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On a problem of Alexandroff concerning the dimension of product spaces II.

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§ 1. Introduction.

Let Q be a class of topological spaces. A topological space X is called *a dimensionally full-valued space for* Q, if, whenever Y is a space of Q, the following equality holds:

$$\dim(X\times Y)=\dim X+\dim Y.$$

Here $\dim X\leqq n$ means that every finite open covering of X has a refinement of order not greater than n.

A sequence $\mathfrak{a}=(q_1, q_2,\cdots, q_i,\cdots)$ of positive integers is called a *k-sequence*[1] if q_i is a divisor of q_{i+1}, $i=1,2,\cdots$, and $q_i>1$ for some i. There exists a natural homomorphism $h(\mathfrak{a}, i)$ from $Z_{q_{i+1}}$ onto Z_{q_i}, $i=1,2,\cdots$, where Z_q means the factor group Z/qZ and Z means the additive group of all integers. Let us denote by $Z(\mathfrak{a})$ the inverse limit group of the inverse system $\{Z_{q_i}: h(\mathfrak{a}, i)\}$. Let (X, A) be a pair of topological spaces. We shall denote by $H_n(X, A: G)$ the n-dimensional Čech homology group of (X, A) with G as a coefficient group based on all open coverings of X. Consider the following property $\boldsymbol{P}$ of an n-dimensional topological space X.

$\boldsymbol{P}$. $\begin{cases}\textit{For every k-sequence } \mathfrak{a} \textit{ there exists a closed subset } A_{\mathfrak{a}} \textit{ of } X \textit{ such that} \\ H_n(X, A_{\mathfrak{a}}: Z(\mathfrak{a}))\neq 0.\end{cases}$

In the first paper under the same title [**10**] we have proved the following theorem.

THEOREM. *Let Q be a class of all compact metric spaces. In order that an n-dimensional compact metric space X be a dimensionally full-valued space for Q, it is necessary and sufficient that X have the property* $\boldsymbol{P}$.

In the proof of this theorem (cf. [**10**, pp. 391–393]) the compactness of X played an essential role. By making use of the unrestricted Čech homology groups we can remove the compactness condition of X from the sufficient condition of the theorem. Throughout this paper *we shall denote by Q the class of all locally compact fully normal spaces*. We shall prove the following theorem.

1) Cf. [**10**, § 1].

THEOREM 1. *An n-dimensional fully normal space X is a dimensionally full-valued space for Q if X has the property **P**.*

By Theorem 1 we can prove the following main theorem of this paper.

THEOREM 2. *In order that an n-dimensional locally compact fully normal space X be a dimensionally full-valued space for Q, it is necessary and sufficient that X has the property **P**.*

Our Theorem 2 is a generalization of the theorem [**10**] refered to above in two respects. Firstly, Theorem 2 does not assume the metrizability of spaces. Secondly, the compactness condition of spaces is weakened to the local-compactness condition; this generalization seems not to be trivial since in the formulation of property $\boldsymbol{P}$ we do not assume the compactness of the closed subset $A_{\mathfrak{a}}$ of X. By the proof of Theorem 1 we can prove the following K. Morita's theorem.

THEOREM 3. (K. Morita [**13**, Theorem 6]). *A 1-dimensional fully normal space X is a dimensionally full-valued space for Q.*

Finally, as a consequence of Theorem 1, we have the following corollary.

COROLLARY. *An n-dimensional fully normal space X which contains a closed subset A such that* $H_n(X, A: Z) \neq 0$ *is a dimensionally full-valued space for Q.*

In Addendum of the previous paper [**10**] we have proved that our property $\boldsymbol{P}$ is equivalent to Boltyanskii's property in compact metric spaces (cf. § 3, Remark). But, in case X is non-compact, we do not know whether our property $\boldsymbol{P}$ is equivalent to Boltyanskii's property even for locally compact fully normal spaces. In § 2 we shall prove several lemmas and introduce the notations used later on. The theorems mentioned above are proved in § 3. In § 4 we shall show that the converse of the corollary is not true even for the case where X is a two-dimensional compact metric space.

§ 2. Lemmas and notations.

A system $\mathfrak{W}$ of subsets in a topological space X is called to be *locally finite* if for each point x of X there exists a neighborhood $U(x)$ such that $U(x)$ intersects a finite number of sets of $\mathfrak{W}$. A normal space is called *fully normal* if every open covering has a locally finite open refinement (cf. [**14**] and [**15**]). Throughout this paper we mean *by a covering a locally finite open covering*. Let X be a fully normal space. A system $\boldsymbol{U} = \{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ of coverings of X is called a *cofinial system* of coverings of X if for each open covering $\mathfrak{U}$ of X there exists a member $\mathfrak{U}_\alpha$ of $\boldsymbol{U}$ such that $\mathfrak{U} < \mathfrak{U}_\alpha$ ($\mathfrak{U}_\alpha$ is a refinement of $\mathfrak{U}$). If $\mathfrak{U}_\alpha < \mathfrak{U}_\beta$ for $\alpha \in \Omega$ and $\beta \in \Omega$, we denote it simply by $\alpha < \beta$. The *order* of a covering is the largest integer n such that there exist $n+1$ members of the covering which has a non-empty intersection. By the *dimension* of X (we denote it by $\dim X$) we mean the least integer n such that every (finite or infinite) open covering has a locally finite refinement of the order n. By [**3**, Theorem 3.5] or [**12**, Theorem 2.1], this dimension is equivalent to the usual Lebesgue dimension. By the *nerve* of a covering we mean the nerve with the Whitehead weak topology (cf. [**16**] or [**5**]). Let K be the nerve of a covering $\mathfrak{U}$. We shall denote the vertex of K corresponding to an element U of $\mathfrak{U}$ by the same notation U. Since X is a normal space, for each covering $\mathfrak{U}$ there exists a canonical mapping[2] of X into the nerve K of the covering $\mathfrak{U}$. Let A be a closed subset of X. Let $\mathfrak{U}$ and $\mathfrak{V}$ be coverings of X such that $\mathfrak{U} > \mathfrak{V}$, and let (K, L) and (M, N) be the pairs of the nerves of $\mathfrak{U}$ and $\mathfrak{V}$ corresponding to (X, A) respectively. A projection of (K, L) into (M, N) defined as usual is continuous (cf. [**5**, § 4]). Let $\{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ be a cofinal system of coverings of X, and let us denote by (K_α, L_α) the pair of the nerves of $\mathfrak{U}_\alpha$ corresponding to (X, A) for $\alpha \in \Omega$ and by $\pi_\alpha{}^\beta$ a projection of (K_β, L_β) into (K_α, L_α) for $\beta > \alpha$. We mean by $H_n(K_\alpha, L_\alpha : G)$ the n-dimensional homology group of finite cycles of (K_α, L_α) with coefficients in G. For each pair $\beta > \alpha$ a projection $\pi_\alpha{}^\beta : (K_\beta, L_\beta) \to (K_\alpha, L_\alpha)$ induces the homomorphism $(\pi_\alpha{}^\beta)_* : H_n(K_\beta, L_\beta : G) \to H_n(K_\alpha, L_\alpha : G)$. The limit group $H_n(X, A : G)$ of the inverse system $\{H_n(K_\alpha, L_\alpha : G) : (\pi_\alpha{}^\beta)_* \mid \alpha < \beta : \alpha \in \Omega \text{ and } \beta \in \Omega\}$ is called the *n-dimensional unrestricted Čech homology group* of (X, A) with coefficients in G (cf. [**3**] or [**4**]). In compact spaces unrestricted Čech homology groups are equal to usual Čech homology groups based on all finite coverings. Let R_1 be the additive group of rational numbers mod 1. The following lemmas are well known (cf. [**11**, § 2] and [**12**, Theorem 3.2]).

LEMMA 1. (*Hopf's extension theorem*). *Let A be a closed subset of an $(n+1)$-dimensional compact space X. In order that a mapping f of A into the n-dimensional sphere S^n be extensible to a mapping of X into S^n, it is necessary and sufficient that the condition $f_* \partial H_{n+1}(X, A : R_1) = 0$ hold, where f_* is the homomorphism of $H_n(A : R_1)$ into $H_n(S^n : R_1)$ induced by the mapping f and ∂ is the boundary homomorphism[3] of $H_{n+1}(X, A : R_1)$ into $H_n(A : R_1)$.*

LEMMA 2. *Let X be a locally compact fully normal space. In order that* $\dim X = n$ *it is necessary and sufficient that*

(1) *there exists a closed subset A of X such that $H_n(X, A : R_1) \neq 0$,*

(2) *for every closed subset A of X and every integer $j > n$ we have $H_j(X, A : R_1) = 0$.*

LEMMA 3. *Let X be a locally compact fully normal space. In order that*

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2) A mapping of X into K is called a *canonical mapping* if the inverse image of the open star of each vertex U is contained in the open set U. Throughout this paper we shall mean *by a mapping a continuous transformation.*

3) Cf. [**6**, Chap. I and Chap. IX].

$\dim X \leqq n$ *it is necessary and sufficient that for every compact subset* A *of* X *we have* $\dim A \leqq n$.

Let X be a topological space and let $\mathfrak{U}$ be a covering of X. Let K be a simplicial complex with the Whitehead weak topology and let $\mathfrak{V}$ be the covering of K consisting of its open stars. By an $(\mathfrak{U}, K)$-mapping of X into K we mean a mapping f of X into K such that $\mathfrak{U} < f^{-1}(\mathfrak{V})$[4]. The following lemma is well known (cf. [**9**, Chap. V, § 8]).

Lemma 4. *Let* X *be a normal space. In order that* $\dim X \leqq n$ *it is necessary and sufficient that for every covering* $\mathfrak{U}$ *of* X *there exist an* n*-dimensional simplicial complex* K *and an* $(\mathfrak{U}, K)$*-mapping of* X *into* K.

The following lemma was proved by K. Morita (cf. [**13**, Theorem 4]).

Lemma 5. *Let* X *be a fully normal space and* Y *a locally compact fully normal space. Then the topological product of* X *and* Y *is fully normal, and we have* $\dim (X \times Y) \leqq \dim X + \dim Y$.

