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A theory of transformation groups on generalized spaces and its applications to Finsler and Cartan spaces.

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One of the main problems in the differential geometry of spaces with given structures is the determination of spaces admitting structure-preserving transformation groups of sufficiently high orders. The problem in generalized spaces, such as non-metric spaces of linear elements, of hyperplane elements or of spreads[1], has been successfully studied, but scarecely the problem in metric spaces, such as Finsler and Cartan spaces[2]. The only one result in Finsler space is due to H. C. Wang [**32**][3], who, by a beautiful group-theoretic method, determined the n-dimensional Finsler spaces admitting a group of motions of order higher than $n(n-1)/2+1$. Now, the author found that this problem could be also treated and solved by the method of tensor calculus for spaces such as Finsler and Cartan spaces, if we could develop the theory of Lie derivatives in the form adapted for the studying of the transformation groups in these spaces, and this could be done from the stand-point of the theory of fibre bundles.

In the present paper we shall give such a development and apply it to determine all the n-dimensional Finsler and Cartan spaces which admit a group of motions of order $n(n-1)/2+1$, for $n\neq 4$.

In Chapter I, we consider a general tensor bundle space to treat Finsler and Cartan spaces simultaneously. For our discussions, we need the theory of linear connections on a tensor bundle space, but such a theory may be obtained by modifying that on spaces of linear elements developed recently by T. Ōtsuki [**26**]. So we refer to [**26**] for the detail. As the modification is very slight, we have noted, as preliminaries, only what will be essentially used in the following. After that, we shall develop the theory of Lie derivatives, as said above, and consider groups of affine transformations on a tensor bundle space.

In Chapter II, we shall state a principle of determining Finsler spaces admitting a transitive group of motions. This principle follows from H. C.

1) [**9**], [**10**], [**11**], [**12**], [**14**], [**15**], [**16**], [**25**], [**36**, Chap. VIII], [**37**], [**38**], [**39**].
2) Cf. [**6**], [**13**], [**15**], [**16**], [**36**, p. 182].
3) His discussions appear also in [**36**, pp. 183-186].

Wang's lemmas [**32**]. By this principle and the results on Riemannian spaces admitting a group of motions of order $n(n-1)/2+1$ due to K. Yano [**35**], we can determine Finsler spaces admitting a group of motions of this order.

Chapter III is devoted to the discussions of spaces of hyperplane elements and of Cartan spaces. By a principle analogous to that in Chapter II, we can also determine Cartan spaces which have similar properties.

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Chapter I. Groups of transformations on generalized spaces.

§ 1. Preliminaries.

Linear connection on a tensor bundle. We consider a tensor bundle[4] $Z=\{X, Y, \alpha(L_n), \tau\}$ of type α, over an n-dimensional differentiable manifold X with an N-dimensional linear space Y as fibre, where α is a linear representation $L_n \rightarrow L_N$ [**28**, p. 23]. Under a coordinate transformation $x^{i'}=x^{i'}(x^i)$ in X, an element $z=(x^i, y^\lambda)\in Z$[5] undergoes the change of components

$$y^{\lambda'}=\Delta_\lambda{}^{\lambda'}(x)y^\lambda, \tag{1.1}$$

where we have put

$$\Delta_\lambda{}^{\lambda'}(x)=\alpha_\lambda{}^{\lambda'}(\Delta(x)), \qquad \Delta(x)=\left(\frac{\partial x^{i'}}{\partial x^i}\right)\in L_n, \tag{1.2}$$

The tensor field over Z, whose components are y^λ at each point $z=(x^i, y^\lambda)$, is called the *intrinsic tensor field* of z and is denoted by y.

Let $B=\{X, L_n, L_n, \pi\}$ be the bundle of n-frames over X, which is a principal bundle, and $\tilde{B}$ the induced bundle $\tau^{-1}B=\{Z, L_n, L_n, \tilde{\pi}\}$. The induced bundle map $\tilde{B}\rightarrow B$ will be denoted by $\tilde{\tau}$. $\tilde{B}$ is equivalent to the bundle of n-frames over Z. An n-frame $b=(x, e_1,\cdots, e_n)\in B$ or $\tilde{b}=(z, e_1,\cdots, e_n)\in\tilde{B}$ may be denoted by local coordinates with $(x^i, a_a{}^i)$ or $(x^i, y^\lambda, a_a{}^i)$ respectively, where $e_a=a_a{}^i X_i$, $(a_a{}^i)\in L_n$, $X_i=\partial/\partial x^i$.

A linear connection on a tensor bundle Z is by definition a linear con-

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4) The notation $Z=\{X, Y, G, \tau\}$ indicates a bundle structure with base space X, fibre Y, structural group G and projection τ; Z indicates simultaneously the space of the bundle.

5) Throughout this paper, Latin indices run from 1 to n, Greek ones from 1 to N and latin capital ones from 1 to a certain integer unless otherwise is stated. Also we adopt the kernel-index-method of J. A. Schouten [**27**, p. 3].

nection belonging to $\tilde{B}$ in the sense of C. Ehresmann [7]. We denote by $\omega=(\omega_j{}^i)$ and $\tilde{\omega}=(\tilde{\omega}_b{}^a)$ the connection forms on Z and on $\tilde{B}$ respectively; their components are related by equations

$$\tilde{\omega}_b{}^a(\tilde{b})=b_i{}^a(da_b{}^i+a_b{}^j\omega_j{}^i) \tag{1.3}$$

for $\tilde{b}=(x^i, y^\lambda, a_a{}^i)$, where $(b_i{}^a)=(a_a{}^i)^{-1}$. We put

$$\omega_j{}^i=\Gamma_j{}^i{}_k dx^k+C_j{}^i{}_\nu dy^\nu . \qquad [O,\ 3.2]^{6)} \tag{1.4}$$

Denoting with $\bar{\alpha}$ the linear homomorphism $L(L_n)\to L(L_N)$ induced by α, we have[7]

$$\begin{aligned} \omega_\mu{}^\lambda&=\bar{\alpha}_\mu{}^\lambda(\omega)=\Gamma_\mu{}^\lambda{}_k dx^k+C_\mu{}^\lambda{}_\nu dy^\nu ,\\ \tilde{\omega}_\beta{}^\alpha&=\bar{\alpha}_\beta{}^\alpha(\tilde{\omega})=b_\lambda{}^\alpha(da_\beta{}^\lambda+a_\beta{}^\mu\omega_\mu{}^\lambda) , \end{aligned} \tag{1.5}$$

where $\Gamma_\mu{}^\lambda{}_k$, $C_\mu{}^\lambda{}_\nu$, $a_\beta{}^\mu$, $b_\lambda{}^\alpha$ have obvious meanings. $C_j{}^i{}_\nu$ and $C_\mu{}^\lambda{}_\nu$ are tensor fields on Z.

The covariant differential of a tensor field T^I of any type β on Z and that of the lift $\tilde{T}^A=b_I^A T^I$, $(b_I^A)=(a_A^I)^{-1}$, of T^I on B are given by

$$\begin{aligned} DT^I&=dT^I+\omega_J^I T^J, & \omega_J^I&=\bar{\beta}_J^I(\omega) ,\\ \tilde{D}\tilde{T}^A&=d\tilde{T}^A+\tilde{\omega}_B^A\tilde{T}^B, & \tilde{\omega}_B^A&=\bar{\beta}_B^A(\tilde{\omega}) , \end{aligned} \tag{1.6}$$

and it is easy to see that the latter is the lift of the former.[7] In particular, the covariant differential of the intrinsic tensor field y^λ on Z and that of its lift $\tilde{y}^\alpha=b_\lambda{}^\alpha y^\lambda$, on $\tilde{B}$ are given by

$$\begin{aligned} Dy^\lambda&=dy^\lambda+(\Gamma_\mu{}^\lambda{}_k dx^k+C_\mu{}^\lambda{}_\nu dy^\nu)y^\mu\\ &=(\delta^\lambda_\nu+C^\lambda{}_\nu)dy^\nu+\Gamma^\lambda{}_k dx^k , \end{aligned} \qquad [O,\ 4.1] \tag{1.7}$$

$$\tilde{D}\tilde{y}^\alpha=d\tilde{y}^\alpha+\tilde{\omega}_\beta{}^\alpha\tilde{y}^\beta=\widetilde{Dy}^\alpha , \qquad [O,\ 4.14,\ 4.15] \tag{1.8}$$

putting $\Gamma^\lambda{}_k=\Gamma_\mu{}^\lambda{}_k y^\mu$, $C^\lambda{}_\nu=C_\mu{}^\lambda{}_\nu y^\mu$. Then we have easily

LEMMA 1.1 [O, Prop. 4.1] *The $n+N$ forms dx^i and Dy^λ on Z are linearly independent if and only if the tensor $\delta^\lambda_\nu+C^\lambda{}_\nu$ forms a regular (N, N)-matrix. Then the $n+N+n^2$ forms*

$$\begin{aligned} \widetilde{dx}^a&=b_i{}^a dx^i ,\\ \widetilde{Dy}^\alpha&=b_\lambda{}^\alpha\{\Gamma^\lambda{}_k dx^k+(\delta^\lambda_\nu+C^\lambda{}_\nu)dy^\nu\} ,\\ \tilde{\omega}_b{}^a&=b_i{}^a\{a_b{}^j\Gamma_j{}^i{}_k dx^k+a_b{}^j C_j{}^i{}_\nu dy^\nu+da_b{}^i\} \end{aligned} \tag{1.9}$$

on $\tilde{B}$ are linearly independent.

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6) [O, 3.2] means that this formula corresponds to the formula (3.2) of the paper [**26**] of T. Ōtsuki. Hereafter we shall indicate the correspondence with [**26**] in this manner.

7) [**3**, p. 53], [**31**].

From (1.8), we have also

LEMMA 1.2. *The linear independence of* $\widetilde{dx}^a$, $\widetilde{Dy}^\alpha$, $\tilde{\omega}_b{}^a$ *is equivalent to that of* $\widetilde{dx}^a$, $d\tilde{y}^a$, $\tilde{\omega}_b{}^a$.

In the rest of this paper, we shall confine ourselves to the case where dx^i and Dy^λ on Z are linearly independent. If we denote by $(M^\lambda{}_\mu)$ the inverse matrix of $(\delta^\lambda{}_\nu + C^\lambda{}_\nu)$ and put

$$\Gamma^*{}_j{}^i{}_k = \Gamma_j{}^i{}_k - C_j{}^i{}_\nu M^\nu{}_\rho \Gamma^\rho{}_k, \qquad \Gamma^*{}_\mu{}^\lambda{}_k = \bar{\alpha}_\mu{}^\lambda(\boldsymbol{\Gamma}_k{}^*), \quad \boldsymbol{\Gamma}_k{}^* = (\Gamma^*{}_j{}^i{}_k) \in L(L_n), \tag{1.10}$$
[O, 4.7]

then we have the relations

$$\omega_j{}^i = \Gamma^*{}_j{}^i{}_k dx^k + C_j{}^i{}_\nu M^\nu{}_\rho Dy^\rho, \tag{1.11}$$
[O, 4.8]

$$\Gamma^{*\lambda}{}_k = M^\lambda{}_\mu \Gamma^\mu{}_k, \tag{1.12}$$
[O, 4.11]

$$\Gamma_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k + C_j{}^i{}_\nu \Gamma^{*\nu}{}_k, \tag{1.13}$$
[O, 4.12]

$$dy^\lambda = M^\lambda{}_\nu Dy^\nu - \Gamma^{*\lambda}{}_k dx^k. \tag{1.14}$$
[O, 4.13]

Under a coordinate transformation, $\Gamma^*{}_j{}^i{}_k$ are changed by the same rule as coefficients of linear connection in ordinary spaces.

