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Ergodic property of recurrent diffusion processes.

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§ 1. Introduction.

The purpose of this paper is to prove an ergodic theorem for recurrent diffusion processes (cf. Definition in § 4). Let $\{X^{(x)}(t, \omega)\}$ be such a process starting from x at $t=0$ and let $f(x)$ and $g(x)$ be any m-integrable functions such that $\int_a^b g(x)dm(x) \neq 0$, where m is Feller's canonical measure of this process. Then

$$\lim_{T\to\infty} \frac{\int_0^T f(X^{(x)}(t, \omega))dt}{\int_0^T g(X^{(x)}(t, \omega))dt} = \frac{\int_a^b f(x)dm(x)}{\int_a^b g(x)dm(x)} \tag{1.1}$$

holds with probability one (cf. Theorem 4.1 in § 4).

Ergodic theorems of this type have been investigated by many authors; by G. Kallianpur and H. Robbins [10] and by Dermann [2] for the Brownian motion, by T. E. Harris [8], by P. Lévy [11] and by K. L. Chung [1] for the recurrent Markov chain with a denumerable number of states and by T. E. Harris and H. Robbins [9] for the recurrent Markov chain with discrete time parameter. Our formula (1.1) is similar to that in [9], but the method of our proof in § 4 is based on the same idea as in [2] and [1]. Recently, G. Maruyama and H. Tanaka [12] treated the same problem by a different method. We learned from Prof. K. Ito that the same result can also be derived directly from the Brownian motion case through the time change which will be given in K. Ito and H. P. McKean Jr.'s forthcoming book. As to this paper, we have obtained many valuable suggestions from the lectures given by Prof. K. Ito at the University of Tokyo. We would like to express our hearty thanks to him and also to Mr. T. Watanabe for his kind discussions.

§ 2. Preliminaries.

Let $X(t, \omega)$ be the so-called Feller's diffusion process with canonical scale $s(x)$, canonical measure $m(x)$ and state space $R=[a, b]=\{x; a\leqq x\leqq b\}$, where a, b are real numbers or $\pm\infty$ (We shall fix the values a and b in this paper.

cf. [6], [7]). We shall use the notation $X^{(x)}(t, \omega), x \in R$ to express the path starting from a real point x at $t=0$. For the space Ω_x of all such paths, a probability measure $P_x\{\ \}$ with $P_x\{\Omega_x\}=1$ is defined. If a non-negative random variable $\tau(\omega)$ defined on Ω_x satisfies the condition: $\{\omega \mid \tau(\omega) \leqq t\} \in B_t^{(x)}$ for any t, where $B_t^{(x)}$ is the Borel field generated by the system of all ω-sets of the form $\{\omega \in \Omega_x \mid X^{(x)}(t, \omega) \in E\}$ $(0 \leqq t \leqq s, E \in \mathfrak{L}_R)$, and $\mathfrak{L}_R$ is the Borel field generated by the intervals in R, then $\tau(\omega)$ is called a Markov time for $X^{(x)}(\cdot, \omega)$ (cf. [4], [5], [13]). As D. Ray [13] proved, Feller's diffusion process has strong Markov property and its path functions $X^{(x)}(t, \omega)$ are continuous in t, for almost all $\omega \in \Omega_x$ with respect to the measure $P_x\{\ \}$ (cf. [4], [5], [6], [7] and [13]).

Now let $\tau_{(x_1, x_2)}(x, \omega) = \min\{t\,;\, X^{(x)}(t, \omega) \in (x_1, x_2)^c\}$ for $a \leqq x_1 < x < x_2 \leqq b$. This is a Markov time and is $B_\tau^{(x)}$-measurable (cf. [4] or [13]).

We define: $$P(x, x_1, x_2) = P_x\{X^{(x)}(\tau_{(x_1, x_2)}(x, \omega), \omega) = x_1\}\,,$$
$$P(x, x_1, x_2) = P_x\{X^{(x)}(\tau_{(x_1, x_2)}(x, \omega), \omega) = x_2\}\,.$$

Then, as E. B. Dynkin [4] proved, we have

$$P(x, x_1, x_2) = \frac{s(x_2) - s(x)}{s(x_2) - s(x_1)} \qquad \text{for any } a < x_1 < x < x_2 < b\,.$$

§ 3. Fundamental theorem.

From now on we assume that $s(x) \equiv x$. Let $x, y, z \in (a, b), x < y < z$ and consider the Markov time $\tau_{(x,z)}(y, \omega)$. Then we have

$$E\{\tau_{(x,z)}(y, \omega)\} < \infty \tag{3.1}$$

(see [4]).

Now, let $\chi_M, M \in \mathfrak{L}_R$, be the indicator of a set M, i.e.

$$\chi_M(x) = \begin{cases} 1, & x \in M, \\ 0, & x \notin M. \end{cases}$$

Then $\chi_M(X^{(x)}(t, \omega))$ is a function measurable in t. Therefore, the following functions are well-defined;

$$\begin{aligned} \bar{T}(\omega\,;\, y \uparrow (x, z), M) &= \int_0^{\tau_{(x,z)}(y,\omega)} \chi_M(X^{(y)}(t, \omega))dt && \text{if } X^{(y)}(\tau_{(x,z)}(y, \omega)) = z\,, \\ &= 0 && \text{if } X^{(y)}(\tau_{(x,z)}(y, \omega)) = x\,, \end{aligned}$$

and

$$\begin{aligned} \bar{T}(\omega\,;\, y \downarrow (x, z), M) &= 0 && \text{if } X^{(y)}(\tau_{(x,z)}(y, \omega)) = z\,, \\ &= \int_0^{\tau_{(x,z)}(y,\omega)} \chi_M(X^{(y)}(t, \omega))dt && \text{if } X^{(y)}(\tau_{(x,z)}(y, \omega)) = x\,. \end{aligned}$$

We have easily the following lemma.

