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On the groups of C. Chevalley.

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Let $\mathfrak{g}$ be a semi-simple Lie algebra over the field C of complex numbers. Then there is a basis of $\mathfrak{g}$ such that every structure constant of $\mathfrak{g}$ with respect to this basis is an integer. This basis generates a Lie algebra $\mathfrak{g}_Z$ over the ring of integers Z. For any field K, the tensor product $\mathfrak{g}_K=\mathfrak{g}_Z\otimes K$ has a structure of the Lie algebra over K. Chevalley has constructed a group G_K which is a subgroup of the group $A(\mathfrak{g}_K)$ of automorphisms of $\mathfrak{g}_K$ and he has proved that, if $\mathfrak{g}$ is simple, the commutator subgroup G_K' of G_K is simple (except for a few exceptional cases). (cf. Chevalley [**3**])

In this note, we shall consider some properties of the groups G_K and determine the automorphisms of G_K with some restrictions on the characteristic of the ground field. Namely, let $\mathfrak{g}$ be a simple Lie algebra over C and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by P the additive group generated by the weights (with respect to $\mathfrak{h}$) of all representations of $\mathfrak{g}$, and by P_r the subgroup of P generated by the roots of $\mathfrak{g}$. Then P/P_r is a finite group isomorphic to the center of the compact simply connected Lie group whose Lie algebra is a real form of $\mathfrak{g}$.[1)] In § 1, we shall introduce the group G_K constructed in Chevalley [**3**], and show that if $\mathfrak{g}$ is simple and K is an infinite field whose characteristic is $\neq 2, 3$ and not a factor of $\mathrm{ord}(P/P_r)$, then G_K is a simple algebraic group whose Lie algebra is $\mathrm{ad}\,\mathfrak{g}_K$ which is isomorphic to $\mathfrak{g}_K$. In § 2, we shall consider the case where $\mathfrak{g}$ is one of the simple Lie algebras of the main four types and show that G_K' is isomorphic to a classical simple group. In §§ 3, 4, we shall consider the birational and biregular automorphisms of the simple algebraic groups G_K of § 1 and show that every such automorphism of G_K is inner except for the type (D_4), and for the type (D_4), the factor group of the group of automorphisms by its normal subgroup of inner automorphisms is the cyclic group of order 3. The automorphisms of the classical groups has been treated by J. Dieudonné [**4**], [**5**] and also that of the unimodular group (the type A) and the symplectic group (the

1) P/P_r is the cyclic group of order $l+1$ if $\mathfrak{g}$ is of the type (A_l), the cyclic group of order 2 if $\mathfrak{g}$ is one of the types (B_l), (C_l) or (E_7), the cyclic group of order 4 if $\mathfrak{g}$ is the type (D_l), $l\geqq 4$ and l is even, the direct product of two cyclic groups of order 2 if $\mathfrak{g}$ is of the type (D_l), $l>4$ and l is odd, the cyclic group of order 3 if $\mathfrak{g}$ is of the type (E_6), and has a unit element only if $\mathfrak{g}$ is one of the types (G_2), (F_4) or (E_8).

type C) has been determined by L. K. Hua and I. Reiner [**6**], [**7**] and [**11**]. Our method is entirely different from that of J. Dieudonné or L. K. Hua and I. Reiner's, and allows to treat the various types of simple groups uniformly. However, in the classical case, if we observe the correspondence between classical simple groups and groups of Chevalley, there are some analogies in the methods of L. K. Hua and I. Reiner's and ours, and as for the automorphisms of $P\Omega_n(K,f)$, our result gives a partial solution (i. e. for the case where f is a quadratic form of maximal index) of a problem which J. Dieudonné has left open (cf. 6° of the last section of [**4**]).

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§ 1. Simple Lie algebras and the groups of Chevalley.

Let $\mathfrak{g}$ be a semi-simple Lie algebra over the field C of complex numbers, and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We call co-weight the elements H of $\mathfrak{h}$ such that $w(H)\in Z$ (the ring of integers) for all $w\in P$. They form a free abelian group $\mathfrak{H}$ of rank l where l is the rank of $\mathfrak{g}$. We denote by $B(X, Y)$ $=\mathrm{Trace}(\mathrm{ad}\,X\cdot \mathrm{ad}\,Y)$, $X, Y\in\mathfrak{g}$, the Killing form on $\mathfrak{g}$. Since the restriction of B to $\mathfrak{h}$ is non-degenerate, for any root r there is a unique element H_r' of $\mathfrak{h}$, such that $B(H_r', H)=r(H)$ for all $H\in\mathfrak{h}$. Then $H_r=2r(H_r')^{-1}H_r'$ is a co-weight and $r(H_r)=2$. The H_r, for all root r, generate the group $\mathfrak{H}$, and there is a system $\{X_r\}$ of root vectors satisfying the following conditions:

(1) $\quad [X_r, X_{-r}]=H_r \qquad$ for all root r,

(2) $\quad [X_r, X_s]=N_{r,s}X_{r+s} \qquad$ if r, s and $r+s$ are roots where $N_{r,s}=\pm(p+1)$ and p is the greatest integer $i\geqq 0$ such that $s-ir$ is a root.[2)]

Let $F: (a_1, a_2,\cdots, a_l)$ be a fundamental root system (f. r. s.) of $\mathfrak{g}$. Set $a_{ij}=-a_i(H_{a_j})$ and $A=(a_{ij})$. Then A is a (l, l)-matrix with integer coefficients such that[3)]

(3) $\quad a_{ii}=-2,\ a_{ij}\geqq 0$ if $i\neq j$, and $a_{ij}=0$ whenever $a_{ji}=0$ $(1\leqq i, j\leqq l)$,

(4) $\quad 0\leqq a_{ij}a_{ji}\leqq 3 \qquad$ if $i\neq j$

(5) $\quad \det A=\mathrm{ord}(P/P_r)$.

We can associate to F a diagram formed by dots and lines defined as follows:

i) The dots of the diagram and the roots of F correspond one to one.

ii) Two dots are not connected if and only if the corresponding roots

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2) Chevalley [**3**, § I, Theorem 1].

3) Chevalley [**3**, § I, IV and V]. The matrices A for any simple Lie algebras can be determined from their root diagrams. (As for the forms of the matrices A, we shall find them in the proof of Proposition 6 in § 2.) If we calculate their determinants, we may have (5) immediately.

r, s of F are orthogonal, i. e. $r(H_s)=0$.

iii) Two dots are connected by a single line if $r(H_s)s(H_r)=1$.

iv) Two dots are connected by a double line with a direction from r to s if $r(H_s)s(H_r)=2$ and $r(H_s)=-2$.

v) Two dots are connected by a triple line with a direction from r to s if $r(H_s)s(H_r)=3$ and $r(H_s)=-3$.

The matrix A determines the root diagram completely and the structure of the diagram depends only on the algebra $\mathfrak{g}$. Reciprocally, the algebra $\mathfrak{g}$ is determined up to an isomorphism by the diagram associated to $\mathfrak{g}$.

If $\mathfrak{g}$ is simple, the diagram associated to it, whose dots we denote by $(S_1, S_2, \cdots, S_l)$, is one of the following:

a) For the type (A_l): S_{i-1} and S_i $(2 \leqq i \leqq l)$ are connected by a single line.

b) For the type (B_l): S_{i-1} and S_i $(2 \leqq i \leqq l-1)$ are connected by a single line, and S_{l-1} and S_l are connected by a double line with a direction from S_{l-1} to S_l.

c) For the type (C_l): S_{i-1} and S_i $(2 \leqq i \leqq l-1)$ are connected by a single line, and S_{l-1} and S_l are connected by a double line with a direction from S_l to S_{l-1}.

d) For the type (D_l): S_{i-1} and S_i $(2 \leqq i \leqq l-1)$, S_{l-2} and S_l are connected by a single line.

e) For the type (E_l), $l=6, 7$ and 8: S_{i-1} and S_i $(i=2, 3, 4$ and $\geqq 6)$, S_3 and S_5 are connected by a single line.

f) For the type (F_4): S_1 and S_2, S_3 and S_4 are connected by a single line and S_2 and S_3 are connected by a double line with a direction from S_2 to S_3.

g) For the type (G_2): S_1 and S_2 are connected by a triple line with a direction from S_1 to S_2.

In the following paragraphs we shall consider the roots of the f. r. s. $(a_1, a_2, \cdots, a_l)$ of the simple Lie algebra numbered so that the diagram of $\mathfrak{g}$ is one of the above by the correspondence $a_i \rightarrow S_i$ $(1 \leqq i \leqq l)$.

We denote by $\mathfrak{g}_Z$ the additive group generated by a basis $(H_1, H_2, \cdots, H_l)$ of $\mathfrak{h}$ and X_r, for all root r. Let K be an arbitrary field. Then the tensor product $\mathfrak{g}_K = \mathfrak{g}_Z \otimes K$ has a structure of Lie algebra over K. We set $\mathfrak{h}_K = \mathfrak{h} \otimes K$, and the elements $H_r \otimes 1_K$, $H_i \otimes 1_K$, $X_r \otimes 1_K$ and $X \otimes 1_K$, where 1_K is the unit element of K and $X \in \mathfrak{g}_Z$, we denote again by H_r, H_i, X_r and X. Then the set H_i $(1 \leqq i \leqq l)$ and X_r (r root) forms a basis of $\mathfrak{g}_K$, which we call a canonical basis.

Lemma 1. *If the characteristic of the field K is not a factor of* $\mathrm{ord}(P/P_r)$, *then* $\mathrm{ad}\, H_{a_i}$ $(1 \leqq i \leqq l)$ *are linearly independent over K and they form a basis of* $\mathrm{ad}\, \mathfrak{h}_K$.

Proof. From the hypothesis of the Lemma, the matrix A is non-singular in K. Set $B=(b_{ij})$ the inverse matrix of the transpose of A and

$$H_i^* = -\sum_{j=1}^{l} b_{ij} H_{a_j} \qquad (1 \leqq i \leqq l)\,. \tag{6}$$

Then $H_i^* \in \mathfrak{h}_K$. For any root $r = \sum_{i=1}^{l} m_i(r) a_i$, we have

$$r(H_i^*) = -\sum_{j=1}^{l} b_{ij} r(H_{a_j}) = -\sum_{j,k=1}^{l} b_{ij} m_k(r) a_k(H_{a_j}) = \sum_{j,k=1}^{l} b_{ij} a_{kj} m_k(r) = m_i(r)\,.$$

Thus

$$(\mathrm{ad}\, H_i^*) X_r = [H_i^*, X_r] = r(H_i^*) X_r = m_i(r) X_r\,. \tag{7}$$

Therefore we may see that the diagonal matrix $\mathrm{ad}\, H_i^*$ $(1 \leqq i \leqq l)$ are linearly independent and also we have our assertion. q. e. d.

Proposition 1. *If the characteristic of the field K is not a factor of* $\mathrm{ord}(P/P_r)$ *and* $\neq 2$, *then* $\mathfrak{g}_K$ *has no center.*

Proof. Since $[H_r, X_r] = r(H_r) X_r$, $r(H_r) = 2$, the operation $\mathrm{ad}\, X_r$ is not null, and we have from Lemma 1, $\mathrm{ad}\, \mathfrak{h}_K$ is isomorphic to $\mathfrak{h}_K$, so the kernel of the adjoint representation of $\mathfrak{g}_K$ is null. q. e. d.

Lemma 2. *If r and s are roots, not all of the following are roots:*

$$r-2s, \quad r-s, \quad r, \quad r+s, \quad r+2s\,.$$

Therefore the absolute values of integers $N_{r,s}$ *of* (2) *do not exceed* 4.

Proof. If all the above are roots, then $2s = (r+2s) - r$ and $2(r+s) = (r+2s) + r$ are not roots. From this we have $(r+2s)(H_r) = 0$.[4] Therefore $r(H_r) = -2s(H_r)$. Similarly $(r-2s) \pm r$ are not roots, and $(r-2s)(H_r) = 0$, i. e. $r(H_r) = 2s(H_r)$. Thus $r(H_r) = 0$, and this is a contradiction. The second assertion is an immediate consequence. q. e. d.

Lemma 3. *If* $\mathfrak{g}$ *is simple, for any two linearly independent roots* r, s, *there exist a f. r. s.* F *and a finite series* $(r_1, r_2, \cdots, r_h)$ *of roots such that* $r = r_1 \in F$, $r_i - r_{i-1}$ are roots $(2 \leqq i \leqq h)$ and $s = r_h$.

Proof. There is a f. r. s. F such that F contains r and that s is a positive root with respect to F.[5] Then there is a finite series $(s_1, s_2, \cdots, s_j)$ of roots such that $s_1 \in F$, $s_i - s_{i-1} \in F$ $(2 \leqq i \leqq j)$ and $s_j = s$.[6] Since r_1 and s_1 are in F and $\mathfrak{g}$ is simple, there is a subset $(u_1, u_2, \cdots, u_k)$ of F such that, in the corresponding diagram of F, dots of u_i and u_{i+1} $(1 \leqq i \leqq k-1)$ are connected by a line and $u_1 = r_1$, $u_k = s_1$. Set $t_i = u_1 + u_2 + \cdots + u_i$ $(1 \leqq i \leqq k)$, $t_{i+k} = u_{i+1} + u_{i+2} + \cdots + u_k$ $(1 \leqq i \leqq k-1)$. Then the series $(t_1, \cdots, t_{2k-2}, s_1, \cdots, s_j)$ has the required property of the Lemma. q. e. d.