Put now

$$DT^I = T^I{}_{|k} dx^k + T^I{}_{\|\nu} M_\rho{}^\nu Dy^\rho, \qquad \text{[O, 9.9]}$$
$$\tilde{D}\tilde{T}^A = \tilde{T}^A{}_{|c} \widetilde{dx}^c + \tilde{T}^A{}_{\|\gamma} \tilde{M}_\xi{}^\gamma \widetilde{Dy}^\xi. \qquad \text{[O, 10.5]} \tag{1.15}$$

Then $T^I{}_{|k}$ are called the first covariant derivatives, and $T^I{}_{\|\nu}$ the second covariant derivatives. They are given explicitly by

$$T^I{}_{|k} = \frac{\partial T^I}{\partial x^k} - \frac{\partial T^I}{\partial y^\nu} \Gamma^{*\nu}{}_k + \Gamma^*{}_J{}^I{}_k T^J, \qquad \text{[O, 9.7]}$$
$$T^I{}_{\|\nu} = \frac{\partial T^I}{\partial y^\nu} + C_J{}^I{}_\nu T^J, \qquad \text{[O, 9.8]} \tag{1.16}$$

and $\tilde{T}^A{}_{|c}$ and $\tilde{T}^A{}_{\|\gamma}$ are respectively their lifts. In particular, we have immediately, from the definition,

$$y^\lambda{}_{|k} = 0, \quad y^\lambda{}_{\|\nu} = \delta^\lambda{}_\nu + C^\lambda{}_\nu. \tag{1.17}$$

Let us denote by X_i, Y_λ, $A_i{}^a$ the vector fields of the natural basis with respect to a coordinates $(x^i, y^\lambda, a_a{}^i)$ and by E_a, F_α, $G_a{}^b$ the dual vector fields of the linearly independent forms (1.9) on $\tilde{B}$. They are related by

$$E_a = a_a{}^i \{X_i - \Gamma^{*\lambda}{}_i Y_\lambda - a_b{}^j \Gamma^*{}_j{}^h{}_i A_h{}^b\}, \quad F_\alpha = a_\alpha{}^\lambda M_\lambda{}^\nu (Y_\nu - a_b{}^j C_j{}^h{}_\nu A_h{}^b), \quad G_a{}^b = a_a{}^i A_i{}^b, \tag{1.18}$$
[O, 5.1–3]

or conversely by

$$
\begin{aligned}
&X_i = b_i{}^a E_a + b_\kappa{}^\alpha \Gamma^\kappa{}_i F_\alpha + b_h{}^a a_b{}^j \Gamma_j{}^h{}_i G_a{}^b , \\
(1.19) \qquad &Y_\lambda = b_\kappa{}^\alpha (\delta^\kappa{}_\lambda + C^\kappa{}_\lambda) F_\alpha + b_h{}^a a_b{}^j C_j{}^h{}_\lambda G_a{}^b , \\
&A_i{}^b = b_i{}^a G_a{}^b .
\end{aligned}
$$

The vectors E_a, F_α are called respectively the basic vectors of the first and the second kind of the given connection, and $G_a{}^b$ are called the fundamental vectors of $\tilde{B}$. In the tangent space $T_{\tilde{b}}(\tilde{B})$ at each point $\tilde{b}$ of $\tilde{B}$, the basic vectors span the horizontal linear subspace of the connection, and the fundamental vectors span the vertical linear subspace. For any vertical vector $V = v_b{}^a G_a{}^b$, we have

$$
(1.20) \qquad \tilde{\omega}_b{}^a(V) = v_b{}^a, \quad \tilde{\omega}_\beta{}^\alpha(V) = \bar{\alpha}_\beta{}^\alpha(\boldsymbol{v}) , \qquad \boldsymbol{v} = (v_b{}^a) \in L(L_n) .
$$

For the lift $\tilde{T}^A$ of a tensor field, we have

$$
(1.21) \qquad E_c \tilde{T}^A = \tilde{T}^A|_c, \quad F_\gamma \tilde{T}^A = \tilde{T}^A\|_\beta \tilde{M}^\beta{}_\gamma, \quad G_a{}^b \tilde{T}^A = -\tilde{\omega}^A_B(G_a{}^b) \tilde{T}^B .
$$

The equations of structure of the connection have the following form, if we use the basic and fundamental vector fields as a basis of the field of tangent spaces:

$$
\begin{aligned}
&[E_c,\ E_d\ ] = \tilde{S}^a{}_{cd} E_a + \tilde{R}^\alpha{}_{0cd} F_\alpha + \tilde{R}^a{}_{bcd} G_a{}^b , \\
&[E_c,\ F_\eta\ ] = \tilde{C}^a{}_{c\delta} \tilde{M}^\delta{}_\eta E_a + \tilde{P}^\alpha{}_{0c\delta} \tilde{M}^\delta{}_\eta F_\alpha + \tilde{P}^a{}_{bc\delta} \tilde{M}^\delta{}_\eta G_a{}^b , \\
&[F_\xi,\ F_\eta\ ] = \tilde{Q}^\alpha{}_{0\gamma\delta} \tilde{M}^\gamma{}_\xi \tilde{M}^\delta{}_\eta F_\alpha + \tilde{Q}^a{}_{b\gamma\delta} \tilde{M}^\gamma{}_\xi \tilde{M}^\delta{}_\eta G_a{}^b , \\
(1.22) \qquad &[G_a{}^b, E_c\ ] = \delta^b_c E_a , \qquad\qquad [\mathrm{O},\ 11.4\text{–}9] \\
&[G_a{}^b, F_\gamma\ ] = \tilde{\omega}_\gamma{}^\alpha(G_a{}^b) F_\alpha , \\
&[G_a{}^b, G_c{}^d] = \delta^b_c G_a{}^d - \delta^d_a G_c{}^b .
\end{aligned}
$$

We notice that, if we apply $[E_c, E_d]$, $[E_c, F_\delta]$ and $[F_\gamma, F_\delta]$ to $\tilde{T}^A$ and use the first three formulas of (1.22), we obtain the so-called Ricci formulas, and Jacobi identities on $[E_c, [E_d, E_e]]$, $[E_c, [E_d, F_\varepsilon]]$, $[E_c, [F_\delta, F_\varepsilon]]$ and $[F_\gamma, [F_\delta, F_\varepsilon]]$ yield the so-called Bianchi identities on ground of (1.22).

Auxiliary connection. In virtue of the transformation law of $\Gamma^*{}_j{}^i{}_k$, the forms $\mathring{\omega}_j{}^i = \Gamma^*{}_j{}^i{}_k dx^k$ define a linear connection on Z, which is called the *auxiliary connection* of the original connection. We denote the corresponding quantities with the auxiliary connection to E_a, F_α, $G_a{}^b$, $R^i{}_{jkl}$, etc. by $\mathring{E}_a$, $\mathring{F}_\alpha$, $\mathring{G}_a{}^b$, $\mathring{R}^i{}_{jkl}$, etc. respectively. From formulas concerning quantities with the original connection, we obtain the corresponding formulas with the auxiliary connection, simply in putting all $C_j{}^i{}_\nu = 0$. Thus we see from (1.16) that the first and the second covariant derivatives with respect to the auxiliary connection are respectively the same as the first covariant derivatives with respect to the original connection and the partial derivatives with respect to y^ν. We have also

$$\overset{\circ}{P}{}^i{}_{jk\omega}=\frac{\partial \Gamma^*{}_j{}^i{}_\kappa}{\partial y^\omega}, \quad \overset{\circ}{Q}{}^i{}_{j\nu\omega}=0, \tag{1.23}$$

$$\begin{aligned} &\overset{\circ}{R}{}^i{}_{jkl}=R^i{}_{jkl}-C_j{}^i{}_\pi M^\pi{}_\kappa R^\kappa{}_{0kl}, \\ &\overset{\circ}{P}{}^i{}_{jk\omega}=P^i{}_{jk\omega}+C_j{}^i{}_{\omega|k}-C_j{}^i{}_\pi M^\pi{}_\kappa(P^\kappa{}_{0k\omega}+C^\kappa{}_{\omega|k}), \end{aligned} \tag{1.24}$$

$$\overset{\circ}{E}_a=E_a, \quad \overset{\circ}{F}_\alpha=a_\alpha{}^\lambda Y_\lambda, \quad \overset{\circ}{G}_a{}^b=G_a{}^b. \tag{1.25}$$

Proper tensor bundle and proper connection. In a tensor bundle $Z=\{X, Y, \alpha(L_n), \tau\}$ of type α, the zero tensor field is a trivial cross-section over X. Identifying it with X, we denote $Z-X$ by Z°. We define an equivalence relation $y\sim y'$ in Y by $y'=ky$, $k\neq 0$. The equivalence relation reduces $Y^\circ=Y-0$ to an $(N-1)$-dimensional projective space P, and we have the natural projection $\rho: Y^\circ\to P$. The above equivalence can be naturally extended in the tensor bundle Z°, and Z° is thereby reduced to a bundle $\bar{Z}=\{X, P, \bar{L}, \sigma\}$, where $\bar{L}$ is the factor group of $\alpha(L_n)$ by a subgroup isomorphic to the multiplicative group of non-zero real numbers. We call $\bar{Z}$ the *proper tensor bundle* over X. Then ρ is extended over Z° and the extension is also denoted by ρ. The projection $\sigma: Z\to X$ induces a principal bundle $\bar{B}=\{\bar{Z}, L_n, L_n, \bar{\pi}\}$. Denoting by $\tilde{B}^\circ$ the portion of $\tilde{B}$ over Z°, we have a communtative diagram

$$\begin{tikzcd} & Z^0 \arrow[dl, "\rho"'] \arrow[dd, "\tau"] & & \tilde{B} \arrow[ll, "\tilde{\pi}"'] \arrow[dl, "\tilde{\rho}"'] \arrow[dd, "\tilde{\tau}"] \\ \bar{Z} \arrow[dr, "\sigma"'] & & \bar{B} \arrow[ll, "\bar{\pi}"'] \arrow[dr, "\tilde{\sigma}"] & \\ & X & & B \arrow[ll, "\pi"] \end{tikzcd} \tag{1.26}$$

$\tilde{\tau}, \tilde{\sigma}, \tilde{\rho}$ being the bundle maps induced respectively by τ, σ, ρ.

A form on Z° is said to be *proper* if it is induced from a form on $\bar{Z}$ by $\rho: Z^\circ\to\bar{Z}$, and a connection of Z° is called *proper* if its connection form is proper. Then we can prove the following lemma.

LEMMA 1.3. *A linear connection on Z° is proper if and only if $\Gamma^*{}_j{}^i{}_\kappa$ and $C_j{}^i{}_\nu$ are homogeneous in y^λ of degree* 0 *and* -1 *respectively and $C_j{}^i{}_\nu$ satisfy*

$$C_j{}^i{}_\nu y^\nu=0. \tag{1.27}$$

Since a dual induced map[8] ρ^* of ρ is commutative with the differential operator d and with the exterior multiplication we have

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8) We denote the tangent bundle of a space X by $T(X)$ and the cotangent bundle by $T^*(x)$. Let φ be a map $X\to X'$. The map induced by φ in the well-known way of $T(X)$ into $T(X')$—the "induced map" of φ—is denoted by φ_*, and the dual of φ_* —the "dual induced map" of φ—by φ^*; $\varphi_*: T(X)\to T(X')$, $\varphi^*: T^*(X')\to T^*(X)$.

LEMMA 1.4. *If a connection is proper, then so are the curvature and torsion forms, and consequently the curvature tensors $R^i{}_{jkl}$, $P^i{}_{jk\omega}$, $Q^i{}_{j\nu\omega}$ are homogeneous in y^λ of degree* 0, -1, -2 *respectively and they satisfy*

$$P^i{}_{jk\omega}y^\omega=0, \qquad Q^i{}_{j\nu\omega}y^\omega=0\,.$$

§ 2. Lie differentiation.

A differentiable transformation φ:

$$x'^i=\varphi^i(x)$$

in the base space X induces a transformation $\bar\varphi$ in the bundle space Z: $\bar\varphi$ is defined by

$$x'^i=\varphi^i(x), \quad y'^\lambda=\alpha_\mu{}^\lambda\Big(\frac{\partial\varphi}{\partial x}\Big)y^\mu, \quad \frac{\partial\varphi}{\partial x}=\Big(\frac{\partial\varphi^i}{\partial x^j}\Big)\in L_n\,,$$

and is called the *extended transformation* of φ in Z. Let $\xi=(\xi^i)$ be a vector field on the base space X generating a (local) one-parameter group φ_t. Then the extended group $\bar\varphi_t$ of φ_t in Z is a one-parameter group with the generating vector field (ξ^i, ξ^λ) such that

$$\xi^\lambda=\bar\alpha_\mu{}^\lambda\Big(\frac{\partial\xi}{\partial x}\Big)y^\mu, \qquad \frac{\partial\xi}{\partial x}=\Big(\frac{\partial\xi^i}{\partial x^j}\Big)\in L(L_n)\,. \tag{2.1}$$

Since $\bar\alpha_\mu{}^\lambda\Big(\dfrac{\partial\xi}{\partial x}\Big)$ are independent of y^λ, we have

$$\frac{\partial\xi^\lambda}{\partial y^\mu}=\bar\alpha_\mu{}^\lambda\Big(\frac{\partial\xi}{\partial x}\Big)\,. \tag{2.2}$$

Now, the *Lie derivative* of a geometric object Ω on Z with respect to a vector field ξ on X is defined by

$$(\pounds\,\Omega)(z)=\lim_{t\to 0}\frac{1}{t}\{\Omega(\bar\varphi_t(z))-\varphi_t'(\Omega(z))\}\,, \tag{2.3}$$

where $\bar\varphi_t$ has the meaning above explained and φ_t' in the last term indicates the transformation induced by φ_t in the bundle of geometric objects of the type of Ω, cf. [**19**], [**22**, p. 30], [**29**], [**36**, p. 20].