LEMMA 3.1. *For any fixed* $x, y, z, \bar{T}(\omega; y\uparrow(x,z), M)$ *and* $\bar{T}(\omega; y\downarrow(x,z), M)$ *are completely additive measure functions with respect to M.*

Furthermore, let

$$T(M; y\uparrow(x,z))=E\{\bar{T}(\omega; y\uparrow(x,z), M)\,|\,X^{(y)}(\tau_{(x,z)}(y,\omega))=z\}\,,$$

$$T(M; y\downarrow(x,z))=E\{\bar{T}(\omega; y\downarrow(x,z), M)\,|\,X^{(y)}(\tau_{(x,z)}(y,\omega))=x\}\,.$$

Then we have the next lemma.

LEMMA 3.2. *For any fixed* $x<y<z$

(1) $T(M; y\uparrow(x,z))$ *and* $T(M; y\downarrow(x,z))$ *are bounded completely additive measure functions with respect to M for fixed* x, y, z.

(2) $T(M; y\uparrow(x,z))=0$ *if* $M\cap(x,z)=\phi$,
$T(M; y\downarrow(x,z))=0$ *if* $M\cap(x,z)=\phi$.

PROOF. (2) is evident by the definition. Therefore, we have only to prove (1). It follows easily from Lemma 3.1. that the above T's are finitely additive measure functions with respect to M. Hence, we will show that they are bounded and completely additive. By definition,

$$\bar{T}(\omega; y\uparrow(x,z), M)\leqq\bar{T}(\omega; y\uparrow(x,z), (x,z))<\tau_{(x,z)}(y,\omega)\,.$$

But, by (3.1),

$$E\{\tau_{(x,z)}(y,\omega)\,|\,X^{(y)}(\tau_{(x,z)}(y,\omega))=x\}\frac{z-y}{z-x}$$
$$+E\{\tau_{(x,z)}(y,\omega)\,|\,X^{(y)}(\tau_{(x,z)}(y,\omega))=z\}\frac{y-x}{z-x}=E\{\tau_{(x,z)}(y,\omega)\}<+\infty\,.$$

Therefore,

$$E\{\bar{T}(\omega; y\uparrow(x,z), M)\,|\,X^{(y)}(\tau_{(x,z)}(y,\omega))=x\}$$
$$\leqq E\{\tau_{(x,z)}(y,\omega)\,|\,X^{(y)}(\tau_{(x,z)}(y,\omega))=x\}$$
$$\leqq\frac{z-x}{z-y}E\{\tau_{(x,z)}(y,\omega)\}<+\infty\,.$$

Since the right hand term is independent of M, T's are bounded. Moreover, since $\bar{T}$'s are dominated by the integrable function $\tau_{(x,z)}(y,\omega)$, the complete additivities of T's follow from those of $\bar{T}$'s.

LEMMA 3.3. *For any* $x, y, z\in[A,B](\subset(a,b))$, *where* $x<y<z$, $T(\cdot\,; y\uparrow(x,z))$ *and* $T(\cdot\,; y\downarrow(x,z))$ *are absolutely continuous with respect to some fixed measure* $\mu=\mu_{[A,B]}(\cdot)$.

PROOF. Take a real number $C\in[A,B]$ arbitrarily and fix it.

Then, we have

$$T(M; C\uparrow(A,B))\geqq T(M; y\uparrow(A,B))\qquad\text{for } y\geqq C$$

and

$$T(M; C\uparrow(A,B))\geqq\frac{B-C}{B-y}\,\frac{y-A}{B-A}\,T(M; y\uparrow(A,B))\quad\text{for } y<C.$$

Since it is clear that

$$T(M;\, y\uparrow(A,B))\geqq T(M;\, y\uparrow(x,z)),$$

$T(\cdot\,;\, y\uparrow(x,z))$ is absolutely continuous with respect to $T(\cdot\,;\, C\uparrow(A,B))$. $T(\cdot\,;\, y\downarrow(x,z))$ is also absolutely continuous with respect to $T(\cdot\,;\, C\downarrow(A,B))$. Therefore, taking

$$\mu(\cdot)=T(\cdot\,;\, C\uparrow(A,B))+T(\cdot\,;\, C\downarrow(A,B)),$$

we can see that our lemma is true. Hence, we can express as follows;

$$T(M;\, y\uparrow(x,z))=\int_M \tau(u;\, y\uparrow(x,z))\mu(du), \tag{3.2}$$

$$T(M;\, y\downarrow(x,z))=\int_M \tau(u;\, y\downarrow(x,z))\mu(du) \tag{3.3}$$

where τ's are density functions of T's with respect to μ. Now, we will prove the following theorem.

THEOREM 3.1. *There exists a measure $n(\cdot)$ on (a,b) such that*

$$\begin{aligned} T(M;\, y\uparrow(x,z)) &= \frac{(z-y)}{(z-x)(y-x)}\int_M (u-x)^2 n(du) \quad \textit{if } M\subset(x,y] \\ &= \frac{1}{(z-x)}\int_M (z-u)(u-x)n(du) \quad \textit{if } M\subset[y,z), \end{aligned} \tag{3.4}$$

$$\begin{aligned} T(M;\, y\downarrow(x,z)) &= \frac{1}{(z-x)}\int_M (z-u)(u-x)n(du) \quad \textit{if } M\subset(x,y], \\ &= \frac{(y-x)}{(z-x)(z-y)}\int_M (z-u)^2 n(du) \quad \textit{if } M\subset[y,z). \end{aligned} \tag{3.5}$$

Furthermore, this measure $n(\cdot)$ is uniquely determined.

PROOF. The uniqueness is evident since the integrands in (3.4) and (3.5) are not zero in (x,z). By the same reason, it is sufficient to prove that, for any two sets of numbers (x,y,z), (x',y',z'), we can contruct a common $n(\cdot)$ satisfying (3.4), and (3.5) in the common domain of definition. Actually we construct a $n(\cdot)$ for any countable dense set in any interval $[A,B]\subset(a,b)$.