Proposition 2. *If* $\mathfrak{g}$ *is simple and the characteristic of the field K is not* 2 *or* 3, *then* $\mathfrak{g}_K$ *is simple over its center.*

4) Chevalley [**3**, § I, VII].
5) Chevalley [**3**, § I, Lemma 1].
6) Chevalley [**3**, § I, Lemma 4].

Proof. Let L be an extension field of K, then $\mathfrak{g}_L=\mathfrak{g}_K\otimes L$. So it is sufficient to see the proposition when the field K is an infinite field.

Let $\mathfrak{a}$ be an ideal containing the center of $\mathfrak{g}_K$, we shall first show that there is a root r such that $X_r\in\mathfrak{a}$. Let $X=H+\sum_r c(r)X_r$, $H\in\mathfrak{h}_K$, be an element of $\mathfrak{a}$, not in the center. If $[X, H_s]=0$ for all root s, then $X=H$, an element of $\mathfrak{h}_K$. Since H is not in the center, there is a root r such that $r(H)\neq 0$. Then $[H, X_r]=r(H)X_r$ and $X_r\in\mathfrak{a}$. If $[X, H_s]\neq 0$ for some root s, then there is a root r such that $c(r)\neq 0$, and since K is infinite, there is an element $H\in\mathfrak{h}_K$ such that $r(H)\neq u(H)$ for all root u not equal to r. Then

$$[\cdots[[X, \underbrace{H]H]\cdots H]}_{k}\sum_r c(r)r(H)^k X_r\in\mathfrak{a}\,.$$

Since the Vandermonde's determinant is not 0, we have $X_r\in\mathfrak{a}$.

Let s be any root linearly independent to r, then by Lemma 3, there is a f. r. s. F and a finite series $(r_1, r_2,\cdots, r_h)$ of roots such that $r_1\in F$, $a(i)=r_i-r_{i-1}$ are roots $(2\leqq i\leqq h)$ and $r_h=s$. Then

$$[\cdots[[X_r, X_{a(2)}]X_{a(3)}]\cdots X_{a(h)}]=cX_s\,,$$

where $c\neq 0$ from Lemma 2 and the condition of the characteristic of K. Therefore $X_s\in\mathfrak{a}$, $[X_s, X_{-s}]=H_s\in\mathfrak{a}$. Thus $X_r, H_r\in\mathfrak{a}$ for all root r and $\mathfrak{a}=\mathfrak{g}_K$. q. e. d.

Theorem 1. *If $\mathfrak{g}$ is simple and the characteristic of the field K is not 2 or 3 and is not a factor of* $\mathrm{ord}(P/P_r)$, *then $\mathfrak{g}_K$ is also simple.*

This follows from Propositions 1 and 2.

Let χ be a homomorphism of P_r into the multiplicative group K^* of the non-zero elements of K. Denote by $h(\chi)$ the automorphism of $\mathfrak{g}_K$ such that $H\to H$ for all $H\in\mathfrak{h}_X$ and $X_r\to\chi(r)X_r$ for all root r. Denote by $\mathfrak{H}_K$ the group formed by $h(\chi)$ for all $\chi\in\mathrm{Hom}(P_r, K^*)$ and by $\mathfrak{H}_K'$ the subgroup of $\mathfrak{H}_K$ formed by $h(\chi)$ for all χ of $\mathrm{Hom}(P_r, K^*)$ which can be extended to the homomorphism of P into K^*.

Proposition 3. *If K is an infinite field whose characteristic is not a factor of* $\mathrm{ord}(P/P_r)$, *then $\mathfrak{H}_K$ is an irreducible algebraic group, whose Lie algebra is* $\mathrm{ad}\,\mathfrak{h}_K$.

Proof. The $\mathfrak{H}_K$ is an irreducible algebraic group of dimension l.[7] Let χ_i be an element of $\mathrm{Hom}(P_r, K^*)$ such that $\chi_i(a_i)=t$, $t\in K^*$, and $\chi_i(a_j)=1$ for $i\neq j$, then the matrix of $h(\chi_i)$ with respect to the canonical basis of $\mathfrak{g}_K$ is the diagonal matrix of degree n (where n is the dimension of $\mathfrak{g}_K$), whose diagonal elements are 1_K, l times, and $t^{m_i(r)}$ $(r=\sum_{i=1}^{l} m_i(r)a_i)$, for all root r, and we denote it by $h_i(t)$. Then $h_i(t)$, $t\in K^*$, form 1-dimensional irreducible subgroup $\mathfrak{H}_i$ of

7) Ono [**8**, Proposition 5].

$\mathfrak{H}_K$. From (7), the matrix of ad H_i^* is diagonal and whose diagonal elements are $0, l$ times, and $m_i(r)$, for all root r. If T is a transcendental element over K, $h_i(T)$ is a generic point of $\mathfrak{H}_i$ over K. Let D_i be a derivation of $L=K(T)$ such that $D_i(T)=T$, then $D_i h_i(T)=(\mathrm{ad}\, H_i^*)h_i(T)$. Thus the Lie algebra of $\mathfrak{H}_i$ is generated by ad H_i^*.[8] Since $\mathfrak{H}_i \subseteqq \mathfrak{H}_K$, ad H_i^* is in the Lie algebra of $\mathfrak{H}_K$ for all i, and ad H_i^* $(1 \leqq i \leqq l)$ generate the l-dimensional Lie algebra ad $\mathfrak{h}_K$. Therefore the Lie algebra of $\mathfrak{H}_K$ is ad $\mathfrak{h}_K$. q. e. d.

For any root r, set $x_r(t)=\exp t(\mathrm{ad}\, X_r)$, $t \in C$. Then there is a matrix $A_r(T) =(A_{r,ij}(T))$, whose coefficients are the polynomials of T with integer coefficients such that the matrix of $x_r(t)$ with respect to the canonical basis of $\mathfrak{g}$ is $A_r(t)$ for all $t \in C$. For $t \in K$, we denote also by $x_r(t)$ the automorphism of $\mathfrak{g}_K$, which is represented by $A_r(t)$ with respect to the canonical basis of $\mathfrak{g}_K$. Denote by $\mathfrak{X}_r$ the group formed by $x_r(t)$, $t \in K$, and by $\mathfrak{U}_K$ (resp. $\mathfrak{B}_K$) the subgroup of $A(\mathfrak{g}_K)$ generated by $\mathfrak{X}_r$, where r runs over all the positive (resp. negative) roots with respect to the regular order of P_r defined by the f. r. s. $(a_1, \cdots, a_l)$.[9] Then we have

PROPOSITION 4. *If K is an infinite field,* $\mathfrak{U}_K$ (resp. $\mathfrak{B}_K$) *is an irreducible algebraic group, whose Lie algebra is* ad $\mathfrak{u}_K$ (resp. ad $\mathfrak{v}_K$) *where* $\mathfrak{u}_K$ (resp. ad $\mathfrak{v}_K$) *is the nilpotent subalgebra of $\mathfrak{g}_K$ generated by X_r, for all positive* (resp. *negative*) *root r.*

PROOF. The $\mathfrak{U}_K$ and $\mathfrak{B}_K$ are irreducible algebraic groups of dimension N, where N is the number of the positive roots.[10] If T is a transcendental element over K, $x_r(T)=(A_{r,ij}(T))$ is a generic point of $\mathfrak{X}_r$ over K. Let D_r be a derivation of $L=K(A_{r,ij}(T))=K(T)$ with respect to T, then we can easily see that $D_r x_r(T)=(\mathrm{ad}\, X_r)x_r(T)$. So the Lie algebra of $\mathfrak{X}_r$ is generated by ad X_r.[8] Therefore the Lie algebra of $\mathfrak{U}_K$ (resp. $\mathfrak{B}_K$) contains ad $\mathfrak{u}_K$ (resp. ad $\mathfrak{v}_K$), and comparing the dimensions of $\mathfrak{U}_K$ and ad $\mathfrak{u}_K$, we have our assertion. q. e. d.

We denote by G_K the subgroup of $A(\mathfrak{g}_K)$ generated by $\mathfrak{H}_K$ and $\mathfrak{X}_r$ for all root r, and by G_K' the subgroup of G_K generated by $\mathfrak{X}_r$ for all root r. Then $\mathfrak{H}_K' \subseteqq G_K'$ and

(8) $$G_K/G_K' \cong \mathfrak{H}_K/\mathfrak{H}_K', \qquad G_K' \cap \mathfrak{H}_K = \mathfrak{H}_K' \text{ [11]},$$

(9) $$h(\chi)x_r(t)h(\chi)^{-1} = x_r(\chi(r)t) \qquad \text{for all root } r, \text{ and } t \in K.$$

If $\mathfrak{g}$ is simple and K is not a finite field of two or three elements, G_K' is the commutator subgroup of G_K and is a simple group.[12]

8) Chevalley [**2**, Proposition 4, p. 132].
9) Chevalley [**3**, p. 20].
10) Ono [**8**, Propositions 2 and 4].
11) Chevalley [**3**, § IV, Lemmas 2 and 4].
12) Chevalley [**3**, § IV, Corollary to Theorem 3].

We call an algebraic group G is simple, when G has no normal algebraic sub-group except for G itself and the group formed by the unit element only. Then we have

THEOREM 2. *If* $\mathfrak{g}$ *is simple and* K *is an infinite field of characteristic* $\neq 2, 3$ *and not a factor of* $\mathrm{ord}(P/P_r)$, *then* G_K *is an irreducible simple algebraic group, whose Lie algebra is* $\mathrm{ad}\,\mathfrak{g}_K$, *which is isomorphic to* $\mathfrak{g}_K$.

PROOF. G_K is an irreducible algebraic group of dimension n, n being the dimension of $\mathfrak{g}_K$.[13] The Lie algebra of G_K contains $\mathrm{ad}\,\mathfrak{g}_K$, whose dimension is equal to n. Therefore the Lie algebra of G_K is $\mathrm{ad}\,\mathfrak{g}_K$. Since $\mathrm{ad}\,\mathfrak{g}_K \cong \mathfrak{g}_K$ is simple, G_K is a simple algebraic group. q. e. d.

For any root r, there is a homomorphism ϕ_r of $SL_2(K)$ onto a subgroup of $A(\mathfrak{g}_K)$, such that $\phi_r\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}=x_{-r}(t)$, $\phi_r\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}=x_r(t)$ for all $t\in K$, and $\phi_r\begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix}$ $=h(\chi_r)$ for all $z\in K^*$, where χ_r is an element of $\mathrm{Hom}(P_r, K^*)$ such that $\chi_r(s)$ $=z^{s(H_r)}$ for all root s.[14] (N. B. In particular if $\mathfrak{g}$ is of the type (A_1), G_K' is the homomorphic image of $SL_2(K)$ by ϕ_r, and isomorphic to $PSL_2(K)$.) We denote by $\omega_r=\phi_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and by $\mathfrak{W}$ the group generated by $\mathfrak{H}_K$ and ω_r for all root r. Then there is one and only one epimorphism ζ of $\mathfrak{W}$ onto the Weyl group W such that if $\omega\in\mathfrak{W}$ and $w=\zeta(\omega)$,

(10) $$\omega X_r \omega^{-1}=X_{w(r)} \qquad \text{for all root } r,$$

(11) $$\omega h(\chi)\omega^{-1}=h(\chi') \qquad \text{for all } \chi\in\mathrm{Hom}(P_r, K^*)$$

where $\chi'\in\mathrm{Hom}(P_r, K^*)$ is defined by $\chi'(r)=\chi(w^{-1}(r))$ for all root r.[15] We denote by $\omega(w)$ the element of $\mathfrak{W}$ such that $\zeta(\omega(w))=w$. If $w\in W$, we denote by E_w' (resp. E_w'') the set of positive root r such that $w(r)>0$ (resp. $w(r)<0$), and denote $(\mathfrak{U}_K)_w'$ (resp. $(\mathfrak{U}_K)_w''$) the group generated by $\mathfrak{X}_r$ for all root $r\in E_w'$ (resp. E_w''). Then G_K is the union of the sets $\mathfrak{U}_K\mathfrak{H}_K\omega(w)(\mathfrak{U}_K)_w''$, for all $w\in W$, and any element s of G_K can be expressed uniquely in the following form

(12) $$s=xh\omega(w)x'', \quad x\in\mathfrak{U}_K, \quad h\in\mathfrak{H}_K, \quad x''\in(\mathfrak{U}_K)_w'', \quad w\in W.$$ [16]

§ 2. Groups of Chevalley constructed from simple Lie algebras of main four types.