In particular, the Lie derivatives of a β-tensor field T^I on Z with respect to a vector field ξ is given by

$$\pounds\,T^I=\frac{\partial T^I}{\partial x^h}\,\xi^h+\frac{\partial T^I}{\partial y^\kappa}\,\xi^\kappa-\bar\beta_J{}^I\Big(\frac{\partial\xi}{\partial x}\Big)T^J \tag{2.4}$$

and is also a β-tensor field on Z. Moreover the Lie derivatives of the first coefficients $\Gamma^*{}_j{}^i{}_k$ of a linear connection, is given by

(2.5)
$$\pounds\Gamma^{*}{}_{j}{}^{i}{}_{k}=\frac{\partial\Gamma^{*}{}_{j}{}^{i}{}_{k}}{\partial x^{h}}\xi^{h}+\frac{\partial\Gamma^{*}{}_{j}{}^{i}{}_{k}}{\partial y^{\kappa}}\xi^{\kappa}-\frac{\partial\xi^{i}}{\partial x^{h}}\Gamma^{*}{}_{j}{}^{h}{}_{k}+\frac{\partial\xi^{h}}{\partial x^{j}}\Gamma^{*}{}_{h}{}^{i}{}_{k}+\frac{\partial\xi^{h}}{\partial x^{k}}\Gamma^{*}{}_{j}{}^{i}{}_{h}+\frac{\partial^{2}\xi^{i}}{\partial x^{j}\partial x^{k}}$$

and is a tensor field on Z of covariant degree 2 and contravariant degree 1, because $\Gamma^{*}{}_{j}{}^{i}{}_{k}$ is a linear geometric object. Clearly we have

(2.6)
$$\pounds\Gamma^{*}{}_{\mu}{}^{\lambda}{}_{k}=\bar{\alpha}_{\mu}{}^{\lambda}(\pounds\Gamma^{*}{}_{k}),\qquad \pounds\Gamma^{*}{}_{k}=(\pounds\Gamma^{*}{}_{j}{}^{i}{}_{k})\in L(L_{n}).$$
$$\pounds\Gamma^{*}{}_{J}{}^{I}{}_{k}=\bar{\beta}_{J}{}^{I}(\pounds\Gamma^{*}{}_{k}),$$

Now let $\tilde{B}$ be the bundle of n-frames over Z. For a transformation φ on X we define a transformation $\tilde{\varphi}$ of $\tilde{B}$ by $\tilde{\varphi}(z, e_{1},\cdots, e_{n})=(\bar{\varphi}(z), \varphi_{*}(e_{1}),\cdots, \varphi_{*}(e_{n}))$ for any point $\tilde{b}=(z, e_{1},\cdots, e_{n})\in\tilde{B}$. By use of local coordinates in $\tilde{B}$, $\tilde{\varphi}$ is represented by

(2.7)
$$x'^{i}=\varphi^{i}(x),\quad y'^{\lambda}=\alpha_{\mu}{}^{\lambda}\Big(\frac{\partial\varphi}{\partial x}\Big)y^{\mu},\quad a'_{a}{}^{i}=\frac{\partial\varphi^{i}}{\partial x^{j}}a_{a}{}^{j}.$$

In particular a one-parameter group of transformation φ_{t} on X yields a one-parameter group of transformations $\tilde{\varphi}_{t}$ on $\tilde{B}$, whose generating vector field Ξ on $\tilde{B}$ is

(2.8)
$$\Xi=\xi^{i}X_{i}+\xi^{\lambda}Y_{\lambda}+\frac{\partial\xi^{i}}{\partial x^{j}}a_{a}{}^{j}A_{i}{}^{a}.$$

$\tilde{\varphi}_{t}$ and Ξ are said to be *induced* on $\tilde{B}$ by φ_{t} or ξ.

Substituting (1.19) into (2.8), we have

(2.9)
$$\Xi=\tilde{\xi}^{a}E_{a}+\tilde{\xi}^{\alpha}{}_{\beta}\tilde{y}^{\beta}F_{\alpha}+\tilde{\xi}^{a}{}_{b}G_{a}{}^{b},$$

where $\tilde{\xi}^{a}$, $\tilde{\xi}^{a}{}_{b}$, and $\tilde{\xi}^{\alpha}{}_{\beta}$ are lifts of ξ^{i}, $\xi^{i}{}_{j}$, and $\xi^{\lambda}{}_{\mu}$ respectively; $\xi^{i}{}_{j}$, $\xi^{\lambda}{}_{\mu}$ being defined as follows:

(2.10)
$$\xi^{i}{}_{j}=\frac{\partial\xi^{i}}{\partial x^{j}}+\Gamma_{j}{}^{i}{}_{k}\xi^{k}+C_{j}{}^{i}{}_{\nu}\xi^{\nu},$$
$$\xi^{\lambda}{}_{\mu}=\bar{\alpha}_{\mu}{}^{\lambda}(\boldsymbol{\xi}),\qquad \boldsymbol{\xi}=(\xi^{i}{}_{j})\in L(L_{n}).$$

These are tensor fields. Namely, by (1.13), the expressions (2.10) are written in the form

(2.11)
$$\xi^{i}{}_{j}=\frac{\partial\xi^{i}}{\partial x^{j}}+\Gamma^{*}{}_{j}{}^{i}{}_{k}\xi^{k}+C_{j}{}^{i}{}_{\nu}\Big(\frac{\partial\xi^{\nu}}{\partial y^{\rho}}+\Gamma^{*}{}_{\rho}{}^{\nu}{}_{k}\xi^{k}\Big)y^{\rho},$$

and these show the tensor character of $\boldsymbol{\xi}=(\xi^{i}{}_{j})$, because

(2.12)
$$\frac{\partial\xi^{i}}{\partial x^{j}}+\Gamma^{*}{}_{j}{}^{i}{}_{k}\xi^{k}=\xi^{i}{}_{|j}+S^{i}{}_{jk}\xi^{k}$$

is a tensor field and the expression in the parentheses of the last term of (2.11) is the representation of (2.12) by $\bar{\alpha}$. From (2.11), we have equations

$$\xi^\lambda{}_\rho y^\rho = (\delta^\lambda{}_\nu + C^\lambda{}_\nu)\left(\frac{\partial \xi^\nu}{\partial y^\rho} + \Gamma^*{}_\rho{}^\nu{}_k \xi^k\right) y^\rho \tag{2.13}$$

or

$$M^\lambda{}_\nu \xi^\nu{}_\rho y^\rho = \left(\frac{\partial \xi^\lambda}{\partial y^\rho} + \Gamma^*{}_\rho{}^\nu{}_k \xi^k\right) y^\rho . \tag{2.13$'$}$$

The equations (2.9) clarifies the geometical meaning of the $L(L_n)$-valued tensor field $\boldsymbol{\xi} = (\xi^i{}_j)$ which appears everywhere in the theory of Lie derivatives in a space with linear connection with torsion, e.g. [**36**, p. 8], that is to say,

THEOREM 2.1. *The components $\tilde{\xi}^a{}_b$ of the lift $\tilde{\boldsymbol{\xi}}$ in $\tilde{B}$ are the vertical components of the induced vector field Ξ on $\tilde{B}$.*

Now we consider Lie differentiation, denoted by $\tilde{\pounds}$, of any geometric object on the principal bundle $\tilde{B}$ with respect to the induced vector field Ξ. Since the components $\tilde{T}^A$ of the lift of a tensor field T are functions on $\tilde{B}$ and Lie differentiation on a function is reduced to an ordinary differentiation, we obtain from (1.21) and (2.9)

$$\tilde{\pounds}\tilde{T}^A = \Xi \tilde{T}^A = \tilde{T}^A{}_{|a}\tilde{\xi}^a + \tilde{T}^A{}_{\|\gamma}\tilde{M}^\gamma{}_\varepsilon \tilde{\xi}^\varepsilon{}_\zeta \tilde{y}^\zeta - \tilde{\xi}^A{}_B \tilde{T}^B . \tag{2.14}$$

It is indeed the lift in $\tilde{B}$ of a β-tensor

$$\begin{aligned}
& T^I{}_{|h}\xi^h + T^I{}_{\|\pi} M^\pi{}_\kappa \xi^\kappa{}_\rho y^\rho - \xi^I{}_J T^J \\
&= \left(\frac{\partial T^I}{\partial x^h} - \frac{\partial T^I}{\partial y^\kappa}\Gamma^{*\kappa}{}_h + \Gamma^*{}_J{}^I{}_h T^J\right)\xi^h + \left(\frac{\partial T^I}{\partial y^\kappa} + C_J{}^I{}_\kappa T^J\right)\left(\frac{\partial \xi^\kappa}{\partial y^\rho} + \Gamma^*{}_\rho{}^\kappa{}_k \xi^k\right) y^\rho \\
&\quad - \left(\bar{\beta}^I_J\left(\frac{\partial \xi}{\partial x}\right) + \Gamma_J{}^I{}_k \xi^k + C_J{}^I{}_\nu \xi^\nu\right) T^J \\
&= \frac{\partial T^I}{\partial x^h}\xi^h + \frac{\partial T^I}{\partial y^\kappa}\xi^\kappa - \bar{\beta}_J^I\left(\frac{\partial \xi}{\partial x}\right) T^J \\
&= \pounds T ,
\end{aligned} \tag{2.14$'$}$$

by (1.16), (2.10) and (2.11). Thus we have established

THEOREM 2.2. *The lift of the Lie derivative of a tensor field T in Z with respect to a vector field ξ is equal to the ordinary derivative of the lift $\tilde{T}$ in B with respect to the induced vector field Ξ, that is,*

$$\widetilde{\pounds T} = \tilde{\pounds}\tilde{T} = \Xi \tilde{T} . \tag{2.15}$$

From (1.17) and (2.14)$'$, we obtain

LEMMA 2.3. *For the intrinsic tensor field y of Z and its lift $\tilde{y}$ in $\tilde{B}$, we have*

$$\pounds y = \tilde{\pounds}\tilde{y} = 0 . \tag{2.16}$$

Since

$$\xi^i{}_{\|\nu} = C_j{}^i{}_\nu \xi^j , \tag{2.17}$$

the Lie derivatives of $C_j{}^i{}_\nu$:

$$\pounds C_j{}^i{}_\nu = C_j{}^i{}_{\nu|h}\xi^h + C_j{}^i{}_{\nu\|\pi} M^\pi{}_\kappa \xi^\kappa{}_\rho y^\rho - \xi^i{}_h C_j{}^h{}_\nu + \xi^h{}_j C_h{}^i{}_\nu + \xi^\kappa{}_\nu C_j{}^i{}_\kappa \tag{2.18}$$

are equal to

$$\pounds C_j{}^i{}_\nu=\xi^i{}_{j\|\nu}-P^i{}_{jl\nu}\xi^l+Q^i{}_{j\nu\omega}M^\omega{}_\kappa\xi^\kappa{}_\rho y^\rho . \tag{2.19}$$

In virtue of (1.21), (1.22) and (2.11), we have furthermore

$$\begin{aligned}\tilde{\pounds}E_c&=[\Xi, E_c]\\&=-\tilde{\Lambda}_\beta{}^\alpha{}_c\tilde{y}^\beta F_\alpha-\tilde{\Lambda}_b{}^a{}_c G^b{}_a ,\end{aligned} \tag{2.20}$$

where we have put

$$\begin{aligned}&\tilde{\Lambda}_b{}^a{}_c=\tilde{\xi}^a{}_{b|c}+\tilde{R}^a{}_{bcd}\tilde{\xi}^d+\tilde{P}^a{}_{bc\eta}\tilde{M}^\eta{}_\delta\tilde{\xi}^\delta{}_\zeta y^\zeta ,\\&\tilde{\Lambda}_\beta{}^\alpha{}_c=\bar{\alpha}^\alpha{}_\beta(\tilde{\boldsymbol{\Lambda}}_c),\qquad \tilde{\boldsymbol{\Lambda}}_c=(\tilde{\Lambda}_b{}^a{}_c)\in L(L_n) .\end{aligned} \tag{2.21}$$

The field $\tilde{\Lambda}_b{}^a{}_c$ on $\tilde{B}$ is the lift of a tensor field $\Lambda_j{}^i{}_k$ on Z:

$$\Lambda_j{}^i{}_k=\xi^i{}_{j|k}+R^i{}_{jkl}\xi^l+P^i{}_{jk\pi}M^\pi{}_\kappa\xi^\kappa{}_\rho y^\rho , \tag{2.22}$$

which are equal to

$$\Lambda_j{}^i{}_k=\pounds\Gamma^*{}_j{}^i{}_k+C_j{}^i{}_\kappa\pounds\Gamma^*{}^\kappa{}_k \tag{2.23}$$

by the comparison with (2.5). Solving (2.23) in $\pounds\Gamma^*{}_j{}^i{}_k$, we have also

$$\pounds\Gamma^*{}_j{}^i{}_k=\Lambda_j{}^i{}_k-C_j{}^i{}_\pi M^\pi{}_\kappa\Lambda^\kappa{}_k , \tag{2.24}$$

where $\Lambda^\kappa{}_k=\Lambda_\mu{}^\kappa{}_k y^\mu$. Similarly, taking account of (2.19), we have,

$$\tilde{\pounds}F_\gamma=[\Xi, F_\gamma]=-(\tilde{\pounds}\tilde{C}^\alpha{}_\pi)\tilde{M}^\pi{}_\gamma F_\alpha-(\tilde{\pounds}\tilde{C}_b{}^a{}_\pi)\tilde{M}^\pi{}_\gamma G^b{}_a . \tag{2.25}$$

Moreover, by (1.21) and (1.22), we have

$$\tilde{\pounds}G^b{}_a=[\Xi, G^b{}_a]=0 . \tag{2.26}$$

In turning to the auxiliary connection, we obtain, by the method stated in § 1, the following formulas, from (2.11), (2.14)′, (2.22) respectively,

$$\begin{aligned}\mathring{\xi}^i{}_j&=\frac{\partial\xi^i}{\partial x^j}+\Gamma^*{}_j{}^i{}_k\xi^k\\&=\xi^i|_j+S^i{}_{jk}\xi^k\\&=\xi^i{}_j-C_j{}^i{}_\nu M^\nu{}_\kappa\xi^\kappa{}_\rho y^\rho ,\end{aligned} \tag{2.27}$$

$$\pounds T^I=T^I|_h\xi^h+\frac{\partial T^I}{\partial y^\kappa}\mathring{\xi}^\kappa{}_\rho y^\rho-\mathring{\xi}^I{}_J T^J \tag{2.28}$$

$$\mathring{\Lambda}_j{}^i{}_k=\pounds\Gamma^*{}_j{}^i{}_k=\mathring{\xi}^i{}_{j|k}+\mathring{R}^i{}_{jkl}\xi^l+\frac{\partial\Gamma^*{}_j{}^i{}_k}{\partial y^\omega}\mathring{\xi}^\omega{}_\rho y^\rho . \tag{2.29}$$

§ 3. Commutation formulas.