By the strong Markov property, we obtain the following relations for $\delta<\gamma<\beta<\alpha$:

$$\begin{aligned} &\frac{\beta-\delta}{\alpha-\delta}T(M;\, \beta\uparrow(\delta,\alpha)) \\ &=\frac{\beta-\gamma}{\alpha-\gamma}T(M;\, \beta\uparrow(\gamma,\alpha))+\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\alpha-\delta)}\{T(M;\, \beta\downarrow(\gamma,\alpha)) \\ &\qquad +T(M;\, \gamma\uparrow(\delta,\alpha))\}, \end{aligned} \tag{3.6}$$

$$T(M;\, \gamma\uparrow(\delta,\alpha))=T(M;\, \gamma\uparrow(\delta,\beta))+T(M;\, \beta\uparrow(\delta,\alpha)). \tag{3.7}$$

For $T(M;\, \gamma\uparrow(\delta,\alpha))$ and $T(M;\, \beta\downarrow\delta,\alpha))$ we have the following dual formulas

(3.6)*, (3.7)*.

$$\frac{\alpha-\gamma}{\alpha-\delta}T(M;\gamma\downarrow(\delta,\alpha))\tag{3.6)*}$$
$$=\frac{\beta-\gamma}{\beta-\delta}T(M;\gamma\uparrow(\delta,\beta))+\frac{(\gamma-\delta)(\alpha-\beta)}{(\beta-\delta)(\alpha-\delta)}\{T(M:\gamma\uparrow(\delta,\beta))$$
$$+T(M:\beta\downarrow\delta,\alpha))\}.$$

$$T(M;\beta\downarrow(\delta,\alpha))=T(M;\beta\downarrow\gamma,\alpha))+T(M;\gamma\downarrow(\delta,\alpha)).\tag{3.7)*}$$

From (3.2), (3.3), (3.6) and (3.7) follows

$$\frac{\beta-\delta}{\alpha-\delta}\tau(u;\beta\uparrow(\delta,\alpha))\tag{3.8}$$
$$=\frac{\beta-\gamma}{\alpha-\gamma}\tau(u;\beta\uparrow(\gamma,\alpha))+\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\alpha-\delta)}\{\tau(u;\beta\downarrow(\gamma,\alpha))$$
$$+\tau(u;\gamma\uparrow(\delta,\alpha))\}$$

for almost all u such that $\gamma\leqq u\leqq\alpha$,

$$\frac{\beta-\delta}{\alpha-\delta}\tau(u;\beta\uparrow(\delta,\alpha))=\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\alpha-\delta)}\tau(u;\gamma\uparrow(\delta,\alpha))\tag{3.9}$$

for almost all u such that $\delta\leqq u\leqq\gamma$,

$$\tau(u;\gamma\uparrow(\delta,\alpha))=\tau(u;\beta\uparrow(\delta,\alpha))\tag{3.10}$$

for almost all u such that $\beta\leqq u\leqq\alpha$,

$$\tau(u;\gamma\uparrow(\delta,\alpha))=\tau(u;\gamma\uparrow(\delta,\beta))+\tau(u;\beta\uparrow(\delta,\alpha))\tag{3.11}$$

for almost all u such that $\delta\leqq u\leqq\beta$.

As before, we have dual formulas (3.8)*, (3.9)*, (3.10)* and (3.11)*.

Without loss of generality, we may suppose that the above results hold for any α,β,γ and $\delta\in S$. Since $T(M;y\uparrow(x,z))$ and $T(M;y\downarrow(x,z))$ are non-negative functions for any $M\in\mathfrak{L}_R$, $\tau(u;y\uparrow(x,z))$ and $\tau(u;y\downarrow(x,z))$ are also non-negative for almost all u and we set $\tau=0$ for exceptional values of u.

By means of (3.10) and (3.10)* we see that

$$\tau(u;\gamma\uparrow(\delta,\alpha))=\sigma(u;(\delta,\alpha))\text{ (independent of }\gamma),\quad\text{if }\gamma\leqq u\leqq\alpha,\tag{3.12}$$

$$\tau(u;\beta\downarrow(\delta,\alpha))=\bar{\sigma}(u;(\delta,\alpha))\qquad\text{if }\delta\leqq u\leqq\beta.\tag{3.12)*}$$

Furthermore, by (3.9)

$$\tau(u;\beta\uparrow(\delta,\alpha))=\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\delta)}\tau(u;\gamma\uparrow(\delta,\alpha))$$

if $\delta\leqq u\leqq\gamma$.

Hence, as $\gamma\rightarrow u+0$, we obtain

$$(3.13)\qquad \tau(u\,;\beta\uparrow(\delta,\alpha))=\frac{(\alpha-\beta)(u-\delta)}{(\alpha-u)(\beta-\delta)}\lim_{\gamma\downarrow u}\tau(u\,;\gamma\uparrow(\delta,\alpha))\quad\text{if}\quad\delta\leqq u<\beta\,,$$

$$=\frac{(\alpha-\beta)(u-\delta)}{(\alpha-u)(\beta-\delta)}\rho'(u\,;(\delta,\alpha))$$

and especially by (3.10),

$$\tau(\beta\,;\beta\uparrow(\delta,\alpha))=\lim_{\beta'\uparrow\beta}\tau(\beta\,;\beta'\uparrow(\delta,\alpha))\,.$$

Hence (3.13) holds for $\delta\leqq u\leqq\beta$.

