) For details about the Schmidt and the trace norm, see e.g. Schatten [**13**].

5) $\mathfrak{D}(H)$ and $\Delta(H)$ denote the domain and the resolvent set of the operator H.

constants a and b independent of n. Furthermore, let $V_n' = V - V_n$ *and*

$$|V_n'|^{1/2}(H_0-\zeta_0)^{-1} \in \boldsymbol{S}, \quad n=1,2,\cdots, \tag{1.13}$$

$$\lim_{n\to\infty} \| |V_n'|^{1/2}(H_0-\zeta_0)^{-1}\|_2 = 0 \tag{1.14}$$

for some $\zeta_0 \in \Lambda(H_0)$. *Put* $H = H_0 + V$, $H_n = H_0 + V_n$ *and* $S(H) = W_+(H, H_0)^* W_-(H, H_0)$. *Then we have*

$$\operatorname*{s-lim}_{n\to\infty} W_\pm(H_n, H_0) = W_\pm(H, H_0), \tag{1.15}$$

$$\operatorname*{w-lim}_{n\to\infty} W_\pm(H_0, H_n) = W_\pm(H_0, H), \tag{1.16}$$

$$\operatorname*{s-lim}_{n\to\infty} S(H_n) = S(H). \tag{1.17}$$

REMARK. When $V \in \boldsymbol{T}_s$, V satisfies the assumptions of Theorem 1, irrespective of H_0. In fact, (1.11) is obvious because $V \in \boldsymbol{B}$; $V \in \boldsymbol{T}_s$ means $|V|^{1/2} \in \boldsymbol{S}$, which implies (1.12) in view of (1.9). Thus Theorem 1 includes Kato's theorem mentioned in § 1 as a special case. When $V \in \boldsymbol{T}_s$, $V_n \in \boldsymbol{T}_s$ and (1.14) is replaced by a stronger condition $\lim \| V_n' \|_1 = 0$, then Theorem 2 is essentially identical with Theorem 2 of Kato [**5**].

Theorem 1 can be proved by a limiting process based on the special case of $V \in \boldsymbol{T}_s$ already proved by Kato. According to Kato [**5**], however, this special case is further reduced to the case of V of finite rank. The proof of such a case was originally given in Kato [**4**] and afterwards simplified in Kato [**6**]. Since the simplified proof was only given in Japanese, we shall restate it in § 3 with his permission. Then in § 4 we shall prove Theorem 1 by reducing it directly to the case of V of finite rank.

5. Application. In the previous paper of the writer[6] we considered an application of Theorems 1 and 2 to a partial differential operators of Schrödinger type in connection with the scattering theory of quantum mechanics. Let E_m be m-dimensional Euclidean space with $m \leqq 3$, $\mathfrak{H} = L^2(E_m)$ and $V(x)$ is a real-valued measurable function belonging to $L^2(E_m) \cap L^1(E_m)$. Consider a partial differential operator $(H_0 u)(x) = -(\Delta u)(x)$, $\Delta = \sum_{i=1}^{m} \partial^2/\partial x_i^2$, and a multiplicative operator $(Vu)(x) = V(x)u(x)$ both properly defined in $L^2(E_m)$. Then according to the previous results, H_0 and V satisfy all the assumptions of Theorem 1 and hence we conclude by Lemma 1.1 and Theorem 1 that the absolutely continuous parts of the differential operators $-\Delta + V(x)$ and $-\Delta$ in $L^2(E_m)$, $m \leqq 3$, are unitarily equivalent. We can also apply Theorem 2 to this problem.[6]

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6) Kuroda [**10**, § 5].

§ 2. Some lemmas.

In this section we collect several lemmas which will be of frequent use in the following.

LEMMA 2.1[7]. *Let $H=\int \lambda dE(\lambda)$ be a self-adjoint operator and let $u \in \mathfrak{M}$ satisfy*

$$d\| E(\lambda)u\|^2/d\lambda = d(E(\lambda)u, u)/d\lambda \leqq m^2 \quad a.e. \tag{2.1}$$

for some constant $m^2 \geqq 0$ (for the meaning of $\mathfrak{M}$ see footnote 2)). Then we have for any $A \in \boldsymbol{S}$

$$\int_{-\infty}^{\infty} \| A \exp(-itH)u\|^2 dt \leqq 2\pi m^2 \| A \|_2^2 . \tag{2.2}$$

LEMMA 2.2. *Let H and u be as in Lemma 2.1; let $A_n \in \boldsymbol{S}$, $n=1,2,\cdots$, $A \in \boldsymbol{S}$; and let $\| A_n - A\|_2 \to 0$, $n \to \infty$. Then we have for every s and t, $-\infty \leqq s, t \leqq +\infty$,*

$$\lim_{n\to\infty} \int_s^t \| A_n \exp(-itH)u\|^2 dt = \int_s^t \| A \exp(-itH)u\|^2 dt . \tag{2.3}$$

PROOF. By the preceeding lemma, the functions $f_n(t) = \| A_n \exp(-itH)u\|$ and $f(t) = \| A \exp(-itH)u\|$ belong to $L^2(-\infty, +\infty)$ and, a fortiori, to $L^2(s, t)$. By virtue of the triangle inequalities and (2.2) we then obtain

$$\int_s^t |f(t)-f_n(t)|^2 dt \leqq \int_s^t \| (A_n - A)\exp(-itH)u\|^2 dt$$

$$\leqq 2\pi m^2 \| A_n - A\|_2^2 \to 0 , \quad n\to\infty ,$$

which means that $f_n \to f$ in $L^2(s, t)$. Then (2.3) follows immediately from the continuity of the norm in $L^2(s, t)$. q. e. d.

As is mentioned in Kato [**5, 6**] the set of such u as stated in Lemma 2.1 is dense in $\mathfrak{M}$ if the number m^2 is varied over all positive numbers. We need a somewhat stronger result.

