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On the local cross-sections in locally compact groups.

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Let G be a topological group, H a closed subgroup of G, p the natural map of G onto the left factor space G/H. e will denote the identity of G. These notations G, H, p, e will keep these meanings throughout the paper. (We can of course deal with the right factor space just as the left factor space. So we limit ourselves on the consideration of the left factor space. The terms like factor space, coset etc. without further qualification will always mean left factor space, left coset etc. in the following.) A continuous map f defined on a neighborhood U of any point in G/H with values in G such that $pf(x)=x$ for each $x\in U$, is called a *local cross-section of H in G* (cf. [7]). It is known that H has a local cross-section in G, if H is a compact Lie group [1] or if G is a locally compact finite-dimensional (separable metric) group [4]. In this paper, we shall prove these facts by actually constructing local cross-sections. These results will be thus proved by a unified method in a simpler way than in the literature and we shall obtain another sufficient condition for the existence of a local cross-section (see Theorem 2 below.). As an application, we obtain a simple proof of a theorem on the dimensions of factor spaces (Theorem 3).

1. The fundamental theorem.

We begin with the following lemma.

LEMMA 1. *H has a local cross-section in G, if there exists a compact subset W of G containing e such that*

1) *WH is a neighborhood of e in G,*
2) $W^{-1}W\cap H=\{e\}$.

The converse is also true if G is locally compact.

PROOF. Suppose that there exists a compact subset $W\ni e$ satisfying 1), 2). Put $e^*=p(e)$, $U=p(W)$, then U is a neighborhood of e^* by 1) and $p'=p\,|\,W$ is the one-to-one map of W onto U by 2). Since W is compact, p' is topological and the inverse map $f=p'^{-1}$ is a local cross-section. To prove the converse, we can suppose without loss of generality that a local cross-section f is defined on a compact neighborhood U of e^* such that $f(e^*)=e$. Put $W=f(U)$. W is a compact set containing e such that WH is a neighborhood of e. To prove 2), let $w_1^{-1}w_2 \in H$, $w_1 = f(u_1)$, $w_2 = f(u_2)$, $u_1, u_2 \in U$, then $p(w_1) = p(w_2)$ and so $u_1 = u_2$ i.e. $w_1 = w_2$. q.e.d.

In the following we shall consider exclusively the case where G is locally compact. Under this condition, the existence of a local cross-section of H in G is equivalent with that of W satisfying the conditions of the above lemma. So we may *call W itself a local cross-section of H.* Then we can formulate our fundamental theorem as follows.

THEOREM 1. *Let G be a locally compact group and H a closed subgroup of G. Then there exists an open subgroup K of G and a compact invariant subgroup Z of K such that the factor group K/Z is a Lie group, and if there exists a compact cross-section of $H \cap Z$ in Z then H has a local cross-section in G.*

PROOF. As G is locally compact, there exists an open compact subgroup K' in the factor group G/G_0 where G_0 is the component of e. Let K be the complete inverse image of K' under the natural map $G \rightarrow G/G_0$, then K is an open subgroup of G. Since the component K_0 of e in K coincides with G_0, K/K_0 is isomorphic to K' and hence compact. Consequently there exists an arbitrarily small compact invariant subgroup Z of K such that K/Z is a Lie group ([5], p. 175).

Let K^* denote the factor group K/Z and π the natural map of K onto K^*. Moreover let $M = H \cap K$, $M^* = \pi(M)$. M^* is a Lie group as a closed subgroup of a Lie group K^*. In K^* choose m one-parameter subgroups ($m = \dim K^*$)

$$x_1^*(t), \cdots, x_n^*(t), x_{n+1}^*(t), \cdots, x_m^*(t);\ |t| \leqq \alpha$$

which can be used as canonical coordinates of the second kind. We can suppose here without loss of generality that

$$x_{n+1}^*(t), \cdots, x_m^*(t);\ |t| \leqq \alpha$$

generate a neighborhood of the identity e^* in M^* ($m - n = \dim M^*$). Then in K choose one-parameter subgroups $\{x_i(t);\ i = 1, 2, \cdots, n\}$ such that

$$\pi x_i(t) = x_i^*(t);\ |t| \leqq \alpha,\ i = 1, 2, \cdots, n \qquad \text{([5], pp. 192)}.$$