Furthermore by (3.11)

$$\frac{(\alpha-\gamma)(u-\delta)}{(\alpha-u)(\gamma-\delta)}\rho'(u\,;(\delta,\alpha))=\frac{(\beta-\gamma)(u-\delta)}{(\beta-u)(\gamma-\delta)}\rho'(u\,;(\delta,\beta))$$

$$+\frac{(\alpha-\beta)(u-\delta)}{(\alpha-u)(\beta-\delta)}\rho'(u\,;(\delta,\alpha))\quad\text{if}\quad\delta\leqq u\leqq\gamma\,.$$

Accordingly

$$\frac{(\alpha-\delta)}{(\alpha-u)(\beta-\delta)}\rho'(u\,;(\delta,\alpha))=\frac{1}{(\beta-u)}\rho'(u\,;(\delta,\beta))\qquad\text{if}\quad\delta\leqq u\leqq\gamma\,.$$

Hence,

$$\rho'(u\,;(\delta,\alpha))=\frac{(\alpha-u)(\beta-\delta)}{(\alpha-\delta)(\beta-u)}\rho'(u\,;(\delta,\beta))\qquad\text{if}\quad\delta\leqq u<\beta\,.$$

Therefore, there exists

$$\lim_{\beta\downarrow u}\frac{\rho'(u\,;(\delta,\beta))}{(\beta-u)}=\rho(u,\delta)\,,$$

which is defined for $u>\delta$.

Thus, we obtain

$$\rho'(u\,;(\delta,\alpha))=\frac{(\alpha-u)(u-\delta)}{(\alpha-\delta)}\rho(u,\delta)\qquad\text{if}\quad\delta\leqq u<\alpha\,.$$

By means of (3.13)

$$(3.14)\qquad \tau(u\,;\beta\uparrow(\delta,\alpha))=\frac{(\alpha-\beta)(u-\delta)^2}{(\alpha-\delta)(\beta-\delta)}\rho(u,\delta)\qquad\text{if}\quad\delta\leqq u\leqq\beta\,.$$

And similarly

$$(3.14)^*\qquad \tau(u\,;\gamma\downarrow(\delta,\alpha))=\frac{(\gamma-\delta)(\alpha-u)^2}{(\alpha-\delta)(\alpha-\gamma)}\bar{\rho}(u\,;\alpha)\qquad\text{if}\quad\gamma\leqq u\leqq\alpha\,.$$

By using (3.11) on $\gamma\leqq u\leqq\beta$

$$(3.15)\qquad \sigma(u\,;(\delta,\alpha))=\sigma(u\,;(\delta,\beta))+\frac{(\alpha-\beta)(u-\delta)^2}{(\alpha-\delta)(\beta-\delta)}\rho(u,\delta)\qquad\text{if}\quad\delta<u\leqq\beta\,.$$

Similarly

$$(3.15)^*\qquad \bar{\sigma}(u\,;(\delta,\alpha))=\bar{\sigma}(u\,;(\gamma,\alpha))+\frac{(\gamma-\delta)(\alpha-u)^2}{(\alpha-\delta)(\alpha-\gamma)}\bar{\rho}(u,\alpha)\qquad\text{if}\quad\gamma\leqq u<\alpha\,.$$

From this, we can conclude that there exist $\lim_{\beta\downarrow u}\sigma(u;(\delta,\beta))$ and $\lim_{\gamma\uparrow u}\bar{\sigma}(u;(\gamma,\alpha))$.

By making use of (3.8) on $\beta\leqq u\leqq\alpha$,

$$\frac{(\beta-\delta)}{(\alpha-\delta)}\sigma(u;(\delta,\alpha))=\frac{(\beta-\gamma)}{(\alpha-\gamma)}\sigma(u;(\gamma,\alpha))+\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\alpha-\delta)}\times\left\{\frac{(\beta-\gamma)(\alpha-u)^2}{(\alpha-\gamma)(\alpha-\beta)}\rho(u,\alpha)+\sigma(u;(\delta,\alpha))\right\}.$$

Therefore,

(3.16) $$\sigma(u;(\delta,\alpha))=\sigma(u;(\gamma,\alpha))+\frac{(\gamma-\delta)(\alpha-u)^2}{(\alpha-\gamma)(\alpha-\delta)}\bar{\rho}(u,\alpha)\qquad\text{for }\beta\leqq u\leqq\alpha.$$

Hence, it holds for $\gamma<u\leqq\alpha$. Similarly

(3.16)* $$\bar{\sigma}(u;(\delta,\alpha))=\bar{\sigma}(u;(\delta,\beta))+\frac{(\alpha-\beta)(u-\delta)^2}{(\alpha-\delta)(\beta-\delta)}\rho(u,\delta)\qquad\text{for }\delta\leqq u<\beta.$$

Next, making use of (3.8) on $\gamma\leqq u\leqq\beta$,

$$\frac{(\alpha-\beta)(u-\delta)^2}{(\alpha-\delta)^2}\rho(u,\delta)\geqq\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\alpha-\delta)}\sigma(u;(\delta,\alpha))\qquad\text{for }\gamma\leqq u\leqq\beta,$$

and

$$\frac{(u-\delta)^2}{(\alpha-\delta)^2}\rho(u,\delta)\geqq\frac{\sigma(u;(\delta,\alpha))}{(\alpha-\gamma)(\alpha-\delta)}(\gamma-\delta)\qquad\text{for }\gamma\leqq u<\alpha.$$

As $\alpha\to u+0$, we have

$$\rho(u,\delta)\geqq\frac{(\gamma-\delta)\lim_{\alpha\downarrow u}\sigma(u;(\delta,\alpha))}{(u-\gamma)(u-\delta)}\qquad\text{for }\gamma<u.$$

Furthermore, as $\gamma\to u-0$,

$$\lim_{\alpha\downarrow u}\sigma(u;(\delta,\alpha))=0.$$

And similarly

$$\lim_{\alpha\uparrow u}\sigma(u;(\delta,\alpha))=0.$$

By (3.15), as $\beta\to u+0$,

(3.17) $$\sigma(u;(\delta,\alpha))=\frac{(\alpha-u)(u-\delta)}{(\alpha-\delta)}\rho(u;\delta)\qquad\text{for }\delta<u<\alpha,$$

and

(3.17)* $$\bar{\sigma}(u;(\delta,\alpha))=\frac{(\alpha-u)(u-\delta)}{(\alpha-\delta)}\bar{\rho}(u;\alpha)\qquad\text{for }\delta<u<\alpha.$$