LEMMA 2.3. *Let H and $E(\lambda)$ be as above and let $f(\lambda)$ be an H-measurable function defined almost everywhere with respect to H (Stone [**14**,* Definition 6.3]*). Let $\mathfrak{L}$ be the set of all elements $u \in \mathfrak{M} \cap \mathfrak{D}(f(H))$ such that $d\| E(\lambda)f(H)u\|^2/d\lambda \leqq m^2$ a.e. and $\{E(l)-E(-l)\}u = u$ for some positive numbers m^2 and l (both depending on u). Then $\mathfrak{L}$ is dense in $\mathfrak{M}$.*

PROOF. Since $\mathfrak{D}(f(H))$ is dense in $\mathfrak{H}$ by hypothesis (see Stone [**14**, Theorem 6.4]), $\mathfrak{M} \cap \mathfrak{D}(f(H)) = P\mathfrak{D}(f(H))$ is dense in $\mathfrak{M}$. Let $u \in \mathfrak{M} \cap \mathfrak{D}(f(H))$ and let $\chi_n(\lambda)$ $(n=1,2,\cdots)$ be the function which is equal to 1 if $|\lambda| < n$ and $d\|E(\lambda)f(H)u\|^2/d\lambda \leqq m^2$ are satisfied and vanishes otherwise. Then, as is easily seen, $u_n = \chi_n(H)u \in \mathfrak{L}$ and $u_n \to u$, $n\to\infty$. This means that $\mathfrak{L}$ is dense in

7) Rosenblum [**12**], Kato [**5**].

$\mathfrak{M}\cap\mathfrak{D}(f(H))$ and consequently in $\mathfrak{M}$. q. e. d.

Next we state several inequalities which play fundamental rôles in the proof of Theorems 1 and 2. We begin with auxiliary propositions.

PROPOSITION 2.1. *Let H and H' be closed operators such that $\mathfrak{D}(H)\supset\mathfrak{D}(H')$. Then $(H-\zeta)(H'-\zeta')^{-1}\in \boldsymbol{B}$ for any complex number ζ and any number ζ' which belongs to the resolvent set of H'.*

PROOF. By hypothesis $(H-\zeta)(H'-\zeta')^{-1}$ is defined everywhere in $\mathfrak{H}$. It is closed because $H-\zeta$ is closed and $(H'-\zeta')^{-1}\in \boldsymbol{B}$. Hence we have $(H-\zeta)(H'-\zeta')^{-1}\in \boldsymbol{B}$ by virtue of Banach's theorem. q. e. d.

PROPOSITION 2.2. *Let H_0, V and H be as in Theorem 1. Then we have $|V|^{1/2}(H_0-\zeta)^{-1}\in \boldsymbol{S}$ for every $\zeta\in\Lambda(H_0)$ and $|V|^{1/2}(H-\zeta)^{-1}\in \boldsymbol{S}$ for every $\zeta\in\Lambda(H)$.*

PROOF. Since $\mathfrak{D}(H)=\mathfrak{D}(H_0)$ we have

$$|V|^{1/2}(H-\zeta)^{-1}=|V|^{1/2}(H_0-\zeta_0)^{-1}(H_0-\zeta_0)(H-\zeta)^{-1}.$$

On the right-hand side $|V|^{1/2}(H_0-\zeta_0)^{-1}\in \boldsymbol{S}$ by (1.12) and $(H_0-\zeta_0)(H-\zeta)^{-1}\in \boldsymbol{B}$ by Proposition 2.1. Hence their product $|V|^{1/2}(H-\zeta)^{-1}$ belongs to $\boldsymbol{S}$ (see § 1, 3). $|V|^{1/2}(H_0-\zeta)^{-1}$ can be treated similarly.

LEMMA 2.4. i) *Let H_0, V and H be as in Theorem* 1 *and in addition we assume that V is self-adjoint. Let $\mathfrak{L}$ be the set of all $u\in\mathfrak{M}_0\cap\mathfrak{D}$ which satisfies*

$$d\|E_0(\lambda)(H_0-i)u\|^2/d\lambda\leqq m^2\quad a.e.\quad and \tag{2.4}$$

$$\{E_0(l)-E_0(-l)\}u=u. \tag{2.5}$$

for some constant $m^2\geqq 0$ and $l\geqq 0$ (both depending on u). Then, if $W_+=W_+(H,H_0)$ exists, we have for any $u\in\mathfrak{L}$

$$\|(U_t-U_s)u\|\leqq C\{\eta(t;u)+\eta(s;u)\}, \tag{2.6}$$

where C and η is given by

$$C=(8\pi m^2\||V|^{1/2}(H_0-i)^{-1}\|_2^2\|(H_0-i)(H-i)^{-1}\|^2)^{1/4}, \tag{2.7}$$

$$\eta(t;u)=\Big[\int_t^\infty\||V|^{1/2}\exp(-itH_0)u\|^2dt\Big]^{1/4},\quad -\infty<t<+\infty. \tag{2.8}$$

(Note that $\eta(t;u)=\Big[\int_t^\infty\||V|^{1/2}(H_0-i)^{-1}\exp(-itH_0)(H_0-i)u\|^2dt\Big]^{1/4}$ is finite by virtue of Proposition 2.2 *and Lemma* 2.1.) *Conversely, if there holds for any $u\in\mathfrak{L}$ the inequality* (2.6) *with some constant C independent of t and s, then W_+ exists.*

ii) *Let, in particular, $V\in \boldsymbol{T}_s$ and $\mathfrak{L}'$ be the set defined as $\mathfrak{L}$ with* (2.4) *replaced by*

$$d\|E_0(\lambda)u\|^2/d\lambda\leqq m^2\quad a.e.. \tag{2.9}$$

Then the same assertions as in i) *hold even if $\mathfrak{L}$ is replaced by $\mathfrak{L}'$ and* (2.7) *by*

$$C=(8\pi m^2\|V\|_1)^{1/4}.\text{[8]} \tag{2.7'}$$

8) The results stated in ii) were previously given by Kato [5].

The similar assertions as i) *and* ii) *hold for* W_- *in place of* W_+.

PROOF. i) Let $u \in \mathfrak{D}$. Then, by integrating the relation $(d/dt)U_t u = i\exp(itH)V\exp(-itH_0)u$, we have

$$(U_t - U_s)u = i\int_s^t \exp(itH)V\exp(-itH_0)u\,dt \tag{2.10}$$

(note that by (1.11) the integrand is strongly continuous in t). Now assume that W_+ exists. Then we have for any $u \in \mathfrak{M}_0 \cap \mathfrak{D}$

$$\|(W_+ - U_s)u\|^2 \leqq 2\Big[\int_s^\infty \| |V|^{1/2}\exp(-itH_0)u\|^2 dt\Big]^{1/2} \times \Big[\int_s^\infty \| |V|^{1/2} W_+ \exp(-itH_0)u\|^2 dt\Big]^{1/2}. \tag{2.11}$$

