Journal of the Mathematical Society of Japan Vol. 11, No. 1. January, 1959

On a certain univalent mapping.

By Kôichi SAKAGUCHI

(Received June 19, 1958)

§ 1. Univalent functions starlike with respect to symmetrical points.

Let $f(z)$ be regular in the unit circle, and suppose that for every r less than and sufficiently close to one and every ζ on $|z|=r$, the angular velocity of $f(z)$ about the point $f(-\zeta)$ is positive at $z=\zeta$ as z traverses the circle $|z|=r$ in the positive direction, viz.

$$\Re\frac{zf'(z)}{f(z)-f(-\zeta)}>0 \qquad \text{for } z=\zeta,\ |\zeta|=r.$$

Then $f(z)$ is said to be starlike with respect to symmetrical points.

Obviously the class of functions univalent and starlike with respect to symmetrical points includes the classes of convex functions and odd functions starlike with respect to the origin.

THEOREM 1. *Let $f(z)=z+\cdots$ be regular in $|z|<1$. Then a necessary and sufficient condition for $f(z)$ to be univalent and starlike with respect to symmetrical points in $|z|<1$ is that*

$$\Re\frac{zf'(z)}{f(z)-f(-z)}>0,\quad |z|<1. \tag{1.1}$$

PROOF. (1) Proof for necessity. We suppose that $f(z)$ is univalent and starlike with respect to symmetrical points. Then

$$f(z)-f(-z)\neq 0,\qquad 0<|z|<1, \tag{1.2}$$

and

$$\Re\{zf'(z)/(f(z)-f(-z))\}>0,\quad |z|=r, \tag{1.3}$$

for every r less than and sufficiently close to one. From (1.2) the function $zf'(z)/(f(z)-f(-z))$ is regular in $|z|<1$, and therefore from (1.3) we have

$$\Re\{zf'(z)/(f(z)-f(-z))\}>0,\quad |z|\leqq r,$$

by virtue of the minimum principle for harmonic functions. Hence (1.1) follows.

(2) Proof for sufficiency. We next suppose that (1.1) holds. Then $f(z)$ is evidently starlike with respect to symmetrical points, and therefore it is sufficient to show that $f(z)$ is univalent in $|z|<1$.

Substituting $-z$ for z in (1.1), we have

$$\Re\{zf'(-z)/(f(z)-f(-z))\} > 0, \quad |z|<1,$$

which combined with (1.1) yields

$$\Re\{z(f'(z)+f'(-z))/(f(z)-f(-z))\} > 0, \quad |z|<1.$$

This shows that the function $f(z)-f(-z)$ is univalent and starlike with respect to the origin for $|z|<1$. Consequently from (1.1) $f(z)$ is close-to-convex for $|z|<1$, and hence $f(z)$ is univalent there [**1**].

We thus complete the proof.

THEOREM 2. *Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be univalent and starlike with respect to symmetrical points in $|z|<1$. Then*

$$|a_n| \leqq 1, \qquad n \geqq 2, \tag{1.4}$$

equality being attained by the function $z/(1+\varepsilon z)$, $|\varepsilon|=1$.

PROOF. From (1.1) we have

$$zf'(z)/(f(z)-f(-z)) \ll (1+z)/2(1-z).$$

On the other hand, since $f(z)-f(-z)$ is odd and starlike with respect to the origin for $|z|<1$,

$$f(z)-f(-z) \ll 2z/(1-z^2).$$

Hence

$$zf'(z) \ll z/(1-z)^2,$$

from which (1.4) follows. The statement concerning equality is evident.

§ 2. Generalization of the condition (1.1)

We can first generalize the condition (1.1) as follows.

THEOREM 3. *Let $f(z)=z+\cdots$ be regular in $|z|<1$, and suppose that for a positive integer k there holds the inequality*

$$\Re\left\{zf'(z)\Big/\sum_{\nu=0}^{k-1}\frac{f(\varepsilon^\nu z)}{\varepsilon^\nu}\right\} > 0, \quad |z|<1, \tag{2.1}$$

where $\varepsilon=e^{2\pi i/k}$. Then $f(z)$ is univalent and close-to-convex in $|z|<1$.