From (3.16) and (3.17)* follows

(3.18) $$\frac{(\alpha-u)(u-\delta)}{(\alpha-\delta)}\rho(u;\delta)=\frac{(\alpha-u)(u-\gamma)}{(\alpha-\gamma)}\rho(u;\gamma)+\frac{(\gamma-\delta)(\alpha-u)^2}{(\alpha-\gamma)(\alpha-\delta)}\bar{\rho}(u;\alpha)\qquad\text{for }\gamma<u\leqq\alpha.$$

Since $\alpha-u>0$, $\gamma-\delta>0$, $\alpha-\gamma>0$ and $\alpha-\delta>0$,

$$\frac{(u-\delta)}{(\alpha-\delta)}\rho(u;\delta)\geqq\frac{(u-\gamma)}{(\alpha-\gamma)}\rho(u;\gamma) \qquad \text{for } \gamma<u<\alpha .$$

As $\alpha\to u+0$ and $\gamma\to u-0$, we can see that

$$\overline{\lim_{\gamma\to u}}\,\rho(u;\gamma\leqq\rho(u,\delta))<+\infty$$

When $\gamma\to u+0$ in (3.17), we have

$$\rho(u;\delta)=\bar{\rho}(\alpha;u)=\varphi(u) \qquad \text{for } \alpha<u<\delta . \tag{3.18}$$

(3.12), (3.16) and (3.18) give us

$$\tau(u;\gamma\uparrow(\delta,\alpha))=\frac{(\alpha-u)(u-\delta)}{(\alpha-\delta)}\varphi(u) \qquad \text{for } \gamma\leqq u<\alpha .$$

And by means of (3.11) and (3.15), we have

$$\tau(u;\gamma\uparrow(\delta,\alpha))=\frac{(\alpha-\gamma)(u-\delta)^2}{(\alpha-\delta)(\gamma-\delta)}\varphi(u) \qquad \text{for } \delta<u\leqq\gamma .$$

Hence, we have only to take $\varphi(u)\mu(du)$ as $n(du)$. This completes the proof.

Now we shall prove that our measure $n(\cdot)$ coincides with the canonical measure. Namely, in a Feller process, the canonical measure is characterized as follows; In any finite interval $[x,z]$, we set

$$p(y)=E\{\tau_{(x,z)}(y,\omega)\} \qquad \text{where } x<y<z .$$

Then

$$\frac{d^+p(y)}{dy}=\lim_{y_1\to y+0}\frac{p(y_1)-p(y)}{y_1-y}$$

exists and $\frac{d^+p(y)}{dy}$ is monotone and continuous from the right. Then the measure

$$m(y_1,y_2)=\frac{d^+p(y_2)}{dy}-\frac{d^+p(y_1)}{dy}$$

is the canonical measure. Now, we will prove $n(\cdot)=m(\cdot)$.

$$\begin{aligned}E\{\tau_{(x,z)}(y;\omega)\}&=\frac{y-x}{z-x}T((x,z);y\uparrow(x,z))+\frac{z-y}{z-x}T((z,x);y\downarrow(x,z))\\&=\frac{(z-y)}{(z-x)^2}\int_{(x,y]}(u-x)^2n(du)+\frac{(y-x)}{(z-x)^2}\int_{(y,z]}(z-u)(u-x)n(du)\\&\quad+\frac{(z-y)}{(z-x)^2}\int_{(x,y]}(u-x)(z-u)n(du)+\frac{(y-x)}{(z-x)^2}\int_{(y,z]}(z-u)^2n(du)\\&=\frac{(z-y)}{(z-x)}\int_{(x,y]}(u-x)n(du)+\frac{(y-x)}{(z-x)}\int_{(y,z]}(z-u)n(du)\end{aligned}$$

$$=\frac{(z-y)}{(z-x)}[(u-x)n(x,u)]_x^y-\frac{(y-x)}{(z-x)}[(z-u)n(x,u)]_y^z$$

$$+\frac{(z-y)}{(z-x)}\int_{(x,y]}n(x,u]du-\frac{(y-x)}{(z-x)}\int_{(y,z]}n(x,u)du\,,$$

$$\frac{d^+p(y)}{dy}=\frac{-1}{(z-x)}\int_{(x,y]}n(x,u]du+\frac{(z-y)}{(z-x)}n(x,y]$$

$$-\frac{1}{(z-x)}\int_{(y,z]}n(x,u]du+\frac{(y-x)}{(z-x)}n(x,y]$$

$$=n(x,y)-\frac{1}{(z-x)}\int_{(x,z]}n(x,u)du\,.$$

Thereore, we have

$$m(y_1,y_2)=\frac{d^+p(y_2)}{dy}-\frac{d^+p(y_1)}{dy}=n(y_1,y_2)\,.$$

§ 4. Ergodic Properties.

We shall define the recurrent Markov process as follows.

DEFINITION. The Markov process is called recurrent, if

$$P_x\{X^{(x)}(t)=y \quad \text{for some } t\}=1$$

for any $x, y\in(a,b)$.

Throughout this section we shall add the assumption of recurrence to A1—A4: that is; A5. Our process is the recurrent one. Then the following lemmais evident.