This is derived from (2.10) by the same method as the one used in Kato [**5**] to derive the inequality (2.6) of [**5**] from (2.2) of [**5**]. We have only to note that the factor $W^* = \operatorname{sign} V$ on the right-hand side of (2.6) of [**5**] can be removed because W^* commutes with $|V|^{1/2}$ and $\|W^*\| \leqq 1$. Now the integrand of the second integral on the right-hand side of (2.11) can be transformed into $\| |V|^{1/2}(H_0-i)^{-1}(H_0-i)(H-i)^{-1}W_+\exp(-itH_0)(H_0-i)u\|^2$, where we use (1.4). Then by virtue of Lemma 2.1, Propositions 2.1 and 2.2 and the relation (1.9) we have for any $u \in \mathfrak{L}$

$$\|(W_+ - U_s)u\| \leqq (8\pi m^2 \| |V|^{1/2}(H_0-i)^{-1}\|_2^2 \|(H_0-i)(H-i)^{-1}\|^2)^{1/4}\eta(s;u). \tag{2.12}$$

From (2.12) and the similar inequality with U_s replaced by U_t we finally obtain (2.6). Conversely, assume that (2.6) holds for every $u \in \mathfrak{L}$ with some constant C. Since the integrals in $\eta(t;u)$ and $\eta(s;u)$ are convergent, the right-hand side of (2.6) tends to zero as $s, t \to +\infty$. This implies that $U_t u$ has a limit as $t \to +\infty$ provided that $u \in \mathfrak{L}$. Since $\mathfrak{L}$ is dense in $\mathfrak{M}_0$ by Lemma 2.3 and $\|U_t\| = 1$, we see that $\text{s-lim}\, U_t P_0 = W_+$ exists. W_- can be treated similarly.

The proof of ii) is much the same as that of i) with simplification due to the fact that $V \in \boldsymbol{T}_s$. We shall not go into details.

§ 3. Perturbation of rank 1.

In this section we prove the following special case of Theorem 1. As is mentioned in § 1, the proof is due to Kato [**6**].

LEMMA 3.1. *Let* H_0 *and* V *be self-adjoint and let* V *be of rank* 1. *Then* $W_\pm = W_\pm(H, H_0)$, $H = H_0 + V$, *exist.*

COROLLARY. *If* V *is self-adjoint and of finite rank,* $W_\pm(H, H_0)$ *exist.*

PROOF. We begin with the special case in which $\mathfrak{M}_0$ can be represented by the function space $L^2(-\infty, \infty)$ in such a way that the part H_{0a} of H_0 in $\mathfrak{M}_0$ is represented by a multiplicative operator: $(H_{0a}u)(x)=xu(x)$. As a self-adjoint operator of rank 1, V is expressible in the form $V=c(\cdot, \varphi)\varphi$, where $\|\varphi\|=1$ and c is a real number. For the moment we assume that $f=P_0\varphi\in\mathfrak{M}_0$ can be represented by a smooth function[9] $f(x)$ of $L^2(-\infty, \infty)$. We now prove the existence of $W_\pm$ under these assumptions.

By virtue of (2.10) we see that, if the integral

$$\int_{-\infty}^{\infty}\| V\exp(-itH_0)u\| dt \tag{3.1}$$

is finite, then $\lim U_tu$, $t\to\pm\infty$, exist. By reference to the expression $V=c(\cdot, \varphi)\varphi$, we have for any $u\in\mathfrak{M}_0$ $\| V\exp(-itH_0)u\|=|c||(\exp(-itH_0)u, \varphi)|=|c||(\exp(-itH_0)P_0u, \varphi)|=|c||(\exp(-itH_0)u, f)|$. Hence, in terms of the representation of $\mathfrak{M}_0$ as $L^2(-\infty, \infty)$, the integral (3.1) is written in the form

$$|c|\int_{-\infty}^{\infty}dt\left|\int_{-\infty}^{\infty}\exp(-itx)u(x)\overline{f(x)}dx\right|, \quad u\in\mathfrak{M}_0. \tag{3.2}$$

If $u(x)$ is smooth, so is $u(x)\overline{f(x)}$ because $f(x)$ is assumed to be smooth. Then the Fourier transform of $u(x)\overline{f(x)}$ tends sufficiently rapidly to zero at infinity and hence (3.2) is finite. Thus we see that $\lim U_tu$, $t\to\pm\infty$, exist for every $u\in\mathfrak{M}_0$ which can be represented by a smooth function of L^2. Since the set of such u is dense in $\mathfrak{M}_0$, it follows that s-lim $U_tP_0=W_\pm$ exist.

When $f=P_0\varphi$ can not be represented by a smooth function, we proceed as follows. Let $f_n(x)$ be a sequence of smooth functions such that $f_n\to f$ in L^2 and $\|f_n\|=\|f\|$ (the existence of such a sequence is well-known). Put $\varphi_n=f_n+(I-P_0)\varphi$ and $V_n=c(\cdot, \varphi_n)\varphi_n$. Then $\varphi_n\to\varphi$ and $\|\varphi_n\|=\|\varphi\|$. We first prove that

$$\lim_{n\to\infty}\| |V|^{1/2}-|V_n|^{1/2}\|_2=0. \tag{3.3}$$

To this end we first observe that $|V|^{1/2}$ is expressible in the form $|V|^{1/2}=|c|^{1/2}(\cdot, \varphi)\varphi$ and similarly for $|V_n|^{1/2}$. Now let ψ_n be a linear combination of φ and φ_n such that $\|\psi_n\|=1$ and $(\psi_n, \varphi)=0$.[10] Then, if u is orthogonal to both φ and ψ_n, we have $(|V|^{1/2}-|V_n|^{1/2})u=0$. By (1.7) we then obtain $\| |V|^{1/2}-|V_n|^{1/2}\|_2^2=\|(|V|^{1/2}-|V_n|^{1/2})\varphi\|^2+\|(|V|^{1/2}-|V_n|^{1/2})\psi_n\|^2=|c|\|\varphi-(\varphi, \varphi_n)\varphi_n\|^2+|c||(\psi_n, \varphi_n)|^2$. Since $\varphi_n\to\varphi$ and $(\psi_n, \varphi)=0$, the right-hand side tends to zero as $n\to\infty$. Thus (3.3) is proved.

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9) We agree for brevity that "smooth" means "sufficiently smooth and tending sufficiently rapidly to 0 as $|x|\to\infty$".

10) Such a ψ_n surely exists for every n. In fact, since we are considering the case in which $f=P_0\varphi$ can not be represented by a smooth function, f_n can not be proportional to f and consequently φ_n to φ.

