

## Note on a $B^*$ -algebra.

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As is well-known I. Gelfand and M. Neumark [1] proved in 1943 that a Banach algebra over the complex number field with a unit 1 and an involution  $*$  satisfying

$$(0.1) \quad x^{**} = x,$$

$$(0.2) \quad (\alpha x)^* = \alpha^* x^* \quad (\alpha^* = \text{the conjugate complex number of } \alpha),$$

$$(0.3) \quad (x+y)^* = x^* y^*,$$

$$(0.4) \quad (xy)^* = y^* x^*$$

is isometric and  $*$ -isomorphic to a  $C^*$ -algebra, (i. e., a uniformly closed self-adjoint algebra acting on a Hilbert space over the complex number field) if and only if it satisfies the following three conditions:

$$(0.5) \quad \|x^* x\| = \|x^*\| \|x\|,$$

$$(0.6) \quad \|x^*\| = \|x\|, \text{ and}$$

$$(0.7) \quad 1+x^* x \text{ has an inverse.}$$

Also they conjectured that

(A) this fact holds without (0.7) and

(B) this fact holds without (0.6) (and (0.7)).

It was pointed out by I. Kaplansky that M. Fukamiya [2] gave implicitly an affirmative answer to the conjecture (A) (See J. A. Schatz' review [3] of [2]), and the assumption of existence of a unit was removed by I. Kaplansky and C. E. Rickart (cf. loc. cit.).

Their answer is very simple and stands on the following three facts:

(0.8) A  $B^*$ -algebra without a unit is isometrically and  $*$ -isomorphically imbeddible into a  $B^*$ -algebra with a unit (I. Kaplansky, C. E. Rickart).

(0.9) The set of non-zero spectra of  $xy$  ( $x, y$  being elements of a Banach algebra) coincides with that of  $yx$ .

(0.10) The set of hermitian elements of a  $B^*$ -algebra with a unit has a semi-ordering  $h \geq 0$  defined by  $h = k^2$  ( $h, k$  being in the set) (M. Fukamiya [2], J. L. Kelley-R. L. Vaught [5]). (cf. I. Kaplansky [4] [8], M. Mimura [6]).

In this note, we shall give a direct proof of the theorem of I. Gelfand and M. Neumark by making no use of (0.7) in § 1. The present proof is not simple, for we make use neither of (0.7) nor of (0.8)–(0.10). In § 2, we shall give an affirmative answer to the conjecture (B).

I wish to express my deep gratitude to Prof. I. Kaplansky for sending me a copy of the appendix of the lecture note [4], to Prof. M. Fukamiya for his pointing me essential errors of my original proof of (B) and for his shortening my original proof of (B), to Prof. M. Nakamura for his valuable suggestions during my preparation of this paper, and to Mr. T. Saito for his shortening my original proof of (B). Also, I express my hearty thanks to Prof. S. Iyanaga for his valuable remarks and to Prof. O. Takenouchi for his kind proof-reading and valuable remarks.

**§ 1. A direct proof of the theorem of I. Gelfand and M. Neumark.**

A Banach algebra over the complex number field  $C$  is called a  $*$ -algebra if it has an involution  $*$  satisfying (0.1)–(0.4). A  $*$ -algebra is called a  $B^*$ -algebra if it satisfies (0.5) and (0.6). A  $*$ -algebra is called  $C$ -symmetric if every maximal commutative<sup>1)</sup>  $*$ -subalgebra of it is  $B^*$ .

The aim of this § is to prove the following theorem.

**THEOREM 1.1.** 1) *A  $C$ -symmetric  $*$ -algebra is homeomorphic and  $*$ -isomorphic to a suitable  $C^*$ -algebra.*

2) *A  $B^*$ -algebra is isometric and  $*$ -isomorphic to a suitable  $C^*$ -algebra.*

Let  $R$  be a  $C$ -symmetric  $*$ -algebra<sup>2)</sup> and  $h$  be an hermitian<sup>3)</sup> element of  $R$ . Denote by  $R_h$  the  $*$ -subalgebra of  $R$  generated by  $h$ . Since  $R_h$  is commutative and contained in a maximal commutative  $*$ -subalgebra of  $R$ , it is  $B^*$ . By virtue of a theorem of I. Gelfand and M.H. Stone  $R_h$  is isometric and  $*$ -isomorphic to  $C(\Omega_h)$  (or  $C'(\Omega_h)$ ),  $\Omega_h$  being the set of spectra<sup>4)</sup> of  $h$  in  $R_h$ . The space  $\Omega_h$  is considered as a bounded closed subset of  $C$ ;  $C(\Omega_h)$  is the  $B^*$ -algebra of complex-valued continuous functions on  $\Omega_h$  and  $C'(\Omega_h)$  is the  $B^*$ -algebra of complex-valued continuous functions  $g$ 's on  $\Omega_h$  with  $g(0) = 0$ , when  $R_h$  has no unit. For such a function  $g$ , we can find an element of  $R_h$  as the inverse image of  $g$  by the canonical  $*$ -isomorphism of  $R_h$  onto  $C(\Omega_h)$  (or  $C'(\Omega_h)$ ). Denote it by  $g(t)/h$  (or  $g(h)$ ).

We denote by  $A$  the set of elements  $a$ 's of  $R$  satisfying the following condition:

(1.1) every spectrum of  $a^*a$  is non-negative.

Now we shall begin with the following

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1) Two elements  $a, b$  of  $R$  are said to be commutative with each other if  $ab = ba$  and  $a*b = ba^*$ .

2) We do not assume that  $R$  has a unit.

3) An element  $h$  of  $R$  is called hermitian if  $h^* = h$ .

4) A non-zero scalar  $\alpha$  is called a (left) spectrum of an element  $a$  of  $R$  (in  $R$ ) if the equation  $xa - ax = \alpha$  has no solution in  $R$ ; 0 is called a (left) spectrum of  $a$  (in  $R$ ) if  $a$  has no inverse. When  $R$  has no unit, of course,  $a$  has no inverse.

LEMMA 1.1. *If  $a^2=0$ , then  $a \in A$  holds.*

PROOF. We have  $a^*a+aa^*=(a+a^*)^2$  and further  $a^*a \cdot aa^*=aa^* \cdot a^*a=0$ . Combining these with the fact that  $a+a^* \in A$ <sup>5)</sup> we finally get  $a \in A$  by an easy computation. q. e. d.