LEMMA 4.1. *The boundary a (b) is reached in finite time from any point x in (a,b) with positive probability, if and only if a (b) is finite. And in this case*

$$P_x\{X^{(x)}(t)=a\ (b)\ \textit{for some}\ t\}=1.$$

LEMMA 4.2. *If a (b) is finite*

$$E(\tau_{[a,x)}(y))<\infty \quad (E(\tau_{(x,b]}(y))<\infty)$$

where

$$\tau_{[a,x)}(y,\omega)=\inf\{t: X^{(y)}(t,\omega)\notin[a,x)\}$$

$$(\tau_{(x,b]}(y,\omega)=\inf\{t: X^{(y)}(t,\omega)\notin(x,b]\})\,.$$

PROOF. By the Lemma 2 in [4], it is sufficient to prove that $P_x\{\tau_{[a,x)}(y)<T\}>\varepsilon>0$ for any $y\in(a,x)$ where ε and T are some positive number independent of y. As our process is recurrent, we shall easily see $P_x\{\tau_{[a,x)}(a)<T\}>\varepsilon$ for some ε and T. But,

$$P_x\{\tau_{[a,x)}(a)<T\}=\int P_x\{\tau_{[a,x)}(a)-\tau_{[a,y)}(a)<T-s\mid\tau_{[a,y)}(a)=s\}dP_s\{\tau_{[a,x)}(a)=s\}$$

$$=\int P_x\{\tau_{[a,x)}(y)<T-s\}dP_s\{\tau_{[a,x)}(a)=s\}$$

$$\leqq \int P_x\{\tau_{[a,x)}(y)<T\}dP_s\{\quad\}$$

$$\leqq P_x\{\tau_{[a,x)}(y)<T\}$$

which proves the lemma.

LEMMA 4.3. *If a (b) is finite, then*

$$\lim_{x\downarrow a}\frac{E\{\tau_{[a,x)}(a)\}}{x-a}=\sigma_1$$

$$\left(\lim_{x\uparrow b}\frac{E\{\tau_{(x,b]}(a)\}}{b-x}=\sigma_2\right)$$

exists and is finite, and $m(a,x)$ $(m(x,b))$ is finite for any $x\in(a,b)$. And extending the m-measure in (a,b) to $[a,b]$ such that

$$\begin{aligned} m(a)&=0 &&\text{if } a=-\infty,\\ &=\sigma_1 &&\text{if } a>-\infty,\\ m(b)&=0 &&\text{if } b=\infty,\\ &=\sigma_2 &&\text{if } b<\infty,\end{aligned}$$

for any $x\leqq y$, $(x,y\in(a,b))$ we have

$$E\Big\{\int_0^{\tau^*}\chi_A(X^{(x)}(t))dt\Big\}=(y-x)m(A)$$

where

$$\tau^*=\inf\{t:X^{(x)}(t)=x,\quad t\geqq\tau_{[a,y)}(x)\}$$

and A is any Borel set in $[a,b]$. (If a or b is infinite, $m(A)$ may be infinity.)

PROOF.

$$E\Big\{\int_0^{\tau^*}\chi_A(X^{(x)}(t))dt\Big\}=E\Big\{\int_0^{\tau_{[a,y)}(x)}\chi_A(X^{(x)}(t))dt\Big\}+E\Big\{\int_{\tau_{[a,y)}(x)}^{\tau^*}\chi_A(X^{(x)}(t))dt\Big\}. \tag{4.1}$$

Let u be any number in (a,x) and

$\tau_0'=\inf\{t:X^{(x)}(t)\notin(u,y)\}$,

$\tau_1=\inf\{t:X^{(x)}(t)\notin(a,y),\ t>\tau_0'\}$ if $X^{(x)}(\tau_0')=u$,

$\tau_1'=\inf\{t:X^{(x)}(t)=u,\ t>\tau_1\}$ if $X^{(x)}(\tau_0')=u$ $X^{(x)}(\tau_1)=a$,

....,

$\tau_k=\inf\{t:X^{(x)}(t)\notin(a,y),\ t>\tau_{k-1}'\}$ if $\begin{cases}X^{(x)}(\tau_0')=u,\\ X^{(x)}(\tau_i)=a,\ i=1,2,\cdots,k-1,\end{cases}$

$\tau_k'=\inf\{t:X^{(x)}(t)=u,\ t>\tau_k\}$ if $\begin{cases}X^{(x)}(\tau_0')=u,\\ X^{(x)}(\tau_i)=a,\ i=1,2,\cdots,k,\end{cases}$

....

Concerning the first term in the right side of (4.1), we have

$$E\Big\{\int_0^{\tau[a,y)^{(x)}}\chi_A(X^{(x)}(t))dt\Big\}$$

$$=E\Big\{\int_0^{\tau_0'}\Big\}+\sum_{k=1}^{\infty}\Big[E\Big\{\int_{\tau_{k-1}'}^{\tau_k}\chi_A(X^{(x)}(t))dt\,|\,X^{(x)}(\tau_0')=u\ \ X^{(x)}(\tau_i)=a\ \ i=1,\cdots,k-1\Big\}$$

$$+\Big\{\int_{\tau_k}^{\tau_k'}\chi_A(X^{(x)}(t))dt\,|\,X^{(x)}(\tau_0')=u\ \ X^{(x)}(\tau_i)=a\ \ i=1,2,\cdots,k\Big\}\Big]$$

$$\times P_r\{X^{(x)}(\tau_0')=u\ \ X^{(x)}(\tau_i)=a\ \ i=1,2,\cdots,k-1\}$$

If a is finite, by using the strong Markov property, we have

$$E\Big\{\int_0^{\tau[a,y)^{(x)}}\chi_A(X^{(x)}(t))dt\Big\}$$

$$=E\Big\{\int_0^{\tau(u,y)^{(x)}}\chi_A(X^{(x)}(t))dt\Big\}+\sum_{k=1}^{\infty}E\Big\{\int_0^{\tau(u,y)^{(u)}}\chi_A(X^{(u)}(t))dt$$

$$+\frac{y-u}{y-a}\int_0^{\tau[a,u)^{(a)}}\chi_A(X^{(a)}(t))dt\Big\}\Big(\frac{y-x}{y-u}\Big)\Big(\frac{y-u}{y-a}\Big)^{k-1}.$$