A topological group G is called to *satisfy the minimal condition* if, whenever $\{G_i \mid i = 1, 2, \cdots\}$ is a decreasing sequence of closed subgroups of G, there exists some integer n such that $G_n = G_{n+1} = \cdots$. The following lemma is easily proved and we omit the proof.

Lemma 6. *Let* $\{G_\alpha : \pi_\alpha{}^\beta\}$ *be an inverse system of compact topological groups over a directed set* $\Omega = \{\alpha\}$[5] *such that each* G_α *satisfies the minimal condition. Let* G *be the limit group of* $\{G_\alpha\}$. *For each* $\alpha \in \Omega$ *there exists an element* β *of* Ω *such that* $\alpha < \beta$ *and* $\pi_\alpha G = \pi_\alpha{}^\beta G_\beta$, *where* π_α *is the projection of* G *into* G_α.

Let q be a positive integer such that $q > 1$ and let us denote the k-sequence $(q, q^2, \cdots, q^i, \cdots)$ by $\mathfrak{a}_q$. There is a natural homomorphism ρ_q from $Z(\mathfrak{a}_q)$ onto Z_q defined by $\rho_q(c) = c_1$, where c_1 is the first coordinate of an element $c = \{c_i \mid i = 1, 2,\}$ of $Z(\mathfrak{a}_q)$.

Lemma 7. *Let* (X, A) *be a pair of* n*-dimensional fully normal spaces. If* $H_n(X, A : Z(\mathfrak{a}_q)) \neq 0$, *then the homomorphism* $(\rho_q)_* : H_n(X, A : Z(\mathfrak{a}_q)) \rightarrow H_n(X, A : Z_q)$ *induced by the homomorphism* $\rho_q : Z(\mathfrak{a}_q) \rightarrow Z_q$ *is non-trivial.*

Proof. Let $\{\mathfrak{U}_\alpha \mid \alpha \in \Omega\}$ be a cofinal system of coverings of X each member of which has the order n; let us denote by (K_α, L_α) the pair of the nerves of $\mathfrak{U}_\alpha$ corresponding to (X, A) for $\alpha \in \Omega$ and by $\pi_\alpha{}^\beta$ a projection of (K_β, L_β) into (K_α, L_α) for $\beta > \alpha$. Let $a = \{a_\alpha \mid \alpha \in \Omega\}$ be a non-zero element of $H_n(X, A : Z(\mathfrak{a}_q))$, where $a_\alpha \in H_n(K_\alpha, L_\alpha : Z(\mathfrak{a}_q))$ for $\alpha \in \Omega$. Since $\dim K_\alpha = n$, we can consider a_α as a cycle of (K_α, L_α) with coefficients in $Z(\mathfrak{a}_q)$ for each $\alpha \in \Omega$. Let $a_\alpha = \sum_i \boldsymbol{t}_{\alpha i} \sigma_{\alpha i}$, $\alpha \in \Omega$, where $\boldsymbol{t}_{\alpha i} \in Z(\mathfrak{a}_q)$ and $\sigma_{\alpha i}$'s are n-simplexes of K_α for each i. Put $a_{\alpha j} = \sum t_\alpha{}^j{}_i \sigma_{\alpha i}$, $j = 1, 2, \cdots$ and $\alpha \in \Omega$, where $t_\alpha{}^j{}_i$ is the j-th coordinate of

4) Let $\mathfrak{V} = \{V\}$ be a covering of a topological space Y and let f be a mapping of X into Y. By $f^{-1}(\mathfrak{V})$ we mean the covering $\{f^{-1}(V)\}$ of X.

5) Cf. [**6**, Chap. VIII].

the element $t_{\alpha i}$ of the inverse limit group $Z(\mathfrak{a}_q)$. Then $a_{\alpha j}$ is a cycle of (K_α, L_α) mod q^j [6] for $j=1,2,\cdots$ and $\alpha\in\Omega$. If $(\rho_q)_*a$ [6a] $=0$, $(\rho_q)_*a_\alpha$ [6a] $=0$ for each $\alpha\in\Omega$. Accordingly we have $a_{\alpha j}\equiv 0$ mod q [7] for $j=1,2,\cdots$ and $\alpha\in\Omega$. Therefore, since $\frac{1}{q}a_{\alpha j}$ [8] is a cycle of (K_α, L_α) mod q^{j-1} for $j=2,3\cdots$, $\frac{1}{q}a_\alpha$ is a cycle of (K_α, L_α) with coefficients in $Z(\mathfrak{a}_q)$. Since $(\pi_\alpha{}^\beta)_*\left(\frac{1}{q}a_\beta\right)=\frac{1}{q}a_\alpha$ for $\beta>\alpha$, $\left\{\frac{1}{q}a_\alpha\,|\,\alpha\in\Omega\right\}$ determines a non-zero element $a(1)$ of $H_n(X, A: Z(\mathfrak{a}_q))$. If $(\rho_q)_*a(1)=0$, by the same argument as above, we can see that $\left\{\frac{1}{q^2}a_\alpha\,|\,\alpha\in\Omega\right\}$ determines a non-zero element $a(2)$ of $H_n(X, A: Z(\mathfrak{a}_q))$. If we could repeat infinitely this process, we should have $a_{\alpha j}\equiv 0$ mod q^i for $i, j=1,2,\cdots$ and $\alpha\in\Omega$. This contradicts $a\neq 0$. Thus there exists an integer i such that the element $a(i)=\left\{\frac{1}{q^i}a_\alpha\,|\,\alpha\in\Omega\right\}$ of $H_n(X, A: Z(\mathfrak{a}_q))$ has a non-zero image under the homomorphism $(\rho_q)_*$.

LEMMA 8. *Let (X, A) be a pair of n-dimensional fully normal spaces such that $H_n(X, A: R_1)\neq 0$. Then there exist a prime number p and an element $\{a_\alpha\,|\,\alpha\in\Omega\}$ of $H_n(X, A: R_1)=\varprojlim\{H_n(K_\alpha, L_\alpha: R_1): (\pi_\alpha{}^\beta)_*\}$ such that for each $\alpha\in\Omega$ the order of a_α is a power of p.*

PROOF. We may assume that $\dim K_\alpha=n$ for each $\alpha\in\Omega$. Let $\{b_\alpha\,|\,\alpha\in\Omega\}$ be a non-zero element of $H_n(X, A: R_1)$. Let q_α be the order of b_α. Let $b_{\alpha_0}\neq 0$ for some $\alpha_0\in\Omega$. Then $q_{\alpha_0}\neq 0$. Let p be a prime number which is a divisor of q_{α_0}. For each $\beta>\alpha_0$, put $q_\beta=p^{\lambda_\beta}\cdot r_\beta$, where λ_β is a positive integer, p and r_β are coprime numbers. If $\alpha_0<\alpha<\beta$, we have $\lambda_\alpha\leqq\lambda_\beta$ and r_α is a divisor of r_β. Put $c_\beta=r_\beta\cdot b_\beta$ for $\beta>\alpha$. Since r_β and p are coprime numbers, c_β is a non-zero element of $H_n(K_\beta, L_\beta: R_1)$. Let us denote by G_β the subgroup of $H_n(K_\beta, L_\beta: R_1)$ generated by the element c_β. Then G_β is a finite group of the order p^{λ_β}. If $\alpha_0<\alpha<\beta$, since r_α is a divisor of r_β, we have $(\pi_\alpha{}^\beta)_*c_\beta=(\pi_\alpha{}^\beta)_*r_\beta\cdot b_\beta=r_\beta\cdot(\pi_\alpha{}^\beta)_*b_\beta=(r_\beta/r_\alpha)\cdot r_\alpha\cdot b_\alpha=(r_\beta/r_\alpha)\cdot c_\alpha$. Thus we have $(\pi_\alpha{}^\beta)_*G_\beta\subset G_\alpha$. Therefore the system $\{G_\alpha: (\pi_\alpha{}^\beta)_*\}$ forms an inverse system. Put $G=\varprojlim\{G_\alpha: (\pi_\alpha{}^\beta)_*\}$.

6) Let q be a positive integer such that $q>1$. By a cycle mod q we mean a cycle with coefficients in Z_q. By a cycle mod 1 we mean a cycle with coefficients in R_1.

6a) These $(\rho_q)_*$ mean the homomorphisms induced by the homomorphism ρ_q between the coefficient groups $Z(\mathfrak{a}_q)$ and Z_q.

7) Let $c=\sum_i t_i\sigma_i$ be an integral chain of (K, L). By $c\equiv 0$ mod q, where q is an positive integer, we mean that $t_i\equiv 0$ mod q for each i.

8) $c=\sum_i g_i\sigma_i$ be a chain of (K, L), where $g_i\in R_1$ or $g_i\in Z$ for each i. Let q be an integer. By $\frac{1}{q}c$ we mean the chain $\sum\frac{1}{q}g_i\sigma_i$ of (K, L).

Assume that $G=0$. Since each G_α is a finite group, there exists $\alpha>\alpha_0$ such that $(\pi_{\alpha_0}{}^{\alpha})^*G_\alpha=0$ by Lemma 6. On the other hand, we have $(\pi_{\alpha_0}{}^{\alpha})_*c_\alpha=r_\alpha\cdot b_{\alpha_0}$. Since r_α and p are coprime numbers and the order of b_{α_0} is $p^{\lambda_{\alpha_0}}r_{\alpha_0}$, we have $(\pi_{\alpha_0}{}^{\alpha})_*c_\alpha=r_\alpha\cdot b_{\alpha_0}\neq0$. This contradicts $(\pi_{\alpha_0}{}^{\alpha})_*G_\alpha=0$. Therefore $G\neq0$. Since an order of every element of G_α is a power of p for each $\alpha\in\Omega$, we can find an element required in the lemma. This completes the proof.

§ 3. Theorems.

THEOREM 1. *An n-dimensional fully normal space X is a dimensionally full-valued space for Q if X has the property* $\boldsymbol{P}$.