Remembering that $P_0\varphi_n = f_n$ is smooth, we see by virtue of the already established part of the lemma that $W_+(H_0+V_n, H_0)$ exist. According to ii) of Lemma 2.4 we then have for any $u \in \mathfrak{L}'$ the inequality similar to (2.6) with U_t, U_s, C and η replaced by $U_t^{(n)}, U_s^{(n)}, C_n$ and η_n respectively, where C_n and η_n are defined by (2.7′) and (2.8) with V replaced by V_n and $U_t^{(n)} = \exp(it(H_0 + V_n))\exp(-itH_0)$. In C_n we can replace $\| V_n \|_1$ by $\| V \|_1$ because $\| V_n \|_1 = |c| = \| V \|_1$. Then take limit as $n \to \infty$ on both sides. By virtue of Lemma 2.2 it follows from (3.3) that η_n converges to η. Since $\| U_t^{(n)} - U_t \| \to 0$[11], the left-hand side converges to that of (2.6). Thus we finally obtain the inequality (2.6) itself. The converse assertion of Lemma 2.4 then ensures the existence of $W_+(H, H_0)$. W_- can be treated similarly.

Suppose now that the function $f(x)$ used above vanishes outside of a Borel set S of real numbers and let $\mathfrak{M}_0'$ be the subspace of $\mathfrak{M}_0 = L^2(-\infty, \infty)$ comprising all functions in L^2 which vanish outside of S. Put $\mathfrak{H}' = \mathfrak{M}_0' \oplus \mathfrak{N}_0$ ($\mathfrak{N}_0$ is by definition the singular subspace of $\mathfrak{H}$ with respect to H_0). Evidently, $\mathfrak{H}'$ is invariant by both H_0 and V and hence by H, too. Let H_0' and H' be the parts of H_0 and H in $\mathfrak{H}'$ respectively. Then, as is easily seen, the existence of $W_\pm(H, H_0)$ proved above implies the existence of $W_\pm(H', H_0')$. Since the restriction of $f(x)$ on S can be varied over all functions of $L^2(S)$, we have the following result: If $\mathfrak{M}_0$ is represented by $L^2(S)$ in such a way that the part H_{0a} of H_0 in $\mathfrak{M}_0$ is given by a multiplicative operator: $(H_{0a}u)(x) = xu(x)$, then $W_\pm(H, H_0)$ exist.

Finally we proceed to the general case. As above, let $V = c(\cdot, \varphi)\varphi$. Let $\mathfrak{H}_0$ be the smallest subspace of $\mathfrak{H}$ containing φ and reducing H_0. Then $\mathfrak{H}_0$ reduces V and hence H and U_t, too. Moreover, the part of U_t in $\mathfrak{H} \ominus \mathfrak{H}_0$ is equal to the identity operator because V is equal to 0 in $\mathfrak{H} \ominus \mathfrak{H}_0$. Thus in order to prove the lemma it suffices to prove the existence of $W_\pm(H', H_0')$, where H' and H_0' are the parts of H and H_0 in $\mathfrak{H}_0$. From the definition of $\mathfrak{H}_0$, however, it follows that H_0' and, a fortiori, the absolutely continuous part H_{0a}' of H_0' have simple spectra (Stone [**14**, Chap. VII]). Denote by S the spectrum of H_{0a}'. Then, as is well known, H_{0a}' is represented by a multiplicative operator in $L^2(S)$: $(H_{0a}'u)(x) = xu(x)$. Thus the general case is reduced to the case already dealt with. q. e. d.

Proof of Corollary. As a self-adjoint operator of finite rank, V is expressible in the form $V = \sum_{k=1}^{r} c_k(\cdot, \varphi_k)\varphi_k$, where r is the rank of V, $(\varphi_k, \varphi_j) = \delta_{kj}$

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11) In fact, $\varphi_n \to \varphi$ implies $\| V - V_n \| \to 0$. Then $\| U_t^{(n)} - U_t \| \to 0$ is a direct consequence of the relation $\exp(-itH_n)\exp(itH) - 1 = i\int_0^t \exp(-itH_n)(V - V_n)\exp(itH)dt$, which is obtained in the same way as (2.10).

and c_k are real numbers different from 0. Put $H_n=H_0+\sum_{k=1}^{n} c_k(\cdot, \varphi_k)\varphi_k$, $n=1,\cdots,r$. Then $H_n-H_{n-1}=c_n(\cdot,\varphi_n)\varphi_n$ is self-adjoint and of rank 1. Hence it follows from Lemma 3.1 that $W_\pm(H_n, H_{n-1})$, $n=1,\cdots,r$, exist. From this we conclude by virtue of (1.6) that $W_\pm(H,H_0)=W_\pm(H_r,H_{r-1})W_\pm(H_{r-1},H_{r-2})\cdots W_\pm(H_1,H_0)$ exist.

§ 4. Proof of the theorems.

1. As was shown by several authors (see e.g. Rellich [11], Kato [7]) the first statement of Theorem 1 that H is self-adjoint is a consequence of the condition (1.11). Therefore we have only to prove the existence of $W_\pm$. We first prove the following

PROPOSITION 4.1. *If Theorem 1 holds true under the additional assumption that V is self-adjoint, then Theorem 1 holds true.*