We denote by $\mathfrak{gl}(n, K)$ the Lie algebra of all (n, n)-matrix with coefficients in K, and by $GL(n, K)$ the group of all non-sigular (n, n)-matrix with coefficients in K. The notation of the classical groups we refer to J. Dieudonné [**5**].

13) Ono [**8**, Theorem 2 and Proposition 7, Corollary 1].
14) Chevalley [**3**, p. 36-37].
15) Chevalley [**3**, Lemma 3, p. 37].
16) Chevalley [**3**, § III, Theorem 2].

Denote by I_n the unit matrix of degree n, by E_{ij} the (n, n)-matrix whose (i, j)-component is 1_K, and all other component is zero, and by $\mathrm{diag}(z_1, \cdots, z_n)$ the diagonal matrix whose (i, i)-component is z_i, $(1 \leqq i \leqq n)$.

Let $\mathfrak{g}$ be a Lie algebra formed by the matrices $X \in \mathfrak{gl}(n, C)$ such that

a) $\quad \mathrm{Trace}\, X = 0, \quad n = l+1$

or such that ${}^tXJ + JX = 0$, where

$$(13) \quad \text{b)} \quad J = \begin{pmatrix} 1 & & \\ & & I_l \\ & I_l & \end{pmatrix}, \quad n = 2l+1,$$

$$\text{c)} \quad J = \begin{pmatrix} & I_l \\ -I_l & \end{pmatrix}, \quad n = 2l,$$

$$\text{d)} \quad J = \begin{pmatrix} & I_l \\ I_l & \end{pmatrix}, \quad n = 2l,$$

and let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ formed by the matrices:

a) $\quad H = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{l+1})$,

b) $\quad H = \mathrm{diag}(1, \lambda_1, \lambda_2, \cdots, \lambda_l, -\lambda_1, -\lambda_2, \cdots, -\lambda_l)$,

c,) d) $\quad H = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_l, -\lambda_1, -\lambda_2, \cdots, -\lambda_l)$,

If we denote by $\lambda_i = \lambda_i(H)$, then λ_i is a linear form on $\mathfrak{h}$.

a) $\quad \lambda_i - \lambda_j$, $(i \neq j, 1 \leqq i, j \leqq l+1)$; $a_i = \lambda_i - \lambda_{i+1}$ $(1 \leqq i \leqq l)$,

b) $\quad \pm\lambda_i \pm \lambda_j$, $\pm\lambda_i$, $(i \neq j, 1 \leqq i, j \leqq 1)$; $a_i = \lambda_i - \lambda_{i+1}$ $(1 \leqq i \leqq l-1)$, $a_l = \lambda_l$,

c) $\quad \pm\lambda_i \pm \lambda_j$, $\pm 2\lambda_i$ $(i \neq j, 1 \leqq i, j \leqq l)$; $a_i = \lambda_i - \lambda_{i+1}$ $(1 \leqq i \leqq l-1)$, $a_l = 2\lambda_l$,

d) $\quad \pm\lambda_i \pm \lambda_j$ $(i \neq j, 1 \leqq i, j \leqq l)$; $a_i = \lambda_i - \lambda_{i+1}$ $(1 \leqq i \leqq l-1)$, $a_l = \lambda_{l-1} + \lambda_l$,

are complete set of roots of $\mathfrak{g}$ with a f. r. s. $(a_1, a_2, \cdots, a_l)$ and the following are corresponding root vectors

a) $\quad X_{\lambda_i - \lambda_j} = E_{ij}$,

b) $\quad X_{\lambda_i + \lambda_j} = E_{i+l+1, j+1} - E_{j+l+1, i+1}, \qquad X_{\lambda_i - \lambda_j} = E_{i+1, j+1} - E_{j+l+1, i+l+1}$,

(14) $\quad X_{\lambda_i} = E_{1, i+l+1} - E_{i+1, 1}$,

c) $\quad X_{\lambda_i + \lambda_j} = E_{i+l, j} + E_{j+l, i}, \qquad X_{\lambda_i + \lambda_j} = E_{ij} - E_{j+l, i+l}, \qquad X_{2\lambda_i} = E_{i, i+l}$,

d) $\quad X_{\lambda_i + \lambda_j} = E_{i+l, j} - E_{j+l, i}, \qquad X_{\lambda_i - \lambda_j} = E_{ij} - E_{j+l, i+l}$,

where $i < j$, $1 \leqq i, j \leqq l$ and $X_{-r} = {}^tX_r$ for all root r. Denote by H_i the element of $\mathfrak{h}$ such that $\lambda_j(H_i) = \delta_{ij}$ $(1 \leqq i, j \leqq l)$, then $H_1, \cdots, H_l, X_r$ for all root r form a basis of $\mathfrak{g}$. For any field K, $H_1 \otimes 1_K, H_2 \otimes 1_K, \cdots, H_l \otimes 1_K, X_r \otimes 1_K$ for all root r which we denote again by $H_1, \cdots, H_l, X_r$ generate a Lie algebra $\mathfrak{g}_K^*$ over K. (N. B. $\mathfrak{g}_K^*$ is different from $\mathfrak{g}_K$ of § 1. cf. proof of Theorem 3.) We also denote by 1 the unit element 1_K of K.

We assume that the characteristic of K is not 2 for b) and d). Denote $x(t, r) = \exp tX_r$ for $t \in K$ and root r, then

$x(t, r)=I_n+tX_r$ for all root r except for $r=\pm\lambda_i$ of b),

$x(t, r)=I_n+tX_r+(t^2/2)X_r^2$ for $r=\pm\lambda_i$ of b).

Moreover denote by $h(z, i)$ $(z\in K^*, 1\leqq i\leqq l)$ the diagonal matrix whose (k, k)-component is $1_K, z$ or z^{-1} according as the (k, k)-component of H_i is 0, 1_K or -1_K. If we denote by G_K^* the subgroup of $GL(n, K)$ formed by those elements s such that $\det s=1$ for a) and ${}^tsJs=J$ for b), c), d) in addition $\det s=1$ for b) and c), then G_K is the group $SL_{l+1}(K), O_{2l+1}^+(K, f), Sp_{2l}(K)$ or $O_{2l}^+(K, f)$ respectively for a), b), c) or d), where f is a quadratic form of maximal index. If K is infinite, G_K^* is an algebraic group, whose Lie algebra is $\mathfrak{g}_K^*$.

PROPOSITION 5. *The matrices $h(z, i)$, $1\leqq i\leqq l, z\in K^*$, and $x(t, r)$, r: root, $t\in K$, generate the group G_K^*, where we assume that the characteristic of K is not* 2 *for b) and d).*

PROOF. We shall prove the Proposition by induction with respect to the rank l of $\mathfrak{g}$. First we shall prove for $l=1$ in (i). Next, assuming that the Proposition holds for the rank less than l, we shall prove for the rank l in (ii).

For a); (i) Let $s=(a_{ij})\in SL_2(K)$, then we may suppose $a_{11}\neq 0$. In fact, if $a_{11}=0$, then $a_{12}\neq 0$ and the (1,1)-component of

$$s\cdot x(-1, \lambda_1-\lambda_2)x(1, \lambda_2-\lambda_1)x(-1, \lambda_1-\lambda_2)$$

is not zero. Thus we have

$$s=x(a_{21}a_{11}^{-1}, \lambda_2-\lambda_1)h(a_{11}, 1)x(a_{12}a_{11}^{-1}, \lambda_1-\lambda_2)\,.$$

(ii) Let $s=(a_{ij})\in SL_{l+1}(K)$, then we may suppose that $a_{11}\neq 0$. In fact, if $a_{11}=0$, there is at least one i such that $a_{1i}\neq 0$ and the (1,1)-component of

$$s\cdot x(-1, \lambda_1-\lambda_i)x(1, \lambda_i-\lambda_1)x(-1, \lambda_1-\lambda_i)$$

is not zero. If we set

$$s'=s\cdot h(a_{11}^{-1}, 1)\cdot\prod_{i=2}^{l} x(-a_{1i}, \lambda_1-\lambda_i)\cdot x(-a_{1l+1}a_{11}, \lambda_1-\lambda_{l+1})$$

$$s''=\prod_{i=2}^{l+1} x(-a_{i1}a_{11}^{-1}, \lambda_i-\lambda_1)\cdot s'\,,$$

then s'' has the following form

$$s''=\begin{pmatrix} 1 & 0\cdots\cdots 0 \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix}.$$

Therefore we may apply induction hypothesis, and we have our assertion.

For b); (i) Let $s=(a_{ij})\in G_K^*$, then from ${}^tsJs=J$, we have

(15) $$a_{11}^2+2a_{31}a_{21}=1$$

(16) $$a_{12}^2+2a_{32}a_{22}=0$$

(17) $$a_{13}^2+2a_{33}a_{23}=0$$

(18) $$a_{12}a_{13}+a_{22}a_{33}+a_{32}a_{23}=1$$

(19) $$a_{11}a_{12}+a_{21}a_{32}+a_{32}a_{22}=0$$

(20) $$a_{11}a_{13}+a_{21}a_{33}+a_{31}a_{23}=0\,.$$

If $a_{22}=0$, then $a_{12}=0$ from (16), $a_{32}a_{33}=1, a_{21}=0$ from (18), (19). Thus the (2,2)-component of $s\cdot x(1, \lambda_1)$ is $-a_{23}/2\neq 0$. Moreover that of $s\cdot x(1, \lambda_1)h(-2a_{23}^{-1}, 1)$ is 1. Therefore we may assume $a_{22}=1$.

If $a_{12}=0$, then $a_{31}=a_{32}=0$ from (16), (19), $a_{33}=a_{11}=1$ from (15), (18) and $a_{13}=-a_{21}, a_{23}=-a_{21}^2/2$ from (17), (20). Therefore $s=x(a_{21}, -\lambda_1)$. If $a_{12}\neq 0$, then (1,2)-component of $x(-a_{12}, \lambda_1)\cdot s$ is zero. From above consequence we have $s=x(a_{12}, +\lambda_1)x(_{21}, -\lambda_1)$.

(ii) Let $s\in G_K^*$ and set

$$s=\begin{pmatrix} a & \mathfrak{a} & \mathfrak{b} \\ {}^t\mathfrak{c} & A & B \\ {}^t\mathfrak{d} & C & D \end{pmatrix}$$

where $A=(a_{ij})$, $B=(b_{ij})$, $C=(c_{ij})$ and $D=(d_{ij})$ are (l, l)-matrices and $\mathfrak{a}=(a_1,\cdots, a_l)$, $\mathfrak{b}=(b_1,\cdots, b_l)$, $\mathfrak{c}=(c_1,\cdots, c_l)$ and $\mathfrak{d}=(d_1,\cdots, d_l)$ are l-dimensional vectors. We also denote the components of $s', s''\in G_K^*$ by $A'=(a_{ij}')$, $B'=(b_{ij}')$ etc. Then

(21) $$a^2+2\sum_{i=1}^{l} a_ib_i=1$$

(22) $$aa_j+\sum_{i=1}^{l} c_ic_{ij}+\sum_{i=1}^{l} d_ia_{ij}=0$$

(23) $$ab_j+\sum_{i=1}^{l} c_id_{ij}+\sum_{i=1}^{l} d_ib_{ij}=0$$

(24) $$\sum_{k=1}^{l} a_{ki}d_{kj}+\sum_{k=1}^{l} c_{ki}b_{kj}+a_ib_j=\delta_{ij} \qquad (1\leqq i, j\leqq l)$$

(25) $$\sum_{k=1}^{l} b_{ki}d_{kj}+\sum_{k=1}^{l} d_{ki}b_{kj}+b_ib_j=0$$

(26) $$\sum_{k=1}^{l} a_{ki}c_{kj}+\sum_{k=1}^{l} c_{ki}a_{kj}+a_ia_j=0\,.$$

If $a_{11}=0$, then $a_{1i}\neq 0$ or $b_{1i}\neq 0$ for some i. Therefore (1,1)-component of A in $s\cdot x(1, \lambda_1-\lambda_i)$ or $s\cdot x(1, -\lambda_1-\lambda_i)$ is not zero, and we may suppose $a_{11}\neq 0$. If we set

$$s'=x(-a_1, -\lambda_1)\prod_{i=2}^{l} x(c_{i1}, \lambda_1+\lambda_i)h(a_{11}^{-1}, 1)\prod_{i=2}^{l} x(-a_{i1}a_{11}^{-1}, \lambda_i-\lambda_1)\cdot s\,,$$

then $a_1'=0$, $c_{i1}'=0$, $a_{i1}'=0$ $(i\geqq 2)$ and $a_{11}'=1$. Moreover $c_{11}'=\sum_{k=1}^{l} c_{k1}a_{k1}=0$, $d_{11}'=\sum_{k=1}^{l} a_{k1}d_{k1}=1$ from (26), (24) for $i=j=1$ respectively.