PROOF. Let Y be an m-dimensional locally compact fully normal space. By Lemmas 3 and 2, there exists a pair (A, B) of compact subsets of Y such that $H_m(A, B: R_1)\neq0$. Let $\boldsymbol{W}=\{\mathfrak{W}_\alpha|\alpha\in\Omega\}$ be a cofinal system of finite coverings of A each member of which has the order m. Let us denote by (M_α, N_α) the pair of the nerves of $\mathfrak{W}_\alpha$ corresponding to (A, B) and by $\pi_\alpha{}^\beta$ a projection of (M_β, N_β) into (M_α, N_α) for $\alpha, \beta\in\Omega$ and $\beta>\alpha$. By Lemma 8 there exist a prime number p and a non-zero element $\{a_\alpha|\alpha\in\Omega\}$ of $H_m(A, B: R_1)=\varprojlim\{H_m(M_\alpha, N_\alpha: R_1): (\pi_\alpha{}^\beta)_*\}$ such that the order of each a_α is a power of p. Since X has the property $\boldsymbol{P}$, there exists a closed subset X_0 such that $H_n(X, X_0: Z(\mathfrak{a}_p))\neq0$, where $\mathfrak{a}_p$ is the k-sequence $(p, p^2,\cdots, p^i,\cdots)$. Let $\boldsymbol{U}=\{\mathfrak{U}_\mu|\mu\in\Gamma\}$ be a cofinal system of coverings of X each member of which has the order n. Let us denote by (K_μ, L_μ) the pair of the nerves of $\mathfrak{U}_\mu$ corresponding to (X, X_0) and by $\delta_\mu{}^\nu$ a projection of (K_ν, L_ν) into (K_μ, L_μ) for $\nu, \mu\in\Gamma$ and $\nu>\mu$. By Lemma 7, there exists an element $\{c_\mu|\mu\in\Gamma\}$ of $H_n(X, X_0: Z(\mathfrak{a}_p))=\varprojlim\{H_n(K_\mu, L_\mu: Z(\mathfrak{a}_p)): (\delta_\mu{}^\nu)_*\}$ such that $(\delta_p)_*\{c_\mu\}\neq0$. Since $\dim K_\mu=n$, we may consider c_μ as a cycle of (K_μ, L_μ) with coefficients in $Z(\mathfrak{a}_p)$ for each $\mu\in\Gamma$. Take an element μ_0 of Γ such that $(\rho_p)_*c_{\mu_0}\neq0$. This means that, if $c_{\mu_0}=\{c_{\mu_0}(i)|i=1, 2,\cdots\}$, where $c_{\mu_0}(i)$ is a cycle of (K_{μ_0}, L_{μ_0}) mod p^i [9], there exists some positive integer j_0 such that $c_{\mu_0}(j)\not\equiv0$ mod p [10] for each $j\geqq j_0$. Take an element α_0 of Ω such that $a_{\alpha_0}\neq0$. We shall prove that the covering $\mathfrak{U}_{\mu_0}\times\mathfrak{W}_{\alpha_0}=\{U\in\mathfrak{U}_{\mu_0}$ and $W\in\mathfrak{W}_{\alpha_0}\}$ of $X\times A$ has no refinement whose order $<m+n$. Let $\mathfrak{W}$ be a refinement of $\mathfrak{U}_{\mu_0}\times\mathfrak{W}_{\alpha_0}$. Since A is compact, there exist a covering $\mathfrak{U}_\mu=\{U_k{}^\mu|k\in\kappa_\mu\}$ of $\boldsymbol{U}$ and coverings $\mathfrak{W}_{\alpha_k}=\{W_l\}$, $k\in\kappa_\mu$, of $\boldsymbol{W}$ such that the covering $\{U_k{}^\mu\times W_l|k\in\kappa_\mu$ and $W_l\in\mathfrak{W}_{\alpha_k}\}$ is a refinement of $\mathfrak{W}$. Obviously, $\mathfrak{U}_\mu$ is a refinement of $\mathfrak{U}_{\mu_0}$. Let S_μ be the subcomplex of K_μ consisting of all closed n-simplexes with a non-zero coefficient in the cycle c_μ of (K_μ, L_μ) with coefficients in $Z(\mathfrak{a}_p)$. Since c_μ is a finite chain, S_μ is a

9) Cf. the proof of Lemma 7.

10) See footnote 7).

finite subcomplex of K_μ. Let $\{U_{k_i}{}^\mu | i=1,2,\cdots,t\}$ be all vertexes of S_μ. Take a covering $\mathfrak{W}_\alpha$ of $\boldsymbol{W}$ which is a common refinement of coverings $\mathfrak{W}_{\alpha_0}$ and $\mathfrak{W}_{\alpha_{k_i}}$, $i=1,2,\cdots,t$. Put $\mathfrak{W}=\{U^\mu_{ki}\times W_l | i=1,2,\cdots,t$ and $W_l\in\mathfrak{W}_\alpha\}$. Let M^* be the nerve of $\mathfrak{W}$ and let N^* be the nerve of $\mathfrak{W}\cap(X\times B\cup X_0\times A)$[11]. By [**1**, Theorem 12.42], there exists a homomorphism into, $\theta:(S_\mu, S_\mu\cap L_\mu)\times(M_\alpha, N_\alpha)$[12] $\to(M^*, N^*)$, whose image is a deformation retract[13] of (M^*, N^*). Let $(M_0{}^*, N_0{}^*)$ be the pair of the nerves of the coverings $\mathfrak{U}_{\mu_0}\times\mathfrak{W}_{\alpha_0}$ corresponding to (X, X_0) $\times(A, B)$. By [**1**, Theorem 12.42], there exists a homeomorphism into, $\theta_0:(K_{\mu_0}, L_{\mu_0})\times(M_{\alpha_0}, N_{\alpha_0})\to(M_0{}^*, N_0{}^*)$, whose image is a deformation retract of $(M_0{}^*, N_0{}^*)$. Define a simplicial mapping π of (M^*, N^*) into $(M_0{}^*, N_0{}^*)$ by $\pi(U, W)=(\delta^\mu_{\mu_0}(U), \pi^\alpha_{\alpha_0}(W))$, where U and W are vertexes of S_μ and M_α respectively. Define a cellular mapping[14] π_0 of $(S_\mu, S_\mu\cap L_\mu)\times(M_\alpha, N_\alpha)$ into $(K_{\mu_0}, L_{\mu_0})\times(M_{\alpha_0}, N_{\alpha_0})$ by $\pi_0(x, y)=(\delta^\mu_{\mu_0}(x), \pi^\alpha_{\alpha_0}(y))$, $(x, y)\in S_\mu\times M_\alpha$. By the definition of θ and θ_0 (cf. [**1**, p. 317]), we have $\pi\theta\simeq\theta_0\pi_0:(S_\mu, S_\mu\cap L_\mu)\times(M_\alpha, N_\alpha)\to(M_0{}^*, N_0{}^*)$[15]. Let i be a positive integer such that the order of the element $a_\alpha=p^i$. Put $i_0=\max(i, j_0)$. Consider the product chain $c_\mu(i_0)\times a_\alpha$[16] of the chain group $C_{m+n}(S_\mu\times M_\alpha: R_1)$. Since $c_\mu(i_0)$ is a cycle of $(S_\mu, S_\mu\wedge L_\mu)$ mod p^{i_0}, a_α is a cycle of (M_α, N_α) mod 1 and the order of a_α is a divisor of p^{i_0}, we see that the chain $c_\mu(i_0)\times a_\alpha$ is a cycle of $(S_\mu, S_\mu\cap L_\mu)\times(M_\alpha, N_\alpha)$ mod 1. Since $c_\mu(i_0)\not\equiv 0$ mod p, we have $c_\mu(i_0)\times a_\alpha\not\equiv 0$ mod 1.[16a] Since $(\delta^\mu_{\mu_0})_* c_\mu(i_0)\equiv c_{\mu_0}(i_0)$ mod p^{i_0}, $(\pi^\alpha_{\alpha_0})_* a_\alpha\equiv a_{\alpha_0}$ mod 1 and the order of a_α is a divisor of p^{i_0}, we have

11) Let $\mathfrak{W}=\{W_i\}$ be a collection of subsets of X and let A be a subset of X. By $\mathfrak{W}\cap A$ we mean the collection $\{W_i\cap A\}$ of subsets of A.

12) Let (X, A) and (Y, B) be pairs of topological spaces. By $(X, A)\times(Y, B)$ we mean the pair $(X\times Y, X\times B\cup A\times Y)$ of spaces.

13) Let (X, A) and (Y, B) be pairs of topological spaces such that $X\subset Y$, $A\subset B$, X and A are closed subsets of Y. It is called that (X, A) is a *deformation retract* of (Y, B) if there exists a homotopy F: $(Y\times I, B\times I)\to(Y, B)$ such that $F|X\times I=$ the identity, $F|Y\times 0=$ the identity, $F(Y\times 1)\subset X$ and $F(B\times 1)\subset A$, where I is the closed interval [0,1].

14) A mapping f of a cell complex K into a cell complex M is called a *cellular mapping* if $f(K^i)\subset M^i$, where K^i means the i-section of K.

15) Let (X, A) and (Y, B) be pairs of topological spaces and let f_0 and f_1 be two mappings of (X, A) to (Y, B). By $f_0\simeq f_1:(X, A)\to(Y, B)$ we mean that there exists a homotopy H: $X\times I\to Y$ such that $H|X\times 0=f_0$, $H|X\times 1=f_1$ and $H(A\times I)\subset B$.

16) Let G_1 and G_2 be two abelian groups paired to a third group G, that is, there exist a function $\phi(g_1, g_2)$ of $G_1\times G_2$ into G which is distributive in both variable and whose values are in G. Let $c=\sum t_j{}^i{}_i\,\sigma_j{}^i{}_i$ be a chain of (K_i, L_i) with coefficients in G_i, $i=1,2$, where $\sigma_j{}^i{}_i$'s are simplexes of K_i, $i=1,2$. By the product chain $c_1\times c_2$ of c_1 and c_2 we understand the chain $\sum\phi(t_{j_1}{}^1, t_{j_2}{}^2)(\sigma_{j_1}{}^1\times\sigma_{j_2}{}^2)$ of the cell complex $(K_1, L_1)\times(K_2, L_2)$ with coefficients in G.