LEMMA 1.2.  *$R$  is commutative if and only if*

$$(1.2) \quad a-ab=0 \text{ implies } ac-acb=0 \text{ for any } c \in R.^{6)}$$

PROOF. The necessity is obvious and so we need only to see the sufficiency. Let  $R$  be a  $C$ -symmetric  $*$ -algebra satisfying (1.2). First of all, we show that any closed left ideal  $I$  of  $R$  is two-sided. For an element  $a$  of  $I$ , we denote by  $u_n(a)$  (or briefly by  $u_n$ ) the element  $g_n(a^*a)$  of  $I$ , where

$$(1.3) \quad g_n(t) = \begin{cases} 1 & (2/n \leq t), \\ \text{linear} & (1/n \leq t \leq 2/n), \\ 0 & (-1/n \leq t \leq 1/n), \\ \text{linear} & (-2/n \leq t \leq -1/n), \\ 1 & (t \leq -2/n). \end{cases}$$

It is easy to see that  $au_n \rightarrow a$  as  $n \rightarrow \infty$ . Further we have  $u_n u_m = u_m u_n = u_n$  if  $2n \leq m$ . Since  $u_n - u_n u_m = 0$  for  $2n \leq m$ , we have, from (1.2),  $u_n c = u_n c u_m$  for any  $c \in R$ , from which it follows that  $au_n c = au_n c u_m \in I$ . As making  $n \rightarrow \infty$ , we get  $ac \in I$  for any  $c \in R$ . This implies that  $I$  is two-sided.<sup>7)</sup> We say that  $I$  is *proper* if  $R/I$  is not  $(0)$  and has a unit. In this sense, for any proper ideal  $I$  of  $R$ , there exists a maximal proper ideal  $J$  of  $R$  containing  $I$ . Next, suppose  $J$  is a maximal proper (left) ideal of  $R$ . Since  $R/J$  constitutes a normed sfield over the complex number field  $C$ , we have  $R/J \cong C$  by the theorem of S. Mazur and I. Gelfand. Denote by  $t_J$  the canonical homomorphism of  $R$  onto  $C$ . Let  $h$  be an hermitian element of  $R$ . If  $h \neq 0$ ,  $h$  has at least one non-zero spectrum, say  $\beta$ . And the closed (left) ideal  $I$  of  $R$  generated by  $(ch - \beta c; c \in R)$  is proper<sup>8)</sup>. Hence there exists a maximal proper ideal  $J$  of

5) For any hermitian element  $h$  of  $R$ , we have  $h \in A$ . In fact, if  $xh - \alpha x = h$  has no solution in  $R$  for  $\alpha \neq 0$ , it has also no solution in  $R_n$ . And there is a simple proof by M. Fukamiya with regard as  $h \in A$  when  $R$  is commutative (cf. [4]).

6) When  $R$  has a unit,  $R$  is commutative if and only if  $ab=0$  implies  $acb=0$  for any  $c \in R$ . The proof is similar.

7) We notice that  $a \in I$  implies  $a^* \in I$  for any closed two-sided ideal  $I$  of  $R$ . In fact, we have  $a^* u_n(a^*) \in I$  by Weierstrass' approximation theorem. Hence we get  $a^* \in I$ .

8) We first show that  $R/I \neq (0)$ . For, otherwise, we can find a sequence  $\{c_n\}$  such that  $c_n h - \beta c_n \rightarrow h$  and hence  $\|ch - \beta c - h\| < |\beta|$  for some  $c \in R$ . Put  $d = (1/\beta)(ch - \beta c - h)$ ,  $x = d - d^2 + d^3 + \dots$  and  $y = x + c - xc$ . Then,  $yh - \beta y = h$ . This is impossible. Hence,  $R/I \neq (0)$ . Denote by  $-$  the canonical homomorphism of  $R$  onto  $R/I$ . Put  $e = (1/\beta)h$ . Then  $\bar{e}$  is hermitian and we have  $\bar{e}\bar{e} = \bar{e}$  for any  $c \in R$ . This means that  $\bar{e}$  is a unit of  $R/I$ . Therefore  $I$  is proper.

$R$  containing  $(ch - \beta c; c \in R)$ . Then we have  $t_J(h) = \beta \neq 0$  by an easy computation. Thus we have  $h = 0$  if and only if  $t_J(h) = 0$  for any maximal proper ideal  $J$  of  $R$ . Now, suppose  $h, k$  are hermitian elements of  $R$ . Put  $x = (1/2i) \cdot (hk - kh)$ . Then, for any maximal proper ideal  $J$  of  $R$ , we have

$$(1.4) \quad t_J(x) = (1/2i)(t_J(h)t_J(k) - t_J(k)t_J(h)) = 0.$$

Hence we get  $x = 0$ , that is,  $hk = kh$ . Since general elements of  $R$  are generated by hermitian elements of  $R$ , this completes the proof. q. e. d.

Denote by  $N$  the set of hermitian elements of  $R$ .

LEMMA 1.3. 1)  $N$  is closed.

2)  $*$  is continuous.

PROOF. THE PROOF OF 1). Let  $\{h_n\}$  be a Cauchy sequence of elements of  $N$ . Suppose its limit is not hermitian, that is,  $h + ik$  for  $h, k \in N$  such as  $k \neq 0$ . Put  $h_n' = h_n - h$ . Then  $h_n'^2 \rightarrow -k^2$  as  $n \rightarrow \infty$ . We may assume that  $\|h_n'\| \leq 1, \|k\| \leq 1$ . Denote by  $P$  the polynomial ring over the complex number field  $C$ . It is not so hard to see that  $\|(1 - h_n'^2)p(h_n'^2)\| \leq \|p(h_n'^2)\|$  for all  $p \in P$  with  $p(0) = 0$ . From this it follows that  $\|(1 + k^2)p(-k^2)\| \leq \|p(-k^2)\|$  for all  $p \in P$  with  $p(0) = 0$ . Since  $k \neq 0$ , we can find a spectre<sup>9)</sup>  $t$  of  $R_k$  with  $t(k) \neq 0$  (and so  $t(k^2) > 0$ ). Since  $\Omega_k$  is compact, there exists a continuous function  $g^{10)}$  on  $\Omega_k$  satisfying (1)  $g(t) = 1$ , (2)  $0 \leq g(t') \leq 1$  for  $t' \in \Omega_k$ , and (3) when  $R_k$  has no unit,  $g(0) = 0$ . Then we get  $\|(1 + k^2)g(k^2)\| \leq 1$  by making use of Weierstrass' approximation theorem. Hence  $1 + t(k^2) \leq 1$ . This is impossible, for  $t(k^2) > 0$ . Thus we get 1).