Next, we set

$$s''=x(-b_1', \lambda_1)\prod_{i=2} x(-b_{i1}', -\lambda_i-\lambda_1)\prod_{i=2}^{l} x(d_{i1}', \lambda_1-\lambda_i)\cdot s',$$

then $b_1''=0$, $b_{i1}''=0$, $d_{i1}''=0$ $(i\geqq 2)$. Moreover, $b_{1i}''=0$, $c_{1i}''=0$ $(i\geqq 2)$ from (25), (26) for $i\geqq 2, j=1$ respectively and $a_{1j}''=0$, $d_{1j}''=0$ $(j\geqq 2)$ from (24) for $i=1$, $j\geqq 2$ and $i\geqq 2, j=1$ respectively. $b_{11}''=0$ from (26) for $i=j=1$ and $c_1''=0$, $d_1''=0$ from (22), (23) for $j=1$ respectively. Thus we have

$$s''=\begin{pmatrix} * & 0 & \mathfrak{a}' & 0 & \mathfrak{b}' \\ 0 & 1 & 0\cdots\cdots 0 & 0\cdots\cdots\cdots 0 \\ & 0 & & \vdots & \\ {}^t\mathfrak{c}' & \vdots & A' & \vdots & B' \\ & \vdots & & 0 & \\ 0 & 0\cdots\cdots\cdots 0 & & 1 & 0\cdots\cdots 0 \\ & \vdots & & 0 & \\ {}^t\mathfrak{d}' & \vdots & C' & \vdots & D' \\ & 0 & & 0 & \end{pmatrix}.$$

Therefore we may apply induction hypothesis and we have our assertion.

For c); (i) If $l=1$, $Sp_2(K)\cong SL_2(K)$. We have our assertion similarly to $SL_2(K)$. (ii) Let $s\in Sp_{2l}(K)$, and set

$$s=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A=(a_{ij})$, $B=(b_{ij})$, $C=(c_{ij})$ and $D=(d_{ij})$ are (l, l)-matrices. We also denote the components of $s', s''\in Sp_{2l}(K)$ by $A'=(a_{ij}')$, $A''(a_{ij}'')$ etc. Then from ${}^t sJs=J$, we have

(27) $$\sum_{k=1}^{l} a_{ki}c_{kj}=\sum_{k=1}^{l} c_{ki}a_{kj}$$

(28) $$\sum_{k=1}^{l} b_{ki}d_{kj}=\sum_{k=1}^{l} d_{ki}b_{kj} \qquad (1\leqq i, j\leqq l)$$

(29) $$\sum_{k=1}^{l} a_{ki}d_{kj}-\sum_{k=1}^{l} c_{ki}b_{kj}=\delta_{ij}.$$

If $a_{11}=0$, then $a_{1i}\neq 0$ or $b_{1i}\neq 0$ for some i, and (1,1)-component of A in $s\cdot x(1, \lambda_1-\lambda_i)x(-1, -\lambda_1+\lambda_i)x(1, \lambda_1-\lambda_i)$ or $s\cdot x(1, 2\lambda_i)x(-1, -2\lambda_i)x(1, 2\lambda_i)$ is not zero. Therefore we may suppose $a_{11}\neq 0$. If we set

$$s'=x(-\sum_{k=1}^{l} a_{k1}c_{k1}, -2\lambda_1)\prod_{j=2}^{l} x(-c_{j1}, \lambda_1+\lambda_j)\cdot h(a_{11}^{-1}, 1)\prod_{j=2}^{l} x(-a_{j1}a_{11}^{-1}, \lambda_j-\lambda_1)\cdot s$$

then $a_{11}'=1$, $a_{i1}'=0$ $(i\geqq 2)$ and $c_{i1}'=0$ $(1\leqq i\leqq l)$. Moreover $d_{11}'=1$ from (29) for $i=j=1$. Next we set

$$s''=\prod_{j=2}^{l} x(d_{j1}', \lambda_1-\lambda_j)x(-\sum_{k=1}^{l} b_{k1}'d_{k1}', 2\lambda_1)\prod_{j=2}^{l} x(b_{j1}', -\lambda_1-\lambda_j)\cdot s'$$

then $b_{j1}''=0$ $(1\leqq j\leqq l)$, $d_{j1}''=0$ $(j\geqq 2)$. Moreover, $b_{1i}''=0$, $c_{1i}''=0$ $(1\leqq i\leqq 0)$ from

(27), (28) for $j=1$ respectively and $a_{1i}''=0, d_{1i}''=0$ $(i\geqq 2)$ from (29) for $i\geqq 2$, $j=1, i=1, j\geqq 2$ respectively. Thus we have

$$s''=\begin{pmatrix} 1 & 0\cdots\cdots 0 & 0 & \cdots\cdots\cdots 0 \\ 0 & & 0 & \\ \vdots & A' & \vdots & B' \\ 0 & & 0 & \\ 0 & \cdots\cdots\cdots 0 & 1 & 0\cdots\cdots 0 \\ \vdots & & 0 & \\ \vdots & C' & \vdots & D' \\ 0 & & 0 & \end{pmatrix}.$$

Therefore we may apply induction hypothesis and we have our assertion.

For d); (i) is trivial and (ii) may be proved similarly to b). q. e. d.

Theorem 3.

a) *If* $\mathfrak{g}$ *is of the type* (A_l), $G_K'\cong PSL_{l+1}(K)$,

b) *If* $\mathfrak{g}$ *is of the type* (B_l), $G_K'\cong P\Omega_{2l+1}(K,f)$,

c) *If* $\mathfrak{g}$ *is of the type* (C_l), $l\geqq 3$, $G_K'\cong PSp_2(K)$,

d) *If* $\mathfrak{g}$ *is of the type* (D_l), $l\geqq 4$, $G_K'\cong P\Omega_{2l}(K,f)$,

where we assume that the characteristic of K *is* $\neq 2$ *for* b), d) *and* f *is a quadratic form of maximal index.*[17]

Proof. Let $\mathfrak{g}$ be a Lie algebra of (13). Then the basis of $\mathfrak{h}$ which is contained in the additive group generated by $H_1,\cdots,H_l$, and X_r, r root, of (14) forms a canonical basis of $\mathfrak{g}$ for a), c) or d) and if we replace $X_{\pm\lambda_i}$ by $\sqrt{2}X_{\pm\lambda_i}$ then they form also a canonical basis of $\mathfrak{g}$ for b). (We assume for a moment $\sqrt{2}\in K$ for b).)

Let P_λ be the additive group generated by λ_i $(1\leqq i\leqq l)$. Since P is generated by a fundamental system of weights which is

a), c) $\quad \Lambda_i=\lambda_1+\lambda_2+\cdots+\lambda_i \quad (1\leqq i\leqq l)$,

b) $\quad \Lambda_i=\lambda_1+\lambda_2+\cdots+\lambda_i \quad (1\leqq i\leqq l-1)$,

$\quad \Lambda_l=(\lambda_1+\lambda_2+\cdots+\lambda_l)/2$

d) $\quad \Lambda_i=\lambda_1+\lambda_2+\cdots+\lambda_i \quad (1\leqq i\leqq l-2)$,

$\quad \Lambda_{l-1}=(\lambda_1+\lambda_2+\cdots+\lambda_{l-1}-\lambda_l)/2$

$\quad \Lambda_l=(\lambda_1+\lambda_2+\cdots+\lambda_{l-1}+\lambda_l)/2$,[18]

P_λ is a subgroup of P, and since P_r is generated by $(a_1,\cdots,a_l)$, P_r is a subgroup of P_λ. Moreover we have

(30)
$$P=P_\lambda\supset P_r \quad \text{for a) and c)}$$
$$P=P_\lambda=P_r \quad \text{for b)}$$
$$P\supset P_\lambda\supset P_r \quad \text{for d)}.$$

17) This result has been obtained by R. Ree [**10**] independently of the author, and J. Dieudonné has supplemented the case of characteristic 2 for (B_l). (cf. Amer. J. Math., 74, 922-923)

18) cf. H. Weyl [**13**].

Therefore if we denote by $\mathfrak{H}_K''$ the subgroup of $\mathfrak{H}_K$ formed by $h(\chi)$ such that $\chi\in \mathrm{Hom}(P_r, K^*)$ is extendible to the homomorphism of P_λ, we have from (30)

(31) $\mathfrak{H}_K \supseteqq \mathfrak{H}_K''=\mathfrak{H}_K'$ for a), c), $\mathfrak{H}_K=\mathfrak{H}_K''\supseteqq\mathfrak{H}_K'$ for b), $\mathfrak{H}_K\supseteqq\mathfrak{H}_K''\supseteqq\mathfrak{H}_K'$ for d).

Denote by $\chi_{z,i}$, $z\in K^*$, $1\leqq i\leqq l$, the element of $\mathrm{Hom}(P_\lambda, K^*)$ such that $\lambda_i\rightarrow z$, $\lambda_j\rightarrow 1$ $(j\neq i)$. Then $\chi_{z,i}$, $1\leqq i\leqq l$, $z\in K^*$, generate the group $\mathrm{Hom}(P_\lambda, K^*)$. We denote again by $\chi_{z,i}$ its restriction to P_r.

Let φ be the homomorphism of G_K^* into $A(g_K)$ defined as follows:

$$h(z,i)\rightarrow h(\chi_{z,i}) \qquad 1\leqq i\leqq l,\ z\in K^*$$

$$x(t,r)\rightarrow x_r(t) \qquad r \text{ root},\ t\in K.$$

If we denote by G_K'' the image of G_K^* by φ, then from (31) we have

$$G_K''=G_K' \text{ for a) and c)},\ G_K''=G_K \text{ for b)},\ G_K\supseteqq G_K''\supseteqq G_K' \text{ for d)}.$$

We can easily see that the kernel of φ is the center of G_K^*, and for b), d) the commutator subgroup of G_K'' is G_K'. Therefore we have our assertion. When $\sqrt{2}\notin K$ and $\mathfrak{g}$ is of b), let $L=K(\sqrt{2})$ and σ be a generator of the Galois group of L/K, then G_L is isomorphic to $P\Omega_{2l+1}(L,f)$ and G_K is isomorphic to the group formed by the element s of G_L such that $s^\sigma=s$[19]. Therefore G_K is isomorphic to $P\Omega_{2l+1}(K,f)$. q. e. d.

§ 3. Involutions in G_K.

We call an automorphism σ of $\mathfrak{g}_K$ an involution if σ^2 is the identity automorphism. Let F: $(a_1,\cdots,a_l)$ be a fundamental root system of $\mathfrak{g}$, as in § 1, and K be a field of characteristic $\neq 2$. Then in $\mathfrak{H}_K$ there are 2^l involutions $h(I)$, $I=(i_1, i_2,\cdots, i_s)$, $1\leqq i_1<i_2<\cdots<i_s\leqq l$, which correspond to $\chi_I\in\mathrm{Hom}(P_r, K^*)$ such that $\chi_I(a_k)=-1$ if $k\in I$ and $\lambda_I(a_k)=1$ if $k\notin I$. As for these involutions we have the following proposition.

Proposition 6. *The involutions in $\mathfrak{H}_K$ are conjugate to one of the following:*

a)	*For the type* (A_l), $l\geqq 1$	$h(1), h(2),\cdots, h(k)$	$k=[l+1/2]$
b)	*For the type* (B_l), $l\geqq 2$	$h(1), h(2),\cdots, h(l)$	
c)	*For the type* (C_l), $l\geqq 3$	$h(1), h(2),\cdots, h(k), h(l)$	$k=[l/2]$
d)	*For the type* (D_l)		
	$l\geqq 4$ *and* l *is even*	$h(1), h(2),\cdots, h(k), h(l-1), h(l)$	$k=l/2$
	$l>4$ *and* l *is odd*	$h(1), h(2),\cdots, h(k), h(l)$	$k=l-1/2$
e)	*For the type* (E_6),	$h(1), h(2)$	
	(E_7),	$h(1), h(4), h(7)$	
	(E_8),	$h(1), h(2)$	
f)	*For the type* (F_4),	$h(1), h(3)$	
g)	*For the type* (G_2),	$h(1)$.	

19) Chevalley [**3**, III p. 46], we denote by $s^\sigma=(a^\sigma{}_{ij})$ when $s=(a_{ij})$.

First, we shall prove some Lemmas.

Lemma 4. *Two elements $h(\chi_1)$, $h(\chi_2)$ in $\mathfrak{H}_K$ are conjugate in G_K if and only if there exists an element w of the Weyl group such that $\chi_1(r)=\chi_2(w(r))$ for all root r.*

In fact, the sufficiency is obvious from (11). If $h(\chi_1)$ and $h(\chi_2)$ are conjugate, then $s\cdot h(\chi_1)=h(\chi_2)\cdot s$ for some s in G_K. If we express s as (12), then from (9) we have

$$\begin{aligned} s\cdot h(\chi_1) &= xh\omega(w)x''h(\chi_1)=xhh(\chi_1{}')\omega(w)\bar{x}'' , \\ h(\chi_2)\cdot s &= h(\chi_2)xh\omega(w)x''=\bar{x}h(\chi_2)h\omega(w)x'' . \end{aligned} \tag{32}$$

Since the expression of the element of G_K as (12) is unique, $\bar{x}=x, \bar{x}''=x''$ and $h(\chi_2)=h(\chi_1{}')$.