16a) Let $c=\sum_i t_i\sigma_i$ be a chain of (K, L) with coefficients in R_1. By $c\equiv 0$ mod 1 we mean that each t_i is an integer.

$$\begin{aligned}(\pi_0)_*(c_\mu(i_0)\times a_\alpha) &\equiv (\delta^\mu_{\mu_0}\times\pi^\alpha_{\alpha_0})_*(c_\mu(i_0)\times a_\alpha)\\ &\equiv (\delta^\mu_{\mu_0})_* c_\mu(i_0)\times(\pi^\alpha_{\alpha_0})_* a_\alpha\\ &\equiv c_{\mu_0}(i_0)\times a_{\alpha_0} \qquad \text{mod } 1.\end{aligned}$$

Since $(\rho_p)_* c_{\mu_0}(i_0)\neq 0$, $a_{\alpha_0}\neq 0$ and $\dim(K_{\mu_0}\times M_{\alpha_0})=m+n$, $c_{\mu_0}(i_0)\times a_{\alpha_0}$ is a non-zero cycle of $(K_{\mu_0}, L_{\mu_0})\times(M_{\alpha_0}, N_{\alpha_0})$ mod 1. Since $\theta_0((K_{\mu_0}, L_{\mu_0})\times(M_{\alpha_0}, N_{\alpha_0}))$ is a deformation retract of (M_0^*, N_0^*), $(\theta_0)_*(c_{\mu_0}(i_0)\times a_{\alpha_0})$ is a non-zero element of $H_{m+n}(M_0^*, N_0^* : R_1)$. Assume that the covering $\mathfrak{W}$ has the order $<m+n$. Let (C, D) be the pair of the nerves of $\mathfrak{W}$ corresponding to $(X, X_0)\times(A, B)$ and let π_1 and π_2 be projections of (M^*, N^*) and (C, D) into (C, D) and (M_0^*, N_0^*) respectively. Then we have $\pi\cong\pi_2\pi_1 : (M^*, N^*)\to(M_0^*, N_0^*)$. Since $\dim C<m+n$, we have $(\theta_0)_*(c_{\mu_0}(i_0)\times a_{\alpha_0})=(\theta_0)_*(\pi_0)_*(c_\mu(i_0)\times a_\alpha)=(\theta_0\pi_0)_*(c_\mu(i_0)\times a_\alpha)=(\pi\theta)_*(c_\mu(i_0)\times a_\alpha)=(\pi_2)_*(\pi_1\theta)_*(c_\mu(i_0)\times a_\alpha)=0$. This contradicts $(\theta_0)_*(c_{\mu_0}(i_0)\times a_{\alpha_0})\neq 0$. Therefore the covering $\mathfrak{W}$ has the order $\geqq m+n$. Since $\mathfrak{W}$ is any refinement of the covering $\mathfrak{U}_{\mu_0}\times\mathfrak{W}_{\alpha_0}$ of $X\times A$, we have $\dim(X\times A)\geqq\dim X+\dim A$. Since $\dim(X\times Y)\leqq\dim X+\dim Y$ by Lemma 5 and $X\times A$ is a closed subset of $X\times Y$, we have $\dim(X\times Y)=\dim X+\dim Y$. This completes the proof.

THEOREM 2. *Let X be an n-dimensional locally compact fully normal space. In order that X is a dimensionally full-valued space for Q, it is necessary and sufficient that X has the property* ***P***.

Before proving Theorem 2 we state the following lemma which is proved easily (cf. [**7**, Theorem 5.1]).

LEMMA 9. *Let (X, A) be a pair of compact spaces. Let G be the limit group of an inverse system $\{G_\alpha | h_\alpha{}^\beta\}$ of abelian groups. Then we have an isomorphism*

$$H_n(X, A : G)\approx \varprojlim\{H_n(X, A : G_\alpha) : (h_\alpha{}^\beta)_*\},$$

where $(h_\alpha{}^\beta)_*$ is the homomorphism of $H_n(X, A : G_\beta)$ into $H_n(X, A : G_\alpha)$ induced by the homomorphism $h_\alpha{}^\beta : G_\beta\to G_\alpha$.

PROOF OF THEOREM 2. The sufficiency of Theorem 2 is a consequence of Theorem 1. To prove the necessity of Theorem 2, it is sufficient to prove the following lemma.

LEMMA 10. *If an n-dimensional locally compact fully normal space X has not the property* ***P***, *there exists a 2-dimensional compactum Y such that* $\dim(X\times Y)=n+1$.

This lemma is proved by a similar way as [**10**, Lemma 18], but for completness we shall give the proof.

PROOF OF LEMMA 10. Since X has not the property ***P***, there exists a k-sequence $\mathfrak{a}=(q_1, q_2,\cdots)$ such that for each pair (A, B) of closed subsets of X $H_n(A, B : Z(\mathfrak{a}))=0$ by [**10**, Lemma 7]. Let $Q(\mathfrak{a})$ be the 2-dimensional compactum constructed in [**10**, § 3, 3]. We shall prove that $\dim(X\times Q(\mathfrak{a}))=n+1$. It is sufficient to prove that $\dim(A\times Q(\mathfrak{a}))=n+1$ for each compact subset A of X

by Lemma 3. Take an n-dimensional compact subset X_0 of X. Let $\boldsymbol{W}=\{\mathfrak{W}_\alpha \in \Omega\}$ be a cofinal system of coverings of X_0 each member of which has the order n. Let us denote by ϕ_α a canonical mapping of X_0 into the nerve M_α of $\mathfrak{W}_\alpha$, $\alpha\in\Omega$, and by $\pi_\alpha{}^\beta$ a projection of M_β into M_α for $\beta>\alpha$. We shall use the same notations as in the proof of [**10**, Lemma 18]. Let $\mathfrak{U}$ be a covering of $X_0\times Q(\mathfrak{a})$. Since X_0 and $Q(\mathfrak{a})$ are compact spaces, there exist an element α_0 of Ω and a positive integer i_0 such that, if $\mathfrak{V}_{i_0}$ is the covering of the simplicial polytope $Q(q_1,\cdots,q_{i_0})$ consisting of the open stars and θ_{i_0} is the projection from $Q(\mathfrak{a})$ onto $Q(q_1,\cdots,q_{i_0})$ (cf. [**10**, § 3, 3]), the covering $\mathfrak{W}_{\alpha_0}\times(\theta_{i_0})^{-1}\mathfrak{V}_{i_0}$ of $X_0\times Q(\mathfrak{a})$ is a star refinement[17] of $\mathfrak{U}$. Let σ be an n-dimensional simplex of M_{α_0} and let μ be a 2-dimensional simplex of $Q(q_1,\cdots,q_{i_0})$. Put $A(\sigma)=\phi_{\alpha_0}^{-1}(\sigma)$, $B(\sigma)=\phi_{\alpha_0}^{-1}(\dot\sigma)$, $C(\mu)=\theta_{i_0}^{-1}(\mu)$ and $D(\mu)=\theta_{i_0}^{-1}(\dot\mu)$. For each $\alpha>\alpha_0$, let us denote by (A_α, B_α) the pair of the subcomplexes of M_α corresponding to $(A(\sigma), B(\sigma))$. For each $j>i_0$, let us denote by (C_j, D_j) the pair of the subcomplexes of $Q(q_1,\cdots,q_j)$ which is the image of $(C(\mu), D(\mu))$ under the projection $\theta_j: Q(\mathfrak{a})\to Q(q_1,\cdots,q_j)$. Since $A(\sigma)$ and $C(\mu)$ are compact sets, we have an isomorphism $H_{n+2}((A(\sigma), B(\sigma))\times(C(\mu), D(\mu)): R_1)\approx \varprojlim \{H_{n+2}((A_\alpha, B_\alpha)\times(C_i, D_i): (\pi_\alpha{}^\beta\times\theta_i{}^j)_* \mid \alpha_0<\alpha<\beta \text{ and } i_0<i<j\}$ by [**10**, Lemma 5], where $\pi_\alpha{}^\beta$ and $\theta_i{}^j$ are the restricted projections $\pi_\alpha{}^\beta\mid A_\beta: (A_\beta, B_\beta)\to(A_\alpha, B_\alpha)$ and $\theta_i{}^j\mid C_j: (C_j, D_j)\to(C_i, D_i)$ respectively. Take an element $a=\{a_{\alpha,i}\mid \alpha>\alpha_0$ and $i=i_0+1, i_0+2,\cdots\}$ of $H_{n+2}(A(\sigma), B(\sigma))\times(C(\mu), D(\mu)): R_1)$, where $a_{\alpha,i}\in H_{n+2}((A_\alpha, B_\alpha)\times(C_i, D_i): R_1)$. By a similar way as in the proof of [**10**, Lemma 18], we have

$$a_{\alpha,i_0+1}=u_\alpha\times\frac{1}{q_{i_0+1}}\,\delta(i_0+1)\,,$$

$$a_{\alpha,i_0+2}=\sum_{h_1=1}^{l_1}\Big(u_{\alpha,h_1}\times\frac{1}{q_{i_0+2}}\,\delta_{h_1}(i_0+2)\Big)\,,$$

$$\vdots \qquad\qquad \vdots$$

$$a_{\alpha,i_0+k}=\sum_{h_1=1}^{l_1}\cdots\sum_{h_{k-1}=1}^{l_{k-1}}\Big(u_{\alpha,h_1\cdots h_{k-1}}\times\frac{1}{q_{i_0+k}}\,\delta_{h_1\cdots h_{k-1}}(i_0+k)\Big)\,,$$

$$\vdots \qquad\qquad \vdots$$

where $u_{\alpha,h_1\cdots h_{k-1}}$ is a cycle of (A_α, B_α) mod q_{i_0+k} and $\delta_{h_1\cdots h_{k-1}}(i_0+k)$ is the fundamental chain with the value ±1 on each 2-simplex of the Möbius band $M_{h_1\cdots h_{k-1}}(q_{i_0+k}/q_{i_0+k-1}, q_{i_0+k})$, $h_1=1,\cdots,l_1,\cdots,h_{k-1}=1,\cdots,l_{k-1}$, of which the complex C_{i_0+k} consists (cf. [**10**, pp. 390 and 396]). Since $(\pi_\alpha^\alpha\times\theta_{i_0+k}^{i_0+k+1})_* a_{\alpha,i_0+k+1}=a_{\alpha,i_0+k}$,

17) Let $\mathfrak{U}=\{U_\alpha\mid\alpha\in\Omega\}$ and $\mathfrak{V}$ be coverings of topological space. It is called that $\mathfrak{U}$ is a star refinement of $\mathfrak{V}$ if the covering $\{\bigcup_{U_\alpha\cap U_\beta\neq\phi} U_\beta\mid\alpha\in\Omega\}$ is a refinement of V (Cf. [**15**, Chap. V]).