THE PROOF OF 2). We introduce a norm  $\|\cdot\|'$  into  $R$  defined by  $\|a\|' = \sup(\|h \cos \theta + k \sin \theta\|; 0 \leq \theta \leq 2\pi)$  for  $a \in R$ , where  $h = (1/2)(a + a^*)$  and  $k = (1/2i)(a - a^*)$ . In view of 1),  $R$  constitutes a Banach space over the complex number field  $C$  with this norm. Denote it by  $R_1$ . Further we see that  $\|a\| \leq 2\|a\|'$ . Hence  $R_1$  is homeomorphic to  $R$  by a theorem of S. Banach. Hence we have  $\|a\|' \leq K\|a\|$  for  $a \in R$  and for some positive number  $K$  independent of  $a$ . Thus we get  $\|a^*\| \leq 2\|a^*\|' = 2\|a\|' \leq 2K\|a\|$ . This implies 2). q. e. d.

Denote by  $N^+$  the set of hermitian elements  $h$ 's of  $R$  with  $h = k^2$  for some  $k \in N$ .

LEMMA 1.4. 1)  $N^+$  is closed.

2)  $A$  is closed.

PROOF. The proof of 1). Let  $\{h_n\}$  be a Cauchy sequence of elements of  $N^+$ . Denote its limit by  $k$ . Assume that  $\|h_n\| \leq 1, \|k\| \leq 1$ . Then we have  $\|(1 - k)p(k)\| \leq \|p(k)\|$  for all  $p \in P$  with  $p(0) = 0$ . Similarly as in the proof of 1), Lemma 1.3 and so every spectrum of  $k$  is non-negative. Hence we get  $\sqrt{k} \in N$  and  $k = (\sqrt{k})^2 \in N^+$ . This shows 1). The statement 2) is an immediate

9) A point  $t$  of  $\Omega_k$  is called a *spectre* of  $R_k$ .

10) The only case which can not be verified by P. Urysohn's theorem is:  $R_k \cong C$ .

consequence of 1) and so the proof will be omitted. q. e. d.

LEMMA 1.5. *There exists an element  $a$  of  $R$  satisfying  $u_n(a)Ru_n(a) \subseteq A$  for  $n \geq 1$ . (The  $u_n$ 's are defined like those in Lemma 1.2.)*

PROOF. If  $R$  is commutative, we have  $A = R$  and so we can choose an arbitrary non-zero element of  $R$  as the element  $a$  in question. On the other hand, if  $R$  is not commutative, from Lemma 1.2 it follows that there exist three elements  $b, c$ , and  $d$  such that  $b - bd = 0$  and  $bc - bcd \neq 0$ . Here we may assume without loss of generality that  $b, d$  are both hermitian. In fact, denoting by  $u_n'(d^*)$  the element  $(1 - g_n(1+t))/(dd^* - d^* - d)$  of  $R$ , we have  $u_n(b) - u_n(b)u_n'(d^*) = 0$  by an easy computation. If  $u_n(b)c - u_n(b)cu_n'(d^*) = 0$  for any  $n$ , we would have  $bu_n(b)c - bu_n(b)cu_n'(d^*) - (bu_n(b)c - bu_n(b)cu_n'(d^*))d = 0$ , from which it follows that  $bc - bcd = 0$  as making  $n \rightarrow \infty$ . This leads to a contradiction. We write  $a$  for  $bc - bcd$ . Since  $b, d$  are both hermitian, we have  $b - db = 0$  and so  $a^2 = bc(b - db)(c - cd) = 0$ . Hence  $a \in A$  by Lemma 1.1. Denote  $u_n(a)$  briefly by  $u_n$  and take  $v_n$  as the element  $h_n(t)/(a^*a)$  of  $R$ , where

$$(1.5) \quad h_n(t) = \begin{cases} t^{-1/2} & (2/n \leq t), \\ \text{linear} & (0 \leq t \leq 2/n), \\ 0 & (t \leq 0). \end{cases}$$

Then we have  $(au_{2n}v_{4n})^*(au_{2n}v_{4n})u_n = u_n$  and  $u_na = 0$ . Hence we have  $(au_{2n}v_{4n}u_nxu_n)^*(au_{2n}v_{3n}u_nxu_n) = (u_nxu_n)^*(u_nxu_n)$  and  $(au_{2n}v_{4n}u_nxu_n)^2 = 0$ . Thus we have from Lemma 1.1  $u_nxu_n \in A$  for any  $x \in R$ . q. e. d.

REMARK. We can find the element  $a$  of  $R$  in Lemma 1.5 such as it is contained in an arbitrary fixed closed ideal  $I$  of  $R$ . In fact, if for any  $a, b \in I$   $a - ab = 0$  implies  $ac - acb = 0$  for any  $c \in R$ ,  $I$  must be commutative.

We may assume without loss of generality that  $\|a^*a\| = 1$  and  $\|u_n\| = 1$ . Since  $A$  is closed by Lemma 1.4, the closure  $R_n$  of  $u_nRu_n$  is contained in  $A$ . Denote by  $N_n$  the set of hermitian elements of  $R_n$ . As to  $N_n$ , we have the following

LEMMA 1.6. *The set  $N_n$  constitutes a semi-ordered linear space over the real number field  $R^{(1)}$  with the semi-order  $a \leq b$  for  $a, b \in N_n$  defined by the relation that every spectrum of  $b - a$  is non-negative.*

PROOF. The others are obvious and so we need only to see the transitive law. Denote  $(\sqrt{1+t}-1)/(a^*a) + u_{4n}$  by  $x$  and  $(tg_{1n}^{-2}(t))^{-1}g_{2n}(t)/(a^*a)^{12}$  by  $x'$ . Then we have  $x'x = xx' = u_{2n}$ . Further we put  $y = bx'$  and denote  $(\sqrt{1+t}-1)/(y^*y)$

11) A linear space over the real number field is called to be *semi-ordered* if it has a semi-ordering compatible with linear operation: (1)  $a, b \geq 0$  imply  $a + b \geq 0$  and (2)  $a \geq 0, \alpha \geq 0$  imply  $\alpha a \geq 0$ .