Lemma 5. *We denote by w_i $(1\leqq i\leqq l)$ the reflexion by a_i. Let I, J be subsets of $\Sigma=(1, 2,\cdots, l)$ such that*

$$\omega(w_i)h(I)\omega(w_i)^{-1}=h(J) .$$

Then (i) *if $i\notin I$, $I=J$,*

(ii) *if $i\in I$, $k\in J$ if and only if $k\in I$, when a_{ki} is even,*

and $k\in J$ if and only if $k\notin I$, when a_{ki} is odd.

In fact, since $w_i(a_j)=a_j+a_{ji}a_i$ and $\chi(w_i(a_j))=\chi(a_j)\chi(a_i)^{a_{ji}}$, we have the Lemma immediately.

In the following we shall denote by $(I)\xrightarrow[i]{}(J)$ for such a pair I, J.

Let $F:(a_1,\cdots, a_l)$ be a f. r. s. of $\mathfrak{g}$ and $(S_1,\cdots, S_l)$ be the dots of the diagram of F. We call a subset $F':(a_{i_1}, a_{i_2},\cdots, a_{i_s})$ of F a connected series when S_{i_k} and $S_{i_{k+1}}$ $(1\leqq k\leqq s-1)$ are connected. A connected series $(a_{i_1}, a_{i_2},\cdots, a_{i_s})$ is called a subsystem of the type (A_s) when S_{i_k} and $S_{i_{k+1}}$ $(1\leqq k\leqq s-1)$ are connected by a single line and S_{i_k} $(2\leqq k\leqq s-1)$ is not connected to any other dot. Moreover we say that a subsystem of the type $(A_s):(a_{i_1},\cdots, a_{i_s})$ has a border a_{i_1} (or a_{i_s}), when S_{i_1} (or S_{i_s}) is not connected to any other dot.

Lemma 6. *Let $F':(a_1, a_2,\cdots, a_h)$ be a subsystem of the type (A_h) of F having a border a_1, then $h(I)$, where $I=(i_1, i_2,\cdots, i_s)$ is a subset of $\Sigma'=(1, 2,\cdots, h)$, is conjugate to $h(k)$ for some $k\in\Sigma'$.*

Proof. Let A' be a (h, h)-matrix whose (i, j)-component is a_{ij}. Then

$$A'=\begin{pmatrix} -2 & 1 & & & & 0 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & 1 & -2 & 1 \\ 0 & & & & 1 & -2 \end{pmatrix}.$$

First we assume that $(a_{i_1}, a_{i_2}, \cdots, a_{i_s})$ is a connected series. If s is odd, from Lemma 5 and the form of the matrix A',

$$(i_1, i_2, \cdots, i_s) \xrightarrow[i_2]{} (i_2, i_4, \cdots, i_s) \xrightarrow[i_4]{} \cdots \xrightarrow[i_{s-1}]{} (i_2, i_3, \cdots, i_{s-1})\,. \tag{33}$$

Repeating this process we have $h(i_1, i_2, \cdots, i_s)$ is conjugate to $h(i_k)$, $k=s-1/2$. If s is even and $i_1>1$, then

$$(i_1, i_2, \cdots, i_s) \xrightarrow[i_1]{} (i_1-1, i_1, i_3, \cdots, i_s) \xrightarrow[i_3]{} \cdots \xrightarrow[i_{s-1}]{} (i_1-1, i_1, i_2, \cdots, i_{s-1})\,. \tag{34}$$

Repeating this process we have $h(i_1, \cdots, i_s)$ is conjugate to $h(1, 2, \cdots, s)$. Then

$$(1, 2, \cdots, s) \xrightarrow[1]{} (1, 3, 4, \cdots, s) \xrightarrow[3]{} (1, 2, 3, 5, \cdots, s) \xrightarrow[5]{} \cdots \xrightarrow[s-1]{} (1, 2, \cdots, s-1)\,. \tag{35}$$

Therefore we may reduce to the case where s is odd.

When the set $(a_{i_1}, a_{i_2}, \cdots, a_{i_s})$ is not a connected series, if we repeat some process of (33), (34) for each brock of the connected series, we may reduce to the case where $(a_{i_1}, a_{i_2}, \cdots, a_{i_s})$ is a connected series. Thus we have our assertion. q. e. d.

Let Σ' be a subset of $\Sigma=(1, 2, \cdots, l)$. We denote by $\mathfrak{W}_{\Sigma'}$ the group generated by the element $\omega(w_k)$, $k \in \Sigma'$. Then we have

LEMMA 7. *Let I, J be two disjoint subsets of $\Sigma=(1, 2, \cdots, l)$. If $h(I)$* (resp, *$h(J)$) is conjugate to $h(i)$* (resp. *$h(j)$) by the element of $\mathfrak{W}_{\Sigma_1}$* (resp. $\mathfrak{W}_{\Sigma_2}$) *and if Σ_1 and J, Σ_2 and I are disjoint respectively, then $h(I, J)$ is conjugate to $h(i, j)$.*

This is an immediate consequence of Lemma 5.

PROOF OF THE PROPOSITION:

a) For the type (A_l): From Lemma 6, every $h(i_1, i_2, \cdots, i_s)$ is conjugate to $h(k)$ for some k. We shall show that $h(i)$ is conjugate to $h(l-i+1)$ for $i<[l/2]$. If $i=1$, then

$$(1) \xrightarrow[1]{} (1,2) \xrightarrow[2]{} (2,3) \xrightarrow[3]{} \cdots \xrightarrow[l-1]{} (l-1, l) \xrightarrow[l]{} (l)\,.$$

If $i>1$, then

$$(i) \xrightarrow[i]{} (i-1, i, i+1) \xrightarrow[i+1]{} (i-1, i+1, i+2) \xrightarrow[i+2]{} \cdots \xrightarrow[l-1]{} (i-1, l-1, l) \xrightarrow[l]{} (i-1, l)\,. \tag{36}$$

Therefore $h(i)$ is conjugate to $h(i-1, l)$. Moreover

$$(i-1, l) \xrightarrow[i-1]{} (i-2, i-1, i, l) \xrightarrow[i]{} \cdots \xrightarrow[l-2]{} (i-2, l-2, l-1, l) \xrightarrow[l-1]{} (i-2, l-1)\,. \tag{37}$$

Repeating this process we have $h(i)$ is conjugate to $h(1, l-i+2)$. Then

$$(1, l-i+2) \xrightarrow[1]{} (1, 2, l-i+2) \xrightarrow[2]{} (2, 3, l-i+2) \xrightarrow[3]{} \cdots \xrightarrow[l-i]{} (l-i, l-i+1, l-i+2) \xrightarrow[l-i+1]{} (l-i+1)\,. \tag{38}$$

Thus we have our assertion.

b) For the type $(B_l): l\geqq 2$. Since the subset $(a_1, a_2, \cdots, a_{l-1})$ of F is a subsystem of the type (A_{l-1}) having a border a_1,from Lemma 6, we have $h(i_1, i_2, \cdots, i_s), i_s\leqq l-1$, is conjugate to $h(k)$ for some k. We shall show that $h(i_1, i_2, \cdots, i_s, l), i_s\leqq l-1$, is conjugate to $h(k)$ for some k, $1\leqq k\leqq l-1$. From Lemma 7, it is enough to consider the involution $h(k, l)$. The matrix A for the type (B_l) has the following form:

$$A=\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & -2 & 2 \\ & & & 1 & -2 \end{pmatrix}.$$

If $k>1$,

$$(k, l)\xrightarrow[k]{}(k-1, k, k+1, l)\xrightarrow[k+1]{}(k-1, k+1, k+2, l)\xrightarrow[k+2]{}\cdots\xrightarrow[l-1]{}(k-1, l-2, l-1)$$

and similarly $(1, l)$ is conjugate to $h(l-2, l-1)$. Thus we may reduce to a case of the first.

c) For the type $(C_l), l\geqq 3$: Since the subset $(a_1, a_2, \cdots, a_{l-1})$ of F is a subsystem of the type (A_{l-1}) having a border a_1, from Lemma 6, we have $h(i_1, i_2, \cdots, i_s), i_s\leqq l-1$, is conjugate to $h(k)$ for some k. We shall show that $h(i_1, i_2, \cdots, i_s, l)$ is conjugate to $h(l)$. From Lemma 7, it is sufficient to consider $h(k, l)$. The matrix A of the type (C_l) has the following form

$$A=\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix}.$$

If $k<l-1$,

$$(k, l)\xrightarrow[l]{}(k, l-1, l)\xrightarrow[l-1]{}(k, l-2, l-1, l)\xrightarrow[l-2]{}\cdots\xrightarrow[k+2]{}(k, k+1, k+2, l)$$
$$\xrightarrow[k+1]{}(k+1, l).$$

Repeating this process we have $h(k, l)$ is conjugate to $h(l-1, l)$ and $(l-1, l)\xrightarrow[l]{}(l)$. Therefore we have our assertion. Finally, we shall show that $h(i)$ is conjugate to $h(l-i), i\leqq[l/2]$. If $i=1$,

$$(1)\xrightarrow[1]{}(1,2)\xrightarrow[2]{}(2,3)\xrightarrow[3]{}\cdots\xrightarrow[l-1]{}(l-2, l-1)\xrightarrow[l-2]{}(l-1).$$

If $i>1$,

$$(i)\xrightarrow[i]{}(i-1, i, i+1)\xrightarrow[i+1]{}(i-1, i+1, i+2)\xrightarrow[i+2]{}\cdots\xrightarrow[l-2]{}(i-1, l-2, l-1)$$
$$\xrightarrow[l-1]{}(i-1, l-1).$$

If we repeat the same process as (37), we have $h(i)$ is conjugate to $h(1, l-i+1)$. Similarly to (38), $h(1, l-i+1)$ is conjugate to $h(l-i)$. Thus we have our assertion.

d) For the type (D_l), $l \geqq 4$: Since the subset $(a_1, a_2, \cdots, a_{l-2})$ of F is a subsystem of the type (A_{l-2}) having a border a_1, from Lemma 6, we have $h(i_1, i_2, \cdots, i_s)$, $i_s \leqq l-2$, is conjugate to $h(k)$ for some k. The matrix of A of the type (D_l) has the following form

$$A=\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & 1 & -2 & 1 & 1 \\ & & & 1 & -2 & \\ & & & 1 & & -2 \end{pmatrix}.$$

(i) $h(i_1, i_2, \cdots, i_s, l-1, l)$ and $h(l-1, l)$ are conjugate to $h(k)$ for some k.

In fact, from Lemma 7, it is enough to consider $h(k, l-1, l)$. If $k < l-3$,

$$(k, l-1, l) \xrightarrow[l-1]{} (k, l-2, l-1, l) \xrightarrow[l-2]{} (k, l-3, l-2) \xrightarrow[l-3]{} \cdots \xrightarrow[k+2]{} (k, k+1, k+2) \xrightarrow[k+1]{} (k+1).$$

If $k = l-3$,

$$(l-3, l-1, l) \xrightarrow[l-1]{} (l-3, l-2, l-1, l) \xrightarrow[l-2]{} (l-2).$$

Finally, $(l-1, l) \xrightarrow[l-1]{} (l-2, l-1, l) \xrightarrow[l-2]{} (l-3, l-2)$, and this is a case of the first.

(ii) $h(i_1, i_2, \cdots, i_s, l)$ and $h(i_1, i_2, \cdots, i_s, l-1)$ are conjugate to $h(l-1)$ or $h(l)$.

In fact, it is enough to see this for $h(k, l-1)$ and $h(k, l)$. If $k=1$,

$$(39) \qquad (1, l) \xrightarrow[1]{} (1, 2, l) \xrightarrow[2]{} (2, 3, l) \xrightarrow[3]{} \cdots \xrightarrow[l-3]{} (l-3, l-2, l) \xrightarrow[l-2]{} (l-2, l-1) \xrightarrow[l-1]{} (l-1),$$

$$(40) \qquad (1, l-1) \xrightarrow[1]{} (1, 2, l-1) \xrightarrow[2]{} (2, 3, l-1) \xrightarrow[3]{} \cdots \xrightarrow[l-3]{} (l-3, l-2, l-1) \xrightarrow[l-2]{} (l-2, l) \xrightarrow[l]{} (l).$$

If $k > 1$,

$$(41) \qquad (k, l) \xrightarrow[k]{} (k-1, k, k+1, l) \xrightarrow[k+1]{} (k-1, k+1, k+2, l) \xrightarrow[k+2]{} \cdots \xrightarrow[l-3]{} (k-1, l-3, l-2, l) \xrightarrow[l-2]{} (k-1, l-2, l-1) \xrightarrow[l-1]{} (k-1, l-1),$$

$$(42) \qquad (k, l-1) \xrightarrow[k]{} (k-1, k, k+1, l-1) \xrightarrow[k+1]{} (k-1, k+1, k+2, l-1) \xrightarrow[k+2]{} \cdots \xrightarrow[l-2]{} (k-1, l-2, l) \xrightarrow[l]{} (k-1, l).$$

Repeating this process, we may reduce to the case where $k=1$.

(iii) If l is odd, $h(l)$ is conjugate to $h(l-1)$.