if we denote by h_i^j a natural homomorphism from Z_{q_j} onto Z_{q_i} for $j > i$, we have $(h_{i_0+k}^{i_0+k+1})_* u_{\alpha,h_1\cdots h_k} = u_{\alpha,h_1\cdots h_{k-1}}$. Let $\alpha_0 < \alpha < \beta$. Since $(\pi_\alpha{}^\beta \times \theta_{i_0+k}^{i_0+k})_* a_{\beta,i_0+k} = a_{\alpha,i_0+k}$, we have $(\pi_\alpha{}^\beta)_* u_{\beta,h_1\cdots h_{k-1}} \equiv u_{\alpha,h_1\cdots h_{k-1}} \bmod q_{i_0+k}$. Let $\alpha_0 < \alpha < \beta$ and $i_0 < i < j$. Define a homomorphism $\mathfrak{P}_{(\alpha,i)}^{(\beta,j)} : H_n(A_\beta, B_\beta : Z_{q_j}) \to H_n(A_\alpha, B_\alpha : Z_{q_i})$ by a composition of homomorphisms $(h_i^j)_* : H_n(A_\beta, B_\beta : Z_{q_j}) \to H_n(A_\beta, B_\beta : Z_{q_i})$ and $(\pi_\alpha{}^\beta)_* : H_n(A_\beta, B_\beta : Z_{q_i}) \to H_n(A_\alpha, B_\alpha : Z_{q_i})$. Since $(A(\sigma), B(\sigma))$ is a pair of compact spaces, we have an isomorphism $H_n(A(\sigma), B(\sigma) : Z(\mathfrak{a})) \approx \varprojlim \{H_n(A_\alpha, B_\alpha : Z_{q_i}) : \mathfrak{P}_{(\alpha,i)}^{(\beta,j)} | \alpha_0 < \alpha < \beta$ and $i_0 < i < j\}$ by Lemma 9. Let $\alpha_0 < \alpha < \beta$. We have $\mathfrak{P}_{(\alpha,i_0+k)}^{(\beta,i_0+k+1)}(u_{\beta,h_1\cdots h_k}) = (\pi_\alpha{}^\beta)_* (h_{i_0+k}^{i_0+k+1})_* u_{\beta,h_1\cdots h_k} = (\pi_\alpha{}^\beta)_* u_{\beta,h_1\cdots h_{k-1}} = u_{\alpha,h_1\cdots h_{k-1}}$. Therefore, a collection $\{u_{\alpha,h_1\cdots h_k} | \alpha_0 < \alpha$ and $k = 1, 2, \cdots\}$ determines an element of the group $\varprojlim \{H_n(A_\alpha, B_\alpha : Z_{q_i})\}$. Since $H_n(A(\sigma), B(\sigma) : Z(\mathfrak{a})) = 0$, each $u_{\alpha,h_1\cdots h_k}$ must be zero. This means that $u_{\alpha,h_1\cdots h_k} \equiv 0 \bmod q_{i_0+k+1}$ for $\alpha > \alpha_0$, $h_1 = 1, \cdots, l_1$, $h_2 = 1, \cdots, l_2, \cdots, h_k = 1, \cdots, l_k$ and $k = 1, 2, \cdots$. Hence, we have $a_{\alpha,i} = 0$ for $\alpha > \alpha_0$ and $i = i_0+1, i_0+2, \cdots$. Thus we can conclude $H_{n+2}((A(\sigma), B(\sigma)) \times (C(\mu), D(\mu)) : R_1) = 0$. By Lemma 1, the restricted mapping $(\phi_{\alpha_0} \times \theta_{i_0}) | (A(\sigma) \times D(\mu) \cup B(\sigma) \times C(\mu))$ is extended to a mapping $\psi(\sigma, \mu)$ of $A(\sigma) \times C(\mu)$ into $(\sigma \times \mu) \cup (\sigma \times \mu)$. Define a mapping ψ of $X_0 \times Q(\mathfrak{a})$ into $(M_{\alpha_0} \times Q(q_1, \cdots, q_{i_0}))^{n+1}$ by $\psi(x, y) = \psi(\sigma, \mu)(x, y)$ for $(x, y) \in A(\sigma) \times C(\mu)$, where L^k means the k-section of the cell complex L. Since the covering $\mathfrak{W}_{\alpha_0} \times (\theta_{i_0})^{-1} \mathfrak{V}_{i_0}$ is a star refinement of $\mathfrak{U}$, the mapping ψ is a $(\mathfrak{U}, K)$-mapping, where K means the k-section of the cell complex $M_{\alpha_0} \times Q(q, \cdots, q_{i_0})$. Since $\mathfrak{U}$ is any covering of $X_0 \times Q(\mathfrak{a})$, we have $\dim(X_0 \times Q(\mathfrak{a})) \leqq n+1$ by Lemma 4. Since $\dim(X_0 \times Q(\mathfrak{a})) \geqq n+1$ by [8], we can conclude that $\dim(X_0 \times Q(\mathfrak{a})) = n+1$. Since X_0 is any n-dimensional compact subset of X, this completes the proof.

By a slight modification of the proof of Theorem 1 we can prove the following lemma.

LEMMA 11. *An n-dimensional fully normal space X is a dimensionally full-valued space for Q if X has the following property* (*):

(*) *There exist a cofinal system $\boldsymbol{U} = \{\mathfrak{U}_\mu | \mu \in \Gamma\}$ of coverings of X and a covering $\mathfrak{U}_{\mu_0}$ of $\boldsymbol{U}$ which satisfy the following condition; for each prime number p there exists a closed subset A_p of X such that, if $\mu > \mu_0$, $0 \neq (\rho_p)_* (\delta_{\mu_0}^\mu)_* : H_n(K_\mu, L_\mu : Z(\mathfrak{a}_p)) \to H_n(K_{\mu_0}, L_{\mu_0} : Z_p)$, where (K_μ, L_μ) is the pair of the nerves of $\mathfrak{U}_\mu$ corresponding to (X, A_p), $\delta_{\mu_0}^\mu$ is a projection of (K_μ, L_μ) into (K_{μ_0}, L_{μ_0}) and ρ_p is a natural homomorphism from $Z(\mathfrak{a})$ onto Z_p.*

LEMMA 12. *A 1-dimensional fully normal space has the property* (*) *mentioned in Lemma* 11.

PROOF. Let X be a 1-dimensional fully normal space. Since Ind X[18] $\geqq 1$

18) By Ind X we mean the dimension of X defined inductively in terms of the boundaries of neighborhoods of closed sets of X (cf. [**2**, p. 102]).

by [**2**, 1.7], there exists a closed subset A such that, whenever U is an open set of X containing A, we have $\bar{U}-U\neq\phi$, where $\bar{U}$ is the closure of U in X. Let x be a point of X. Let $\mathfrak{U}$ be a covering of X. By $A\sim x$ in $\mathfrak{U}$ we shall mean that there exists a finite number of elements U_i of $\mathfrak{U}$, $i=1,2,\cdots,n$, such that $U_1\cap A\neq\phi$, $x\in U_n$ and $U_i\cap U_{i+1}\neq\phi$, $i=1,2,\cdots,n-1$. Since the set $\bigcup\{x\,|\,A\sim x$ in $\mathfrak{U}\}$ is a closed and open set containing A, we have $A\sim x$ for each $x\in X$. Take a point x_0 of $X-A$. Let $\{\mathfrak{U}_\mu'\,|\,\mu\in\Gamma\}$ be a cofinal system of coverings of X each member of which has the order 1. Let $\mathfrak{U}_\mu'=\{U_{\mu k}'\,|\,k\in\kappa_\mu\}$, $\mu\in\Gamma$. We may assume that there exists an open set $U_{\mu k_0}'$ of $\mathfrak{U}_\mu'$ such that $U_{\mu k_0}'\cap A=\phi$, $x_0\in U_{\mu k_0}'$ and $x_0\notin U_{\mu k}'$ for $k\neq k_0$. By [**12**, Theorem 1.1], there exists a covering $\mathfrak{V}_\mu=\{V_{\mu k}\,|\,k\in\kappa_\mu\}$ such that $\bar{V}_{\mu k}\subset U_{\mu k}'$ for each $k\in\kappa_\mu$. Put $U_{\mu_0}=X-\bigcup_{k\neq k_0}\bar{V}_{\mu k}$, $U_{\mu k_0}=V_{\mu k_0}-x_0$ and $U_{\mu k}=V_{\mu k}$ for $k\neq k_0$. Then $\{\mathfrak{U}_\mu=\{U_{\mu 0}, U_{\mu k_0}, U_{\mu k}$ for $k\in\kappa_\mu\}\,|\,\mu\in\Gamma\}$ forms a cofinal system $\boldsymbol{U}$ of coverings of X each member of which has the order 1. Let $(K_\mu, L_\mu\cup U_{\mu_0})$ be the pair of the nerves of $\mathfrak{U}_\mu$ corresponding to $(X, A\cup x_0)$, $\mu\in\Gamma$, where $U_{\mu 0}$ means the vertex corresponding to the open set $U_{\mu 0}$ containing x_0. Since $A\sim x_0$ in $\mathfrak{U}_\mu$ for each $\mu\in\Gamma$, the group $H_1(K_\mu, L_\mu\cup U_{\mu 0}:Z)$ contains a non-zero cycle z_μ such that the 1-simplex $(U_{\mu 0}, U_{\mu k_0})$ of K_μ appears in z_μ with the coefficient ± 1, $\mu\in\Gamma$. Let ρ be the homomorphism of Z into $Z(\mathfrak{a}_p)$ defined by $\rho(1)=\{h_i(1)\,|\,i=1,2,\cdots\}$, where h_i is a natural projection of Z into $Z_{p^i}=Z/p^iZ$, $i=1,2,\cdots$. The image $\tilde{z}_\mu$ of z_μ under the induced homomorphism $(\rho)_*$ is a non-zero element of $H_1(K_\mu, L_\mu\cup U_{\mu 0}:Z(\mathfrak{a}_p))$. Let $\mathfrak{U}_\nu$ be a refinement of $\mathfrak{U}_\mu$ and let δ_μ^ν be a projection of $(K_\nu, L_\nu\cup U_{\nu 0})$ into $(K_\mu, L_\mu\cup U_{\mu 0})$. By the construction of the coverings $\{\mathfrak{U}_\mu\}$, the image of z_ν under the induced homomorphism $(\delta_\mu^\nu)_*: H_1(K_\nu, L_\nu\cup U_{\nu_0}:Z)\rightarrow H_1(K_\mu, L_\mu\cup U_{\mu 0}:Z)$ is a cycle which has the coefficient ± 1 on the 1-dimensional simplex $(U_{\mu 0}, U_{\mu k_0})$ of K_μ. Therefore we have $(\rho_p)_*(\delta_\mu^\nu)_*\tilde{z}_\nu\neq 0$, where $(\delta_\mu^\nu)_*: H_1(K_\nu, L_\nu\cup U_{\nu 0}:Z(\mathfrak{a}_p))\rightarrow H_1(K_\mu, L_\mu\cup U_{\mu 0}:Z(\mathfrak{a}_p))$ and $(\rho_p)_*: H_1(K_\mu, L_\mu\cup U_{\mu 0}:Z(\mathfrak{a}_p))\rightarrow H_1(K_\mu, L_\mu\cup U_{\mu 0}:Z_p)$. This shows that, if we put $A_p=A\cup x$ for each prime number p and $\mathfrak{U}_{\mu 0}=$ any covering of $\boldsymbol{U}, X$ has the property (*). This completes the proof.