12) Here we consider  $\infty \cdot 0$  as 0.

$+u_{2n}$  by  $z$ . Then we have  $x, y, z \in R_{4n}$  and  $u_{2n}^2 + a^*a + b^*b = (xz)^*(xz)$  by an easy computation. Thus we get the assertion. q. e. d.

Let  $K$  be the linear space over the real number field generated by  $u_2$  and  $N_1$ . By Lemma 1.6,  $K$  constitutes an  $N$ -space over the real number field with  $u_2$  as its order unit.<sup>13)</sup> Moreover we can see that  $K$  is admissible<sup>14)</sup> by an easy computation. Hence, by the extension theorem of Hahn and Banach, there exists at least one positive<sup>15)</sup> linear functional (say  $f'$ ) of  $K$  with  $f'(u_1^2) > 0$ . As to  $f'$ , we have the following

LEMMA 1.7. *It holds that  $f'(u_1c^*cu_1) \geq 0$  for  $c \in R$ .*

PROOF. Denote  $(1 - \sqrt{1-t})/u_2^2$  by  $v$ . Since  $(cu_1 - vcu_1)^2 = 0$ , we have, by Lemma 5.1,  $cu_1 - vcu_1 \in A$  and so  $f'(u_1c^*cu_1 - u_1c^*u_2^2cu_1) \geq 0$ . On the other hand, from  $u_2cu_1 \in A$  it follows that  $f'(u_1c^*u_2^2cu_1) \geq 0$ . Hence we get  $f'(u_1c^*cu_1) = 0$ . q. e. d.

We define a functional  $f$  of  $R$  by  $f(c) = f'(u_1(1/2)(cc^*)u_1) + if'(u_1(1/2i)(c - c^*)u_1)$  for  $c \in R$ . Then  $f$  is a state<sup>16)</sup> of  $R$  by Lemma 1.7. Denote by  $\Gamma^*$  the set of states of  $R$  and, for  $f \in \Gamma^*$ , we denote by  $M_f$  the set of elements  $c$ 's of  $R$  with  $f(c^*c) = 0$ , which is a closed left ideal of  $R$ . Denote by  $\pi$  the canonical homomorphism of  $R$  onto  $R/M_f$  and denote by  $H_f$  the completion of the unitary space  $R/M_f$  over the complex number field  $\mathbb{C}$  with the inner product  $(\bar{b}, \bar{c}) = f(c^*b)$ . It is easy to see that  $R$  is represented on  $\mathfrak{H}_f$ .<sup>17)</sup> Its kernel will be denoted by  $I_f$ . Similarly we denote by  $\mathfrak{H}$  the direct sun of  $(\mathfrak{H}_f; f \in \Gamma^*)$ . Then  $R$  is also represented on  $\mathfrak{H}$ .<sup>18)</sup> Its kernel will be denoted by  $I$ . Then we have  $I = \bigcap (I_f; f \in \Gamma^*)$ , which must be  $(0)$ . For, otherwise, from the remark of Lemma 1.5, we have a state  $f$  of  $R$  with  $f(u^2) > 0$  for some hermitian element  $u$  of  $I$ . This leads to a contradiction. Thus  $R$  is  $*$ -isomorphic to a self-adjoint algebra acting on  $H$ , say  $S$ .

The following lemma is due to I. E. Segal.

LEMMA 1.8. *Let  $S_1$  be a commutative  $B^*$ -algebra and  $S_2$  be a commutative, not necessarily complete  $B^*$ -algebra. Suppose  $S_1$  is  $*$ -isomorphic to  $S_2$ . Then  $S_1$  is isometric to  $S_2$ .*

13) We mean by an  $N$ -space over the real number field the semi-ordered linear space over the real number field with an order unity  $e$ :  $N = (a; a \leq ne$  for some  $n$ ).

14) With the same terminologies as in 13),  $N$  is called *admissible* if  $a \leq \inf(\varepsilon; a \leq \varepsilon e)$  for any  $a \in N$ .

15) A linear functional  $f$  of a semi-ordered linear space is called to be *positive* if  $a \geq 0$  implies  $f(a) \geq 0$ .

16) A linear functional  $f$  of  $a^*$ -algebra  $R$  is called a *state* of  $R$  if  $f(a^*a) \geq 0$  and  $f(a^*) = \overline{f(a)}$ .

17) cf. [1], [2] and [7].

18) cf. [1], [2] and [7].

PROOF. Denote by  $\mathcal{Q}_i$  the spectrum<sup>19)</sup> of  $S_i$ . Then we may assume without loss of generality that  $\mathcal{Q}_2$  is contained in  $\mathcal{Q}_1$ . If  $\bar{\mathcal{Q}}_2 \neq \mathcal{Q}_1$  ( $\bar{\mathcal{Q}}_2$  being the closure of  $\mathcal{Q}_2$  in  $\mathcal{Q}_1$ ), there exists a non-zero continuous function  $g$  defined on  $\mathcal{Q}_1$  such that  $\mu(g) = 0$  for  $\mu \in \mathcal{Q}_2$ . Of course, this is a contradiction. Hence we get  $\bar{\mathcal{Q}}_2 = \mathcal{Q}_1$ . For  $c_i \in S_i$ , the norm of  $c_i$  is expressed as  $\|c_i\| = \sup(|\mu(c_i)|; \mu \in \mathcal{Q}_i)$ . Thus we obtain the assertion. q. e. d.

Now we are in a position to prove Theorem 1.1.

THE PROOF OF THEOREM 1.1. Let  $h$  be an hermitian element of  $R$  and let  $R'$  be a maximal commutative  $*$ -subalgebra of  $R$  containing  $h$ . In view of Lemma 1.8,  $R'$  is isometrically and  $*$ -isomorphically imbedded into  $S$  by the  $*$ -isomorphism obtained before. Hence  $N$  is isometrically imbedded into  $S$  by that  $*$ -isomorphism. From this it follows that  $S$  is closed by making use of Lemma 1.3. Denote by  $\|\cdot\|_0$  the norm of  $S$  and by  $\|\cdot\|_0'$  the norm of  $S$  defined by  $\|a\|_0' = \sup(\|h \cos \theta + k \sin \theta\|; 0 \leq \theta \leq 2\pi)$ , where  $h = (1/2)(a + a^*)$ ,  $k = (1/2i)(a - a^*)$ . Moreover we use the norm of  $R$  and the norm of  $R$  cited in Lemma 1.3. Then these four norms are equivalent (that is, they induce homeomorphic topologies) to each others. This shows 1). Since  $N$  is isometrically imbedded into  $S$ ,  $R$  is isometric to  $S$  by (0.5) and (0.6) if  $R$  is  $B^*$ . This implies 2). q. e. d.