Let

$$V = \{x_1^*(t_1) x_2^*(t_2) \cdots x_m^*(t_m);\ |t_j| \leqq \alpha,\ j = 1, 2, \cdots, m\}$$

which is a neighborhood of e^* in K^*. For every sufficient small positive number β, $W^{*-1} W^* \subset V$ where

$$W^* = \{x_1^*(t_1) x_2^*(t_2) \cdots x_n^*(t_n);\ |t_i| \leqq \beta,\ i = 1, 2, \cdots, n\}.$$

The set W^* is a local cross-section of M^* in K^* and is covered by the compact set

$$W=\{x_1(t_1)x_2(t_2)\cdots x_n(t_n)\,;\ |t_i|\leqq\beta,\ i=1,2,\cdots,n\}\,.$$

Now suppose that the group Z has a compact cross-section of $H\cap Z$ i.e. there is a compact subset D containing the identity e such that

$$Z=D(H\cap Z)\ \text{and}\ D^{-1}D\cap(H\cap Z)=\{e\}\,.$$

Let $\widetilde{W}=WD$. Then $\widetilde{W}$ is a local cross-section of M in K.

In fact, it is obvious that $\widetilde{W}$ is a compact set containing e. Next from the fact

$$WMZ=WZM=WD(H\cap Z)M=\widetilde{W}M$$

it follows that $\widetilde{W}M$ is a neighborhood of e in K. Finally the condition $\widetilde{W}^{-1}\widetilde{W}\cap M=\{e\}$ is proved as follows.

If $\tilde{w}^{-1}\tilde{w}'\in M$, $\tilde{w}=wd$, $\tilde{w}'=w'd'$, $w,w'\in W$, $d,d'\in D$, then $\pi(\tilde{w}^{-1}\tilde{w}')=(\pi(w))^{-1}\pi(w')\in W^{*-1}W^{*}\cap M^{*}$. And so $\pi(w)=\pi(w')$. Let

$$w=x_1(t_1)x_2(t_2)\cdots x_n(t_n),\ w'=x_1(t_1')x_2(t_2')\cdots x_n(t_n')\,,$$

then

$$x_1^*(t_1)x_2^*(t_2)\cdots x_n^*(t_n)=x_1^*(t_1')x_2^*(t_2')\cdots x_n^*(t_n')\,.$$

Consequently it follows from the properties of canonical coordinates of the second kind that $t_1=t_1'$, $t_2=t_2'$, $\cdots$, $t_n=t_n'$ i.e. $w=w'$. Therefore $\tilde{w}^{-1}\tilde{w}'=d^{-1}d'\in D^{-1}D\subset Z$. Now $\tilde{w}^{-1}\tilde{w}'\in M$ and so $\tilde{w}^{-1}\tilde{w}'\in H\cap Z$. But since $D^{-1}D\cap(H\cap Z)=\{e\}$, it follows that $\tilde{w}^{-1}\tilde{w}'=e$.

Finally $\widetilde{W}$ is also a local cross-section of H in G, since K is open in G and $\widetilde{W}^{-1}\widetilde{W}$ is contained in K.

2. Applications.

We are going to make use of the above results to derive some consequences on local cross-sections of subgroups and dimensions of factor spaces.

As the simplest case, suppose $H\cap Z$ is a direct factor of Z. Then the condition of Theorem 1 is satisfied. The trivial case where H is a direct factor of G, and the case where $H\cap Z=\{e\}$ or Z, are contained in this case. If there are no arbitrarily small non-trivial subgroups in H, there is a Z such that $H\cap Z=\{e\}$ and so H has a local cross-section in G. This gives an extension of Gleason's results so far as we are concerned with a locally compact group G. If all arbitrarily small subgroups of G are in H, there is a Z such that $H\cap Z=Z$. Moreover, when Z is 0-dimensional, the condition of Theorem 1 is satisfied as will be shown below (although $H\cap Z$ is not always the direct factor of Z). This contains the case where G is a locally compact finite-dimensional group.