Let ξ_1, ξ_2 be two vector fields on X, and Ξ_1, Ξ_2 the vector fields on B induced by them respectively. From the definition (2.8) of induced vector fields, we have

$$\tag{3.1} \begin{aligned} \tilde{\text{£}}_1 \Xi_2 &= [\Xi_1, \Xi_2] \\ &= (\text{£}_1 \xi_2{}^i) X_i + (\text{£}_1 \xi_2)^\lambda Y_\lambda + \frac{\partial}{\partial x^p} (\text{£}_1 \xi_2{}^i) a_a{}^p A_i{}^a , \end{aligned}$$

where we have used the equations

$$\tag{3.2} \text{£}_1 \xi_2{}^i = [\xi_1, \xi_2]^i = \xi_1{}^j \frac{\partial \xi_2{}^i}{\partial x^j} - \xi_2{}^j \frac{\partial \xi_1{}^i}{\partial x^j}$$

and

$$\tag{3.3} \begin{aligned} (\text{£}_1 \xi_2)^\lambda &= \bar{\alpha}_\mu{}^\lambda \Big(\frac{\partial}{\partial x} [\xi_1, \xi_2] \Big) y^\mu \\ &= \xi_1{}^j \frac{\partial \xi_2{}^\lambda}{\partial x^j} + \xi_1{}^\mu \frac{\partial \xi_2{}^\lambda}{\partial y^\mu} - \xi_2{}^j \frac{\partial \xi_1{}^\lambda}{\partial x^j} - \xi_2{}^\mu \frac{\partial \xi_1{}^\lambda}{\partial y^\mu} , \end{aligned}$$

cf. [**30**]. Hence we have

LEMMA 3.1. *The Lie derivative $\tilde{\text{£}}_1 \Xi_2 = [\Xi_1, \Xi_2]$ is the induced vector field on $\tilde{B}$ of the Lie derivative $\text{£}_1 \xi_2 = [\xi_1, \xi_2]$.*

Denoting by £_{12} and $\tilde{\text{£}}_{12}$ the Lie differentiations in Z and $\tilde{B}$ with respect to the vector fields $[\xi_1, \xi_2]$ and $[\Xi_1, \Xi_2]$ respectively, it follows immediately from the above lemma

THEOREM 3.2. *For the lift $\tilde{T}$ of a tensor field T, we have*

$$\tag{3.4} (\tilde{\text{£}}_1 \tilde{\text{£}}_2 - \tilde{\text{£}}_2 \tilde{\text{£}}_1) \tilde{T} = \tilde{\text{£}}_{12} \tilde{T} ,$$

and consequently for a tensor field T on Z

$$\tag{3.5} (\text{£}_1 \text{£}_2 - \text{£}_2 \text{£}_1) T = \text{£}_{12} T .$$ [9)]

Next, compute Jacobi identities

$$[\Xi_1, [\Xi_2, E_c]] - [\Xi_2, [\Xi_1, E_c]] + [E_c, [\Xi_1, \Xi_2]] = 0 ,$$

$$[\Xi_1, [\Xi_2, F_\gamma]] - [\Xi_2, [\Xi_1, F_\gamma]] + [F_\gamma, [\Xi_1, \Xi_2]] = 0 ,$$

using (2.20), (2.25) and (3.1). In the first equation we pass them to the auxiliary connection. Then we obtain

THEOREM 3.3. *We have*

$$\tag{3.6} \tilde{\mathring{\varLambda}}_{12b}{}^a{}_c = \text{£}_1 \tilde{\mathring{\varLambda}}_{2b}{}^a{}_c - \text{£}_2 \tilde{\mathring{\varLambda}}_{1b}{}^a{}_c ,$$

i. e.

$$\tag{3.7} \text{£}_{12} \Gamma^*{}_j{}^i{}_k = (\text{£}_1 \text{£}_2 - \text{£}_2 \text{£}_1) \Gamma^*{}_j{}^i{}_k ,$$ [10)]

and

$$\tag{3.8} \text{£}_{12} C_j{}^i{}_\nu = (\text{£}_1 \text{£}_2 - \text{£}_2 \text{£}_1) C_j{}^i{}_\nu .$$

(3.8) may be also obtained as an application of (3.5) on $C_j{}^i{}_\nu$.

From (1.21), (2.14), (2.20), (2.25), we obtain

9), 10) These formulas have been proved by different methods for more general geometric objects, [**16**], [**22**, p. 114], [**29**], [**30**], [**34**], [**36**, p. 28].

THEOREM 3.4. *We have*

$$
\begin{aligned}
&\tilde{\pounds}(\tilde{T}^A{}_{|c})-(\tilde{\pounds}\tilde{T}^A)_{|c}=\tilde{\Lambda}_B{}^A{}_c\tilde{T}^B-\tilde{T}^A{}_{\|\gamma}\tilde{M}^\gamma{}_\xi\tilde{\Lambda}^\xi{}_c\,,\\
&\tilde{\pounds}(\tilde{T}^A{}_{\|\gamma})-(\tilde{\pounds}\tilde{T}^A)_{\|\gamma}=(\tilde{\pounds}\tilde{C}_B{}^A{}_\gamma)\tilde{T}^B\,,
\end{aligned}
\tag{3.9}
$$

where

$$\Lambda_B{}^A{}_c=\bar{\beta}_B{}^A(\boldsymbol{\Lambda}_c),\ \boldsymbol{\Lambda}_c=(\Lambda_b{}^a{}_c)\in L(\mathfrak{L}_n)\,.$$

Turning to the auxiliary connection, we obtain

COROLLARY 3.5. *We have*

$$
\begin{aligned}
&\pounds(T^I{}_{|k})-(\pounds T^I)_{|k}=(\pounds\Gamma^*{}^I{}_{Jk})T^J-\frac{\partial T^I}{\partial y^\kappa}(\pounds\Gamma^*{}^\kappa{}_k)\,,\\
&\pounds\left(\frac{\partial T^I}{\partial y^\nu}\right)-\frac{\partial}{\partial y^\nu}(\pounds T^I)=0\,;
\end{aligned}
\tag{3.10}
$$

The second of these equations means that Lie differentiation commutes with the partial differentiation with respect to the second coordinates y^ν.

Using (1.20), (1.22), (2.20), (2.25), (2.26) repeatedly in the following Jacobi identities

$$
\begin{aligned}
&[\Xi,[E_c,E_d]]+[E_c,[E_d,\Xi]]+[E_d,[\Xi,E_c]]=0\,,\\
&[\Xi,[E_c,F_\delta]]+[E_c,[F_\delta,\Xi]]+[F_\delta,[\Xi,E_c]]=0\,,\\
&[\Xi,[F_\gamma,F_\delta]]+[F_\gamma,[F_\delta,\Xi]]+[F_\delta,[\Xi,F_\gamma]]=0\,,
\end{aligned}
\tag{3.11}
$$

we find the following commutation formulas

$$
\begin{aligned}
\tilde{\Lambda}_b{}^a{}_{c|d}-\tilde{\Lambda}_b{}^a{}_{d|c}&=\tilde{\pounds}\tilde{R}^a{}_{bcd}-\Lambda_b{}^a{}_e\tilde{S}_c{}^e{}_d-(\tilde{\pounds}\tilde{C}_b{}^a{}_\zeta)\tilde{M}^\zeta{}_\varepsilon\tilde{R}^\varepsilon{}_{0cd}\\
&\quad+\tilde{P}^a{}_{bc\zeta}\tilde{M}^\zeta{}_\varepsilon\tilde{\Lambda}^\varepsilon{}_d-\tilde{P}^a{}_{bd\zeta}\tilde{M}^\zeta{}_\varepsilon\tilde{\Lambda}^\varepsilon{}_c\,,\\
\tilde{\Lambda}_b{}^a{}_{c\|\delta}-(\tilde{\pounds}\tilde{C}_b{}^a{}_\delta)_{|c}&=\tilde{\pounds}\tilde{P}^a{}_{bc\delta}-\tilde{\Lambda}_b{}^a{}_e\tilde{C}_c{}^e{}_\delta-(\tilde{\pounds}\tilde{C}_b{}^a{}_\zeta)\tilde{M}^\zeta{}_\varepsilon(\tilde{P}^\varepsilon{}_{0c\delta}+\tilde{C}^\varepsilon{}_{\delta|c})\\
&\quad-\tilde{Q}^a{}_{b\delta\zeta}\tilde{M}^\zeta{}_\varepsilon\tilde{\Lambda}^\varepsilon{}_c\,,\\
(\tilde{\pounds}\tilde{C}_b{}^a{}_\gamma)_{\|\delta}-(\tilde{\pounds}\tilde{C}_b{}^a{}_\delta)_{\|\gamma}&=\tilde{\pounds}\tilde{Q}^a{}_{b\gamma\delta}+(\tilde{\pounds}\tilde{C}_b{}^a{}_\varepsilon)(\tilde{C}_\gamma{}^\varepsilon{}_\delta-\tilde{C}_\delta{}^\varepsilon{}_\gamma)\,.
\end{aligned}
\tag{3.12}
$$

§ 4. Group of affine transformations.

In this paragraph we deal only with spaces with proper connections.

Clearly any transformation φ of X leaves invariant the vectorial form dx, and consequently the induced transformation $\tilde{\varphi}$ leaves invariant n forms $\widetilde{dx}^a$ on $\tilde{B}$. $\tilde{\varphi}$ leaves also invariant the lift $\tilde{y}$ of the intrinsic tensor field y, therefore N functions $\tilde{y}^\alpha$ on $\tilde{B}^\circ$, and furthermore $d\tilde{y}^\alpha$, because the dual induced map $\tilde{\varphi}^*$ commutes with the differential operator d.

Now denote by H the map of a given proper connection in a principal bundle $\tilde{B}^\circ$. A transformation φ on X is said to be *affine*, if the induced transformation $\tilde{\varphi}$ on $\tilde{B}^\circ$ preserves the connection H, [**24**, p. 66], i.e. the commutativity

$$\tilde{\varphi}_* \circ H_{\tilde{b}} = H_{\tilde{\varphi}(\tilde{b})} \circ \tilde{\varphi}_* \tag{4.1}$$

holds at any point $\tilde{b} \in \tilde{B}^\circ$. Then we can prove the following

THEOREM 4.1. *The induced transformation $\tilde{\varphi}$ of any transformation φ on X leaves invariant the $n+N$ forms $\widetilde{dx}^a$ and $d\tilde{y}^\alpha$. In order that φ is affine, it is necessary and sufficient that the induced transformation $\tilde{\varphi}$ leaves invariant the n^2 forms $\tilde{\omega}_b{}^a$; then $\tilde{\varphi}$ leaves also invariant the N forms $\widetilde{Dy}^\alpha$.*

PROOF. The second part is proved as follows. We may regard a point $\tilde{b} \in \tilde{B}^\circ$ as an admissible map $L_n \to \tilde{B}^\circ$. Let $I_{\tilde{b}}$ be the identity map of $T_{\tilde{b}}(\tilde{B}^\circ)$ onto itself at $\tilde{b}$. Then the map H is related to the form $\tilde{\omega}$ by $H_{\tilde{b}} = I_{\tilde{b}} - \tilde{b}_* \circ \tilde{\omega}_{\tilde{b}}$. Therefore, if φ is an affine transformation, then, from the commutativity (4.1), we have $\tilde{\varphi}_* \circ (I_{\tilde{b}} - \tilde{b}_* \circ \tilde{\omega}_{\tilde{b}}) = (I_{\tilde{\varphi}(\tilde{b})} - \tilde{\varphi}(\tilde{b})_* \circ \tilde{\omega}_{\tilde{\varphi}(\tilde{b})}) \circ \tilde{\varphi}_*$. Since $\tilde{\varphi}_* \circ \tilde{b}_* = \tilde{\varphi}(\tilde{b})_*$, we have $\tilde{\varphi}(\tilde{b})_* \circ \tilde{\omega}_{\tilde{b}} = \tilde{\varphi}(\tilde{b})_* \circ \tilde{\omega}_{\tilde{\varphi}(\tilde{b})} \circ \tilde{\varphi}_*$ or $\tilde{\omega}_{\tilde{b}} = \tilde{\omega}_{\tilde{\varphi}(\tilde{b})} \circ \tilde{\varphi}_*$, from which follows the relation $\tilde{\varphi}^* \tilde{\omega} = \tilde{\omega}$ as an $L(L_n)$-valued form $\tilde{\omega}$ on $\tilde{B}^\circ$. The last part follows from $\tilde{\varphi}^*(\widetilde{D}\tilde{y}^\alpha) = \tilde{\varphi}^* H^* d\tilde{y}^\alpha = H^* \tilde{\varphi}^* d\tilde{y}^\alpha = H^* d\tilde{y}^\alpha = \widetilde{D}\tilde{y}^\alpha$.