By making use of Lemma 9 the proof of Lemma 12 shows that the following lemma holds.

LEMMA 13. *A 1-dimensional locally compact fully normal space has the property* $\boldsymbol{P}$.

The following theorem is a consequence of Lemmas 11 and 12.

THEOREM 3. *A 1-dimensional fully normal space is a dimensionally full-valued space for* Q.

The following lemma is proved by a similar way as in the proof of [**10**, Lemma 20] and we omit the proof.

LEMMA 14. *If an n-dimensional fully normal space contains a closed subset*

A such that $H_n(X, A: Z)\neq 0$, then X has the property $\boldsymbol{P}$[19].

By Lemma 14 and Theorem 1 we have the following corollary.

COROLLARY 1. *If an n-dimensional fully normal space X contains a closed subset A such that $H_n(X, A: Z)\neq 0$, then X is a dimensionally full-valued space for Q.*

The following corollary which is a generalization of [**10**, Corollary 2] is a consequence of Corollary 1 and [**10**, Lemmas 21–23].

COROLLARY 2. *The following spaces are dimensionally full-valued spaces for Q.*

1) *Finite or infinite polytopes with the Whitehead weak topology.*

2) *Two dimensional locally compact* ANR*'s.*[20]

3) *M-dimensional* ANR*'s containing points which are* HL^{m-1} *and* $(m-1)$-HS[21].

4) *Finite dimensional and locally compact* ANR*'s which have the property Δ in the sense of Borsuk*[22].

REMARK. Consider the following properties of an n-dimensional fully normal space X.

$\boldsymbol{P}_1$. *For every prime number p and every k-sequence $\mathfrak{a}$ each member of which is a power of p there exists a closed subset $A_{\mathfrak{a}}$ of X such that $H_n(X, A_{\mathfrak{a}}: Z(\mathfrak{a}))\neq 0$.*

$\boldsymbol{P}_2$. *For every prime number p there exists a closed subset A_p of X such that $H_n(X, A_p: Z(\mathfrak{a}_p))\neq 0$, where $\mathfrak{a}_p$ is the k-sequence $(p, p^2, \cdots, p^i, \cdots)$.*

By a similar way as [**10**, Lemmas 2 and 3 in Addendum], we can prove that the three properties $\boldsymbol{P}$, $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ of an n-dimensional fully normal space are equivalent. Therefore we have

THEOREM 2′. *In order that an n-dimensional locally compact fully normal*

19) In this case we can prove easily that X has the property (*) mentioned in Lemma 11, too.

20) A metric space X is called an ANR if, whenever X is a closed subset of a metric space Y, there exists a mapping from some neighborhood of X in Y into X which keeps X point-wise fixed.

21) Let E^{j+1} be a $(j+1)$-cell whose boundary is a j-sphere S^j. A point x_0 of a topological space is called HL^k if for each neighborhood U of x_0 there exists a neighborhood V of x_0 such that any mapping $f: S^j\rightarrow V-x_0$ is extensible to a mapping $F: E^{j+1}\rightarrow U-x_0$ for $j=0, 1, \ldots, k$. A point x_0 of a topological space is called k-HS if there exists a neighborhood U of x_0 such that for any neighborhood V of x_0 there exists a mapping $f: S^k\rightarrow V-x_0$ which has no extension $F: E^{k+1}\rightarrow U-x_0$. (Cf. Y. Kodama, On homotopically stable points and product spaces, Fund. Math., **44** (1957), 171–185.)

22) A topological space X is said to *have the property Δ* if for each point x of X and each neighborhood U of x there exists a neighborhood V of x such that every compact subset A of V is contractible in a subset of U of the dimension $\leqq \dim A+1$. (Cf. K. Borsuk, Ensembles dont les dimensions modulaires de Alexandroff coincident avec la dimension de Menger-Urysohn, Fund. Math., **27** (1936), 77–93.)

space X be a dimensionally full-valued space for Q, it is necessary and sufficient that X have any one property of $\boldsymbol{P}$, $\boldsymbol{P}_1$ *and* $\boldsymbol{P}_2$.

In [**10**, Addendum], we have proved that our property P is equivalent to the following Boltyanskii's property for n-dimensional compact metric spaces.

B. *For every prime number p there exists a pair (A_p, B_p) of closed subsets of X such that $H^n(A_p, B_p : Q_p) \neq 0$, where Q_p means the additive group of all rational numbers of the form m/p^k reduced modulo 1 and $H^n(A, B: G)$ means the n-dimensional unrestricted Čech cohomology group of (A, B) with coefficients in G.*

But we do not know whether Boltyanskii's property $\boldsymbol{B}$ is equivalent to our property $\boldsymbol{P}$ even for locally compact fully normal spaces, since it seems that the duality between the unrestricted Čech homology groups and cohomology groups does not hold generally.

§ 4. Examples.

Let $\mathfrak{p} = (p_1, p_2, \cdots)$ be a sequence of positive integers. We shall construct a 2-dimensional continuum $R(\mathfrak{p})$ for each $\mathfrak{p}$. Let E be a 2-cell whose boundary is a 1-sphere S. For a positive integer q, let us denote by $N(q)$ a polytope obtained from E by identifying points on S corresponding to each other under the rotation of angle $2\pi/q$. Let f be the identification mapping. We shall call $f(S)$ the "boundary" of $N(q)$. The boundary of $N(q)$ is a 1-sphere. In general, $N(q)$ is a 2-dimensional curvilinear polytope. We shall consider $N(q)$ as a simplicial polytope with a fixed triangulation. Let T be the boundary of $N(q)$. Let us give an orientation to each 2-simplex of $N(q)$ such that the integral chain $c(N(q))$ which has the value 1 on each 2-simplex is a cycle relative to T. Obviously $H_2(N(q), T: Z) \approx Z$ and $c(N(q))$ is a generator of $H_2(N(q), T: Z)$. We call $c(N(q))$ the fundamental chain of $N(q)$. The following lemma is proved easily by a similar way as in the proof of [**10**, Lemma 14].

LEMMA 15. *Let f be a topological mapping from the boundary T of $N(q)$ onto the 1-sphere S which is the boundary of the 2-cell E and let $F: (N(q), T) \to (E, S)$ be an extension of f.*[23] *If F_* is the induced homomorphism of $H_2(N(q), T: Z)$ into $H_2(E, S: Z)$, we have $F_*(c(N(q)) = q \cdot \nu$, where ν is a generator of $H_2(E, S: Z)$.*

Put $R(p_1) = N(p_1)$. Let us replace every triangle τ of $R(p_1)$ by $N_\tau(p_2)$ such that $N_\tau(p_2) \cap N_{\tau'}(p_2) = T \cap T'$, where each $N_\tau(p_2)$ is a topological image[23a)]

23) Since E is contractible in itself, it is obvious that there exists at least one extension F of f.

23a) By a *topological image* of a topological space X we mean a space homeomorphic to X.