In fact, $h(l)$ is conjugate to $h(1, l-1)$ from (40), $h(1, l-1)$ is conjugate to $h(2, l)$ from (41), and $h(2, l)$ is conjugate to $h(3, l-1)$ from (42). Repeating this process we have $h(l)$ is conjugate to $h(l-2, l-1)$ and $(l-2, l-1)\xrightarrow[l-1]{}(l-1)$. Thus $h(l)$ is conjugate to $h(l-1)$. (N. B. From the same consequence as above we may see that, if l is even, $h(l)$ is not conjugate to $h(l-1)$.)

(iv) $h(i)$ is conjugate to $h(l-i)$, $2\leqq i\leqq[l/2]$.

In fact, if $i>2$

$$(43)\qquad (i)\xrightarrow[i]{}(i-1, i, i+1)\xrightarrow[i+1]{}(i-1, i+1, i+2)\xrightarrow[i+2]{}\cdots\xrightarrow[l-3]{}(i-1, l-3, l-2)$$
$$\xrightarrow[l-2]{}(i-1, l-2, l-1, l)\xrightarrow[l-1]{}(i-1, l-1, l),$$
$$(i-1, l-1, l)\xrightarrow[i-1]{}(i-2, i-1, i, l-1, l)\xrightarrow[i]{}(i-2, i, i+1, l-1, l)\xrightarrow[i+1]{}\cdots$$
$$\xrightarrow[l-3]{}(i-2, l-3, l-2, l-1, l)\xrightarrow[l-2]{}(i-2, l-2).$$

Then repeating the same process as (37), (38), we have $h(i)$ is conjugate to $h(l-i)$. If $i=2, h(2)$ is conjugate to $h(1, l-1, l)$ from (43), and

$$(1, l-1, l)\xrightarrow[1]{}(1, 2, l-1, l)\xrightarrow[2]{}(2, 3, l-1, l)\xrightarrow[3]{}\cdots$$
$$\xrightarrow[l-3]{}(l-3, l-2, l-1, l)\xrightarrow[l-2]{}(l-2).$$

Thus we have our assertion.

e) For the type (E_l), $l=6, 7$ and 8: The matrix A of the type (E_l) has the following form

$$A=\begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & 1 & & \\ & & 1 & -2 & & & \\ & & 1 & & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & \cdot & \cdot & \cdot \end{pmatrix}.$$

(i) $h(2)$ is conjugate to $h(3)$. In fact,

$$(2)\xrightarrow[2]{}(1, 2, 3)\xrightarrow[3]{}(1, 3, 4, 5)\xrightarrow[1]{}(1, 2, 3, 4, 5)\xrightarrow[2]{}(2, 4, 5)$$
$$\xrightarrow[4]{}(2, 3, 4, 5)\xrightarrow[3]{}(3).$$

(ii) $h(4)$ is conjugate to $h(5)$. In fact,

$$(4)\xrightarrow[4]{}(3, 4)\xrightarrow[3]{}(2, 3, 5)\xrightarrow[2]{}(1, 2, 5)\xrightarrow[5]{}(1, 2, 3, 5, 6)\xrightarrow[3]{}(1, 3, 4, 6)$$

$$\xrightarrow[4]{} (1,4,6) \xrightarrow[1]{} (1,2,4,6) \xrightarrow[2]{} (2,3,4,6) \xrightarrow[3]{} (3,5,6) \xrightarrow[5]{} (5)\,.$$

(iii) $h(1)$ is conjugate to $h(6)$. In fact, for $l=6$,

$$(1) \xrightarrow[1]{} (1,2) \xrightarrow[2]{} (2,3) \xrightarrow[3]{} (3,4,5) \xrightarrow[5]{} (4,5,6) \xrightarrow[6]{} (4,6)\,,$$

and similarly $h(1)$ is conjugate to $h(4,l)$ for $l=7$ and 8. On the other hand $(6) \xrightarrow[6]{} (5,6)$ for $l=6$, and similarly $h(l)$ is conjugate to $h(5,l)$ for $l=7$ and 8. Since $h(4)$ is conjugate to $h(5)$ from (ii), $h(4,l)$ is conjugate to $h(5,l)$ from Lemma 7. Therefore $h(1)$ is conjugate to $h(6)$.

(iv) For the type (E_6), $h(3)$ is conjugate to $h(4)$. In fact,

$$(3) \xrightarrow[3]{} (2,3,4,5) \xrightarrow[2]{} (1,2,4,5) \xrightarrow[1]{} (1,4,5) \xrightarrow[5]{} (1,3,4,5,6) \xrightarrow[3]{} (1,2,3,6)$$

$$\xrightarrow[2]{} (2,6) \xrightarrow[6]{} (2,5,6) \xrightarrow[5]{} (2,3,5) \xrightarrow[3]{} (3,4) \xrightarrow[4]{} (4)\,.$$

(v) For the type (E_7), $h(3)$ is conjugate to $h(6)$. In fact,

$$(3) \xrightarrow[3]{} (2,3,4,5) \xrightarrow[4]{} (2,4,5) \xrightarrow[5]{} (2,3,4,5,6) \xrightarrow[6]{} (2,3,4,6,7) \xrightarrow[3]{} (3,5,6,7)$$

$$\xrightarrow[5]{} (5,7) \xrightarrow[7]{} (5,6,7) \xrightarrow[6]{} (6)\,.$$

(vi) For the type (E_8), $h(6)$ is conjugate to $h(5)$ and $h(3)$ is conjugate to $h(7), h(8)$. We shall show the first two consequences for examples.

$$(6) \xrightarrow[6]{} (5,6,7) \xrightarrow[7]{} (5,7,8) \xrightarrow[8]{} (5,8) \xrightarrow[5]{} (3,5,6,8) \xrightarrow[6]{} (3,6,7,8) \xrightarrow[7]{} (3,7)$$

$$\xrightarrow[3]{} (2,3,4,5,7) \xrightarrow[5]{} (2,4,5,6,7) \xrightarrow[6]{} (2,4,6) \xrightarrow[4]{} (2,3,4,6) \xrightarrow[3]{} (3,5,6) \xrightarrow[5]{} (5)\,.$$

$$(3) \xrightarrow[3]{} (2,3,4,5) \xrightarrow[4]{} (2,4,5) \xrightarrow[5]{} (2,3,4,5,6) \xrightarrow[3]{} (3,6) \xrightarrow[6]{} (3,5,6,7)$$

$$\xrightarrow[5]{} (5,7) \xrightarrow[7]{} (5,6,7,8) \xrightarrow[6]{} (6,8) \xrightarrow[8]{} (6,7,8) \xrightarrow[7]{} (7)\,.$$

From (i)$\cdots$(vi), we have the following:

For the type (E_6): $h(1)$ and $h(6)$; $h(2), h(3), h(4)$ and $h(5)$ are conjugate.

For the type (E_7): $h(1), h(2), h(3)$ and $h(6)$; $h(4)$ and $h(5)$ are conjugate.

For the type (E_8): $h(1), h(4), h(5)$ and $h(6)$; $h(2), h(3), h(7)$ and $h(8)$.
are conjugate.

Since (a_5, a_6, a_7, a_8) is a subsystem of the type (A_4) having a border a_8, $h(i_1,\cdots,i_s)$, $5\leqq i_1<\cdots<i_s\leqq 8$, is conjugate to $h(i)$ for some i.

On the other hand, since $(a_1, a_2, a_3, a_4, a_5)$ is a subsystem of the type (D_5), similarly to the proof for the type (D_5), $h(i_1,\cdots,i_s)$, $1\leqq i_1<\cdots<i_s\leqq 5$, is conjugate to $h(i)$ for some i. Therefore, from Lemma 7, any $h(i_1,\cdots,i_s)$ is conjugate to $h(i,j)$ for $1\leqq i\leqq 4$, $5\leqq j\leqq 8$. We may easily see that $h(i,j)$ is conjugate to $h(k)$ for some k.

f) For the type (F_4): Since (a_1, a_2, a_3), (a_4, a_3, a_2) form the f.r.s. of the types $(B_3), (C_3)$ respectively, $h(i_1,\cdots,i_s)$ is conjugate to $h(i)$ for some i, if $i_1\neq 1$ or $i_s\neq 4$. If $i_1=1$ and $i_s=4$, we may reduce to the case where $i_1\neq 1$ or $i_s\neq 4$. Finally, $h(3)$ and $h(4)$, $h(1)$ and $h(2)$ are conjugate respectively. In fact,

$$(3) \xrightarrow[3]{} (3,4) \xrightarrow[4]{} (4),$$

$$(1) \xrightarrow[1]{} (1,2) \xrightarrow[2]{} (2,3) \xrightarrow[3]{} (2,3,4) \xrightarrow[4]{} (2,4) \xrightarrow[2]{} (1,2,3,4) \xrightarrow[3]{} (1,2,3) \xrightarrow[2]{} (2).$$

g) For the type (G_2): $(1) \xrightarrow[1]{} (1,2) \xrightarrow[2]{} (2)$. Thus any involution is conjugate to $h(1)$. q.e.d.

Let K be an infinite field of characteristic $\neq 2$. We denote by $\mathfrak{N}_i$ the normalizer of $h(i)$ in G_K, which is an algebraic subgroup of G_K, and by $(\mathfrak{N}_i)_0$ the irreducible component of the identity of $\mathfrak{N}_i$. When we express the root r by f.r.s.: $r=\sum_{i=1}^{l} m_i(r)a_i$, let E_i be the set of roots such that $m_i(r)$ is even. Then we have the following:

Lemma 8. *Let K be an infinite field whose characteristic is not* 2, *then the Lie algebra $\mathfrak{n}_i$ of the group $(\mathfrak{N}_i)_0$ is generated by $\bar{\mathfrak{h}}$, the Lie algebra of $\mathfrak{H}_K$, and* ad X_r, $r\in E_i$.

Proof. Let $\mathfrak{n}_i'$ be the Lie algebra formed by the element $\bar{X}\in\bar{\mathfrak{G}}$, the Lie algebra of G_K, such that $h(i)\bar{X}h(i)^{-1}=\bar{X}$. Since $\bar{\mathfrak{G}}$ is generated by ad X_r and $\bar{\mathfrak{h}}$, and also

$$h(i)\bar{H}h(i)^{-1}=\bar{H} \qquad \text{for all } \bar{H}\in\bar{\mathfrak{h}}$$

$$h(i)(\mathrm{ad}\,X_r)h(i)^{-1}=(-1)^{m_i(r)}\,\mathrm{ad}\,X_r \qquad \text{for all root } r,$$

$\mathfrak{n}_i'$ is generated by $\bar{\mathfrak{h}}$ and ad X_r, $r\in E_i$. It is obvious that $\mathfrak{n}_i'\supseteqq\mathfrak{n}_i$. On the other hand $(\mathfrak{N}_i)_0$ contains $\mathfrak{H}_i$ $(1\leqq i\leqq l)$ and $\mathfrak{X}_r, r\in E_i$ (cf. (9) of § 1). Therefore its Lie algebra $\mathfrak{n}_i$ contains $\bar{\mathfrak{h}}$ and ad $X_r, r\in E_i$ (cf. proof of Propositions 3 and 4). Thus $\mathfrak{n}_i'=\mathfrak{n}_i$. q.e.d.

Proposition 7. *If K is an infinite field whose characteristic is not* 2, *then the Lie algebra $\mathfrak{n}_i$ is isomorphic to the following, where we denote by $\mathfrak{g}(*)$ the Lie algebra which is isomorphic to the Lie algebra of G_K constructed from the simple Lie algebra of the type* $(*)$, *and by $\mathfrak{a}_1$ the* 1-*dimensional commutative Lie algebra.*

a) *For the type* $(A_l), l\geqq 2$

$$\mathfrak{n}_1\cong\mathfrak{a}_1\oplus\mathfrak{g}(A_{l-1})$$

$$\mathfrak{n}_i\cong\mathfrak{a}_1\oplus\mathfrak{g}(A_{i-1})\oplus\mathfrak{g}(A_{l-i}) \qquad 2\leqq i\leqq[l+1/2].$$

b) *For the type* $(B_l), l\geqq 2$

$$\mathfrak{n}_i\cong\mathfrak{g}(D_i)\oplus\mathfrak{g}(B_{l-i}) \qquad 1\leqq i\leqq l-1$$

$$\mathfrak{n}_l\cong\mathfrak{g}(D_l),$$

where $\mathfrak{g}(D_1)=\mathfrak{a}_1$, $\mathfrak{g}(D_2)=\mathfrak{g}(A_1)\oplus\mathfrak{g}(A_1)$ *and* $\mathfrak{g}(D_3)=\mathfrak{g}(A_3)$.

c) *For the type* $(C_l), l\geqq 3$

$$\mathfrak{n}_i \cong \mathfrak{g}(C_i) \oplus \mathfrak{g}(C_{l-i}) \qquad 1 \leqq i \leqq [l/2]$$
$$\mathfrak{n}_l \cong \mathfrak{a}_1 \oplus \mathfrak{g}(A_{l-1}),$$

where $\mathfrak{g}(C_i) = \mathfrak{g}(A_i)$ *for* $i = 1, 2$.

d) *For the type* $(D_l), l \geqq 4$

$$\mathfrak{n}_i \cong \mathfrak{g}(D_i) \oplus \mathfrak{g}(D_{l-i}) \qquad 1 \leqq i \leqq [l/2]$$
$$\mathfrak{n}_l \cong \mathfrak{a}_1 \oplus \mathfrak{g}(A_{l-1}),$$

where $\mathfrak{g}(D_i)$ *for* $i = 1, 2, 3$, *have the same meaning as in* b).

e) *For the type* $(E_l), l = 6, 7, 8$.