PROOF. Substituting $\varepsilon^\mu z$ for z in (2.1), we have

$$\Re\left\{zf'(\varepsilon^\mu z)\Big/\sum_{\nu=0}^{k-1}\frac{f(\varepsilon^\nu z)}{\varepsilon^\nu}\right\} > 0, \quad |z|<1,$$

where μ is an integer. Hence

$$\Re\left\{\sum_{\mu=0}^{k-1} zf'(\varepsilon^\mu z)\Big/\sum_{\nu=0}^{k-1}\frac{f(\varepsilon^\nu z)}{\varepsilon^\nu}\right\} > 0, \quad |z|<1,$$

which shows that the function $\sum_{\nu=0}^{k-1}\{f(\varepsilon^\nu z)/\varepsilon^\nu\}=kz+\cdots$ is univalent and starlike with respect to the origin for $|z|<1$. Consequently from (2.1) $f(z)$ is univalent and close-to-convex there.

Next we shall generalize the condition (2.1). For this purpose we prepare a lemma.

LEMMA. *Let $f(z)$ be regular in $|z|<1$, and let $s(z)=z+\cdots$ be regular and starlike with respect to the origin there, and suppose that*

$$\Re \frac{(zf'(z))'}{s'(z)} > 0, \quad |z|<1. \tag{2.2}$$

Then $f(z)$ is univalent and close-to-convex in $|z|<1$.

Accordingly if $zf'(z)$ is univalent and close-to-convex in $|z|<1$, then $f(z)$ is also univalent and close-to-convex there.

PROOF. From (2.2), for an arbitrary r in $0<r<1$, there exists a positive number $a(r)$ such that

$$|(zf'(z))'/s'(z)-a|<a, \quad |z|\leqq r.$$

Hence if we put $g(z)=(zf'(z))'/s'(z)-a$, then

$$(zf'(z))'-as'(z)=g(z)s'(z), \quad |g(z)|<a, \quad |z|\leqq r.$$

We denote by L the linear segment $\overline{0,\ s(z)}$, $|z|\leqq r$, on the w-plane, and denote by L_0 the image curve of L under $z=s^{-1}(w)$. By integrating both sides of the above equality along L_0, we have

$$|zf'(z)-as(z)|=\left|\int_{L_0} g(z)s'(z)dz\right|=\left|\int_L g(z)dw\right|$$
$$<\int_L a|dw|=a|s(z)|.$$

Consequently

$$|zf'(z)/s(z)-a|<a, \quad |z|\leqq r,$$

so that

$$\Re\{zf'(z)/s(z)\}>0, \quad |z|<1.$$

Hence $f(z)$ is univalent and close-to-convex in the unit circle.

THEOREM 4. *Let $f(z)=z+\cdots$ be regular in $|z|<1$, and put $F_0(z)=f(z)$, $F_1(z)=zf'(z)$, $F_2(z)=z(zf'(z))',\cdots$. If for a non-negative integer n, there holds the inequality*

$$\Re\left\{zF_n'(z)/\sum_{\nu=0}^{k-1}\frac{F_n(\varepsilon^\nu z)}{\varepsilon^\nu}\right\}>0, \quad |z|<1, \tag{2.3}$$

where k is a positive integer and $\varepsilon=e^{2\pi i/k}$, then $f(z)$ is univalent and close-to-convex in $|z|<1$.

PROOF. When (2.3) holds, by the preceding theorem $F_n(z)$ is univalent and close-to-convex in $|z|<1$. Consequently from the lemma the functions $F_{n-1}(z)$, $F_{n-2}(z),\cdots, F_0(z)=f(z)$ are all univalent and close-to-convex there.

Quite similarly we have the following theorem as a criterion for p-valence by using the theory of multivalently close-to-convex functions [2].

THEOREM 5. *Let* $f(z)=z^p+\cdots$ *be regular in* $|z|<1$, *and put* $F_0(z)=f(z)$, $F_1(z)=zf'(z)$, $F_2(z)=z(zf'(z))',\cdots$. *If for a non-negative integer* n, *there holds the inequality*

$$\Re\left\{zF_n'(z)/\sum_{\nu=0}^{k-1}\frac{F_n(\varepsilon^\nu z)}{\varepsilon^{\nu p}}\right\}>0, \quad |z|<1, \tag{2.4}$$

where k *is a positive integer and* $\varepsilon=e^{2\pi i/k}$, *then* $f(z)$ *is* p*-valent and close-to-convex in* $|z|<1$.

Nara Gakugei University.

References

[1] W. Kaplan, Close-to-convex schlicht functions, Michigan Math. J., 1 (1952), 169-185.

[2] T. Umezawa, Multivalently close-to-convex functions, Proc. Amer. Math. Soc., 8 (1957), 869-874.