of $N(p_2)$, T and T' are the boundaries of $N_\tau(p_2)$ and $N_{\tau'}(p_2)$ respectively. We have a 2-dimensional simplicial complex $R(p_1, p_2) = \bigcup_\tau N_\tau(p_2)$. Let Δ_1 be the 1-section of $R(p_1)$. We may consider Δ_1 as a subset of $R(p_1, p_2)$. There exists a projection $\phi_1{}^2$ from $R(p_1, p_2)$ onto $R(p_1)$ such that the restricted mapping $\phi_1{}^2 | \Delta_1$ is topological. The integral chain $c(p_1, p_2) = \sum_\tau c(N_\tau(p_2))$ is a cycle of $R(p_1, p_2)$ relative to the boundary T of $R(p_1)$, where $c(N_\tau(p_1))$ is the fundamental chain of $N_\tau(p_2)$, and $c(p_1, p_2)$ is a generator of the group $H_2(R(p_1, p_2), T: Z)$ which is isomorphic to Z. Moreover, by Lemma 15, we have $(\phi_1{}^2)_* c(p_1, p_2) = p_2 \cdot c(p_1)$, where $c(p_1)$ is the fundamental chain of $R(p_1)$. Let us suppose that for some i we have constructed the following 2-dimensional simplicial polytope $R(p_1, \cdots, p_i)$: (1) $R(p_1, \cdots, p_i)$ contains the 1-section Δ_{i-1} of $R(p_1, \cdots, p_{i-1})$, (2) there exists a projection ϕ^i_{i-1} from $R(p_1, \cdots, p_i)$ onto $R(p_1, \cdots, p_{i-1})$ such that the restricted mapping $\phi^i_{i-1} | \Delta_{i-1}$ is topological, (3) $H_2(R(p_1, \cdots, p_i), T: Z) \approx Z$, (4) the integral chain $c(p_1, \cdots, p_i)$ which has the value 1 on each 2-simplex of $R(p_1, \cdots, p_i)$ is a generator of $H_2(R(p_1, \cdots, p_i), T: Z)$ and $(\phi^i_{i-1})_* c(p_1, \cdots, p_i) = p_i \cdot c(p_1, \cdots, p_{i-1})$. Let us replace every triangle μ of $R(p_1, \cdots, p_i)$ by $N_\mu(p_{i+1})$ such that $N_\mu(p_{i+1}) \cap N_{\mu'}(p_{i+1}) = T_\mu \cap T_{\mu'}$, where $N_\mu(p_{i+1})$ is a topological image of $N(p_{i+1})$, T_μ and $T_{\mu'}$ are the bouudaries of $N_\mu(p_{i+1})$ and $N_{\mu'}(p_{i+1})$ respectively. We have a 2-dimensional simplicial complex $R(p_1, \cdots, p_{i+1}) = \bigcup_\mu N_\mu(p_{i+1})$. If Δ_i is the 1-section of $R(p_1, \cdots, p_i)$, we may consider Δ_i as a subset of $R(p_1, \cdots, p_{i+1})$. There exists a projection ϕ^{i+1}_i from $R(p_1, \cdots, p_{i+1})$ onto $R(p_1, \cdots, p_i)$ such that the restricted mapping $\phi^{i+1}_i | \Delta_i$ is topological. Obviously $H_2(R(p_1, \cdots, p_{i+1}), T: Z) \approx Z$ and the integral chain $c(p_1, \cdots, p_{i+1}) = \sum_\mu c(N_\mu(p_{i+1}))$ is a generator of $H_2(R(p_1, \cdots, p_{i+1})$, where $c(N_\mu(p_{i+1}))$ is the fundamental chain of $N_\mu(p_{i+1})$. Moreover, by Lemma 15, we have $(\phi^{i+1}_i)_* c(p_1, \cdots, p_{i+1}) = p_{i+1} \cdot c(p_1, \cdots, p_i)$. Put $R(\mathfrak{p}) = \varprojlim \{R(p_1, \cdots, p_i) : \phi^i_{i-1}\}$. Let ϕ_i be the projection from $R(\mathfrak{p})$ onto $R(p_1, \cdots, p_i)$. We shall call the boundary of $R(p_1)$ the "boundary" of $R(\mathfrak{p})$.

Lemma 16. *For each sequence* $\mathfrak{p}$ *of positive integers the space* $R(p)$ *is a 2-dimensional continuum.*

Proof. Let $\mathfrak{p} = (p_1, \cdots, p_i, \cdots)$. Put $q_i = p_1 \cdot p_2 \cdot \cdots \cdot p_i$ for $i = 1, 2, \cdots$. Let T be the boundary of $R(\mathfrak{p})$. By the continuity theorem of Čech homology groups (cf. [**6**, Chap. X]), we have an isomorphism $H_2(R(\mathfrak{p}), T: R_1) \approx \varprojlim \{H_2(R(p_1, \cdots, p_i), T: R_1) : (\phi^{i+1}_i)_*\}$. Consider the collection $\left\{\frac{1}{q_i} c(p_1, \cdots, p_i) | i = 1, 2, \cdots\right\}$, where $c(p_1, \cdots, p_i)$ is a generator of the group $H_2(R(p_1, \cdots, p_i), T: Z)$. Since $(\phi^{i+1}_i)_* c(p_1, \cdots, p_{i+1}) = p_{i+1} \cdot c(p_1, \cdots, p_i)$, we have $(\phi^{i+1}_i)_* \left(\frac{1}{q_{i+1}} c(p, \cdots, p_{i+1})\right) = \frac{1}{q_i} c(p_1, \cdots, p_i)$ for $i = 1, 2, \cdots$. Therefore $\left\{\frac{1}{q_i} c(p_1, \cdots, p_i)\right\}$ determines a non-zero element of $H(R(\mathfrak{p}), T: R_1)$. By Lemma 2 we have $\dim R(\mathfrak{p}) \geqq 2$. Since $\dim R(\mathfrak{p}) \leqq 2$ by [**10**, Lemma

12], we have $\dim R(\mathfrak{p})=2$.

The following lemma shows that the converse of Corollary 1 is not true.

LEMMA 17. *There exists a 2-dimensional continuum X such that* (i) *X has the property* $\boldsymbol{P}$, (ii) *for each pair (A, B) of closed subsets we have* $H_2(A, B: Z)=0$.

PROOF. Let p be a prime number. Let $\mathfrak{p}(p)$ be the sequence $(p, p, \cdots)$. Let us prove that the continuum $R(\mathfrak{p}(p))$ has the following properties: (1) $H_2(R(\mathfrak{p}(p)), T: Z(\mathfrak{a}_q))\neq 0$ for each prime number $q\neq p$, where T is the boundary of $R(\mathfrak{p}(p))$, (2) $H_2(A, B: Z)=0$ for each pair (A, B) of closed subsets. Let us denote by R_i the 2-dimensional simplicial polytope $R(\overbrace{p,\cdots,p}^{i\text{-fold}})$, $i=1,2,\cdots$. Put $\phi_i{}^j=\phi_i^{i+1}\cdots\phi_{j-1}^j$, $j>i$, where ϕ_i^{i+1} is the projection from R_{i+1} onto R_i. Let $h_i{}^j$ be a natural homomorphism from Z_{q^j} onto Z_{q^i}, $j>i$. For $j>i$ and $j'>i'$, define a homomorphism $\mathfrak{P}_{(i,i')}^{(j,j')}: H_2(R_j, T: Z_{q^{j'}})\rightarrow H_2(R_i, T: Z_{q^{i'}})$ by a composition of the homomorphisms $(h_{i'}^{j'})_*: H_2(R_j, T: Z_{q^{j'}})\rightarrow H_2(R_j, T: Z_{q^{i'}})$ and $(\phi_i{}^j)_*: H_2(R_j, T: Z_{q^{i'}})\rightarrow H_2(R_i, T: Z_{q^{i'}})$. By Lemma 9 we have an isomorphism $H_2(R(\mathfrak{p}(p)), T: Z(\mathfrak{a}_q))\approx \varprojlim \{H_2(R_i, T: Z_{q^{i'}}): \mathfrak{P}_{(i,i')}^{(j,j')}\}$. Put $c_i=c(\overbrace{p,\cdots,p}^{i\text{-fold}})$, $i=1,2,\cdots$. Since c_i is an integral cycle relative to T, we may consider c_i as a cycle relative to T mod p^j, $j=1,2,\cdots$ and $i=1,2,\cdots$. Let $j>i$ and $j'>i'$. Since p and q are coprime numbers, we have $\mathfrak{P}_{(i,i')}^{(j,j')}c_j\equiv(\phi_i{}^j)_*(h_{i'}^{j'})_*c_j\equiv(\phi_i{}^j)_*c_j\equiv p^{(j-i)}\cdot c_i\not\equiv 0 \mod q^{i'}$. Accordingly we have $0\neq\mathfrak{P}_{(i,i')}^{(j,j')}H_2(R_j, T: Z_{q^{j'}})\subset H_2(R_i, T: Z_{q^{i'}})$. Since $H_2(R_i, T: Z_{q^{i'}})$ is a finite group for $i=1,2,\cdots$ and $i'=1,2,\cdots$, we can conclude that $H_2(R(\mathfrak{p}(p)), T: Z(\mathfrak{a}_q))\neq 0$ by Lemma 6. This completes the proof of (1). To prove (2), by [**10**, Lemma 7], it is sufficient to prove that $H_2(R(\mathfrak{p}(p)), A: Z)=0$ for each closed subset A of $R(\mathfrak{p}(p))$. Put $A_i=\phi_i(A)$, $i=1,2,\cdots$, where ϕ_i is the projection from $R(\mathfrak{p}(p))$ onto R_i. Let $\bar{A}_i$ be the smallest closed subcomplex of the simplicial polytope R_i containing A_i. Then the projection ϕ_i^{i+1} maps $\bar{A}_{i+1}$ into $\bar{A}_i$, $i=1,2,\cdots$. Since $(R(\mathfrak{p}(p)), A)=\varprojlim\{(R_i, \bar{A}_i): \phi_i^{i+1}\}$[24], by the continuity theorem of Čech homology groups, we have an isomorphism $H_2(R(\mathfrak{p}(p)), A: Z)\approx\varprojlim\{H_2(R_i, \bar{A}_i: Z): (\phi_i^{i+1})_*\}$. Take a 2-simplex σ of $R_k-\bar{A}_k$ for some k. Put $\sigma_j=(\phi_k{}^j)^{-1}\sigma$, $j>k$. Let $a=\{a_i \mid i=1,2,\cdots\}$ be any element of $H_2(R(\mathfrak{p}(p)), A: Z)$, where $a_i\in H_2(R_i, \bar{A}_i: Z)$, $i=1,2,\cdots$. Since a_i is an integral cycle, for each $j>k$ a_j has the same integral coefficient t_j on each 2-simplex of σ_j. Let $j'>j>k$. Since $(\phi_j{}^{j'})_* t_{j'}\cdot\sigma_{j'}=t_{j'}\cdot(\phi_j{}^{j'})_*\sigma_{j'}=t_{j'}\cdot p^{(j'-j)}\cdot\sigma_j=t_j\cdot\sigma_j$[25] by Lemma 15, we have

24) Let (X, A) be a pair of topological spaces and let $\{(X_\alpha, A_\alpha): \pi_\alpha{}^\beta\}$ be an inverse system of pairs of topological spaces. By $(X, A)=\varprojlim\{(X_\alpha, A_\alpha): \pi_\alpha{}^\beta\}$ we mean that $X=\varprojlim\{X_\alpha: \pi_\alpha{}^\beta\}$ and $A=\varprojlim\{A_\alpha: \pi_\alpha{}^\beta \mid A_\beta\}$.

25) In this case, we mean by $t_j\cdot\sigma_j$ the integral chain which has the integral coefficient t_j on each 2-simplex of σ_j and by $(\phi_j{}^{j'})_*$ the chain homomorphism induced by $\phi_j{}^{j'}$.