$\mathfrak{n}_1 \cong \mathfrak{a}_1 \oplus \mathfrak{g}(D_5), \mathfrak{g}(A_1) \oplus \mathfrak{g}(D_6), \mathfrak{g}(D_8)$ *respectively for* $l = 6, 7, 8$.

$\mathfrak{n}_2 \cong \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_5), \mathfrak{g}(A_1) \oplus \mathfrak{g}(E_7)$ *respectively* $l = 6, 8$.

$$\mathfrak{n}_4 \cong \mathfrak{g}(A_7) \qquad \textit{for } l = 7.$$
$$\mathfrak{n}_7 \cong \mathfrak{a}_1 \oplus \mathfrak{g}(E_6) \qquad \textit{for } l = 7.$$
$$\mathfrak{n}_8 \cong \mathfrak{b}(A_1) \oplus \mathfrak{g}(E_7) \qquad \textit{for } l = 8.$$

f) *For the type* (F_4),

$$\mathfrak{n}_1 \cong \mathfrak{g}(A_1) \oplus \mathfrak{g}(C_3), \qquad \mathfrak{n}_3 \cong \mathfrak{g}(B_4).$$

g) *For the type* (G_2),

$$\mathfrak{n}_1 \cong \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_1).$$

Proof.

a) For the type (A_l): The positive roots of the type (A_l) are formed by the following:

$$a_i \ (1 \leqq i \leqq l) \text{ and } a_i + a_{i+1} + \cdots + a_k \ (1 \leqq i < k \leqq l).$$ [20]

Among the roots of E_i $(i > 1)$, $(a_1, \cdots, a_{i-1})$ and $(a_{i+1}, \cdots, a_l)$ form the f.r.s. of the types (A_{i-1}) and (A_{l-i}) respectively and these systems are orthogonal each other. The roots of E_1 form a system of roots of the type (A_{l-1}) with a f.r.s. $(a_2, \cdots, a_l)$.

b) For the type (B_l): The positive roots of the type (B_l) are formed by the roots of the type (A_l) and the following:

$$a_i + a_{i+1} + \cdots + a_{k-1} + 2a_k + 2a_{k+1} + \cdots + 2a_l, \ 1 \leqq i < k \leqq l.$$ [21]

The positive roots of the type (D_l) are formed by the roots of the type (A_l) and the following:

$$a_i + a_{i+1} + \cdots + a_{l-2} + a_l \qquad 1 \leqq i \leqq l-2$$
$$a_i + a_{i+1} + \cdots + a_{k-1} + 2a_k + \cdots + 2a_{l-2} + a_{l-1} + a_l, \quad 1 \leqq i < k < l-1.$$ [22]

Among the roots of E_i, $(a_1, \cdots, a_i, r)$, where $r = a_{i-1} + 2a_i + 2a_{i+1} + \cdots + 2a_l$, and

20) Seligmann [**12**, p. 43].
21) ibid. p. 51.
22) ibid. p. 48.

$(a_{i+1}, a_{i+2}, \cdots, a_l)$ form the f. r. s. of the types (D_i) and (B_{l-i}) respectively and these systems are orthogonal each other.

c) For the type (C_l): The positive roots of the type (C_l) are formed by the roots of the type (A_l) and the following:

$$a_i+a_{i+1}+\cdots+a_{k-1}+2a_k+\cdots+2a_{l-1}+a_l\,, \qquad 1\leqq i<k\leqq l-1$$

$$2a_i+2a_{i+1}+\cdots+2a_{l-1}+a_l \qquad 1\leqq i\leqq l-1\,.$$ [23]

Among the roots of E_i $(i\leqq l-1)$, $(a_1, a_2, \cdots, a_{i-1}, r)$ where $r=2a_i+2a_{i+1}+\cdots+2a_{l-1}+a_l$, and $(a_{i+1}, a_{i+2}, \cdots, a_l)$ form the f. r. s. of the types (C_i) and (C_{l-i}) respectively and these systems are orthogonal each other. The roots of E_l form a system of roots of the type (A_{l-1}) with a f. r. s. $(a_1, a_2, \cdots, a_{l-1})$.

d) For the type (D_l), $l\geqq 4$: Among the roots of E_i $(i\leqq l-1)$, $(a_1, a_2, \cdots, a_i, r)$ where $r=a_{i-1}+2a_i+2a_{i+1}+\cdots+2a_{l-1}+a_l$, and $(a_{i+1}, a_{i+2}, \cdots, a_l)$ form the f. r. s. of the types (D_i) and (D_{l-i}) respectively and these systems are orthogonal to each other. The roots of E_l form a system of roots of the type (A_{l-1}) with a f. r. s. $(a_1, a_2, \cdots, a_{l-1})$.

e) For the type (E_l): For the form of roots of the types (E_6), (E_7) and (E_8), we refer to Seligman [**12**]. (N. B. The numbering of the f. r. s. in [**12**] of p. 59 is different from ours.) Among the roots of E_1, $(a_6, a_5, a_3, a_4, a_2)$, $(a_7, a_6, a_5, a_3, a_4, a_2)$ and $(r, a_8, a_7, a_6, a_5, a_3, a_4, a_2)$, where $r=2a_1+3a_2+4a_3+2a_4+3a_5+2a_6+a_7$, form the f. r. s. of the types (D_5), (D_6) and (D_8) respectively for the types (E_6), (E_7) and (E_8), and for the type (E_7), r is orthogonal to them.

Among the roots of E_2, (a_4, a_3, a_5, a_6, r), $(a_4, a_3, a_5, a_6, r, a_7)$ and $(a_4, a_3, a_5, a_6, r, a_7, a_8)$, where $r=a_1+2a_2+2a_3+a_4+a_5$, form the f. r. s. of the types (A_5), (D_6) and (E_7) respectively for the types (E_6), (E_7) and (E_8), and a_1 is orthogonal to them.

Among the roots of E_4, $(a_1, a_2, a_3, a_5, a_6)$, $(a_1, a_2, a_3, a_5, a_6, a_7, r)$ and $(a_1, a_2, a_3, a_5, a_6, a_7, r, a_8)$, where $r=a_1+2a_2+3a_3+2a_4+2a_5+a_6$, form the f. r. s. of the types (A_5), (A_7) and (D_8) respectively for the types (E_6), (E_7) and (E_8), and for the type (E_6), r is orthogonal to them.

Finally, among the roots of E_l, $(a_1, a_2, \cdots, a_5)$, $(a_1, a_2, \cdots, a_6)$ and $(a_1, a_2, \cdots, a_7)$ form the f. r. s. of the types (D_5), (E_6) and (E_7) respectively for the types (E_6), (E_7) and (E_8), and for the type (E_8), $2a_1+4a_2+6a_3+3a_4+5a_5+4a_6+3a_7+a_8$ is orthogonal to them.

f) For the type (F_4): For the form of the roots of the type (F_4), we refer to Seligman [**12**, p. 57]. Among the roots of E_1, (a_4, a_3, a_2) form the f. r. s. of the type (C_3) and the root $2a_1+3a_2+4a_3+2a_4$ is orthogonal to them. Among the roots of E_3, (a_2, a_1, r, a_4), where $r=a_2+2a_3$, form the f. r. s. of the type (B_4).

23) ibid. p. 54.

g) For the type (G_2): The positive roots of the type (G_2) are formed by

$$a_1, a_2, a_1+a_2, a_1+2a_2, a_1+3a_2, 2a_1+3a_2 .$$

Among the roots of E_1 and E_2, a_2 and $2a_1+3a_2, a_1$ and a_1+2a_2 are orthogonal each other and form the f. r. s. of the types (A_1) respectively.

From these facts and Lemma 8, we can easily see the Proposition. q. e. d.

LEMMA 9. *If K is an infinite field whose characteristic is not* 2 *or* 3 *and is not a factor of* $\mathrm{ord}(P/P_r)$, *then* $\mathrm{ad}\,\mathfrak{h}_K$ *is a Cartan subalgebra of* $\mathrm{ad}\,\mathfrak{g}_K$. $\mathfrak{H}_K$ *is a Cartan subgroup of* G_K.[24]

PROOF. Since $|r(H_s)|\leqq 3$, for any root s there exists a root r such that $[H_r, X_s]=s(H_r)X_s$, $s(H_r)\neq 0$. Therefore the normalizer of $\mathfrak{h}_K$ in $\mathfrak{g}_K$ is $\mathfrak{h}_K$ itself. Thus $\mathfrak{h}_K$ is a Cartan subagebra of $\mathfrak{g}_K$. Since $\mathrm{ad}\,\mathfrak{g}_K\cong\mathfrak{g}_K$, $\mathrm{ad}\,\mathfrak{h}_K$ is a Cartan subalgebra of $\mathrm{ad}\,\mathfrak{g}_K$.

If $s\in G_K$ is in the normalizer of $\mathfrak{H}_K$ and if we express s as (12), then (32) holds for all $h(\chi_1)\in\mathfrak{H}_K$. Thus $x''=\prod_{r>0} x_r(t_r)=\bar{x}''=\prod_{r>0} x_r(\chi_1(r)t_r)$ for all $\chi_1\in \mathrm{Hom}(P_r, K^*)$. Therefore $x''=\bar{x}''=1$ and similarly $x=\bar{x}=1$, i. e. $s\in\mathfrak{W}_K$. Conversely, the element of $\mathfrak{W}_K$ is in the normalizer of $\mathfrak{H}_K$. (cf. (11)). Thus the normalizer of $\mathfrak{H}_K$ in G_K is $\mathfrak{W}_K$. The factor group $\mathfrak{W}_K/\mathfrak{H}_K$ is the finite group (isomorphic to the weyl group) (cf. § 1). Therefore $\mathfrak{H}_K$ is the connected component of the identity of $\mathfrak{W}_K$, and $\mathfrak{H}_K$ is a Cartan subgroup of G_K.[25] q. e. d.

THEOREM 4. *If K is an algebraically closed field whose characteristic is $\neq 2$, then the involutions in G_K in the Proposition* 6 *form a system of complete representatives of the non-conjugate involutions in G_K.*

PROOF. Since an involution in G_K is semi-simple, it is contained in a Cartan subgroup of G_K. Moreover Cartan subgroups of G_K are conjugate.[26] Since $\mathfrak{H}_K$ is a Cartan subgroup of G_K from Lemma 9, it is sufficient to consider the involutions in $\mathfrak{H}_K$. If two involutions in $\mathfrak{H}_K$ are conjugate each other, then their normalizers are isomorphic. Thus we have our assertion except for the types (D_l) where l is even from Propositions 6 and 7.

For the type (D_l) where $l\geqq 4$ and l is even, the normalizers of $h(l-1)$ and $h(l)$ are isomorphic to $\mathfrak{a}_1\oplus\mathfrak{g}(A_{l-1})$ and for the type (D_4), the normalizers of $h(1), h(3)$ and $h(4)$ are isomorphic to $\mathfrak{a}_1\oplus\mathfrak{g}(A_3)$. However the remark of the proof of d) of Proposition 6, we see that $h(l-1)$ and $h(l)$ for $l\geqq 6$ and $h(1)$, $h(3)$ and $h(4)$ for $l=4$, are non-conjugate each other. (Remark; These involutions are conjugate each other by suitable "outer" automorphisms.) q. e. d.

24) A Cartan subgroup of G is a maximal nilpotent subgroup such that all its subgroup of finite index is also finite index in its normalizer.

25) Borel [**1**, p. 75].

26) Borel [**1**, (20.4) and (20.5)].

§ 4. Automorphisms of G_K.

We denote by $G_K(*)$ the group of Chevalley constructed from the simple Lie algebra of the type $(*)$. Let K be an infinite field, we denote by $A(G_K)$ the group of birational and biregular automorphisms of G_K and by $I(G_K)$ the normal subgroup of $A(G_K)$ formed by inner automorphisms of G_K.

Lemma 10. *If K is infinite field, every birational and biregular automorphism of $G_K(A_1)$ is inner.*

Proof. We shall identify $G_K(A_1)$ to $PSL_2(K)$ by the homomorphism ϕ_r (cf. § 1). Any automorphism of $PSL_2(K)$ is induced by an automorphism of $SL_2(K)$.[27] Moreover any automorphism of $SL_2(K)$ have the following form:

$$A \to PA^{\sigma}P^{-1}, \qquad A \in SL_2(K),$$

where P is a non-singular matrix and σ an automorphism of $SL_2(K)$ induced by an automorphism of K.[28] If such an automorphism is rational over K, then σ is an identity automorphism. Thus any birational and biregular automorphism of $SL_2(K)$ is

$$A \to PAP^{-1}, \text{ where } P \text{ is a non-singular matrix.}$$

Since the non-singular matrix P is a product of the matrices of the types $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, if we identify the automorphisms of $PSL_2(K)$ to $G_K(A_1)$, the automorphisms induced by these matrices are, for G_K, the inner automorphisms induced by $x_{-r}(u)$, $x_r(t)$ and $h(\chi)$ where $\chi(r)=a^2$, and r is a root, respectively. Therefore we have our assertion. q. e. d.