In order to make free use of dimension theory *it is assumed hereafter that G is separable metric.*

LEMMA 2. *Let Z be a 0-dimensional compact group, X a closed subgroup of Z. Then there is a cross-section of X in Z i.e. there exists a compact set Y containing the identity e such that $Z=YX$ and $Y^{-1}Y\cap X=\{e\}$.*

PROOF. Z is the limit group of a countable number of finite groups Z_i with the homomorphisms π_i of Z_{i+1} onto Z_i. Let p_i be the projection of Z on Z_i. Let $X_i=p_i(X)$. We choose now an element from each coset of X_1 in Z_1; from X_1 we choose the identity e_1; and let Y_1 denote the set of the elements of Z_1 thus chosen. It is obvious that $Z_1=Y_1X_1$, $Y_1^{-1}Y_1\cap X_1=\{e_1\}$. In each coset of X_2 in Z_2 there is an element which is mapped into Y_1 by π_1. Choose such an element from each coset of X_2 and from X_2 the identity e_2, and let Y_2 denote the set of the chosen elements. Then $Z_2=Y_2X_2$, $Y_2^{-1}Y_2\cap X_2=\{e_2\}$ and $\pi_1(Y_2)=Y_1$. Repeating this process we obtain finally a cross-section $\lim Y_i$ of X. q.e.d.

Now we suppose that G is a locally compact finite-dimensional group. Then we can choose a 0-dimensional group as Z in Theorem 1, by the structure theorem on finite-dimensional compact groups ([6], p. 213). Therefore the condition of Theorem 1 is satisfied. This gives us another proof of a theorem first proved by Mostert [4].

Thus we obtain the following theorem as a corollary of Theorem 1.

THEOREM 2. *Let G be a locally compact group and H a closed subgroup of G. There is a local cross-section of H in G in each of the following cases:*

1) *G is finite-dimensional,*
2) *there are no arbitrarily small non-trivial subgroups in H,*
3) *all arbitrarily small subgroups of G are in H.*

As another application of Theorem 1, we give a simple proof of the following theorem which was first proved by Yamanoshita [8].

THEOREM 3. *Let G be a locally compact group, H a closed subgroup of G. Then*

$$\dim G=\dim H+\dim G/H.$$

PROOF. If G is infinite-dimensional, we can suppose without loss of generality that G/H is finite-dimensional. Then the natural map p of G onto G/H is a closed mapping on a compact neighborhood of any point in G. So, if H were finite-dimensional, then G would be finite-dimensional ([3], pp. 92, 93). Therefore H is infinite-dimensional and the theorem holds.

If G is finite-dimensional, we can choose a 0-dimensional group as Z and there is a local cross-section $\widetilde{W}$ of H in G—here we use the same notation as in the proof of Theorem 1. The image of $\widetilde{W}=WD$ under p is a compact neighborhood in G/H. Let U denote it. Since $p^{-1}(U)$ is homeomorphic to the product space $H\times U$ ([7]), we have

$$\dim G=\dim p^{-1}(U)=\dim(H\times U)=\dim(V\times WD),$$

where V is any compact neighborhood of any point in H. Now as a one-to-one continuous image of a compact set, the euclidian cube W^* is homeomorphic to W and WD is homeomorphic to $W \times D$. On the other hand the logarithmic law:

$$\dim(A \times B) = \dim A + \dim B$$

holds if B is 0-dimensional or if A is compact and B is 1-dimensional ([2]). Therefore

$$\dim(V \times WD) = \dim(V \times W \times D) = \dim V + \dim(W \times D)$$

i. e.

$$\dim G = \dim V + \dim WD = \dim H + \dim(G/H).$$

3. Groups without local cross-sections.

We see from Theorems 1 and 2 that, if there are no local cross-sections of H in a locally compact group G, there must be arbitrarily small infinite-dimensional compact subgroups Z_ν each having infinitely many points both in H and outside H. Moreover such Z_ν has no cross-sections of $H \cap Z_\nu$.

After [7] (p. 33) there is an unpublished example of Hanner which provides a compact abelian group of infinite-dimension and a closed 0-dimensional subgroup without a local cross-section. The following is another example of the same kind of groups.

Example. Let K be the direct product of infinitely many circle groups $\{K_\mu\}$. Let D be a closed subgroup of K such that the projection of D under the natural map $K \to K_\mu$ is a non-trivial finite group for infinitely many μ. (Here we can give D any dimension as we please.) Then K is not locally homeomorphic to the product space of K/D and D. Hence a local cross-section of D does not exist in K (cf. [7]).

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