But

$$E\Big\{\int_0^{\tau(u,y)^{(x)}}\chi_A(X^{(x)}(t))dt\Big\}=\frac{x-u}{y-u}\,T(A\,;\,x\uparrow(u,y))+\frac{y-x}{y-u}\,T(A\,;\,x\downarrow(u,y))\,;$$

$$E\Big\{\int_0^{\tau(a,y)^{(u)}}\chi_A(X^{(x)}(t)dt\Big\}=\lim_{v\downarrow a}\Big(\frac{u-v}{y-v}\,T(A\,;\,u\uparrow(v,y)+\frac{y-u}{y-v}\,T(A\,;\,u\downarrow(v,y))\Big),$$

the Theorem 3.1 gives that

$$E\Big\{\int_0^{\tau[a,y)^{(x)}}\chi_A(X^{(x)}(t))dt\Big\}=\frac{y-x}{u-a}\int_{(a,u]\cap A}(v-a)m(dv)+(y-x)m((u,x]\cap A)$$

$$+\int_{(x,y]\cap A}(y-v)m(dv)+\frac{(y-x)(y-a)}{(y-u)(u-a)}\int_0^{\tau[a,u)^{(a)}}\chi_A(X^{(a)}(t))dt\,. \tag{4.2}$$

For the left side of (4.2) is bounded by $E\{\tau_{[a,y)}(x)\}$ (cf. Lemma 4.2), putting $A=(a,b]$, we can prove $m(a,x)$ is finite, and $\lim_{u\to\infty}\frac{\tau_{[a,u)}(a)}{u-a}=\sigma_1$ exists and is finite.

Therefore

$$E\Big\{\int_0^{\tau[a,y)^{(x)}}\chi_{A\cap(a,y)}(X^{(x)}(t))dt\Big\}=\lim_{u\downarrow a}E\Big\{\int_0^{\tau(a,y)^{(x)}}\chi_{A\cap(u,y)}(X^{(x)}(t))dt\Big\}$$

$$=(y-x)m((a,x]\cap A)+\int_{(x,y]\cap A}(y-v)m(dv)\,.$$

On the other hand

$$E\Big\{\int_0^{\tau[a,y)^{(x)}}\chi_{\{a\}}(X^{(x)}(t))dt\Big\}=\lim_{u\downarrow a}E\Big\{\int_0^{\tau[a,y)^{(x)}}\chi_{[a,u)}(X^{(x)}(t))dt\Big\}=\sigma_1=m(a)\,.$$

Finally, we have

$$(4.3)\qquad E\Big\{\int_0^{\tau_{[a,y)}(x)}\chi_A(X^{(x)}(t))dt\Big\}=(y-x)m([a,x]\cap A)+\int_{(x,y]\cap A}(y-v)m(dv)\,.$$

If a is infinite, then $P_x\{X^{(x)}(\tau_{(a,y)}(x))=a\}=0$. Therefore $\tau_{[a,y)}(x)=\lim_{u\downarrow a}\tau_{(u,y)}(x)$ with probability one.

Thus

$$(4.3)'\qquad E\Big\{\int_0^{\tau_{[a,y)}(x)}\chi_A(X^{(x)}(t))dt\Big\}=\lim_{u\downarrow a}\Big\{\frac{x-u}{y-u}T(A\,;x\uparrow(u,y))+\frac{u-x}{y-u}T(A\,;x\downarrow(u,y))\Big\}$$

$$=(y-x)m((a,x]\cap A)+\int_{(x,y]\cap A}(y-v)m(dv)$$

(which may be infinity if A is not a bounded set).

Concerning the 2nd term in the right side of (4.1), the strong Markov property gives

$$E\Big\{\int_{\tau_{[a,y)}(x)}^{\tau^*}\chi_A(X^{(x)}(t))dt\Big\}=E\Big[E\Big\{\int_{\tau_{[a,y)}(x)}^{\tau^*}\chi_A(X^{(x)}(t))dt\,|\,\tau_{[a,y)}(x)\Big\}\Big]$$

$$=E\Big\{\int_0^{\tau_{(x,b]}(y)}\chi_A(X^{(y)}(t))dt\Big\}\,.$$

And by the similar argument as above, we have

$$(4.4)\qquad E\Big\{\int_{\tau_{[a,y)}(x)}^{\tau^*}\chi_A(X^{(x)}(t))dt\Big\}=\int_{(x,b]\cap A}(v-x)m(dv)+m((y,b]\cap A)\,.$$

From (4.3) and (4.4), the lemma has proved.

REMARK. If a is finite for example, it can be shown that the corresponding boundary condition of the process at a is given by $\dfrac{f^+(a)}{Af(a)}=\sigma_1=m(a)$, where A is a infinitesimal operator of the process.

LEMMA 4.4. *If $f(x)$ is any m-summable and integrable function, then*

$$E\Big\{\int_0^{\tau^*(\omega)}f(X^{(x)}(t,\omega))dt\,|\,x\Big\}=(y-x)\int_a^b f(x)dm(x)\,.$$

PROOF. By the usual approximation procedure our lemma follows easily from Lemma 4.3.

THEOREM 4.1. *Under the assumption* A1—A5, *if $f(x)$ and $g(x)$ are any two real-valued m-summable functions and $\int_a^b g(x)dm(x)\neq 0$, then we have for almost all $\omega\in\Omega_x$.*

$$\lim_{T\to\infty}\frac{\int_0^T f(X^{(x)}(t,\omega))dt}{\int_0^T g(X^{(x)}(t,\omega))dt}=\frac{\int_a^b f(x)dm(x)}{\int_a^b g(x)dm(x)}$$

where m is extended canonical measure; i.e. $m(a)=\lim_{u\downarrow a}\frac{E(\tau_{[a,u)}(a))}{u-a}$ $m(b)=\lim_{u\uparrow b}\frac{E(\tau_{(u,b]})}{u-b}$.