By virtue of Theorem 1.1, following corollaries are reduced to the well-known results for the case of  $C^*$ -algebras.

COROLLARY 1. *Let  $R$  be a  $C$ -symmetric  $*$ -algebra. Then  $R$  satisfies (A). (I. Kaplansky-M. Fukamiya-J. L. Kelley-R. L. Vaught).*

COROLLARY 2. *Let  $R$  be a  $C$ -symmetric  $*$ -algebra. Then  $N$  has an intrinsic semi-ordering  $a \geq 0$  for  $a \in N$  defined by  $a = h^2$  for some  $h \in N$  (loc. cit.)*

COROLLARY 3. *Let  $R$  be a  $B^*$ -algebra. Then  $R$  is isometrically and  $*$ -isomorphically imbedded into a  $B^*$ -algebra with a unit (I. Kaplansky [4]).*

## § 2. An affirmative answer to (B).

We shall say, for a while, that a  $*$ -algebra  $R$  is  $B'^*$  if it satisfies the condition:

$$(2.1) \quad \|a^*a\| = \|a^*\| \|a\| \text{ for all } a \in R.$$

19) We say that a homomorphism  $t$  of a normed ring  $R$  over  $C$  onto  $C$ , if  $R$  has a unit, (into  $C$ , if  $R$  has no unit) is a *spectre* of  $R$  and the set of spectres of  $R$  is the *spectrum* of  $R$ . We notice that every spectre  $t$  of  $R$  is continuous. In fact, if  $t(a) \neq 0$ , the equation  $xa - t(a)x = a$  has no solution in  $R$  and so  $t(a)$  is a spectrum of  $a$ , and then  $|t(a)| \leq \|a\|$  by a theorem of I. Gelfand. From this it follows that the spectrum of  $R$  is compact with respect to the usual Stone topology. If  $R$  is a not necessarily complete normed ring, the spectrum of the completion of  $R$  is called the spectrum of  $R$ .

It is obvious<sup>20)</sup> that a  $B^*$ -algebra is always  $B'^*$ . The aim of this § is, conversely, to give an affirmative answer to (B), that is, to prove the following

THEOREM 2.1. *Every  $B'^*$ -algebra is necessarily  $B^*$ .*

Let  $R$  be a  $B'^*$ -algebra. The following lemma is well known and is included only for completeness.

LEMMA 2.1.  *$R$  is  $C$ -symmetric.*

PROOF. By the definition of  $C$ -symmetricity we need only to see that  $R$  is  $B^*$  if it is commutative. Suppose  $R$  is commutative. Then it is obvious that  $\|a\|_\infty \leq \|a\|$ ,  $\|a^*\|_\infty \leq \|a^*\|$ , and  $\|a^*\|_\infty = \|a\|_\infty$ . Moreover we have from (2.1)  $\|a^*a\|_\infty = \|a^*a\|$  and from the commutativity of  $R$   $\|a^*a\|_\infty \leq \|a^*\|_\infty \|a\|_\infty$ . From these it follows that  $\|a^*\| = \|a\|$ . Thus we get  $\|a^*a\| = \|a^*\| \|a\| = \|a\|^2$ . q. e. d.

Combining this lemma with Corollary 1 of Theorem 1.1, we may use (A) for the present case. From now on we denote by  $\beta(a)$  the positive number determined by  $\|a^*\| = \beta(a)\|a\|$  for a non-zero element  $a$  of  $R$  and we put  $\beta(0) = 1$ .<sup>21)</sup> We recall two elements  $a, b$  of  $R$  to be orthogonal to each other if  $b^*a = ab^* = 0$  and mean by  $a, b, \dots (\perp)$  the fact that  $a, b, \dots$  are mutually orthogonal, non-zero elements of  $R$ .

The following lemma is a key to prove the theorem.

LEMMA 2.2.  *$a, b (\perp)$  implies  $\beta(a) = \beta(b) = \beta(a+b)$ .*

PROOF. Suppose  $0 \neq \|b^*b\| \leq \|a^*a\|$ . Then we have  $\|a^*\| \|a\| = \|a^*a\| = \|a^*(a+b)\| \leq \|a^*\| \|a+b\|$ , that is,  $\|a\| \leq \|a+b\|$ . Similarly we have  $\|a^*\| \leq \|(a+b)^*\|$ . On the other hand, we have  $\|a^*\| \|a\| \leq \|(a+b)^*\| \|(a+b)\| = \|(a+b)^*(a+b)\| = \|a^*a + b^*b\| \leq \|a^*a\| = \|a^*\| \|a\|$ . Thus we get  $\|a\| = \|a+b\|$  and  $\|a^*\| = \|(a+b)^*\|$ . This shows that  $\beta(a) = \beta(a+b)$ .

If we select a positive scalar  $\alpha$  such that  $\|(\alpha a)^*(\alpha a)\| = \|b^*b\|$ , we similarly get  $\beta(b) = \beta(\alpha a + b) = \beta(\alpha a) = \beta(a)$ . Thus we reach the assertion. q. e. d.

In order to prove the theorem, we now prepare a tool to eliminate the trivial cases. A projection  $e$  of  $R$  is called *minimal* if it is non-zero and if every element  $a$  of  $eRe$  is written as  $a = \beta e$  for some scalar  $\beta$ . Moreover an element  $a$  of  $R$  is called *primitive* if  $a^*a$  is written as  $a^*a = \beta e_*$ , where  $\beta$  is a non-zero scalar and  $e_*$  is a minimal projection of  $R$ . In this case,  $\beta$  is

20) If  $a \neq 0$ ,  $\|a\|^2 = \|a^*a\| \leq \|a^*\| \|a\|$  implies  $\|a\| \leq \|a^*\|$ . Similarly we have  $\|a^*\| \leq \|a\|$ . Thus we get  $\|a^*\| = \|a\|$ .