PROOF. Let H_0, V and H be as in Theorem 1. Consider the Hilbert space $\hat{\mathfrak{H}}=\mathfrak{H}_1\oplus\mathfrak{H}_2$, $\mathfrak{H}_1=\mathfrak{H}_2=\mathfrak{H}$ and denote the norm in $\hat{\mathfrak{H}}$ by $|||\ |||$. Every $u\in\hat{\mathfrak{H}}$ is uniquely expressible in the form $u=u_1+u_2$, $u_1\in\mathfrak{H}_1$ and $u_2\in\mathfrak{H}_2$. Then $|||u|||^2=\|u_1\|^2+\|u_2\|^2$. Let A and B be operators in $\mathfrak{H}$ and let $A\oplus B$ denote the operator in $\hat{\mathfrak{H}}$ defined as follows: $u\in\mathfrak{D}(A\oplus B)$ if and only if $u_1\in\mathfrak{D}(A)\subset\mathfrak{H}_1$ and $u_2\in\mathfrak{D}(B)\subset\mathfrak{H}_2$, and $(A\oplus B)u=Au_1+Bu_2$ for such a u. Obviously $\mathfrak{H}_1$ and $\mathfrak{H}_2$ reduce $A\oplus B$ and the parts of $A\oplus B$ in $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are identical with A and B respectively. Let now $\hat{H}_0=H_0\oplus H_0$ and $\hat{V}=V\oplus(-V)$. Then, as is easily seen, $\hat{H}_0$ is self-adjoint, $\hat{V}$ is symmetric and $\hat{H}=\hat{H}_0+\hat{V}=(H_0+V)\oplus(H_0-V)$ is self-adjoint. Furthermore, we see that the part of $U_t(\hat{H},\hat{H}_0)$ in $\mathfrak{H}_1$ is identical with $U_t(H,H_0)$. Thus we finally see that $W_\pm(H,H_0)$ and $W_\pm(H_0,H)$ exist if $W_\pm(\hat{H},\hat{H}_0)$ and $W_\pm(\hat{H}_0,\hat{H})$ exist. On the other hand, $\hat{V}$ has the deficiency index (m,m), $m=0,1,\cdots$ or ∞[12] and hence $\hat{V}$ has a self-adjoint extension $\hat{V}'$. We shall prove that all the assumptions of Theorem 1 are satisfied for $\hat{H}_0$ and $\hat{V}'$ in place of H_0 and V; then we see by hypothesis that $W_\pm(\hat{H}_0+\hat{V}',\hat{H}_0)=W_\pm(\hat{H},\hat{H}_0)$ and $W_\pm(\hat{H}_0,\hat{H}_0+\hat{V}')=W_\pm(\hat{H}_0,\hat{H})$ exist (note that $\hat{H}_0+\hat{V}'=\hat{H}_0+\hat{V}=\hat{H}$ because $\mathfrak{D}(\hat{V})\supset\mathfrak{D}(\hat{H}_0)$ and $\hat{V}'\supset\hat{V}$) and we complete the proof of the proposition. Let now $u\in\mathfrak{D}(\hat{H}_0)$ and $u=u_1+u_2$, $u_1\in\mathfrak{H}_1$ and $u_2\in\mathfrak{H}_2$. Then a simple calculation gives $|||\hat{V}'u|||^2=|||\hat{V}u|||^2=\|Vu_1\|^2+\|Vu_2\|^2\leqq(a|||\hat{H}_0u|||+b|||u|||)^2$, which implies that $\hat{H}_0$ and $\hat{V}'$ satisfy (1.11). Since H_0 and V satisfy (1.12), we easily obtain $|\hat{V}|^{1/2}(\hat{H}_0-\zeta_0)^{-1}\in\boldsymbol{S}$. On the other hand, $\hat{V}'\supset\hat{V}$ implies that $\mathfrak{D}(|\hat{V}'|)\supset\mathfrak{D}(|\hat{V}|)$ and $|||\,|\hat{V}'|u|||=|||\,|\hat{V}|u|||$ for each $u\in\mathfrak{D}(|\hat{V}|)$.[13] From this it follows that $\mathfrak{D}(|\hat{V}'|^{1/2})\supset\mathfrak{D}(|\hat{V}|^{1/2})$ and $|||\,|\hat{V}'|^{1/2}u|||\leqq|||\,|\hat{V}|^{1/2}u|||$.[14] Thus we see that $|\hat{V}'|^{1/2}(\hat{H}_0-\zeta_0)^{-1}\in\boldsymbol{S}$, which shows that $\hat{H}_0$ and $\hat{V}'$ satisfy (1.12).

12) Achieser and Glasmann [1, Anhang I].
13) von Neumann [15].
14) Heinz [2], Kato [8].

2. We now assume that V is self-adjoint and prove Theorem 1 under this additional assumption. To simplify the description we assume throughout subsections 2—4 that H_0, V and H satisfy all the assumptions of Theorem 1 and in addition V is self-adjoint. Furthermore we put $\| |V|^{1/2}(H_0-i)^{-1}\|_2 = K$, which is finite by virtue of Proposition 2.2.

PROPOSITION 4.2. *There exist non-negative constants a' and b' such that*

$$\| Vu \| \leqq a' \| Hu \| + b' \| u \| \quad \text{for each} \quad u \in \mathfrak{D}. \tag{4.1}$$

If $0 \leqq a < 1/2$, a' can be taken as $0 \leqq a' < 1$.

PROOF. It follows from (1.11) that $\| Vu \| \leqq a(\| Hu \| + \| Vu \|) + b\| u \|$. Remembering $0 \leqq a < 1$, we then obtain $\| Vu \| \leqq a(1-a)^{-1} \| Hu \| + b(1-a)^{-1} \| u \|$. Hence we have only to put $a' = a(1-a)^{-1}$ and $b' = b(1-a)^{-1}$.

PROPOSITION 4.3. *Let $V \geqq 0$ or $V \leqq 0$. Then there exists a sequence $\{V_n\}$ of self-adjoint operators of finite rank having the following properties:*

$$\| |V_n|^{1/2}(H_0-i)^{-1} \|_2 \leqq K; \tag{4.2}$$

$$\| (H_0-i)(H_n-i)^{-1} \| \leqq (1+b)(1-a)^{-1}+1 \equiv M, \tag{4.3}$$

where $H_n = H_0 + V_n$;

$$\operatorname*{s-lim}_{n\to\infty} \exp(itH_n) = \exp(itH); \tag{4.4}$$

$$\limsup_{n\to\infty} \int_s^\infty \| |V_n|^{1/2} \exp(-itH_0)u \|^2 dt \tag{4.5}$$

$$\leqq \int_s^\infty \| |V|^{1/2} \exp(-itH_0)u \|^2 dt, \quad u \in \mathfrak{L}^{15)}.$$

PROOF. For brevity we assume that $V \geqq 0$. The other case can be treated similarly. Put

$$A_n = V^{1/2}(1-in^{-1}H_0)^{-1} = inV^{1/2}(H_0+in)^{-1}, \qquad n = 1, 2, \cdots, \tag{4.6}$$

$$V_n' = A_n^* A_n = (1+in^{-1}H_0)^{-1} V (1-in^{-1}H_0)^{-1}. \tag{4.7}$$

Since $A_n \in \boldsymbol{S}$ by Proposition 2.2, we have $V_n' \in \boldsymbol{T}_s$ (see § 1, 3). Moreover, V_n' is evidently positive definite. Hence V_n' is expressible in the form $V_n' = \sum_{k=1}^\infty \lambda_k^{(n)}(\cdot, \varphi_k^{(n)})\varphi_k^{(n)}$, where $\{\lambda_k^{(n)}\}$ is a sequence of positive eigenvalues of V_n' (degenerate eigenvalues being repeated) and $\{\varphi_k^{(n)}\}$ is a sequence of corresponding eigenvectors. By (1.10) we have $\sum_k \lambda_k^{(n)} < \infty$. We now choose for each n a natural number r_n such that $\sum_{k=r_n+1}^\infty \lambda_k^{(n)} < n^{-1}$ and put

$$V_n = \sum_{k=1}^{r_n} \lambda_k^{(n)}(\cdot, \varphi_k^{(n)})\varphi_k^{(n)}. \tag{4.8}$$

15) A detailed consideration shows that lim sup can be replaced by lim. Nevertheless, we confine ourselves to verifying (4.5) which is sufficient for later purpose.