In virtue of the duality, it follows at once

THEOREM 4.2. *In order that a transformation φ on X is affine, it is necessary and sufficient that the induced transformation $\tilde{\varphi}$ leaves invariant all the basic vector fields of the first and the second kinds and the fundamental vector fields.*

Now let us prove the following theorem

THEOREM 4.3. *The group G of all affine transformations on X with a proper linear connection is a Lie group.*[11)]

Let $L_n{}^+$ be the subgroup of L_n of matrices of positive determinants and $\hat{X}$ the right factor space $B/L_n{}^+$ of the principal bundle B by $L_n{}^+$. $\hat{X}$ is a double covering of X. We denote the covering map $\hat{X} \to X$ by κ and the non-identical covering transformation of $\hat{X}$ by ε; $\kappa \circ \varepsilon = \kappa$. Let $\hat{Z}$ be the induced tensor bundle of Z by κ, and $\hat{\varepsilon}$ the map induced from ε by κ. $\hat{Z}$ is then a double covering of Z, and $\hat{\varepsilon}$ gives the non-identical covering transformation of $\hat{Z}$. The space of the bundle B has also a structure of the principal bundle: $\{\hat{X}, L_n{}^+, L_n{}^+, \times\}$. Since $\tilde{B}$ is the bundle induced from B by the projection τ, the bundle space $\tilde{B}$ has also a structure $\{\hat{Z}, L_n{}^+, L_n{}^+, \times\}$. Therefore the original connection on Z with the components $\tilde{\omega}_b{}^a$ on $\tilde{B}$ defines a proper connection on $\hat{Z}$, and we can speak of affine transformations on $\hat{X}$. Let $\hat{G}$ be the group of all affine transformations on $\hat{X}$. Then we have clearly

LEMMA 4.4. *A transformation φ on X induces a transformation $\hat{\varphi}$ on $\hat{X}$ satisfying the commutativity condition $\hat{\varphi} \circ \varepsilon = \varepsilon \circ \hat{\varphi}$. Conversely, if a transformation*

11) This is a generalization of a theorem due to Myers-Steenrod-Nomizu-Kobayashi, [**21**], [**23**], [**17**], [**18**].

$\hat{\varphi}$ on $\hat{X}$ satisfies this condition, then $\hat{\varphi}$ can be regarded as a transformation induced from a transformation on X.

LEMMA 4.5. *If a transformation $\tilde{\varphi}$ on $\tilde{B}^{\circ}$ leaves the $n+N$ forms $\widetilde{dx}^a$ and $d\tilde{y}^{\alpha}$ invariant, then there is a transformation $\hat{\varphi}$ on $\hat{X}$ which induces $\tilde{\varphi}$.*

PROOF. Let $\tilde{\varphi}$ be represented by the following functions by use of coordinate neighborhoods of any point $\tilde{b}\in\tilde{B}^{\circ}$ and the image $\tilde{\varphi}(\tilde{b})$:

$$x'^i=\varphi^i,\ y'^{\lambda}=\psi^{\lambda},\ a'_a{}^i=\chi_a{}^i \tag{4.2}$$

depending on $x^i, y^{\lambda}, a_a{}^i$. From the invariance of $\widetilde{dx}^a=b_i{}^a dx^i$ follow the equations

$$b'_i{}^a\Big(\frac{\partial\varphi^i}{\partial x^j}dx^j+\frac{\partial\varphi^i}{\partial y^{\lambda}}dy^{\lambda}+\frac{\partial\varphi^i}{\partial a_b{}^j}da_b{}^j\Big)=b_i{}^a dx^i\,,$$

and hence we have

$$\frac{\partial\varphi^i}{\partial y^{\lambda}}=0,\ \frac{\partial\varphi^i}{\partial a_b{}^j}=0,\ \chi_a{}^i=\frac{\partial\varphi^i}{\partial x^j}a_a{}^j\,. \tag{4.3}$$

By the first two of these, the functions φ^i are independent of y^{λ} and $a_a{}^i$, and therefore $x'^i=\varphi^i$ define a transformation $\hat{\varphi}$ on $\hat{X}$, because Y° and L_n^+ are connected unless Z is a bundle of scalar densities. In a similar manner, we have equations $\psi^{\lambda}=\bar{\alpha}_{\mu}{}^{\lambda}\Big(\dfrac{\partial\varphi}{\partial x}\Big)y^{\mu}$ from the invariance of $d\tilde{y}^{\alpha}=d(b_{\lambda}{}^{\alpha}y^{\lambda})$. These equations together with the last equations of (4.3) show that the transformation $\tilde{\varphi}$ is induced from $\hat{\varphi}$ in $\tilde{B}^{\circ}$.

LEMMA 4.6. *The group $\hat{G}$ of all affine transformations on $\hat{X}$ is a Lie group.*

PROOF. In virtue of Theorem 4.1 and Lemma 4.5, $\hat{G}$ is isomorphic to the group $\tilde{G}$ of all transformations in $\tilde{B}^{\circ}$, which leave the $n+N+n^2$ linearly independent forms $\widetilde{dx}^a$, $d\tilde{y}^{\alpha}$, $\tilde{\omega}_b{}^a$ invariant. By a well known theorem due to S. Kobayashi [**17**], [**18**], $\tilde{G}$ is a Lie group and hence so is $\hat{G}$.

PROOF OF THEOREM 4.3. By Theorem 4.1 and Lemma 4.5, the group G is isomorphic to a subgroup $\{\hat{\varphi}\,|\,\hat{\varphi}\circ\varepsilon=\varepsilon\circ\hat{\varphi}\}$ of $\hat{G}$. The subgroup, being closed in $\hat{G}$, is a Lie group and consequently so is G. The proof is completed.

Now let φ_t be a 1-parameter group of affine transformations in X. Then, the induced vector field Ξ in $\tilde{B}^{\circ}$ should satisfy, by Theorem 4.2, $[\Xi, E_c]=0$ and $[\Xi, F_{\gamma}]=0$, cf. [**24**, p. 67]. From (2.20) and (2.25), it follows that equations $\tilde{A}_b{}^a{}_c=0$ and $\tilde{\mathfrak{L}}\tilde{C}_b{}^a{}_{\gamma}=0$ should hold. Thus we have

THEOREM 4.7. *In order that a one-parameter group of transformations is affine, it is necessary and sufficient that the generating vector field ξ^i satisfies the differential equations*

$$\begin{gathered}\xi^i{}_{|j}=\xi^i{}_j-S^i{}_{jk}\xi^k-C_j{}^i{}_{\pi}M^{\pi}{}_{\omega}\xi^{\omega}{}_{\rho}y^{\rho},\ \xi^i{}_{\|\mu}=C_h{}^i{}_{\mu}\xi^h\,,\\ A_j{}^i{}_k=\xi^i{}_{j|k}+R^i{}_{jkl}\xi^l+P^i{}_{jk\omega}M^{\omega}{}_{\kappa}\xi^{\kappa}{}_{\rho}y^{\rho}=0\,,\\ \mathfrak{L}C_j{}^i{}_{\nu}=\xi^i{}_{j\|\nu}-P^i{}_{jl\nu}\xi^l+Q^i{}_{j\nu\omega}M^{\omega}{}_{\kappa}\xi^{\kappa}{}_{\rho}y^{\rho}=0\,.\end{gathered} \tag{4.4}$$

We shall call a vector field ξ^i satisfying these differential equations an *infinitesimal affine transformation.*

By (2.23) and (2.24), $\Lambda_j{}^i{}_k=0$ is equivalent to $\pounds\Gamma^*{}_j{}^i{}_k=0$. Thus, turning to the auxiliary connection, we have

COROLLARY 4.8. *In order that a vector field ξ^i is an infinitesimal affine transformation, it is necessary and sufficient that ξ^i satisfies a mixed system*[12] *of differential equations*

$$\text{(4.5)}\qquad \begin{aligned} &\xi^i{}_{|j}=\mathring{\xi}^i{}_j-S^i{}_{jk}\xi^k\,,\\ &\frac{\partial\xi^i}{\partial y^\mu}=0\,,\\ &\mathring{\xi}^i{}_{j|k}=-\mathring{R}^i{}_{jkl}\xi^l-\mathring{P}^i{}_{jk\omega}\mathring{\xi}^\omega{}_\rho y^\rho\,,\\ &\frac{\partial\mathring{\xi}^i{}_j}{\partial y^\nu}=\mathring{P}^i{}_{jl\nu}\xi^l\,, \end{aligned}$$

with an associated system

$$\text{(4.6)}\qquad \pounds C_j{}^i{}_\nu=\xi^h C_j{}^i{}_{\nu|h}+\mathring{\xi}^\kappa{}_\rho y^\rho\frac{\partial C_j{}^i{}_\nu}{\partial y^\kappa}-\mathring{\xi}^i{}_h C_j{}^h{}_\nu+\mathring{\xi}^h{}_j C_h{}^i{}_\nu+\mathring{\xi}^\kappa{}_\nu C_j{}^i{}_\kappa=0\,.$$

Chapter II. Spaces of linear elements and Finsler spaces.

§ 5. Groups of affine transformations on spaces of linear elements.

Throughout this chapter, let Z be the tangent bundle $T(X)=\{X, Y^n, L_n, \tau\}$ of an n-dimensional space X, whose fibre space is an n-dimensional vector space Y^n. We have now $n=N$, and α is the identity automorphism of L_n and consequently the homomorphism $\bar{\alpha}$ is also the identity automorphism of $L(L_n)$ in the notations of the preceding chapter. A point $z\in T(X)$ will be now denoted by (x^i, y^i) in a coordinate neighborhood, whereas we have hitherto denoted it by (x^i, y^λ). We obtain formulas concerning the tangent bundle $T(X)$ in replacing simply the Greek indices in a number of formulas in the preceding chapter by Latin indices. Each formula thus obtained will be indicated with the same number as in the preceding chapter marked with an asterisk below, like $(4.4)_*$.

Now let

$$\text{(5.1)}\qquad \omega_j{}^i=\Gamma^*{}_j{}^i{}_k dx^k+C_j{}^i{}_k Dy^k$$

define a proper linear connection on $Z^\circ=T^\circ(X)$. By Lemma 1.3_*, the coefficients $\Gamma^*{}_j{}^i{}_k$ and $C_j{}^i{}_k$ are homogeneous in y^i of degree zero and -1 respectively. $C_j{}^i{}_k$ satisfy furthermore the equations

12) [8].

