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On exceptional values of entire function.

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1. Let $f(z)$ be an entire function of order $\rho(0<\rho<\infty)$. We say that α is an e. v. V,[1] (exceptional value in the sense of Valiron) if

$$\Delta(\alpha)=1-\liminf_{r\to\infty}\frac{N(r,\alpha)}{T(r,f)}>0 .$$

Let S denote the set of all increasing real functions $\psi(x)$ such that $\log x=o(\psi(x))$. We shall call α an e. v. S, if

$$\liminf_{r\to\infty}\frac{\log M(r,f)}{n(r,\alpha)\psi(r)}>0 \tag{1}$$

for some $\psi\subset S$.

We prove:

THEOREM 1. *If α is an e. v. S, then it is an e. v. V also with $\Delta(\alpha)=1$.*

PROOF. Let $\rho(r)$ be a Lindelöf proximate order for $f(z)$. Then

$$\lim_{r\to\infty}\rho(r)=\rho ,$$

$$\lim_{r\to\infty}(r\log r\rho'(r))=0 ,$$

$$\log M(r,f)\leqq r^{\rho(r)} \qquad \text{for all } r\geqq r_0$$

and $\log M(r,f)=r^{\rho(r)}$ for an infinity of r. Now

$$\rho(2r)-\rho(r)=\int_r^{2r}\rho'(t)dt=o\left(\int_r^{2r}\frac{dt}{t\log t}\right)=o\left(\frac{1}{\log r}\right)$$

and

$$\begin{aligned}\log M(2r,f)&\leqq(2r)^{\rho(2r)}<Ar^{\rho(2r)}\\&=Ae^{\{\rho(2r)-\rho(r)\}\log r}r^{\rho(r)}\\&<A_1r^{\rho(r)}\\&=A_1\log M(r,f) \qquad \text{for arbitrarily large } r\end{aligned}$$

and since

$$\log M(r,f)\leqq 3T(2r,f)$$

we get

1) See [1], p. 227

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)} < \infty .$$

Hence from (1)

$$\limsup_{r\to\infty} \frac{T(r,f)}{n(r,\alpha)\psi(r)} > 0 .$$

Further,

$$N(r,\alpha)=\int_{r_0}^{r} \frac{n(t,\alpha)}{t}\,dt + O(1) \leqq An(r,\alpha)\log r .$$

Thus

$$\limsup_{r\to\infty} \frac{T(r,f)}{N(r,\alpha)} = \infty$$

as

$$\log r = o(\psi(r)) .$$

Hence α is an e. v. V with $\Delta(\alpha)=1$.

2. Let E denote the set of increasing functions $\phi(x)$ such that

(2) $$\int_{A}^{\infty} \frac{dx}{x\phi(x)}$$

is convergent. We say that α is an e. v. E, if

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{n(r,\alpha)\phi(r)} > 0$$

for some $\phi \subset E$. See S. M. Shah [1].

We observe here that in section 1 for defining e. v. S we did not assume the convergence of (2), for instance $\psi(x)$ in section 1 can be equal to $\log x \log\log x$.

Let

$$m(r,f)=m(r)=\operatorname*{Min}_{|z|=r} |f(z)| .$$

We prove:

Theorem 2. *If $f(z)$ be an entire function having '0' as an e. v. E, then*

(3) $$\lim_{r\to\infty} m(r)=0 .$$

The above theorem explores a class of entire functions for which (3) holds. It covers a fortiori the classes having e. v. P (Picard) and e. v. B (Borel), because e. v. P or e. v. B $\Rightarrow$ e. v. E. The theorem is important in that (3) does not hold for all entire functions. There exist classes of entire functions for which

$$\limsup_{r\to\infty} m(r)=\infty .$$

As a matter of fact P. Erdös and A. J. Macintyre [2] have proved that if there be a strictly increasing sequence of integers λ_n satisfying

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1}-\lambda_n} < \infty ,$$

then for the entire function

$$f(z) = \sum_{0}^{\infty} a_n z^{\lambda_n}$$

we have

$$\limsup_{r\to\infty} \frac{m(r)}{M(r)} = 1 .$$

Further for all entire functions of order $\rho (0 \leqq \rho < \infty)$, there is the classical result of Littlewood namely

$$m(r) > \{M(r)\}^{c(\rho)-\varepsilon}$$

for arbitrarily large r, the constant $c(\rho)$ depending only on ρ. For $0<\rho<1$ the best possible value of $c(\rho)$ was discovered to be $\cos \pi\rho$ by Wiman and Valiron, (see Hayman [3]).

PROOF OF THEOREM 2. Suppose, if possible, that

$$\limsup_{r\to\infty} m(r) \geqq a > 0$$

and choose d such that $|d|<a$, then $|f(z)|>|d|$ on a sequence of circles, so by Rouché's theorem $f(z)$ and $f(z)-d$ will have the same number of zeros in $|z| \leqq r$, where r runs through a sequence. So

$$n(r, 0) \sim n(r, d)$$

for a sequence of circles.

Further since 0 is an e. v. E

$$\log M(r, f) \sim K r^{\rho} \quad (0<K<\infty) ,$$

$$N(r, 0) = o(T(r, f)) .$$

Hence

$$N(r, 0) = O(\log M(r, f)) = o(r^{\rho}) .$$

Also

$$N(r, 0) + N(r, d) > H(k) \log M\left(\frac{r}{k}\right) \qquad (k>1,\ d \neq 0) .$$

3. For an entire function $f(z)$, we call α to be an e.v.N (in the sense of Nevanlinna) if

$$\delta(\alpha)=1-\limsup_{r\to\infty}\frac{N(r,\alpha)}{T(r,f)}>0\,.$$

We prove:

THEOREM 3. *If $f(z)$ be an entire function having α as an e.v.N, then $m(r,f)=O(1)$.*

We see here that the above theorem gives an information less precise than the one given by theorem 2; but at the same time we know that e.v.N is a much stronger exceptional value than e.v.E, as e.v.E $\Rightarrow$ e.v.N, but not the converse (see [1]).

PROOF OF THEOREM 3. Let in the terminology of Nevanlinna

$$p\left(r,\ \frac{1}{f-\alpha}\right)=p(r,\alpha)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{f(re^{i\theta})-\alpha}\right|d\theta\,.$$

New, if possible, let $m(r,f)$ be not bounded, then there will exist a sequence of circles $|z|=r$ through which $m(r,f)\to\infty$. Hence $p(r,\alpha)=O(1)$ for a sequence of r. Now

$$p(r,\alpha)+N(r,\alpha)=T(r,f)+O(1)\,.$$

So

$$\limsup_{r\to\infty}\frac{N(r,\alpha)}{T(r,f)}=1\,.$$

Thus $\delta(\alpha)=0$ giving a contradiction against that α is an e.v.N. This completes the proof.

We observe finally that theorem 2 could have been proved, if we had assumed the asymptotic property of $f(z)$. But in the proof we have not made any appeal to its asymptotic behaviour.

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References

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[4] G. Valiron, Lectures on the General Theory of Integral Functions, Chel. Pub. Co., (1949).