Then V_n is self-adjoint and of finite rank. We shall prove that V_n thus defined satisfies (4.2)—(4.5). To this end we first note the following relations which are direct consequences of (4.7) and (4.8):

$$\| V_n u \| \leqq \| V_n' u \|, \quad \| V_n - V_n' \| \leqq \| V_n - V_n' \|_1 < n^{-1}, \tag{4.9}$$

$$\| V_n^{1/2} u \|^2 \leqq \| V_n'^{1/2} u \|^2 = (V_n' u, u) = (A_n^* A_n u, u) = \| A_n u \|^2. \tag{4.10}$$

By virtue of (4.10), (1.7) and (1.9) we now obtain

$$\begin{aligned} \| V_n^{1/2}(H_0 - i)^{-1} \|_2 &\leqq \| V_n'^{1/2}(H_0 - i)^{-1} \|_2 = \| A_n (H_0 - i)^{-1} \|_2 \\ &= \| V^{1/2}(H_0 - i)^{-1}(1 - in^{-1}H_0)^{-1} \|_2 \leqq \| V^{1/2}(H_0 - i)^{-1} \|_2 = K, \end{aligned}$$

where we used the inequality $\| (1 - in^{-1}H_0)^{-1} \| \leqq 1$. This proves (4.2). By (4.9) and (1.11) we have for any $u \in \mathfrak{D}$

$$\begin{aligned} \| V_n u \| &\leqq \| V_n' u \| = \| (1 + in^{-1}H_0)^{-1} V (1 - in^{-1}H_0)^{-1} u \| \leqq \| V (1 - in^{-1}H_0)^{-1} u \| \\ &\leqq a \| H_0 (1 - in^{-1}H_0)^{-1} u \| + b \| (1 - in^{-1}H_0)^{-1} u \| \leqq a \| H_0 u \| + b \| u \|. \end{aligned}$$

Hence, by using the relations $H_0 = H_n - V_n$, $\| (H_n - i)^{-1} \| \leqq 1$ and $\| H_n (H_n - i)^{-1} \| \leqq 1$, we have $\| H_0 (H_n - i)^{-1} u \| \leqq \| H_n (H_n - i)^{-1} u \| + \| V_n (H_n - i)^{-1} u \| \leqq a \| H_0 (H_n - i)^{-1} u \| + (1 + b) \| u \|$. Remembering that $0 \leqq a < 1$, we finally obtain $\| H_0 (H_n - i)^{-1} u \| \leqq (1 + b)(1 - a)^{-1} \| u \|$, from which (4.3) follows immediately.

In order to prove (4.4) we show that $\lim V_n u = V u$ for each $u \in \mathfrak{D}$. By (4.7) and (4.9) we have for any $u \in \mathfrak{D}$

$$\begin{aligned} \| V_n u - V u \| &\leqq \| (V_n - V_n') u \| + \| (V_n' - V) u \| \\ &\leqq n^{-1} \| u \| + \| (1 + in^{-1}H_0)^{-1} V \{ (1 - in^{-1}H_0)^{-1} - 1 \} u \| \\ &\quad + \| \{ (1 + in^{-1}H_0)^{-1} - 1 \} V u \|. \end{aligned}$$

The first and the third terms on the right-hand side tend to zero as $n \to \infty$, because $\text{s-lim}(1 + in^{-1}H_0)^{-1} = 1$. Since $u \in \mathfrak{D}$, we see by (1.11) that the second term is majorized by $\| V \{ (1 - in^{-1}H_0)^{-1} - 1 \} u \| \leqq a \| \{ (1 - in^{-1}H_0)^{-1} - 1 \} H_0 u \| + b \| \{ (1 - in^{-1}H_0)^{-1} - 1 \} u \|$, which also tends to zero for the same reason as above. Thus we obtain $\lim V_n u = V u$ for each $u \in \mathfrak{D}$. From this we have for any $u \in \mathfrak{H}$ and any non-real ζ $\{ (H_n - \zeta)^{-1} - (H - \zeta)^{-1} \} u = (H_n - \zeta)^{-1} (V - V_n)(H - \zeta)^{-1} u \to 0$, $n \to \infty$, i. e. $\text{s-lim}(H_n - \zeta)^{-1} = (H - \zeta)^{-1}$. According to the general theory of semi-groups of operators in a Banach space, this strong convergence of the resolvent implies (4.4). This can be proved by the same method as given in the proof of Theorem 15.4.1 of Hille [**3**].

In order to prove (4.5) it suffices to prove the relation

$$\lim_{n \to \infty} \int_s^\infty \| V_n'^{1/2} \exp(-itH_0) u \|^2 dt = \int_s^\infty \| V^{1/2} \exp(-itH_0) u \|^2 dt, \tag{4.11}$$

$u \in \mathfrak{L}$, because $\| V_n^{1/2} u \| \leqq \| V_n'^{1/2} u \|$ by (4.10). Since $u \in \mathfrak{L}$ we see, on putting

$F(l)=E_0(l)-E_0(-l)$, that there exists $l>0$ such that $F(l)u=u$ or equivalently $u\in F(l)\mathfrak{H}$. Let H_0' be the part of H_0 in $F(l)\mathfrak{H}$. Then $u\in F(l)\mathfrak{H}$ implies that $v=\exp(-itH_0)(H_0-i)u\in F(l)\mathfrak{H}$ and $(1-in^{-1}H_0)^{-1}v=(1-in^{-1}H_0')^{-1}v$. By (4.10) and (4.6) we therefore obtain

$$\text{(4.12)}\qquad \int_s^\infty \| V_n'^{1/2}\exp(-itH_0)u\|^2dt=\int_s^\infty \| A_n\exp(-itH_0)u\|^2dt$$
$$=\int_s^\infty \| V^{1/2}(H_0-i)^{-1}(1-in^{-1}H_0')^{-1}\exp(-itH_0)(H_0-i)u\|^2dt\,.$$

Since H_0' is bounded by definition, we have $p_n=\|(1-in^{-1}H_0')^{-1}-1\|\to 0$, $n\to\infty$, and consequently

$$\text{(4.13)}\qquad \lim_{n\to\infty}\| V^{1/2}(H_0-i)^{-1}(1-in^{-1}H_0')^{-1}-V^{1/2}(H_0-i)^{-1}\|_2$$
$$\leqq \lim_{n\to\infty} p_n \| V^{1/2}(H_0-i)^{-1}\|_2=0\,.$$

On the other hand $u\in\mathfrak{L}$ implies $d\|E_0(\lambda)(H_0-i)u\|/d\lambda\leqq m^2$ (see (2.4)). Hence, by taking account of (4.13), we can apply Lemma 2.2 to the integral on the right-hand side of (4.12), with the result

$$\lim_{n\to\infty}\int_s^\infty \| V_n'^{1/2}\exp(-itH_0)u\|^2dt=\int_s^\infty \| V^{1/2}(H_0-i)^{-1}\exp(-itH_0)(H_0-i)u\|^2dt$$
$$=\int_s^\infty \| V^{1/2}\exp(-itH_0)u\|^2dt\,,$$

which proves (4.5). q. e. d.