Proposition 8. *If $\mathfrak{g}$ is a simple Lie algebra not of the type (D_4) and K is an algebraically closed field whose characteristic is not 2, then $A(G_K)=I(G_K)$.*

Proof. We shall prove the proposition by induction with respect to the rank of $\mathfrak{g}$. From Lemma 10, the Proposition holds for the rank 1.

i) For the type (A_l), we assume that the proposition holds for the group $G_K(A_m), m<l$. Let σ be an automorphism of the group $G(A_l)$ and we shall show that σ is the identity automorphism except for an inner automorphism. Since Cartan subgroups $\mathfrak{H}_K$ and $\mathfrak{H}_K^{\sigma}$ are conjugate each other, we may suppose that $\mathfrak{H}_K=\mathfrak{H}_K^{\sigma}$. Moreover since the normalizers of $h(1)$ and $h(1)^{\sigma}$ are isomorphic, from Propositions 6 and 7, $h(1)$ and $h(1)^{\sigma}$ are conjugate each other. Thus we may suppose also that $h(1)=h(1)^{\sigma}$. Then $\mathfrak{N}_1=\mathfrak{N}_1^{\sigma}$ and $(\mathfrak{N}_1)_0=(\mathfrak{N}_1)_0^{\sigma}$. Denote by G_1 the irreducible subgroup of $\mathfrak{N}_1$ generated by $\mathfrak{H}_i$ $(i \geqq 2)$ and

27) J. Dieudoone̒ [4], Supplemented note by L. K. Hua, Theorem 4, p. 109.
28) L. K. Hua [6, Appendix].

$\mathfrak{X}_r, r \in E_1$, which is isomorphic to $G_K(A_{l-1})$. The Lie algebra of G_1 which is an ideal of $\mathfrak{n}_1$ isomorphic to $\mathfrak{g}_K(A_{l-1})$ is generated by $\operatorname{ad} H_i^*$ $(i \geqq 2)$ and $\operatorname{ad} X_r$, $r \in E_1$ (cf. Proof of Propositions 3 and 4). Denote by $\mathfrak{A}$ the irreducible subgroup of $\mathfrak{N}_1$ generated by $\mathfrak{H}_1$. The Lie algebra of $\mathfrak{A}$ is generated by $\operatorname{ad} H_1^*$. We may easily see that G_1 and $\mathfrak{A}$ are normal in $\mathfrak{N}_1$, and $\mathfrak{N}_1$ is the direct product of G_1 and $\mathfrak{A}$. Since $\dim \mathfrak{A}=1$, and $\dim G_1 \geqq 3, G_1 \cap G_1^{\sigma}$ is not $\{1\}$. Since G_1^{σ} and G_1 are normal in $\mathfrak{N}_1, G_1 \cap G_1^{\sigma}$ is a non-trivial normal subgroup of G_1. Therefore we have that $G_1=G_1^{\sigma}$. Since $\mathfrak{A} \subseteqq \mathfrak{H}_K$, we have also that $\mathfrak{A}=\mathfrak{A}^{\sigma}$. By induction hypothesis, we may suppose that σ fixes the elements of G_1. Then since $h(2) \in G_1, \mathfrak{N}_2^{\sigma}=\mathfrak{N}_2$. If we denote by G_2 the subgroup of $\mathfrak{N}_2$ generated by $\mathfrak{H}_1, \mathfrak{X}_{a_1}$ and $\mathfrak{X}_{-a_1}$, then G_2 is a normal subgroup of $\mathfrak{N}_2$ isomorphic to $G_K(A_1)$, and $G_2^{\sigma}=G_2$ follows from $G_2^{\sigma} \cap G_2 \supseteqq \mathfrak{H}_1$. From Lemma 10, σ induces an inner automorphism. Since σ fixes the irreducible subgroup $\mathfrak{H}_1$ of G_2, σ might be an inner automorphism induced by the element $h(\chi) \in G_2$ (cf. (9) of § 1). Thus we may suppose that σ fixes the elements of G_1 and G_2 at the same time. Since G_K is generated by G_1 and G_2,[29] we have our assertion.

ii) For the type $(B_l), (C_l)$ and (F_4), we may prove the Proposition by the same way as in i).

iii) For the type $(D_l), l \geqq 5$, since the proposition is true for the type (A_4), if we replace $\mathfrak{N}_1$ and $\mathfrak{N}_2$ in the proof of i) by $\mathfrak{N}_4$ and $\mathfrak{N}_3$, we may prove for the type (D_5). Thus we have our assertion by induction with respect to the rank, similarly to i).

iv) For the type $(E_l), l=6, 7$ and 8, Since the Proposition is true for the type $(D_l), l \geqq 5$, we have our assertion by induction similarly to i).

v) For the type (G_2), in this case $\mathfrak{N}_1$ is isomorphic to $G_1 \times G_2$, where G_i $(i=1, 2)$ is isomorphic to $G_K(A_1)$. If the automorphism induced in $\mathfrak{N}_1$ transforms G_i into G_j $(i \neq j, i, j=1, 2)$, then if we adjoin the inner automorphism induced by a suitable $\omega(w)$, we have $G_i^{\sigma}=G_i$ $(i=1, 2)$. Therefore we may suppose σ fixes the elements of both G_1 and G_2, and we have our assertion.

q. e. d.

LEMMA 11. *If K is an infinite field (not necessarily algebraically closed) and if $\mathfrak{g}$ is of the type (D_4), there is an element of $A(G_K)$ of order 3 which is not inner.*

PROOF. From the proof of Theorem 3, G_K'' $(G_K \supseteqq G_K'' \supseteqq G_K')$ is isomorphic to $PO_8^+(K, f), f$ being a quadratic form of maximal index. Since G_K'' is dense in G_K (in the sense of Zariski topology), the rational automorphism of $PO_8^+(K, f)$ induces a rational automorphism of G_K.[30] On the other hand, J. Dieudonné has noted that there is an automorphism of $PO_8^+(K, f)$ which is

29) Chevalley [3, Lemma 4, p. 38].

30) See for example Borel [1, Proposition 5.2].

not induced by any automorphism of $O_8^+(K,f)$.[31] We may see that this automorphism is rational. Moreover, since the inner automorphisms of $PO_8^+(K,f)$ are induced by the inner automorphisms of $O_8^+(K,f)$, it is not inner. Thus we have our assertion. q. e. d.

Proposition 9. *If $\mathfrak{g}$ is of the type (D_4) and K is the algebraically closed field of characteristic $\neq 2$, then $A(G_K)/I(G_K)$ is the cyclic group of order* 3.

Proof. Let σ be an automorphism of G_K, which is a representative of a co-set of $A(G_K)/I(G_K)$. We may suppose that $\mathfrak{H}_K{}^\sigma=\mathfrak{H}_K$, and $h(1)^\sigma=h(1), h(3)$ or $h(4)$. If $h(1)^\sigma=h(1)$, then $\mathfrak{N}_1{}^\sigma=\mathfrak{N}_1$. Denote by G_1 (resp. $\mathfrak{A}$) the normal subgroup of $\mathfrak{N}_1$ generated by $\mathfrak{H}_i$ $(i>1)$ and $\mathfrak{X}_r, r\in E_1$ (resp. $\mathfrak{H}_1$), then we may suppose that G_1 and $\mathfrak{A}$ are transformed into itself by σ. Since the automorphisms of the group of the type (A_3) are all inner, σ might be inner by the same consequence of the Proposition 8. If each $h(1), h(3)$ and $h(4)$ are not fixed by σ, then σ induces a cyclic permutation of $h(1), h(3)$ and $h(4)$. When two such automorphisms σ_1 and σ_2 induce the same permutation of $(h(1), h(3), h(4))$, $\sigma_1\sigma_2^{-1}$ is inner from above consequence, so the number of co-sets of $A(G_K/I(G_K)$ is at most 3. On the other hand, from Lemma 11 there is an outer automorphism of order 3. Therefore we have $A(G_K)/I(G_K)$ is the cyclic group of order 3. q. e. d.

Theorem 5. *If $\mathfrak{g}$ is simple and K is an infinite field whose characteristic is not* 2, *then*

$$A(G_K)=I(G_K) \qquad \textit{except for the type } (D_4)\,,$$

$$A(G_K)/I(G_K) \textit{ is the cyclic group of order } 3 \textit{ for the type } (D_4)\,.$$

Proof. Let σ be an element of $A(G_K)$. If L is an algebraically closed extension field of K, then $G_L=(G_K)^L$ [32] and σ induces one and only one automorphism σ^L of G_L[33]. From Proposition 8, if $\mathfrak{g}$ is not of the type (D_4), σ^L is an inner automorphism induced by an element s of G_L. We shall show $s\in G_K$, and it follows that σ is an inner automorphism of G_K. Let $d\sigma^L$ be the differential of σ^L, then

$$d\sigma^L:\ \mathrm{ad}\,X\to s(\mathrm{ad}\,X)s^{-1}=\mathrm{ad}(sX)\,, \qquad X\in\mathfrak{g}_L\,.$$

Since $(d\sigma)^L=d\sigma^L$ [34], we have

$$d\sigma:\ \mathrm{ad}\,X\to \mathrm{ad}(sX)\,, \qquad X\in\mathfrak{g}_K\,.$$

Therefore $sX\in\mathfrak{g}_K$ for all $X\in\mathfrak{g}_K$, and s must be in $GL(n,K)$. From Theorem 1 of [8], $s\in G_L\cap GL(n,K)=G_K$.

31) Dieudonné [4, p. 60].
32) Ono [8, Theorem 2].
33) Chevalley [3, Proposition 9, p. 109].
34) Chevalley [3, Definition 1 and its note, p. 138].

If $\mathfrak{g}$ is of the type (D_4), the mapping $\sigma \rightarrow \sigma^L$ is an isomorphism of $A(G_K)$ into $A(G_L)$. From above consequence, we have that σ is inner if and only if σ^L is inner. Therefore from Proposition 9 the order of $A(G_K)/I(G_K)$ is at most 3. From Lemma 11, we have that $A(G_K)/I(G_K)$ is the cyclic group of order 3. q. e. d.

REMARK. If we prove the Theorem 3 for the group over complex number field, then by the theory of reduction mod $\mathfrak{p}$ of linear algebraic groups (cf. Ono [8], [9]), we have the Theorem for general case. As for the Theorem 5, if K is the complex number field, $A(G_K)/I(G_K)$ can be identified to the subgroup of $A(\mathfrak{g})/I(\mathfrak{g})$, which is the cyclic group of order 2 for the type (A_l), $l \geqq 2$, (D_l), $l \geqq 5$, and (E_6), the symmetric group of degree 3 for the type (D_4), and has a unit element only for all other types. Therefore, in the case where $A(\mathfrak{g})/I(\mathfrak{g})=\{1\}$, we have $A(G_K)=I(G_K)$ immediately. It is also desirable that we would have the Theorem for general case by the theory of reduction mod. $\mathfrak{p}$, after it is proved only for the case where K is the complex number field.

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References

[1] A. Borel, Groupes linéaires algébriques, Ann. of Math., **64** (1956), 20-82.
[2] C. Chevalley, Théorie des groupes de Lie, Tome II, Hermann, Paris, 1951.
[3] C. Chevalley, Sur certains groupes simples, Tohoku Math. J., **7** (1955), 14-66.
[4] J. Dieudonné, On the automorphism of classical groups, Mem. of Amer. Math. Soc., **2** (1951).
[5] J. Dieudonné, La géométrie des groupes classiques, Ergebnisse der Math., **5** (1955).
[6] L. K. Hua, On the automorphisms of the symplectic group over any field, Ann. of Math., **49** (1948), 739-759.
[7] L. K. Hua and I. Reiner, Automorphisms of the unimodular groups, Trans. Amer. Math. Soc., **71** (1951), 331-348.
[8] T. Ono, Sur les groupes de Chevalley, J. Math. Soc. Japan, **10** (1958), 307-313.
[9] T. Ono, Sur la reduction modulo $\mathfrak{p}$ des groupes linéaires algébriques, Osaka Math, J., **10** (1958), 57-73.
[10] R. Ree, On some simple groups defined by Chevalley, Trans. Amer. Math. Soc., **84** (1957) 392-400.
[11] I. Reiner, Automorphisms of symplectic modular group, ibid., **80** (1955), 35-50.
[12] G. Seligman, On Lie algebras of prime characteristic, Mem. of Amer. Math. Soc., **19** (1956).
[13] H. Weyl, The structure and representation of continuous groups, Lecture Note (1934-1935).