PROOF. Firstly consider the case where $f(x)\geqq 0$ for all x.

Let

$$t_1(\omega)=\inf\{t\,;\,X^{(x)}(t,\omega)=y\},\quad t_2(\omega)=\inf\{t\,;\,t>t_1(\omega),\,X^{(x)}(t,\omega)=x\},$$
$$t_3(\omega)=\inf\{t\,;\,t>t_2(\omega),\,X^{(x)}(t,\omega)=y\},\cdots.$$

Since $\tau^*(\omega)$ is finite with probability 1, the sequence $\{t_i(\omega)\}$ is denumerable for almost all $\omega\in\Omega_x$.

Let

$$Y_i(\omega)=\int_{t_{2i-2}(\omega)}^{t_{2i}(\omega)} f(X^{(x)}(t,\omega))dt\,.$$

Then, the random variables $Y_i(\omega)$ are independent and have the same distribution function by the stationary strong Markov property of $X^{(x)}(t,\omega)$.

Now, let $n(T;\omega)$ be the largest even number such that $t_{n(T,\omega)}\leqq T$, then

$$\sum_{i=1}^{n(T,\omega)} Y_i\leqq\int_0^T f(X^{(x)}(t,\omega))dt\leqq\sum_{i=1}^{n(T,\omega)+1} Y_i\,.$$

Since the process $X^{(x)}(t,\omega)$ is recurrent, we have $P_x\{\lim_{n\to\infty} n(T,\omega)=\infty\}=1$.

Therefore, it follows from the strong law of large numbers of Kolmogoroff and Khintchine and from Lemma 4.4 that

$$P_x\Big\{\lim_{T\to\infty}\frac{\sum_{i=1}^{n(T,\omega)} Y_i}{n(T,\omega)}=(y-x)\int_a^b f(x)dm(u)\Big\}=1\,.$$

But since Y_i is finite with probability 1,

$$P_x\Big\{\lim_{T\to\infty}\frac{\int_0^T f(X^{(x)}(t,\omega))dt}{n(T,\omega)}=(y-x)\int_a^b f(x)dm(x)\Big\}=1\,. \tag{4.5}$$

Thus (4.5) holds for any m-summable function $f(x)$, since any function can be expressed by the difference of positive valued functions. The same arguments can be applied for $g(x)$.

As we can easily see,

$$E\{\tau^*(\omega)\}=(y-x)\int_a^b dm(x)\,.$$

Hence, $E\{\tau^*(\omega)\}$ is finite for any x,y if and only if $m([a,b])$ is finite. The case $E\{\tau^*(\omega)\}<\infty$ corresponds to positive states, and $E\{\tau^*(\omega)\}=\infty$ corresponds for null states in Markov chains with a denumerable number of states.

In particular, when $E\{\tau^*(\omega)\}<\infty$, we have the following theorem.

THEOREM 4.2. *If $f(x)$ is any m-summable function and $\int_a^b dm(x)<\infty$, then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X^{(x)}(t,\omega))dt=\frac{1}{m([a,b])}\int_a^b f(x)dm(x).$$

PROOF. We shall use the same notation as in the Theorem 4.1. We remark that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X^{(x)}(t,\omega))dt=\lim_{T\to\infty}\frac{n(T,\omega)}{T}\frac{1}{n(T,\omega)}\int_0^T f(X^{(x)}(t,\omega))dt. \tag{4.6}$$

But, since $T-t_{n(T,\omega)}$ is finite with probability 1, by the strong law of large numbers we have

$$\begin{aligned}\lim_{T\to\infty}\frac{T}{n(T,\omega)}&=\lim_{T\to\infty}\frac{t_2+(t_4-t_2)+\cdots+(t_{2n}-t_{2n-2})+\cdots+(T-t_{2n})}{n(T,\omega)}\\&=E\{\tau^*(\omega)\}=m([a,b])(y-x)\end{aligned} \tag{4.7}$$

with probability one.

Hence, from (4.6), (4.7), and Theorem 4.1 follows the theorem.

Now we note that

$$E\left\{\int_0^T f(X^{(x)}(t,\omega))dt\,|\,x\right\}=\int_0^T dt\int_a^b f(y)P(t,x,dy).$$

Then, putting $f(x)=\chi_E(x)$ by means of Theorem 4.2, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T P(t,x,E)dt=\frac{1}{m([a,b])}\int_E dm(x),$$

for any Borel measurable set E.

Let

$$F_T(x)=\frac{1}{T}\int_0^T P(t,y,[a,x))dt \quad\text{and}\quad F(x)=\frac{1}{m([a,b])}\int_a^x dm(x).$$

Then, $F_T(x), F(x)$ are monotone non-decreasing and bounded. $\lim_{T\to\infty}F_T(x)=F(x)$ holds at all continuity points of $F(x)$, and $F(a)-F(b)=1=\lim_{T\to\infty}[F_T(a)-F_T(b)]$. Therefore, since $P(t,x,E)$ is bounded,

$$\begin{aligned}\int_a^b P(t,x,E)dm(x)&=m([a,b])\int_a^b P(t,x,E)dF(x)\\&=\lim_{T\to\infty}m([a,b])\int_a^b P(t,x,E)dF_T(x)\\&=\lim_{T\to\infty}m([a,b])\int_a^b P(t,x,E)\frac{1}{T}\int_0^T P(s,y,dx)ds\\&=\lim_{T\to\infty}m([a,b])\frac{1}{T}\int_0^T P(s+t,y,E)ds\end{aligned} \tag{4.8}$$

$$=\lim_{T\to\infty} m([a, b]) \frac{1}{T}\left(\int_0^T P(t, y, E)dt - \int_0^t P(t, y, E)dt\right)$$
$$=m(E) .$$

In (4.8) we used the Chapman-Kolmogoroff's equation and Fubini's theorem. From equation (4.8), we have the following corollary.

COROLLARY. *$m(\cdot)$ is an invariant measure.*

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