3. The case $0\leqq a<1/2$.

PROPOSITION 4.4. *Let $V\geqq 0$ or $V\leqq 0$. Then $W_\pm(H,H_0)$ exist.*

PROOF. By hypothesis there exists such a sequence $\{V_n\}$ as described in Proposition 4.3. Since V_n is self-adjoint and of finite rank, we see by Corollary to Lemma 3.1 that $W_+(H_n,H_0)$ exists for every n. Hence, by virtue of i) of Lemma 2.4 we obtain for any $u\in\mathfrak{L}$ the inequality similar to (2.6) with U_t, U_s, C and η replaced by $U_t^{(n)}$, $U_s^{(n)}$, C_n and η_n respectively, where $U_t^{(n)}=\exp(itH_n)\exp(-itH_0)$, and C_n and η_n are defined by (2.7) and (2.8) with V and H replaced by V_n and H_n. By virtue of (4.2) and (4.3), however, we can replace in this inequality C_n by $(8\pi m^2K^2M^2)^{1/4}$. Then by taking superior limit as $n\to\infty$ on both sides and considering (4.4) and (4.5) we obtain for any $u\in\mathfrak{L}$ the inequality (2.6) with $C=(8\pi m^2K^2M^2)^{1/4}$. Then the converse assertion in i) of Lemma 2.4 proves the existence of $W_+(H,H_0)$. W_- can be treated similarly. q. e. d.

PROPOSITION 4.5. *If to the assumption of Theorem 1 we add the assumptions that V is self-adjoint and $0\leqq a<1/2$, then $W_\pm(H,H_0)$ and $W_\pm(H_0,H)$ exist.*

PROOF. For the moment we assume that $V\geqq 0$ or $V\leqq 0$. Then the existence of $W_\pm(H, H_0)$ is ensured by the preceeding proposition. On the other hand, since $0 \leqq a < 1/2$ by hypothesis, we have the inequality (4.1) with constant a' such that $0 \leqq a' < 1$. From this and Proposition 2.2 it follows by virtue of Proposition 4.4 that $W_\pm(H_0, H) = W_\pm(H - V, H)$ also exist.

To treat the general case we decompose V in the form $V = V_+ - V_-$, where $V_\pm$ have the properties $V_\pm \geqq 0$ and

$$(4.14)\qquad \begin{cases} \mathfrak{D}(V_\pm) \supset \mathfrak{D}(V),\ \mathfrak{D}(V_\pm^{1/2}) \supset \mathfrak{D}(|V|^{1/2}), \\ \| V_\pm u \| \leqq \| Vu \|, \quad \text{for every} \quad u \in \mathfrak{D}(V), \\ \| V_\pm^{1/2} u \| \leqq \| |V|^{1/2} u \|, \quad \text{for every} \quad u \in \mathfrak{D}(|V|^{1/2}). \end{cases}$$

From this it follows that (1.11) and (1.12) hold true for $V_\pm$ in place of V. Since $0 \leqq a < 1/2$, we then see by the part of the proposition already proved that $W_\pm(H_0 + V_+, H_0)$ and $W_\pm(H_0, H_0 + V_+)$ exist. Next we prove the existence of $W_\pm(H, H_0 + V_+)$ and $W_\pm(H_0 + V_+, H)$. If this is done, we can see by (1.6) that $W_\pm(H, H_0) = W_\pm(H, H_0 + V_+) W_\pm(H_0 + V_+, H_0)$ and $W_\pm(H_0, H) = W_\pm(H_0, H_0 + V_+) W_\pm(H_0 + V_+, H)$ all exist and the proof of the proposition is complete. Now by (4.14), Propositions 4.2 and 2.2 we have $\| V_- u \| \leqq \| Vu \| \leqq a' \| Hu \| + b' \| u \|$, $0 \leqq a' < 1$, and $V_-^{1/2}(H - i)^{-1} \in \boldsymbol{S}$. Proposition 4.4 therefore ensures the existence of $W_\pm(H_0 + V_+, H) = W_\pm(H + V_-, H)$. In order to prove the existence of $W_\pm(H, H_0 + V_+) = W_\pm(H_0 + V_+ - V_-, H_0 + V_+)$ we first note that

$$(4.15)\qquad V_-^{1/2}(H_0 + V_+ - i)^{-1} = V_-^{1/2}(H_0 - i)^{-1}(H_0 - i)(H_0 + V_+ - i)^{-1} \in \boldsymbol{S}.$$

Furthermore, from (1.11) with V replaced by V_+ it follows that $\| (H_0 + V_+) u \| \geqq \| H_0 u \| - \| V_+ u \| \geqq (1 - a) \| H_0 u \| - b \| u \|$, $u \in \mathfrak{D}$, and consequently $\| H_0 u \| \leqq (1 - a)^{-1} \| (H_0 + V_+) u \| + b(1 - a)^{-1} \| u \|$. Hence by (4.14) and (1.11) we obtain for any $u \in \mathfrak{D}$

$$(4.16)\qquad \| V_- u \| \leqq \| Vu \| \leqq a(1 - a)^{-1} \| (H_0 + V_+) u \| + \{ab(1 - a)^{-1} + b\} \| u \|.$$

Since $0 \leqq a < 1/2$ is assumed, we have $0 \leqq a(1 - a)^{-1} < 1$. By referring once more to Proposition 4.4, we therefore see from (4.15) and (4.16) that $W_\pm(H_0 + V_+ - V_-, H_0 + V_+)$ exist, as we wished to prove. q. e. d.