21) According to [4] [9], we have already known the following facts: (a)  $*$  is continuous, (b)  $R$  is homeomorphic and isomorphic to a  $C^*$ -algebra, and (c) (R. Kadison) if  $a$  is regular, then  $\beta(a) = 1$ . The facts (a) and (b) are contained in Lemma 1.3 and Theorem 1.1. As to the fact (c), we shall use it, to prove the theorem, in a weaker sense: if  $u$  is unitary, then  $\beta(u) = 1$ . And this fact is an immediate consequence of Lemma 2.1.



positive by (A) and  $w = (\sqrt{\beta})^{-1}a$  is a primitive partial isometry<sup>22)</sup>, and then we get a polar decomposition  $a = w(\sqrt{\beta}e_*)$ , because  $(w - we_*)^*(w - we_*) = 0$  and so  $w = we_*$ . This decomposition is unique in the sense that  $\beta > 0$  and  $w = we_*$ . Put  $e = ww^*$ . Since  $w = we_*$ , we have  $e^2 = e (= e^*)$  and hence  $e$  is a projection of  $R$ . We call  $e_*$  the initial projection of  $a$  (denoted by  $e_*(a)$ ) and  $e$  the final projection of  $a$  (denoted by  $e(a)$ ).

LEMMA 2.3. *If  $a$  is a primitive element of  $R$ , then  $e(a)$  is also minimal.*

PROOF. Suppose  $b$  is an arbitrary element of  $R$ . Since  $w^*bw$  is contained in  $e_*Re_*$ , it is written as  $w_*bw = \beta e_*$ . Hence we have  $ebe = w(w^*bw)w^* = w(\beta e_*)w^* = \beta e$ . This shows that  $e$  is minimal. q. e. d.

If  $a$  is primitive, so is  $a^*$  by Lemma 2.3.

LEMMA 2.4. *If  $a$  is primitive and  $ba \neq 0$  for  $b \in R$ , then  $ba$  is also primitive.*

PROOF. Since  $(ba)^*(ba)$  is contained in  $e_*(a)Re_*(a)$ , it is written as  $(ba)^*(ba) = \beta e_*(a)$  for some scalar  $\beta$ . The scalar  $\beta$  is not zero, for  $ba \neq 0$ . Hence  $ba$  is primitive. q. e. d.

LEMMA 2.5. *Let  $w$  be a primitive partial isometry of  $R$ . Then we have  $e(w)Re_*(w) = Cw$ .*

PROOF. Suppose  $a$  is an element of  $e(w)Re_*(w)$ . Since  $w^*a$  is contained in  $e_*(w)Re_*(w)$ , it is written as  $w^*a = \beta e_*(w)$  and so we have  $a = ww^*a = \beta w \in Cw$ . The converse is obvious and thus we obtain  $e(w)Re_*(w) = Cw$ . q. e. d.

LEMMA 2.6. *Let  $a$  be a primitive element of  $R$ . Denote  $e(a) + e((1 - e(a))e_*(a))$  briefly by  $f$ . Then we have the following two propositions:*

- 1)  $a \in fRf$ .
- 2)  $fRf$  is isomorphic to  $(C)_1$  or  $(C)_2$ .<sup>23)</sup>

PROOF. Suppose  $a$  is primitive. Denote  $e_*(a)$ ,  $e(a)$  by  $e_*$ ,  $e$ , and  $e((1 - e(a))e_*(a))$  by  $e'$ . Since  $e'(1 - e)e_* = (1 - e)e_*$ , we have  $(e + e')e_* = e_* + e'ee_*$ . By noticing that  $e'e = 0$ , we get  $fe_* = e_*$  or  $e_*f = e_*$ , that is,  $af = ae_*f = ae_* = a$ . On the other hand, it is obvious that  $fe = e$  and so  $fa = fea = ea = a$ . Thus we get the first assertion.

If  $eae' = e'ae = 0$ , we have  $a = faf = eae + e'ae' = \alpha e + \beta e'$  ( $\alpha, \beta$ , being scalars). This shows that  $a$  is normal<sup>24)</sup> and so  $e_* = e = f$ . Therefore  $fRf = eRe = Ce \cong (C)_1$ .

On the other hand, if there is an element  $b (= a$  or  $a^*)$  of  $R$  such that  $ebe' \neq 0$ ,  $ebe'$  is primitive by Lemma 2.4 and so written as  $ebe' = \beta w$ , where  $\beta$  is a non-zero scalar and  $w$  is a primitive partial isometry of  $fRf$ . We write  $e_{11}$ ,  $e_{22}$ ,  $e_{12}$ , and  $e_{21}$  for  $e$ ,  $e'$ ,  $w$ , and  $w^*$  respectively. Then  $e_{ij}$ 's form a complete

22) An element  $w$  of  $R$  is called a partial isometry of  $R$  if  $w^*w$  is a projection of  $R$ . In this case,  $ww^*$  is also a projection of  $R$ .

23) We denote by  $(C)_r$  the metric algebra of degree  $r$  over  $C$ .

24) An element  $a$  of  $R$  is called to be normal if  $a^*a = aa^*$ .

set of matrix units of degree 2. In view of Lemma 2.5, every element  $c$  of  $fRf$  is written as  $c = \sum_{i,j=1}^2 2r_{ij}e_{ij}$  ( $r_{ij}$  being a scalar). Thus we get the second assertion. q. e. d.

LEMMA 2.7. *If  $a$  is primitive, then we have  $\beta(a) = 1$ .<sup>25)</sup>*

PROOF. In view of Lemma 2.6, we can assume without loss of generality that  $R$  is  $*$ -isomorphic to  $(C)_1$  or  $(C)_2$ . As the assertion is quite obvious in the latter case, we restrict the proof only in the former case. Denote by  $a = w(\beta e_*)$  the polar decomposition of  $a$ . An easy computation is used to find a partial isometry  $w'$  of  $R$  orthogonal to  $w$ . Then  $u = w + w'$  is a unitary<sup>26)</sup> of  $R$  and so it satisfies  $\beta(u) = 1$  by Lemma 2.1. Hence we have  $\beta(a) = \beta(\beta w) = \beta(w) = 1$  by Lemma 2.2. q. e. d.

Now we enter the last lemma for the proof of Theorem 2.1.