(5.2) $$C_j{}^i{}_k y^k = 0 .$$

Let us now seek for the condition that a space X with a proper connection admits locally a group of affine transformations of maximum order n^2+n, that is, the condition of complete integrability of $(4.4)_*$, or of $(4.5)_*$ with $(4.6)_*$. In order that $(4.5)_*$ admits n^2+n independent solutions, $(4.6)_*$ should be satisfied by any ξ^i and any $\xi^i{}_j$ and consequently we should have $C_j{}^i{}_{k|h}=0$ and

(5.3) $$\frac{\partial C_j{}^i{}_k}{\partial y^h} y^p - \delta^i_h C_j{}^p{}_k + \delta^p_j C_h{}^i{}_k + \delta^p_k C_j{}^i{}_h = 0 .$$

Contracting over k and p and taking account of (5.2), we have

(5.4) $$-C_j{}^i{}_h - \delta^i_h C_j{}^k{}_k + C_h{}^i{}_j + nC_j{}^i{}_h = 0 ,$$

and, contracting further over i and j, $C_i{}^i{}_k=0$ for $n>1$. Contracting over i and h in (5.4), we have $C_j{}^i{}_i=0$. Consequently $C_h{}^i{}_j=(1-n)C_j{}^i{}_h=(1-n)^2C_h{}^i{}_j$ and hence $C_j{}^i{}_k=0$ for $n>2$. For $n=2$, from (5.4) it follows immediately that $C_j{}^i{}_k+C_k{}^i{}_j=0$ and hence $C_1{}^i{}_1=C_2{}^i{}_2=0$. Consequently $C_i{}^i{}_k=0$ imply $C_1{}^i{}_2=-C_1{}^i{}_1=0$. Thus all $C_j{}^i{}_k$ vanish for $n\geqq 2$. Moreover, by Corollary 3.5_* and $(1.23)_*$, we have

$$\frac{\partial}{\partial y^l} \pounds \Gamma^*{}^i_{jk} = \pounds \frac{\partial \Gamma^*{}_j{}^i{}_k}{\partial y^l} = \pounds \mathring{P}^i{}_{jkl} ,$$

and, from the integrability conditions of $(4.5)_*$,

$$\pounds P^i{}_{jkl} = \xi^h P^i{}_{jkl|h} + \xi^h{}_p y^p \frac{\partial P^i{}_{jkl}}{\partial y^h} - \xi^i{}_h P^h{}_{jkl} + \xi^h{}_j P^i{}_{hkl} + \xi^h{}_k P^i{}_{jkl} + \xi^h{}_l P^i{}_{jkl} = 0 .$$

Since, for the complete integrability, these equations should be satisfied by any ξ^i and $\xi^i{}_j$, we have equations

$$\frac{\partial P^i{}_{jkj}}{\partial y^h} y^p - \delta^i_h P^p{}_{jkl} + \delta^p_j P^i{}_{hkl} + \delta^p_k P^i{}_{jhl} + \delta^p_l P^i{}_{jkh} = 0 .$$

Contracting over h and p and considering the homogeneity of $P^i{}_{jkl}$ of degree -1 in y^i, we have at once $P^i{}_{jkl}=\partial \Gamma^*{}_j{}^i{}_k/\partial y^l=0$, i. e., $\Gamma^*{}_j{}^i{}_k$ are independent of y^i. Hence, by a well-known theorem [**8**, p. 234], [**16**], [**34** p. 20], [**36**, pp. 94 and 194], we obtain

THEOREM 5.1. *A necessary and sufficient condition that a space of dimension $n\geqq 2$ with a proper linear connection admits a group of affine transformations of maximum order n^2+n is that the space is locally affinely flat.*

§ 6. Groups of motions in Finsler spaces.

A *Finsler space* X is a metric space in which the length of a tangent vector $z=(x^i, y^i)$ of X at a point x is given by

(6.1) $$|z|^2=2F(x, y)=g_{ij}(x, y)y^i y^j$$

or the arc length s of a curve by

$$s=\int\sqrt{2F(x, dx)}=\int\sqrt{g_{ij}dx^i dx^j}\,,$$

where the *fundamental function* $F(x, y)$ is a function of x^i and y^i, homogeneous in y^i of degree two and positive valued for any non-zero vector $z=(x^i, y^i)$, $(y^i)\neq 0$. $F(x, y)$ may be regarded as a function defined on $T^\circ(X)$. The functions

(6.2) $$g_{ij}(x, y)=\frac{\partial^2 F}{\partial y^i \partial y^j}$$

are homogenous of degree zero in y^i and constitute the components of a symmetric tensor field on $T^\circ(X)$ associated with $\tilde{B}^\circ$. The tensor field g_{ij} is supposed to be positive definite and called the *metric tensor* of the Finsler space. E. Cartan [**5**, pp. 10–16] has introduced a metric connection in such spaces. We notice that this connection is proper in our sense and has torsion $C_{ijk}=\frac{1}{2}\partial^3 F/\partial y^i \partial y^j \partial y^k$.

A transformation φ on a space X with Finsler metric is called a *motion* if the induced transformation φ_* in $T^\circ(X)$ leaves the fundamental function F invariant: $F(\varphi_*(z))=F(z)$ for any z.

Suppose now that a Finsler space admits an effective group G of motions of order r, and denote the motion corresponding to an element $g\in G$ by φ_g. Taking an arbitrary point $x_0\in X$, let G_0 be the isotropic subgroup of G at x_0 and denote the induced map of φ_g at x_0 on $T_{x_0}(X)$ by $(\varphi_{g*})_{x_0}$. G_0 is of order $r'\geqq \max(r-n, 0)$. Then the map γ, defined by $\gamma(g)=(\varphi_{g*})_{x_0}$, $g\in G_0$, is a linear representation of G_0 into L_n.

In determining the Finsler space with completely integrable equations of Killing, H. C. Wang [**32**] proved, by a group-theoretic method, the following lemmas and theorem.

Lemma W. 1. *The linear representation γ is faithful.*

Lemma W. 2. *$\gamma(G_0)$ is conjugate to a subgroup O' of the orthogonal group O_n in L_n.*

Theorem W. 3.[13] *If an n-dimensional Finsler space, $n\neq 4$, admits an effective group of motions of order $r>n(n-1)/2+1$, the space is a Riemannian space of constant curvature.*

We are now going to determine n-dimensional Finsler spaces admitting a group G of motions of order $r=n(n-1)/2+1$ for $n\neq 4$.

13) This theorem can be also proved by tensor calculas in the same way as we shall prove Theorem 8.1.

First we notice the following

LEMMA 6.1. *If a transformation group G operates transitively on a space X and $\gamma(G_0)$ at a point x_0 is conjugate to a subgroup O' of the orthogonal group O_n, then we can introduce in X a Riemannian metric with respect to which G is a group of motions.*

PROOF. By assumption $\gamma(G_0)$ leaves invariant a Euclidean metric on the tangent space $T_{x_0}(X)$ at x_0, and the Riemannian metric defined from the Euclidean metric by the transitivity of G on X is a required one. q. e. d.

Combining Lemma 6.1 with Lemma W. 2, we can state the following

PRINCIPLE 6.2. *The problem to determine Finsler spaces admitting a transitive group G of motions is reduced to determining Riemannian spaces admitting the group G as group of motions and to finding, in these spaces, Finsler metrics which are left invariant under G.*

Reterning to our problem, let q' be the dimension of the orbit of a point x by G and $q=\max\{q' \mid x \in X\}$; q is usually called the *generic rank* of the group G. Now suppose G is intransitive on $X, q<n$, and we shall show that this leads to a contradiction under our hypothesis. The order of the isotropic subgroup G_0 of G at any point x_0 would be equal to $r'=r-q' \geqq r-q>(n-1)\cdot(n-2)/2$, and so the order of $\gamma(G_0)$ must be also equal to r' by Lemma W. 1. Then the subgroup O' of the orthogonal group O_n, to which $\gamma(G_0)$ is conjugate in L_n, have to coincide with the orthogonal group O_n itself for $n \neq 4$ [**20**, Lemma 4]. By the same argument as in the proof of Theorem W. 3, we can see that the space must be then Riemannian. In virtue of a lemma due to K. Yano [**51**], the group G of order $n(n-1)/2+1$ must be then transitive on X; this contradicts our assumption. Thus G is transitive on X.

Hence, by Lemma 6.1, there exists a Riemannian metric on X with respect to which G is a group of motions. K. Yano [**51**, Theorem 9] showed that an n-dimensional Riemannian space ($n \neq 4$) admitting a group of motions of order $n(n-1)/2+1$ is one of the following:

(I) the product of a straight line and an $(n-1)$-dimensional Riemannian space of constant curvature,

(II) a space of negative constant curvature.

We have to find Finsler metrics in such spaces which are invariant under G.

Case (I). There exists a coordinate neighborhood of X where r generating vector fields of G are given by

$$
\begin{aligned}
& X_1, \\
(6.3) \qquad & \left(1-\frac{K}{4} v\right) X_a+\frac{K}{2} x^a x^b X_b, \qquad (a, b=2,3, \cdots, n) \\
& x^a X_b-x^b X_a,
\end{aligned}
$$

where K is the constant curvature of the $(n-1)$-dimensional space and $v=\sum_{a=2}^{n} x^a x^a$. The corresponding vector fields of the first extended group in $T(X)$ are given by

$$
\begin{aligned}
&X_1\,,\\
(6.4)\qquad &\Big(1-\frac{K}{4}v\Big)X_a+\frac{K}{2}x^a x^b X_b+\frac{K}{2}(y^a x^b+x^a y^b)Y_b-\frac{K}{2}x^b y^b Y_a\,,\\
&x^a X_b-x^b X_a+y^a Y_b-y^b Y_a\,.
\end{aligned}
$$

The fundamental function F is an absolute invariant of the extended group, that is, all the derivatives of F with respect to the vector fields (6.4) vanish. Therefore we see that F is independent of x^1, and that F contains the variables x^a and y^a $(a=2,\cdots,n)$ only as a function of $v=\sum_{a=2}^{n} x^a x^a$, $w=\sum_{a=2}^{n} y^a y^a$ and $t=\sum_{a=2}^{n} x^a y^a$. Hence F may be regarded as a function of y^1, v, w, t: $F=F(y^1, v, w, t)$. We assume that F is differentiable in y^1, v, w, t. F satisfies moreover

$$2\Big(1+\frac{K}{4}v\Big)x^a\frac{\partial F}{\partial v}+Kwx^a\frac{\partial F}{\partial w}+\Big\{\Big(1+\frac{K}{4}v\Big)y^a+\frac{K}{2}tx^a\Big\}\frac{\partial F}{\partial t}=0\,.$$

Multiplying these equations by x^a and y^a and summing up with respect to $a=2,\cdots,n$ we have

$$2\Big(1+\frac{K}{4}v\Big)v\frac{\partial F}{\partial v}+Kvw\frac{\partial F}{\partial w}+\Big\{\Big(1+\frac{K}{4}v\Big)t+\frac{K}{2}tv\Big\}\frac{\partial F}{\partial t}=0\,,$$

$$2\Big(1+\frac{K}{4}v\Big)t\frac{\partial F}{\partial v}+Ktw\frac{\partial F}{\partial w}+\Big\{\Big(1+\frac{K}{4}v\Big)w+\frac{K}{2}t^2\Big\}\frac{\partial F}{\partial t}=0\,,$$

from which we have

$$\Big(1+\frac{K}{4}v\Big)(t^2-vw)\frac{\partial F}{\partial t}=0\,.$$

Hence $\partial F/\partial t=0$ in our domain except in the surface $x^2=\cdots=x^n, y^2=\cdots=y^n$. However, since we have supposed F to be differentiable, we have $\partial F/\partial t=0$ in all our domain. Hence F is a solution of a differential equation

$$\Big(1+\frac{K}{4}v\Big)\frac{\partial F}{\partial v}+\frac{K}{2}w\frac{\partial F}{\partial w}=0\,.$$

Since $w\Big/\Big(1+\frac{K}{4}v\Big)^2$ is a soluton of this equation, a general fundamental function is given by a function

$$F=f\left((y^1)^2,\ \frac{w}{\Big(1+\frac{K}{4}v\Big)^2}\right),\tag{6.5}$$

where $f(t_1, t_2)$ is a homogeneous function of degree one in two variables t_1 and t_2.

Case (II). There exists a coordinate system of X where r vector fields generating G are given by

$$-\frac{1}{k}X_1+x^bX_b\,,\qquad X_a\,,\qquad x^aX_b-x^bX_a\,, \qquad (a, b=2, 3,\cdots, n) \tag{6.6}$$

where $-k^2$ is the negative constant curvature of the space X. The vector fields of the extended group in $T(X)$ of G are

$$-\frac{1}{k}X_1+x^bX_b+y^bY_b\,,\qquad X_a\,,\qquad x^aX_b-x^bX_a+y^aY_b-y^bY_a\,. \tag{6.7}$$

By the same reason as in the former case, we see that the fundamental function F is independent of x^a, and that F contains y^a only as a function of $w=\sum_{a=2}^{n} y^ay^a$, so that F may be expressed as $F=F(x^1, y^1, w)$. We assume again the differentiability of $F(x^1, y^1, w)$. F satisfies moreover a differential equation

$$-\frac{1}{k}\frac{\partial F}{\partial x^1}+2w\frac{\partial F}{\partial w}=0\,.$$

Since $e^{2kx^1}w$ is a solution of this equation, a general fundamental function is given by a function

$$F=f((y^1)^2,\ e^{2kx^1}w)\,, \tag{6.8}$$

where f has the same property as in Case (I). Concluding the above discussions, we can state the following

THEOREM 6.3. *A necessary and sufficient condition that an n-dimensional Finsler space* ($n\neq 4$) *admits a group of motions of order* $n(n-1)/2+1$ *is that the fundamental function F is given by either* (6.5) *or* (6.8) *in a suitable coordinate system.*

Chapter III. Spaces of hyperplane elements and Cartan spaces.

§ 7. Space of hyperplane elements.