4. We shall next remove the additional assumption $0 \leqq a < 1/2$. To this end put $\alpha_n = 2^n(2^n + 1)^{-1}$, $n = 0, 1, \cdots$. Then $1/2 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots < 1$ and $\alpha_n \to 1$, $n \to \infty$. Proposition 4.5 shows that $W_\pm(H, H_0)$ and $W_\pm(H_0, H)$ exist if $0 \leqq a < 1/2 = \alpha_0$. Moreover, since $0 \leqq a < 1$ by hypothesis there exists an n such that $a < \alpha_n$. In order to prove the existence of $W_\pm$ it therefore suffices to prove the following

PROPOSITION 4.6. *Let n be an arbitrarily fixed non-negative integer and assume that $W_\pm(H, H_0)$ and $W_\pm(H_0, H)$ exist if $0 \leqq a < \alpha_n$. Then $W_\pm(H, H_0)$ and $W_\pm(H_0, H)$ exist if $0 \leqq a < \alpha_{n+1}$.*

Proof. Let $0 \leqq a < \alpha_{n+1}$ and let $H' = H_0 + (1/2)V$. From (1.11) and (1.12) we get

$$\| (1/2)Vu \| \leqq (a/2) \| H_0 u \| + (b/2) \| u \|, \quad u \in \mathfrak{D}, \tag{4.17}$$

$$| (1/2)V |^{1/2} (H_0 - i)^{-1} \in \boldsymbol{S}. \tag{4.18}$$

Since $a/2 < 1/2 \leqq \alpha_n$, it then follows by the assumption of the proposition that $W_\pm(H', H_0)$ and $W_\pm(H_0, H')$ exist. On the other hand, in the same way as in the proof of Propositions 4.2 and 2.2, we obtain from (4.17) and (4.18) that $\| (1/2)Vu \| \leqq a(2-a)^{-1} \| H'u \| + b' \| u \|$, $u \in \mathfrak{D}$, and $| (1/2)V |^{1/2} (H' - i)^{-1} \in \boldsymbol{S}$. Since $0 \leqq a < \alpha_{n+1}$ by hypothesis, we easily obtain $0 \leqq a(2-a)^{-1} < \alpha_n$. Again, by virtue of the assumption of the proposition, we then see that $W_\pm(H' + (1/2)V, H') = W_\pm(H, H')$ and $W_\pm(H', H)$ exist. Thus, by virtue of (1.6) we finally see that $W_\pm(H, H_0) = W_\pm(H, H') W_\pm(H', H_0)$ and $W_\pm(H_0, H) = W_\pm(H_0, H') W_\pm(H', H)$ exist. q. e. d.

Thus we proved Theorem 1 under the additional assumption that V is self-adjoint. Then by virtue of Proposition 4.1 the proof of Theorem 1 is complete.

5. Proof of Theorem 2. In the first place, we note that (1.16) and (1.17) are consequences of (1.15). This is seen as follows. By virtue of (1.5), (1.16) follows immediately from (1.15). It then follows from (1.15) and (1.16) that $S(H_n)$ converge weakly to $S(H)$. Since $S(H_n)$ is equal to zero in $\mathfrak{H} \ominus \mathfrak{M}_0$ and unitary in $\mathfrak{M}_0$ (see Lemma 1.1), this weak convergence implies the strong convergence of $S(H_n)$, that is (1.17). Thus we have only to prove (1.15).

Since $W_\pm(H_n, H_0)$ and $W_\pm(H_0, H)$ exist by hypothesis, we see by (1.6) that $W_\pm(H_n, H)$ exist. Moreover we have $W_\pm(H_n, H_0) = W_\pm(H_n, H) W_\pm(H, H_0)$. This implies that, in order to prove (1.15), it suffices to prove

$$\operatorname*{s-lim}_{n \to \infty} W_\pm(H_n, H) = P. \tag{4.19}$$

Now by the similar argument as in the proof of Proposition 4.1 we easily see that, in order to prove Theorem 2, it suffices to prove (4.19) under the additional assumption that V_n' has a self-adjoint extension $\tilde{V}_n'$. Let now $\mathfrak{L}$ be befined as in Lemma 2.4 with H_0 replaced by H and let $W_\pm^{(n)} = W_\pm(H_n, H)$. Then, as in the proof of that lemma, we have for any $u \in \mathfrak{L}$ the inequality similar to (2.12) with H_0, H and $W_\pm$ replaced by H, H_n and $W_\pm^{(n)}$. Then, by estimating $\eta(s; u)$ in the same way as we obtain (2.12) and putting $s = 0$, we have for any $u \in \mathfrak{L}$

$$\| W_+^{(n)} u - u \| \leqq (4\pi m^2 \| | \tilde{V}_n' |^{1/2} (H - i)^{-1} \|_2^2 \| (H - i)(H_n - i)^{-1} \|)^{1/2}. \tag{4.20}$$

By hypothesis, however, there exist constants a, b such that $0 \leqq a < 1$, $0 \leqq b$ and $\| V_n u \| \leqq a \| H_0 u \| + b \| u \|$, $u \in \mathfrak{D}$, $n = 1, 2, \cdots$. Hence by means of the similar calculations as in the proof of (4.3) we have $\|(H_0-i)(H_n-i)^{-1}\| \leqq (1+b)(1-a)^{-1}+1$. This implies that $\|(H-i)(H_n-i)^{-1}\| = \|(H-i)(H_0-i)^{-1}(H_0-i)(H_n-i)^{-1}\|$ are bounded in n. Moreover, we have as in the proof of Proposition 4.1 $\||\tilde{V}_n'|^{1/2}(H_0-\zeta_0)^{-1}\|_2 \leqq \||V_n'|^{1/2}(H_0-\zeta_0)^{-1}\|_2$. Hence by (1.14) we have $\||\tilde{V}_n'|^{1/2}(H-i)^{-1}\|_2 \leqq \||\tilde{V}_n'|^{1/2}(H_0-\zeta_0)^{-1}\|_2\|(H_0-\zeta_0)(H-i)^{-1}\| \to 0$, $n\to\infty$. Thus we see by (4.20) that s-lim $W_+^{(n)}u=u$ for $u\in\mathfrak{L}$. Since $\mathfrak{L}$ is dense in $\mathfrak{M}$ and $\|W_+^{(n)}P\|\leqq 1$, we finally see that s-lim $W_+^{(n)} =$ s-lim $W_+^{(n)}P=P$. W_- can be treated similarly.

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Added in proof. On page 258 we proved that lim $V_n u = Vu$, $u\in\mathfrak{D}$, implies (4.4). But this fact is a special case of Theorem 5.2 of a recent work of H. F. Trotter, Approximation of semi-groups of operators, Pacif. J. Math., 8 (1958), 887-919.