$t_j = t_{j'} \cdot p^{(j'-j)}$ for each $j' > j$. Therefore t_j is zero for $j > k$. Since σ is any 2-simplex of $R_k - \bar{A}_k$, we have $a_i = 0, i = 1, 2, \cdots$. Since a is any element of $H_2(R(\mathfrak{p}(p)), A : Z)$, we have $H_2(R(\mathfrak{p}(p)), A : Z) = 0$. This completes the proof of (2). To complete the proof of the lemma, let p and q be two different prime numbers. Let T and T' be the boundaries of $R(\mathfrak{p}(p))$ and $R(\mathfrak{p}(q))$ respectively, and let f be a topological mapping of T into T'. Let us denote by X the space obtained from $R(\mathfrak{p}(p)) + R(\mathfrak{p}(q))$[26] by identifying points on $T + T'$ corresponding to each other under the homeomorphism f. Let g be the identification mapping and put $S = g(T + T')$. Let r be a prime number. We have $p \neq r$ or $q \neq r$. Let $p \neq r$. Since $H_2(R(\mathfrak{p}(p)), T : Z(\mathfrak{a}_r)) \neq 0$ and $H_2(R(\mathfrak{p}(p)) + R(\mathfrak{p}(q)), T + T' : Z(\mathfrak{a}_r)) \approx H_2(X, S : Z(\mathfrak{a}_r))$ by the map excision theorem [**17**], we have $H_2(X, S : Z(\mathfrak{a}_r)) \neq 0$. Similarly, if $q \neq r$, we have $H_2(X, S : Z(\mathfrak{a}_r)) \neq 0$, too. Put $X_1 = g(R(\mathfrak{p}(p)))$ and $X_2 = g(R(\mathfrak{p}(q)))$. Let A be a closed subset of X. If $H_2(X, A : Z) \neq 0$ we have $H_2(X, A \cup S : Z) \neq 0$ by [**10**, Lemma 7]. On the other hand, since $H_2(X, A \cup S : Z) \approx H_2(X_1, X_1 \cap A : Z) + H_2(X_2, X_2 \cap A : Z)$ and $H_2(X_1, X_1 \cap A : Z) = H_2(X_2, X_2 \cap A : Z) = 0$, $H_2(X, A \cap S : Z)$ must be zero. Therefore we have $H_2(X, A : Z) = 0$ for each closed subset A of X. By [**10**, Lemma 7], this shows that the continuum X has the property (ii) mentioned in the lemma. This completes the proof.

Lemma 18. *For each prime number p, there exists a 2-dimensional continuum $X(p)$ such that* (i) *there exists a closed subset A of $X(p)$ such that $H(X(p), A : Z(\mathfrak{a}_p)) \neq 0$,* (ii) *for any prime number $q \neq p$ and any pair (A, B) of closed subsets of $X(p)$ we have $H_2(A, B : Z(\mathfrak{a}_q)) = 0$.*

Proof. Let $\mathfrak{p}_p = \{p_1, \cdots, p_i, \cdots\}$ be a sequence consisting of all positive integers of the form q^k, where q ranges over all prime numbers except p and k ranges over all positive integers. Put $X(p) = R(\mathfrak{p}_p)$. Let T be the boundary of $R(\mathfrak{p}_p)$. Since each member p_i of the sequence $\mathfrak{p}_p$ and p are coprime numbers, we can see by a similar way as in the proof of Lemma 18 that $H_2(X(p), T : Z(\mathfrak{a}_p)) \neq 0$. To prove that $X(p)$ has the property (ii) mentioned in the lemma, let q be a prime number different from p. Let A be a closed subset of $X(p)$. Put $R_i = R(p_1, \cdots, p_i)$ and $A_i = \phi_i(A), i = 1, 2, \cdots$, where ϕ_i is the projection from $X(p)$ onto R_i. Let $\bar{A}_i$ be the smallest subcomplex of R_i containing $A_i, i = 1, 2, \cdots$. By Lemma 9 and the continuity theorem of Čech homology groups, we have an isomorphism $H_2(X(p), A : Z(\mathfrak{a}_q)) \approx \varprojlim \{H_2(R_i, \bar{A}_i : Z_{q^{i'}}) : \mathfrak{P}_{(i,i')}^{(j,j')} \mid j > i \text{ and } j' > i'\}$, where $\mathfrak{P}_{(i,i')}^{(j,j')}$ is a composition of the homomorphisms $(h_{i'}^{j'})_* : H_2(R_j, \bar{A}_j : Z_{q^{j'}}) \to H_2(R_j, \bar{A}_j : Z_{q^{i'}})$ and $(\phi_i^j)_* : H_2(R_j, \bar{A}_j : Z_{q^{i'}}) \to H_2(R_i, \bar{A}_i : Z_{q^{i'}})$. Assume that $H_2(X(p), A : Z(\mathfrak{a}_q)) \neq 0$. Let $\{a_{i,i'} \mid i = 1, 2, \cdots \text{ and } i' = 1, 2, \cdots\}$

26) Let $\{X_\alpha \mid \alpha \in \Omega\}$ be a collection of topological spaces. By $\sum_{\alpha \in \Omega} X_\alpha$ we understand a topological space X such that X is an union of topological images X_α''s of X_α's and $X_\alpha' \cap X_\beta' = \phi$, $\alpha \neq \beta$.

be a non-zero element of $H_2(X(p), A: Z(\mathfrak{a}_q))$, where $a_{i,i'} \in H_2(R_i, \bar{A}_i : Z_{q^{i'}})$, $i = 1, 2, \cdots$ and $i' = 1, 2, \cdots$. Let $a_{i,i'} \neq 0$. There exist integers i_0 and j_0 such that $i_0 > i$, $j_0 \geqq i'$ and the i-th member p_{i_0} of the sequence $\mathfrak{p}_p = q^{j_0}$. Take any 2-simplex σ of $R_{i_0-1} - \bar{A}_{i_0-1}$. Put $\tau = (\phi^{i_0}_{i_0-1})^{-1}\sigma$. Since $a_{i_0,i'}$ is a cycle mod $q^{i'}$, $a_{i_0,i'}$ must have the same coefficient t on each 2-simple of τ, where $t \in Z_{q^{i'}}$. Let $\tilde{t}$ be an integer such that $\rho(\tilde{t}) = t$, where ρ is a natural homomorphism from Z onto $Z_{q^{i'}}$. Suppose that $a_{i_0-1,i'}$ has the coefficient s on the 2-simplex σ, where $s \in Z_{q^{i'}}$. Let $\tilde{s}$ be an integer such that $\rho(\tilde{s}) = s$. Since $j_0 \geqq i'$, we have $\tilde{s}\cdot\sigma \equiv (\phi^{i_0}_{i_0-1})_*\tilde{t}\cdot\tau \equiv \tilde{t}\cdot(\phi^{i_0}_{i_0-1})_*\tau \equiv \tilde{t}\cdot q^{j_0}\cdot\sigma \equiv 0$[27] mod $q^{i'}$. Therefore we have $s = 0$. Since σ is any 2-simplex of $R_{i_0-1} - \bar{A}_{i_0-1}$, $a_{i_0-1,i'}$ must be zero. Since $\mathfrak{P}^{(i_0-1,i')}_{(i,i')} a_{i_0-1,i'} = a_{i,i'}$, this contradicts $a_{i,i'} \neq 0$. Thus, we have $H_2(X(p), A: Z(\mathfrak{a}_q)) = 0$. By [**10**, Lemma 7], we see that the continuum $X(p)$ has the property (ii) mentioned in the lemma. This completes the proof.

LEMMA 19. *There exists a 2-dimensional continuum which has the property* $\boldsymbol{P}$ *but not the property* (∗) *mentioned in Lemma* 11.

PROOF. First, let us remark that in compact spaces the property (∗) is equivalent to the following property (∗∗).

(∗∗) *For each prime number p there exist a closed subset A_p of X and a covering $\mathfrak{U}$ of X such that $0 \neq (\phi)_* H_n(X, A_p : Z(\mathfrak{a}_p)) \subset H_n(K, L: Z(\mathfrak{a}_p))$, where (K, L) is the pair of the nerves of $\mathfrak{U}$ corresponding to (X, A) and ϕ is a canonical mapping of (X, A) into (K, L).*

Thus, to prove the lemma, it is sufficient to construct a 2-dimensional continuum X which has the property $\boldsymbol{P}$ but not the property (∗∗). Let $(p_1, p_2, \cdots)$ be a sequence of all prime numbers. Put $X' = x_0 + \sum_{i=1}^{\infty} X(p_i)$[28], where x_0 is one point space and $X(p_i)$ is the continuum constructed in Lemma 18, $i = 1, 2, \cdots$. Let x_i be a point on the boundary of $X(p_i)$. $i = 1, 2, \cdots$. Let X be a continuum obtained from X' by retopologizing X such that x_0 is the topological limit of a sequence $\{X'(p_i)\}$, where $X'(p_i)$ is the subspace, homeomorphic to $X(p_i)$, of X', and by identifying the set $\sum_{i=1}^{\infty} x_i$ with the point x_0. Let f be the identification mapping. Put $X_i = f(X'(p_i))$, $i = 1, 2, \cdots$, and $\bar{x} = f(x_0)$. Let $\mathfrak{U} = \{U_i \mid i = 1, 2, \cdots, k\}$ be a covering of X such that $\bar{x} \in U_1$ and $\bar{x} \notin \bigcup_{i=2}^{k} \bar{U}_i$. Put $V = X - \bigcup_{i=2}^{k} \bar{U}_i$. There exists an integer i_0 such that, if $i \geqq i_0$, $X_i \subset V$. Let A be a closed subset of X. Let (K, L) be the pair of the nerves of $\mathfrak{U}$ corresponding to (X, A) and let ϕ be a canonical mapping of (X, A) into (K, L). Since $\phi(\bigcup_{i=i_0}^{\infty} X_i) = U_1$ and $H_2(X_k, X_k \cap A : Z(\mathfrak{a}_{p_j})) = 0$ for $k < i_0 \leqq j$ by Lemma 18,

27) Cf. footnotes 7) and 25).
28) See footnote 26).

we have $(\phi)_* H_2(X, A : Z(\mathfrak{a}_{p_j})) = 0$, $j = i_0, i_0+1, \cdots$. Since $\mathfrak{U}$ is any covering of X, the continuum X has not the property (**). Since it is obvious that X has the property $\boldsymbol{P}$, this completes the proof.

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