In this chaptar, we take, as Z in Chapter I the cotangent space, $T^*(X)=\{X, Y^{*n}, L_n, \tau\}$ of a space X, where Y^{*n} is the dual vector space to Y^n. We have again $n=N$ in the notations of Chapter I, but now the automorphism

α of L_n maps each matrix to the transpose of its inverse, and consequently $\bar{\alpha}$ has the effect

$$\bar{\alpha}_i{}^j(\boldsymbol{g}) = -g_i{}^j \tag{7.1}$$

for any element $\boldsymbol{g}=(g_j{}^i)\in L(L_n)$. A point $z\in T^*(X)$ will be denoted by (x^i, u_i). In replacing suitably the indices—in replacing the Greek indices by Latin ones and interchanging the super- and subscripts—and changing occasionally the signs, we obtain formulas on $T^*(X)$ from some formulas of Chapter I. We shall indicate these formulas by asterisk placed upward like (1.3)*.

We obtain in particular as (1.3)* and (1.5)*,

$$\omega_j{}^i = \Gamma^*{}_j{}^i{}_k dx^k + C_j{}^{ik} Du_k\,, \tag{7.2}$$

$$\bar{\alpha}_i{}^j(\omega) = -\omega_i{}^j,\ \bar{\alpha}_i{}^j(\Gamma_k) = -\Gamma_i{}^j{}_k,\ \bar{\alpha}_i{}^j(C^k) = -C_i{}^{jk}\,, \tag{7.3}$$

The covariant derivatives of a tensor field T^I are given as (1.16)* by

$$\begin{aligned} T^I{}_{|k} &= \frac{\partial T^I}{\partial x^k} + \frac{\partial T^I}{\partial u_h}\Gamma^*{}_{hk} + \Gamma^*{}_J{}^I{}_k T^J\,, \\ T^I{}^{\|k} &= \frac{\partial T^I}{\partial u_k} + C_J{}^{Ik} T^J\,, \end{aligned} \tag{7.4}$$

and the Lie derivative of T^I are given as (2.14)* by

$$\pounds T^I = T^I{}_{|h}\xi^h - T^I{}^{\|l} M^h{}_l \xi^p{}_h u_p - \xi^I{}_J T^J\,, \tag{7.5}$$

where

$$\xi^i{}_j = \frac{\partial \xi^i}{\partial x^j} + \Gamma^*{}_j{}^i{}_k \xi^k - C_j{}^{ik}\Big(\frac{\partial \xi^l}{\partial x^k} + \Gamma^*{}_k{}^l{}_p \xi^p\Big) u_l\,, \tag{7.6}$$

which is obtained as (2.11)*. In particular, for the intrinsic covariant vector field $u=(u_j)$ on $T^*(X)$, we have

$$\begin{aligned} &u_{j|k}=0,\quad u_j{}^{\|k} = \delta_j^k - C_j{}^k,\quad C_j{}^k = C_j{}^{ik} u_i\,, \\ &\pounds u_j = 0\,. \end{aligned} \tag{7.7}$$

By Lemma 1.3*, a linear connection (7.1) is proper if and only if $\Gamma^*{}_j{}^i{}_k$ and $C_j{}^{ik}$ are homogeneous in u_j of degree zero and minus one respectively and $C_j{}^{ik}$ satisfy

$$C_j{}^{ik} u_k = 0\,. \tag{7.8}$$

Then Lemma 1.4* shows that the curvature tensors $R^i{}_{jkl}$, $P^i{}_{jk}{}^l$, $Q^i{}_j{}^{kl}$ are homogeneous in u_j of degree 0, -1, -2 resepctively and the last two satisfy

$$P^i{}_{jk}{}^l u_l = 0,\quad Q^i{}_j{}^{kl} u_l = 0\,. \tag{7.9}$$

By the same argument as in the proof of Theorem 5.1, we can prove the following

THEOREM 7.1. *A space of hyperplane elements with a proper connection admits a group of affine transformations of maximum order $n+n^2$ if and only* *if it is affinely flat.*

§ 8. Groups of motions in Cartan spaces.

A *Cartan space* is the space in which the volume of a hypersurface is given by $(n-1)$-ple integral

$$\int L(x, \mathfrak{u})dv\,, \tag{8.1}$$

where $\mathfrak{u}=(\mathfrak{u}_j)$ is the covariant vector density field of weight -1 tangent to the hypersurface, $L(x, \mathfrak{u})$ a scalar function of x^i and $\mathfrak{u}_j$, which is positive homogeneous in $\mathfrak{u}_j$ of degree one, and dv the volume element of the hypersurface.

Now let $\mathfrak{L}(x, u)$ be the function of the point $z=(x, u)\in T^*(X)$ defined as follows. In each coordinate system (x, u) are given by their components (x^i, u_j). Let $\mathfrak{u}$ be the vector density which has the same components u_j as u in the same coordinate system. Then we put

$$\mathfrak{L}(x^i, u_j)=L(x, \mathfrak{u})\,.$$

$\mathfrak{L}(x^i, u_j)$ is no longer a scalar function on $T^*(X)$, but it is easily shown that $\mathfrak{L}(x^i, u_j)$ is the component of a scalar density defined on $T^*(X)$ of weight -1, positive homogeneous in u_j of degree 1 and positive valued for $u\neq 0$. It is obvious that such scalar densities $\mathfrak{L}(x, u)$ is in one-one corespondence with scalar functions $L(x, \mathfrak{u})$. We put $\mathfrak{F}=\frac{1}{2}\mathfrak{L}^2$ and call $\mathfrak{F}$ the *fundamental scalar density.*

Then the contravariant metric tensor field g^{ij} is defined by

$$g^{ij}=\frac{1}{\mathfrak{g}}\frac{\partial^2\mathfrak{F}}{\partial u_i\partial u_j}\,, \tag{8.2}$$

where

$$\mathfrak{g}=\left(\det\left|\frac{\partial\mathfrak{F}}{\partial u_i\partial u_j}\right|\right)^{\frac{1}{n-1}}. \tag{8.3}$$

g^{ij} are homogeneous in u_j of degree zero and the determinant $|g^{ij}|$ is equal to $1/\mathfrak{g}$. The covariant metric tensor g_{ij} is the reciparocal system of g^{ij}. From (8.2) and the homogeneity of $\mathfrak{F}$, we have

$$g^{ij}u_j=\frac{1}{\mathfrak{g}}\frac{\partial\mathfrak{F}}{\partial u_i}\,,\quad g^{ij}u_iu_j=\frac{2\mathfrak{F}}{\mathfrak{g}}\,. \tag{8.4}$$

A linear connection on $T^*(X)$ is said to be *metric* if the covariant derivatives of the metric tensor with respect to it vanish:

$$(8.5)\qquad \begin{aligned} g_{ij|k} &= \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ij}}{\partial u_h}\Gamma^*_{hk} - \Gamma^*{}_i{}^h{}_k g_{hj} - \Gamma^*{}_i{}^h{}_k g_{ih} = 0\,, \\ g^{ij}\|^k &= \frac{\partial g^{ij}}{\partial u_k} + g^{ih}C_h{}^{jk} + g^{hj}C_h{}^{ik} = 0\,. \end{aligned}$$

A remarkable metric connection is characterized by conditions [4], [9, p. 21 and supplément]

$$(8.6)\qquad \Gamma^*{}_j{}^i{}_k = \Gamma^*{}_k{}^i{}_j, \quad C^{ijk} = C^{jik}\,,$$

where $C^{ijk} = g^{ih}C_h{}^{jk}$. The second coefficients C^{ijk} of this metric connection are uniquely determined by the fundamental scalar density $\mathfrak{F}$, as follows: From the second equations of (8.5) and (8.6), we have

$$(8.7)\qquad C^{ijk} = -\frac{1}{2}\frac{\partial g^{ij}}{\partial u_k} = \frac{1}{2}g^{ij}\frac{1}{\mathfrak{g}}\frac{\partial \mathfrak{g}}{\partial u_k} - \frac{1}{2}\frac{1}{\mathfrak{g}}\frac{\partial^3 \mathfrak{F}}{\partial u_i \partial u_j \partial u_k}\,.$$

Putting $C^k = g_{ij}C^{ijk} = C_i{}^{ik}$, we have

$$(8.8)\qquad C^k = -\frac{1}{2}g_{ij}\frac{\partial g^{ij}}{\partial u_k} = \frac{1}{2}\frac{1}{\mathfrak{g}}\frac{\partial \mathfrak{g}}{\partial u_k}$$

and

$$(8.9)\qquad C^{ijk} = g^{ij}C^k - \frac{1}{2}\frac{1}{\mathfrak{g}}\frac{\partial^3 \mathfrak{F}}{\partial u_i \partial u_j \partial u_k}\,.$$

From these equations, we have further

$$(8.10)\qquad C_j{}^{ik}u_i = u_j C^k, \quad C^{ijk}u_k = 0, \quad C^k u_k = 0\,,$$

$$(8.11)\qquad C_k{}^{ik} = (2-n)C^i\,,$$

and

$$\begin{aligned} \det|\delta_j^k - C_j{}^{ik}u_i| &= \det|\delta_j^k - u_j C^k| \\ &= 1 - u_j C^j = 1\,, \end{aligned}$$

that is, the matrix $(\delta_j^k - C_j{}^k)$ is unimodular. Hence the inverse matrix $(M_j{}^i)$ of $(\delta_j^k - C_j{}^k)$ is equal to $(\delta_j^k - u_j C^k)$.

Moreover we can see that, if a symmetric tensor

$$(8.12)\qquad \begin{aligned} H^{ij} &= g^{ij} + 2\frac{\mathfrak{F}}{\mathfrak{g}}C_k C^{kij} \\ &= g^{ij} + 2\frac{\mathfrak{F}}{\mathfrak{g}}C^i C^j - \frac{\mathfrak{F}}{\mathfrak{g}^2}C_k\frac{\partial^3 \mathfrak{F}}{\partial u_i \partial u_j \partial u_k} \end{aligned}$$

is of rank n, then the first coefficients $\Gamma^*{}_j{}^i{}_k$ of the metric connection satisfying (8.6) are completely determined. A Cartan space is said to be *regular* if the symmetric tensor H^{ij} has the rank n. We shall deal only with regular Cartan spaces in the following. We notice that the metric connection satisfying (8.6) of a regular Cartan space is proper in our sense. The tensor H^{ij} satisfies

(8.13) $$H^{ij}u_j = g^{ij}u_j\,.$$

A transformation φ on X is called a *motion* of the Cartan space if the induced transformation φ^* on $T^*(X)$ leaves invariant the fundamental scalar density $\mathfrak{F}$. In order that a vector field ξ^i on X is an infinitesimal motion it is necessary and sufficient that ξ^i satisfy an equation

(8.14) $$\pounds\mathfrak{F}=0\,.$$

Then, by Lemma 3.5*, $\pounds\mathfrak{g}=0$ and $\pounds g^{ij}=0$. Conversely, if $\pounds g^{ij}=0$, then $\pounds\mathfrak{g}=0$ and, by Lemma 2.3* and (8.4), we have $\pounds\mathfrak{F}=0$. Hence the equation (8.14) is equivalent to

(8.15) $$\pounds g^{ij} = -(\xi^{ij}+\xi^{ji})=0\,,$$

where we have put $\xi^{ij}=\xi^i{}_h g^{hj}$. (8.15) are Killing equations in the Cartan space.

Moreover, applying Theorem 3.4* on g_{ij}, we obtain

$$g_{ih}A_j{}^h{}_k + g_{hj}A_i{}^h{}_k = 0\,,$$

or, substituting (2.23)*,

$$g_{ih}\pounds\Gamma^*{}_j{}^h{}_k + g_{jh}\pounds\Gamma^*{}_i{}^h{}_k - 2C_{ij}{}^h\pounds\Gamma^*_{hk} = 0\,.$$

Since $\Gamma^*{}_j{}^i{}_k$ are symmetric in j and k, we obtain equations

(8.16) $$\pounds\Gamma^*{}_j{}^i{}_k = C_j{}^{ih}\pounds\Gamma^*{}_{hk} + C_k{}^{ih}\pounds\Gamma^*{}_{hj} - C_{jk}{}^h g^{il}\pounds\Gamma^*{}_{hl}\,.$$

Contracting these equations by u_i, we have

$$\pounds\Gamma^*{}_{jk} = u_j C^h\pounds\Gamma^*{}_{hk} + u_k C^h\pounds\Gamma^*{}_{hi} - C_{jk}{}^h u^l\pounds\Gamma^*{}_{hl}\,,$$

and, contracting further by $u^k = g^{ki}u_i$,

$$u^k\pounds\Gamma^*{}_{jk} = u_k u^k C^h\pounds\Gamma^*{}_{hj} = 2\,\frac{\mathfrak{F}}{\mathfrak{g}}\,C^h\pounds\Gamma^*{}_{hj}\,.$$

From the last two equations, we have

(8.17) $$\pounds\Gamma^*{}_{jk} + \Big(2\,\frac{\mathfrak{F}}{\mathfrak{g}}\,C_{lk}{}^h - u_j\delta^h_k - u_k\delta^h_j\Big)C^l\pounds\Gamma^*{}_{hl} = 0$$

and, contracting by C^k,

(8.18) $$\Big(g^{ij} + 2\,\frac{\mathfrak{F}}{\mathfrak{g}}\,C^{jki}C_k - u^iC^j\Big)C^l\pounds\Gamma^*{}_{jl} = 0\,.$$

Since we have

$$\begin{aligned}\det\left|g^{ij}+2\,\frac{\mathfrak{F}}{\mathfrak{g}}\,C^{jki}C_k - u^iC^j\right| &= \det|H^{ij}-u^iC^j|\\ &=(1-u_iC^i)\det|H^{ij}|\\ &=\det|H^{ij}|\neq 0\end{aligned}$$

because of the regularity of the space, we have $C^l\pounds\Gamma^*{}_{jl}=0$. From (8.17) and

(8.16), we obtain $\pounds\Gamma^{*}{}_{j}{}^{i}{}_{k}=0$, which are equivalent to equations

$$A_{j}{}^{i}{}_{k}=\xi^{i}{}_{j|k}+R^{i}{}_{jkl}\xi^{l}-P^{i}{}_{jk}{}^{l}M^{h}{}_{l}\xi^{p}{}_{h}u_{p}=0\,.$$

by (2.23)* and (2.24)*.