LEMMA 2.8.  *$a, b, c (\perp)$  implies  $\beta(a) = 1$ .*

PROOF. We write  $h$  for  $a^*a$  and  $v$  for  $bb^*$ . Then we have

$$(2.2) \quad va = 0.$$

If  $hw = 0$ , we have  $a^*av = 0$  and so  $(av)^*(av) = 0$  and then  $av = 0$ . Hence we have  $a, v (\perp)$  and  $\beta(v) = 1$ .<sup>27)</sup> Thus we obtain  $\beta(a) = 1$  by Lemma 2.2. Therefore we may assume that  $hw \neq 0$ .

First we see Lemma 2.8 under the following assumption:

$$(2.3) \quad hv \text{ is primitive.}$$

Since  $hw \neq 0$ , we have  $h\sqrt{v} \neq 0$  and so  $a^*avh = hwh \neq 0$ . Hence we get  $avh \neq 0$ . Moreover  $hw$  is primitive and so  $vh$  is also primitive. Hence  $avh$  is primitive by Lemma 2.4. Since  $avh = a(va^*)a$ , we have  $avh, b (\perp)$ . Hence we have  $\beta(a) = \beta(b) = \beta(avh) = 1$  by Lemma 2.2 and Lemma 2.7.

Next we see Lemma 2.8 for the case that  $hw$  is not primitive. We write  $d$  for  $vh$ . Then  $d$  is non-zero and not primitive with  $hw$ .

We first show that there exist two hermitian elements  $v_1, v_2$  of  $R$  enjoying the following three conditions:

$$(2.4) \quad v_1, v_2, d^*d \text{ are mutually commutative.}$$

$$(2.5) \quad v_1v_2 = 0.$$

$$(2.6) \quad dv_1 \neq 0, dv_2 \neq 0.$$

If  $d^*d$  has at least two non-zero spectra (say  $\beta_1, \beta_2; 0 < \beta_1 < \beta_2$ ), we can take  $f_i(t)/d^*d$  as  $v_i$ , where  $f_i(t)$  is the function defined by

25) This lemma is due to M. Fukamiya. Owing to this lemma, proofs of Lemma 2.8 and the theorem become much simpler than original ones. His proof is elegant: Suppose  $R$  has a unit. Since  $ae = eae$  is written as  $\beta e$  ( $\beta$  being a scalar),  $a = ea$  has at most two spectra with  $ae$ . Hence  $a - \lambda$  is regular but for two exceptional  $\lambda$ 's. If  $a - \lambda$  is regular, we have  $\|(a - \lambda)^*\| = \|a - \lambda\|$  by R. Kadison's result. As making  $\lambda \rightarrow 0$ , we get  $\|a^*\| = \|a\|$ , that is,  $\beta(a) = 1$ .

26) An element  $u$  of  $R$  is called to be unitary if  $u^*u = 1$ .

27) For  $v$  is hermitian.

$$f_1(t) = \begin{cases} 0 & (t \leq (1/2)(\beta_1 + \beta_2)), \\ \text{linear} & ((1/2)(\beta_1 + \beta_2) \leq t \leq \beta_2), \\ 1 & (\beta_2 \leq t), \end{cases}$$

$$f_2(t) = \begin{cases} 1 & (t \leq \beta_1), \\ \text{linear} & (\beta_1 \leq t \leq (1/2)(\beta_1 + \beta_2)), \\ 0 & ((1/2)(\beta_1 + \beta_2) \leq t). \end{cases}$$

The verification of (2.4)–(2.6) is easy and so it will be omitted. On the other hand, if  $d^*d$  has exactly one non-zero spectrum (say  $\beta$ ), we write  $w$  for  $(\sqrt{\beta})^{-1}d$ . Then  $w$  is a partial isometry of  $R$  and  $d = w(\sqrt{\beta}w^*w)$  is a polar decomposition of  $d$ . Write  $e_*$  for  $w^*w$ . Since  $d$  is not primitive,  $e_*$  is not minimal. Hence we can find an element  $b$  of  $e_*Re_*$ , which is never written as  $b = \alpha e_*$  for any scalar  $\alpha$ . In this case, either  $\Re(b)$  or  $\Im(b)$ <sup>28)</sup> is never written as  $\alpha e_*$  for any scalar  $\alpha$  and so we may assume that  $b$  is hermitian. By taking, if necessary,  $\gamma e_* - b$  (for some scalar  $\gamma$ ) in place of  $b$ , we may assume moreover that  $b$  is positive and has an inverse in  $e_*Re_*$ . Since  $b$  is never written as  $\alpha e_*$  for any scalar  $\alpha$ ,  $b$  has at least two non-zero spectra.<sup>29)</sup> We construct  $v_1, v_2$  by taking  $b$  in place of  $d^*d$  as before. Then it is not so hard to see that these  $v_1, v_2$  satisfy (2.4)–(2.6).

We put  $v_2' = dv_2d^*$ ,  $k = vv_2'v$ , and  $l = ahv_1h$ .

We see that

$$(2.7) \quad k \neq 0.$$

For, otherwise,  $vv_2'v = 0$  and so  $(vd\sqrt{v_2})(vd\sqrt{v_2})^* = 0$ . Hence we get  $vd\sqrt{v_2} = 0$ . This shows that  $v^2hv_2 = 0$  and so  $(vhv_2)^*(vhv_2) = 0$  and then  $vhv_2 = 0$ . That is,  $dv_2 = 0$ . This contradicts to (2.6). Therefore we must have (2.7).

Further we see that

$$(2.8) \quad l \neq 0.$$

For, otherwise, we have  $ahv_1h = 0$  and so  $(ah\sqrt{v_1})(ah\sqrt{v_1})^* = 0$ . Hence we have  $ah\sqrt{v_1} = 0$  and so  $ahv_1 = 0$ . This implies that  $a^*ahv_1 = h^2v_1 = 0$ . From this it follows that  $(hv_1)^*(hv_1) = 0$  and so  $hv_1 = 0$ . Hence we get  $dv_1 = 0$ . This contradicts to (2.6). Therefore we must have (2.8).<sup>30)</sup>

Since  $l = a(hv_1a^*)a$ ,  $l$  is orthogonal to  $c$  and so

$$(2.9) \quad \beta(c) = \beta(l).$$

In view of (2.2) we have  $kl = vv_2'(va)hv_1h = 0$ . On the other hand, we have  $lk = ahv_1hvv_2'v = ah(v_1d^*dv_2d^*)v = ah(v_1v_2d^*dd^*)v = 0$ . From these it follows that

28)  $\Re(b) = (1/2)(b + b^*)$ ,  $\Im(b) = (1/2i)(b - b^*)$ .

29) For, otherwise, we have  $b = ae_1$ ,  $e_1$  being a projection of  $e_*Re_*$  with  $e_1 \neq e_*$ . This contradicts to the existence of the inverse of  $b$  in  $e_*Re_*$ .

30) This proof of (2.7) is due to T. Saito.

$k$  is orthogonal to  $l$  and so in view of (2.7) and (2.8) we have

$$(2.10) \quad \beta(l) = \beta(k).$$

Since  $k$  is hermitian, we have

$$(2.11) \quad \beta(k) = 1.$$

In view of (2.9)–(2.11), we get  $\beta(c) = 1$  and so  $\beta(a) = 1$ . Thus we reach the assertion. q. e. d.

We are now in a position to prove the theorem.

THE PROOF OF THEOREM 2.1. Let  $a$  be an arbitrary (but fixed) non-zero element of  $R$ .<sup>31)</sup> We shall see the assertion for this  $a$  by the induction from 3 to 1 with respect to  $r$ , which appears in the following statement:

$$(2.12) \quad a^*a \text{ has at least } r \text{ non-zero spectra.}$$

Suppose first  $a^*a$  has at least three non-zero spectra (say  $\beta_1, \beta_2, \beta_3$ ;  $0 < \beta_1 < \beta_2 < \beta_3$ ). Consider following three functions:

$$h_1(t) = \begin{cases} 1 & (t \leq \beta_1), \\ \text{linear} & (\beta_1 \leq t \leq (1/2)(\beta_1 + \beta_2)), \\ 0 & ((1/2)(\beta_1 + \beta_2) \leq t), \end{cases}$$

$$h_2(t) = \begin{cases} 0 & (t \leq (1/2)(\beta_1 + \beta_2)), \\ \text{linear} & ((1/2)(\beta_1 + \beta_2) \leq t \leq \beta_2), \\ 1 & (t = \beta_2), \\ \text{linear} & (\beta_2 \leq t \leq (1/2)(\beta_2 + \beta_3)), \\ 0 & ((1/2)(\beta_2 + \beta_3) \leq t), \end{cases}$$

$$h_3(t) = \begin{cases} 0 & (t \leq (1/2)(\beta_2 + \beta_3)), \\ \text{linear} & ((1/2)(\beta_2 + \beta_3) \leq t \leq \beta_3), \\ 1 & (\beta_3 \leq t). \end{cases}$$

Write  $v_i$  for  $h_i(t)/a^*a$  and  $a_i$  for  $av_i$  ( $1 \leq i \leq 3$ ). Then  $a_i$ 's are non-zero and mutually orthogonal. Hence we have from Lemma 8  $\beta(a_i) = 1$ . Therefore we have  $\|a^*a\| = \|a^*av_1\|^2 = \|a_1^*a_1\| = \|a_1\|^2 \leq \|a\|^2 \|v_1\|^2 = \|a\|^2$ , that is,  $\|a^*a\|^{1/2} \leq \|a\|$ . Similarly we have  $\|a^*a\|^{1/2} \leq \|a^*\|$ .<sup>32)</sup> On the other hand, we have  $\|a^*a\|^{1/2} = \|a^*\| \|a\|$ . Thus we get  $\|a\| = \|a^*a\|^{1/2} = \|a^*\|$ , that is,  $\beta(a) = 1$ .

Next we assume that  $a^*a$  has at least  $r$  ( $< 3$ ) (eventually exactly  $r$ ) non-zero spectra and that the assertion holds for every element  $b$  of  $R$ , for which  $b^*b$  has at least  $r+1$  non-zero spectra. In this case,  $a^*a$  is expressed as  $a^*a = \sum_{j=1}^r \alpha_j e_{*j}$ , where  $\alpha_j$ 's are scalars and  $e_{*j}$ 's are projections of  $R$ .<sup>33)</sup> Write

31) When  $a = 0$ , the assertion is trivially valid and so this case is omitted.

32)  $aa^*$  has also at least three non-zero spectra.

33) We notice that  $\Omega_{a^*a}$  is a finite set.

$a_j$  for  $ae_{*j}$ . Then we have  $a_1, \dots, a_r (\perp)$  and  $a = a_1 + \dots + a_r$ . Hence we have  $\beta(a_1) = \dots = \beta(a_r) = \beta(a)$  by Lemma 2.2.

If  $e_{*j}$  (say  $e_{*1}$ ) is not minimal, then there exists an hermitian element  $h$  of  $e_{*1}Re_{*1}$ , which has at least two non-zero spectra. Moreover, it necessary, by taking  $\beta e_{*1} - h$  in place of  $h$  (for a sufficiently large positive scalar  $\beta$ ), we may assume without loss of generality that  $h$  is positive and has at least two non-zero spectra distinct from  $\sqrt{\alpha_j/\alpha_1}$  ( $2 \leq j \leq r$ ). Put  $b = ah + \sum_{j=2}^r a_j$ . Then  $ah, \dots, a_r (\perp)$  and  $b^*b$  has at least  $r+1$  non-zero spectra. Hence we have  $\beta(b) = 1$  and so, if  $r \geq 2$ ,  $\beta(a_2) = 1$ , that is,  $\beta(a) = 1$ . On the other hand, if  $r = 1$ , we take  $c = w\sqrt{h}$  in place of  $b$ , where  $w = (\sqrt{\alpha_1})^{-1}a$  is a partial isometry of  $R$ . Then  $c^*c$  has at least two non-zero spectra and so  $\beta(c) = 1$ . Hence we have  $\|\sqrt{h}\| = \|w\sqrt{h}\| \leq \|w\| \|\sqrt{h}\|$  and so  $1 \leq \|w\|$ . Similarly we get  $1 \leq \|w^*\|$ . On the other hand, we have  $1 = \|w^*w\| = \|w^*\| \|w\|$ . Hence we get  $\|w^*\| = \|w\| = 1$  and so  $\beta(a) = \beta(\sqrt{\alpha_1}w) = \beta(w) = 1$ .

Thus we may assume without loss of generality that  $e_{*1}$  (and so all  $e_{*j}$ 's) is minimal. So  $a_j$  is primitive by Lemma 2.4 and then  $\beta(a_j) = 1$  by Lemma 2.7. This implies that  $\beta(a) = 1$ . This completes our assertion. q. e. d.

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