Now the complete integrability condition of the Killing equations (8.15) is equivalent, by Theorem 4.7*, to that of a mixed system of linear differential equations

$$\begin{aligned}
&\xi^{i}{}_{|j}=\xi^{i}{}_{j}+C_{j}{}^{il}M^{h}{}_{l}\xi^{p}{}_{h}u_{p}\,,\\
&\xi^{i}\|^{j}=C_{h}{}^{ij}\xi^{h}\,,\\
&\xi^{i}{}_{j|k}=-R^{i}{}_{jkl}\xi^{l}+P^{i}{}_{jk}{}^{l}M^{h}{}_{l}\xi^{p}{}_{h}u_{p}\,,\\
&\xi^{i}{}_{j}\|^{k}=P^{i}{}_{jl}{}^{k}\xi^{l}+Q^{i}{}_{j}{}^{kl}M^{h}{}_{l}\xi^{p}{}_{h}u_{p}
\end{aligned}$$

with (8.15) as the associated system.

From the complete integrability of this mixed system follows that the equations

$$\pounds C^{ijk}=C^{ijk}{}_{|h}\xi^{h}-C^{ijk}\|^{l}M^{h}{}_{l}\xi^{p}{}_{h}u_{p}-\xi^{i}{}_{h}C^{hjk}-\xi^{j}{}_{h}C^{ihk}-\xi^{k}{}_{h}C^{ijh}=0 \tag{8.19}$$

should be satisfied by any ξ^{i} and any ξ^{ij} satisfying (8.15). Therefore $C^{ijk}{}_{|h}=0$ and

$$\begin{aligned}
&C^{ijk}\|^{l}M^{m}{}_{l}g_{mh}u_{p}+\delta^{i}_{p}C_{h}{}^{jk}+\delta^{j}_{p}C_{h}{}^{ik}+\delta^{k}_{p}C^{ij}{}_{h}\\
&\quad=C^{ijk}\|^{l}M^{m}{}_{l}g_{mp}u_{h}+\delta^{i}_{h}C_{p}{}^{ij}+\delta^{j}_{h}C_{p}{}^{ik}+\delta^{k}_{h}C^{ij}{}_{p}
\end{aligned}$$

should hold. Contracting k and p and taking account of (7.7), (8.10), (8.11), we have

$$(n-2)C^{ij}{}_{h}+C_{h}{}^{ji}+C_{h}{}^{ij}=C^{ij}{}_{k}\|^{l}M^{k}{}_{l}u_{h}+(2-n)(\delta^{i}_{h}C^{j}+\delta^{j}_{h}C^{i}) \tag{8.20}$$

and, contracting further by g_{ij},

$$(n-2)C_{h}=C_{k}\|^{l}M^{k}{}_{l}u_{h}\,.$$

Since $u_{h}C^{h}=0$ and $u_{h}u^{h}=2\mathfrak{F}/\mathfrak{g}\neq 0$, we obtain $C_{k}\|^{l}M^{k}_{l}=0$ and hence $C_{h}=0$ for $n>2$. Thus C^{ijk} are symmetric in all indices and by (8.20) we have

$$nC^{ijh}=C^{ij}{}_{k}\|^{l}M^{k}{}_{l}u^{h}\,.$$

Since $C^{ijh}u_{h}=0$, we have $C^{ij}{}_{k}\|^{l}M^{k}{}_{l}=0$ and consequently $C^{ijk}=0$, which means that g^{ij} are independent of u_{j}, that is, the metric is Riemannian. Thus we have established

THEOREM 8.1. *If a regular Cartan space admits a group of motions of maximum order $n(n+1)/2$ for $n>2$, then it is a Riemannian space of constant curvature.*

§ 9. Cartan spaces admitting a group of motions of order $r \geqq n(n-1)/2+1$.

We shall prove in this last paragraph an analogue of Theorem 6.3 on Cartan spaces. However we must impose a condition on the Cartan space; i.e., we suppose that the representation $\gamma: G_0 \to L_n$ of the isotropic subgroup G_0 of motions at any point $x \in X$ is faithful. In the case of Finsler spaces, the corresponding proposition (Lemma W. 1. in § 6) is proved by means of the normal coordinates, on which we have no knowledge in Cartan spaces. The author has the conjecture that this condition might be satisfied by any Cartan space, but he has not been successful in proving it. So we must restrict ourselves to the consideration of regular Cartan spaces satisfying this condition.

We define a correspondence η of the cotangent space $T^*_x(X)$ to the tangent space $T_x(X)$ at each point $x \in X$ by equations

$$y^i = g^{ij} u_j. \tag{9.1}$$

This correspondence is a differentiable homeomorphism of $T^{*\circ}(X)$ onto $T^\circ(X)$, because

$$\left|\frac{\partial y^i}{\partial u_j}\right| = |g^{ij} - 2C^{ikj} u_k| = |g^{ij} - 2u^i C^j| = |g^{ij}| \neq 0$$

at any point. If we define a scalar function

$$F(x, y) = \frac{\mathfrak{F}(x, \eta^{-1}(y))}{\mathfrak{g}(x, \eta^{-1}(y))},$$

then F is homogeneous of degree 2 in y^i and we may consider a Finsler space with F as fundamental function and having the same base space X. If φ is a motion of the Cartan space, then g^{ij} is invariant under φ^* and we have $\varphi_* \circ \eta = \eta \circ \varphi^*$. Moreover $\mathfrak{F}$ and $\mathfrak{g}$ are invariant under φ^* and hence F is also invariant under φ_*. Therefore, by Lemma W. 2 in § 6, $\gamma(G_0)$ is conjugate to a subgroup O' of the rotation group O_n in L_n.

If the group G of motions of a Cartan space is of order $r > n(n-1)/2+1$, then $\gamma(G_0)$ is conjugate to the subgroup O_n in L_n, and hence is of order $n(n-1)/2$ and so G_0 has also the same order. Therefore, at x_0, the equations (8.19) with $\xi^i(x_0) = 0$ should be satisfied by arbitrary ξ^{ij} satisfying $\xi^{ij} + \xi^{ji} = 0$. Repeating the discussion at the end of the previous paragraph, we have $C^{ijk} = 0$ at x_0 for $n = 3$ or $n \geqq 5$. The point x_0 being arbitrary, we have $C^{ijk} = 0$ on X, and see that the space is Riemannian and consequently of constant curvature.

If the group G is of order $r = n(n-1)/2+1$, then under our assumption at the bigining of this paragraph, we can verify that G acts transitively on X.

Hence, by the same reasoning as in § 6, we obtain an exact analogue of Principle 6.2, and we can determine Cartan spaces admitting a group of motions of order $n(n-1)/2+1$ as follows.

We can find a coordinate neighborhood in X where r generating vector fields of G are given by (6.3) or (6.6) as in Case (I) or (II) of § 6.

Case (I). Corresponding to (6.3) the generating vector fields of the extended group of G in $T^*(X)$ are given by

$$X_1\,,$$
$$\left(1-\frac{K}{4}v\right)X_a+\frac{K}{2}x^a x^b X_b+\frac{K}{2}(u_a x^b-x^a u_b)U^b-\frac{K}{2}x^b u_b U^a\,,$$
$$x^a X_b-x^b X_a-u_b U^a+u_a U^b\,, \qquad (a, b=2, 3, \cdots, n)$$

where $U^a=\partial/\partial u_a$ and $v=\sum_{a=2}^{n} x^a x^a$. Since the fundamental scalar density $\mathfrak{F}$ of weight two is invariant under the extended group, it is a solution of differential equations

$$\frac{\partial\mathfrak{F}}{\partial x^1}=0\,,$$

$$\left(1-\frac{K}{4}v\right)\frac{\partial\mathfrak{F}}{\partial x^a}+\frac{K}{2}x^a x^b\frac{\partial\mathfrak{F}}{\partial x^b}+\frac{K}{2}(u_a x^b-x^a u_b)\frac{\partial\mathfrak{F}}{\partial u_b} -\frac{K}{2}x^b u_b\frac{\partial\mathfrak{F}}{\partial x_a}+K(n-1)x^a\mathfrak{F}=0\,, \tag{9.2}$$

$$x^a\frac{\partial\mathfrak{F}}{\partial x^b}-x^b\frac{\partial\mathfrak{F}}{\partial x^a}-u_b\frac{\partial\mathfrak{F}}{\partial u_a}+u_a\frac{\partial\mathfrak{F}}{\partial u_b}=0\,.$$

It follows that $\mathfrak{F}$ is independent of x^1, and that $\mathfrak{F}$ contains the variables x^a and u_a only as a function of $v=\sum_{a=2}^{n} x^a x^a$, $w^*=\sum_{a=2}^{n} u_a u_a$ and $t^*=\sum_{a=2}^{n} x^a u_a$. So we may express $\mathfrak{F}=\mathfrak{F}(u^1, v, w^*, t^*)$. We assume the differentiability of $\mathfrak{F}(u^1, v, w^*, t^*)$. From the second equations of (9.2), we can see that $\mathfrak{F}$ is independent of t^* as in Case (I), § 6, and $\mathfrak{F}$ is a solution of a differential equation

$$\left(1+\frac{K}{4}v\right)\frac{\partial\mathfrak{F}}{\partial v}-\frac{K}{2}w^*\frac{\partial\mathfrak{F}}{\partial w^*}+\frac{K}{2}(n-1)\mathfrak{F}=0\,.$$

Since $w^*\Big/\left(1+\frac{K}{4}v\right)^{2n-4}$ is a solution of this equation, a general fundamental scalar density is given by

$$\mathfrak{F}=f\left((u_1)^2,\quad w^*\Big/\left(1+\frac{K}{4}v\right)^{2n-4}\right), \tag{9.3}$$

where $f(t_1, t_2)$ is a homogeneous function of degree one in two variables t_1 and t_2.

Case (II). $\mathfrak{F}$ is a solution of differential equations

$$
\begin{aligned}
&-\frac{1}{k}\frac{\partial\mathfrak{F}}{\partial x^1}+x^b\frac{\partial\mathfrak{F}}{\partial x^b}-u_b\frac{\partial\mathfrak{F}}{\partial u_b}+2(n-1)\mathfrak{F}=0\,, \\
&\frac{\partial\mathfrak{F}}{\partial x^a}=0\,, \qquad (9.4)\\
&x^a\frac{\partial\mathfrak{F}}{\partial x^b}-x^b\frac{\partial\mathfrak{F}}{\partial x^a}-u_b\frac{\partial\mathfrak{F}}{\partial u_a}+u_a\frac{\partial\mathfrak{F}}{\partial u_b}=0\,.
\end{aligned}
$$

It follows that $\mathfrak{F}$ is independent of x^a, and that $\mathfrak{F}$ contains u_a only as a function of $w^*=\sum_{a=2}^{n}u_a u_a$. Consequently we may express $\mathfrak{F}=\mathfrak{F}(x^1, u_1, w^*)$. From the first equation of (9.4), $\mathfrak{F}$ is a solution of a differential equation

$$\frac{1}{k}\frac{\partial\mathfrak{F}}{\partial x^1}+2w^*\frac{\partial\mathfrak{F}}{\partial w^*}-2(n-1)\mathfrak{F}=0\,.$$

Since $e^{2k(n-2)x^1}w^*$ is a solution of this equation, a general fundamental scalar density $\mathfrak{F}$ is given by

$$\mathfrak{F}=f((u_1)^2,\quad e^{2k(n-2)x^1}w^*)\,. \tag{9.5}$$

Gathering the above results, we have established

THEOREM 9.1. *Under the assumption at the biginning of this paragraph, a necessary and sufficient condition that an n-dimensional Cartan space* ($n\geqq 3, n\neq 4$) *admits a group of motions of order* $n(n-1)/2+1$ *is that the fundamental scalar densitv* $\mathfrak{F}$ *is given by either* (9.3) *or* (9.5) *in a suitable coordinate system.*

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